Microlocal Analysis on Morrey Spaces

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1. Introduction

The spaces now called Morrey spaces were introduced by C. B. Morrey to study regularity properties of solutions to quasilinear elliptic PDE, but since then they have been useful in other areas of PDE. Before saying more on this, let us first define the Morrey spaces $M^p_q(\mathbb{R}^n)$.

If $1 \le q \le p < \infty$ and $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$, we say $f \in M^p_q(\mathbb{R}^n)$ provided

$$R^{-n} \int_{B_R} |f(x)|^q \, dx \le C R^{-nq/p}, \tag{1.1}$$

for all balls B_R of radius $R \le 1$ in $\mathbb{R}^n$. If we set $\delta_R f(x) = f(Rx)$, the left side of (1.1) is equal to $\int_{B_1} |\delta_R f(x)|^q dx$, so an equivalent condition is

$$\|\delta_R f\|_{L^q(B_1)} \le C' R^{-n/p}, \tag{1.2}$$

for all balls B_1 of radius 1, and for all $R \in (0, 1]$. It follows from Hölder's inequality that

$$L^p_{\mathrm{unif}}(\mathbb{R}^n) = M^p_p(\mathbb{R}^n) \subset M^p_q(\mathbb{R}^n) \subset M^p_1(\mathbb{R}^n).$$

We also say $f \in \mathcal{M}^p_q(\mathbb{R}^n)$ provided (1.1) holds for all $R < \infty$.

Morrey used these spaces to study inhomogeneous equations

$$\sum \partial_j a^{jk}(x) \partial_k u = f, \tag{1.3}$$

on a domain in $\mathbb{R}^n$, when $a^{jk}(x)$ are bounded and measurable and (1.3) is elliptic. Using a clever dilation argument and the DeGiorgi-Nash-Moser estimates on solutions to the homogeneous version of (1.3), Morrey was able to show that, if $p = n + \delta$, with small $\delta > 0$, and $f = \sum \partial_j g_j$, with $g_j \in L^p$, then $\nabla u \in M^p_2$. Hölder continuity of the solution u is then a consequence of Morrey's lemma:

$$\nabla u \in M^p_1(\mathbb{R}^n),\ p > n \Longrightarrow u \in C^r(\mathbb{R}^n), \quad r = 1 - \frac{n}{p}. \tag{1.4}$$

In fact, (1.4) is a special case of the following:

$$M^p_1(\mathbb{R}^n) \subset C^{-n/p}_*(\mathbb{R}^n). \tag{1.5}$$

Here, $C^r_*(\mathbb{R}^n)$ is a Zygmund space, which can be defined as follows. Pick $\Psi_0 \in C^\infty_0(\mathbb{R}^n)$, such that $\Psi_0(\xi) = 1$ for $|\xi| \le 1$, 0 for $|\xi| \ge 2$, set $\Psi_k(\xi) = \Psi_0(2^{-k}\xi)$, and then set $\psi_0 = \Psi_0$, $\psi_k = \Psi_k - \Psi_{k-1}$ for $k \ge 1$. The family $\{\psi_k\}$ is called a Littlewood-Paley partition of unity. For any $r \in \mathbb{R}$, one defines

$$C^r_*(\mathbb{R}^n) = \{u : \|\psi_k(D)u\|_{L^\infty} \le C 2^{-kr}\}. \tag{1.6}$$

It is not hard to show that, for $r \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, $C^r(\mathbb{R}^n) = C^r_*(\mathbb{R}^n)$. To see that (1.5) holds, one can check from the definition (1.1) that

$$
\begin{aligned}
f \in M^p_1(\mathbb{R}^n) &\Longleftrightarrow \big\| e^{t\Delta} |f| \big\|_{L^\infty} \le C t^{-n/2p} \\
&\Longrightarrow \| e^{t\Delta} f \|_{L^\infty} \le C t^{-n/2p},
\end{aligned}
\tag{1.7}
$$

for $t \in (0,1]$. From this one readily deduces that, if $u \in M^p_1(\mathbb{R}^n)$, then (1.6) holds, with $r = -n/p$.

In recent times, Morrey spaces have been incorporated into techniques of microlocal analysis, and it is our purpose to carry out this development further in this article.

In §2 we recall some known results about the action of pseudodifferential operators (ψDOs) on Morrey spaces. We define "Morrey scales," spaces $M^{p,s}_q(\mathbb{R}^n)$, for $s \in \mathbb{R}$, and make note of the consequent action of ψDOs on these spaces. We also extend to Morrey scales E. Stein's theorem on the action of ψDOs with symbols in $S^m_{1,1}$.

This is useful for applications of the paradifferential operator calculus of J.-M. Bony and Y. Meyer. We recall Meyer's formula for the action of a smooth function F on a function u (possibly taking values in $\mathbb{R}^\ell$). More details can be found in [Mey], or in [T1]. We have

$$
F(u) = M(u; x, D)u + F(u_0),
\tag{1.8}
$$

where $u_0 = \Psi_0(D)u$ and

$$
\begin{aligned}
M(u; x, \xi) &= \sum_{k \ge 0} m_k(x) \psi_{k+1}(\xi), \\
m_k(x) &= \int_0^1 F'(\Psi_k(D)u + \tau \psi_{k+1}(D)u)\, d\tau.
\end{aligned}
\tag{1.9}
$$

A straightforward calculation using the chain rule shows that

$$
u \in L^\infty(\mathbb{R}^n) \Longrightarrow M(u; x, \xi) \in S^0_{1,1}.
\tag{1.10}
$$

We recall that, for $0 \le \delta \le 1$, $m \in \mathbb{R}$,

$$
p(x,\xi) \in S^m_{1,\delta} \Longleftrightarrow |D^\beta_x D^\alpha_\xi p(x,\xi)| \le C_{\alpha\beta} \langle \xi \rangle^{m - |\alpha| + \delta|\beta|},
\tag{1.11}
$$

where $\langle \xi \rangle^2 = 1 + |\xi|^2$. If $p(x,\xi) \in S^m_{1,0}$ has an asymptotic expansion in terms homogeneous of degree $m - j$, $j \ge 0$, we say $p(x,\xi) \in S^m$, or sometimes, for emphasis, $p(x,\xi) \in S^m_{cl}$.

A further ingredient in paradifferential operator calculus is the process of "symbol smoothing." Given a symbol $M(x,\xi) \in S^m_{1,1}$, write

$$
M(x,\xi) = M^\#(x,\xi) + M^b(x,\xi),
\tag{1.12}
$$

where, with $J_k = \Psi_0(\varepsilon_k D_x)$, and $\varepsilon_k \searrow 0$,

$$
M^\#(x,\xi) = \sum_{k \ge 0} J_k M(x,\xi)\, \psi_{k+1}(\xi).
\tag{1.13}
$$

Choices most frequently made are

$$J_k = \Psi_0(2^{-\delta k} D_x), \quad \text{or} \quad J_k = \Psi_{k-3}(D_x). \tag{1.14}$$

In these respective cases, one gets

$$M^{\#}(x,\xi) \in S^m_{1,\delta}, \quad \text{or} \quad M^{\#}(x,\xi) \in \mathcal{B}S^m_{1,1}, \tag{1.15}$$

where

$$\mathcal{B}S^m_{1,1} = \{p(x,\xi) \in S^m_{1,1} : \hat{p}(\eta,\xi) \text{ is supported in } |\eta| \le \rho|\xi|\}, \tag{1.16}$$

for some $\rho < 1$. For fixed $\rho < 1$, the class (1.16) will be denoted $\mathcal{B}_\rho S^m_{1,1}$. One can show that, if $M(x,\xi) = M(u;x,\xi)$ is given by (1.9), then, for $r > 0$,

$$u \in C^r \Longrightarrow M^b(x,\xi) \in S^{-r\delta}_{1,\delta}, \quad \text{or} \quad \mathcal{B}S^{-r}_{1,1}, \tag{1.17}$$

in the two respective cases of (1.14). Thus, the action of ψDOs with symbols in these various classes are significant for nonlinear analysis. For example, we extend to Morrey scales Moser estimates on nonlinear functions $F(u)$, and also Rauch's lemma.

In §3 we apply Morrey space analysis in its traditional context: analysis of quasilinear elliptic PDE. We analyze a family of such equations, containing as an important example the system relating the metric tensor of a Riemannian manifold to its Ricci tensor, in harmonic coordinates. The analysis involves a combination of paradifferential operator calculus and integration by parts arguments. The specific application to the Ricci tensor is given in §4.

In §5 we resume the internal development of analysis on Morrey spaces. We extend a commutator estimate of T. Kato and G. Ponce [KP] to the Morrey scale setting. We also extend to "microlocal" Morrey scales a comutator estimate of M. Beals [Be], and we recall some known results on commutators $[P, M_f]$, when $f \in$ bmo, and sketch a proof of this given in [AT]. One ingredient in these commutator estimates is the decomposition

$$fv = T_f v + T_v f + R(f,v), \tag{1.18}$$

where

$$T_f v = \sum_{k \ge 4} \Psi_{k-4}(D) f \cdot \psi_k(D) v \tag{1.19}$$

is Bony's paraproduct. This is an example of (1.8)–(1.14), with $F(u_1,u_2) = u_1 u_2$.

In §6 we recall and extend some work of [CFL1-2] and [DR1-2] on a class of pseudodifferential operators whose symbols $p(x,\xi)$ are bmo in x, and a subalgebra whose symbols have x-dependence in vmo $\cap L^\infty$. Here, bmo$(\mathbb{R}^n)$ is the "local" version of BMO$(\mathbb{R}^n)$, with norm

$$\|u\|_{\text{bmo}} = \|u\|_{\text{BMO}} + \|\Psi_0(D)u\|_{L^\infty}. \tag{1.20}$$

The seminorm $\|u\|_{\mathrm{BMO}}$ is give by $\sup_r \eta_u(r)$, where

$$\eta_u(r) = \sup_{\operatorname{diam} B = \rho \le r} \rho^{-n} \int_B |u(x) - u_B|\, dx. \tag{1.21}$$

Here, B runs over all balls of diameter ρ, and u_B stands for the mean value of u on B. The subspace $\mathrm{VMO}(\mathbb{R}^n)$ consists of $u \in \mathrm{BMO}$ such that $\eta_u(r) \to 0$ as $r \to 0$, and vmo consists of $u \in \mathrm{VMO}$ such that $\Psi_0(D)u \in L^\infty(\mathbb{R}^n)$. It is known (cf. [Sar]; see also [CFL]) that VMO is the closure in BMO of the space of uniformly continuous functions on $\mathbb{R}^n$, or equivalently of the space

$$\mathcal{B}^\infty = \{u \in L^\infty(\mathbb{R}^n) : D^\alpha u \in L^\infty(\mathbb{R}^n),\ \forall\, \alpha\}. \tag{1.22}$$

Similarly, vmo is the closure of $\mathcal{B}^\infty$ in bmo. Clearly $\mathrm{vmo} \cap L^\infty = \mathrm{VMO} \cap L^\infty$.

In §7 we derive some Morrey space estimates for solutions to wave equations. In §8 we discuss spaces of conormal distributions and variants.

2. Morrey scales

Since the work [P] it has been known that, if $0 \le \delta < 1$,

$$P \in OPS^0_{1,\delta}(\mathbb{R}^n) \Longrightarrow P : M^p_q(\mathbb{R}^n) \to M^p_q(\mathbb{R}^n), \quad 1 < q \le p < \infty. \tag{12.1}$$

Thus, when $1 < q \le p < \infty$, it is reasonable to consider the scale of spaces

$$M^{p,s}_q(\mathbb{R}^n) = \Lambda^{-s} M^p_q(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : \Lambda^s u \in M^p_q(\mathbb{R}^n)\}, \tag{2.2}$$

where

$$\big(\Lambda^s u\big)\hat{\ }(\xi) = \big(1 + |\xi|^2\big)^s\, \hat{u}(\xi). \tag{2.3}$$

Clearly the standard Sobolev space $H^{s,p}(\mathbb{R}^n) \subset M^{p,s}_q(\mathbb{R}^n)$. It follows from (2.1) that, given $s, m \in \mathbb{R},\ \delta \in [0,1),\ 1 < q \le p < \infty$,

$$P \in OPS^m_{1,\delta}(\mathbb{R}^n) \Longrightarrow P : M^{p,s}_q(\mathbb{R}^n) \to M^{p,s-m}_q(\mathbb{R}^n). \tag{2.4}$$

Since such P map C^s_* to C^{s-m}_* for all $s \in \mathbb{R}$, we see that (1.5) implies

$$M^{p,s}_q(\mathbb{R}^n) \subset C^{s-n/p}_*(\mathbb{R}^n). \tag{2.5}$$

Similarly we can define

$$\mathcal{M}^{p,s}_q(\mathbb{R}^n) = \Lambda^{-s} \mathcal{M}^p_q(\mathbb{R}^n), \tag{2.6}$$

and we have

$$P \in OPS^m_{1,\delta}(\mathbb{R}^n) \Longrightarrow P : \mathcal{M}^{p,s}_q(\mathbb{R}^n) \to \mathcal{M}^{p,s-m}_q(\mathbb{R}^n), \tag{2.7}$$

provided $1 < q \le p < \infty$. We will mainly use the spaces $M^{p,s}_q(\mathbb{R}^n)$, and occasionally refer to the fact that analogous results hold for $\mathcal{M}^{p,s}_q(\mathbb{R}^n)$.

We mention some further results, which will be useful in our development. The following proposition was established in Theorem 3.8 of [T2]. A number of cases had appeared earlier, e.g., in [Ad], [CF], [P].

Proposition 2.1. *If* $1 < p_1 < p_2 < \infty$ *and*

$$m = -\beta = -n\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big) < 0, \tag{2.8}$$

then, for $1 < q_1 \le p_1 < \infty,\ 0 \le \delta < 1,$

$$P \in OPS^m_{1,\delta}(\mathbb{R}^n) \Longrightarrow P : M^{p_1}_{q_1}(\mathbb{R}^n) \rightarrow M^{p_2}_{q_2}(\mathbb{R}^n), \tag{2.9}$$

with

$$\frac{q_2}{q_1} = \frac{p_2}{p_1}, \quad \text{if also} \ \ p_1 \le n, \tag{2.10}$$

and otherwise (2.9) holds provided $q_2/q_1 < p_2/p_1$. *Furthermore,*

$$P \in OPS^m_{1,\delta}(\mathbb{R}^n) \Longrightarrow P : M^{p_1}_1(\mathbb{R}^n) \rightarrow M^{p_2}_{q_2}(\mathbb{R}^n), \quad \text{for} \ \ q_2 < \frac{p_2}{p_1}. \tag{2.11}$$

In addition, (2.4), (2.9), and (2.11) hold for $P \in OP\mathcal{B}S^m_{1,1}(\mathbb{R}^n)$*; in particular,*

$$P \in OP\mathcal{B}S^m_{1,1}(\mathbb{R}^n) \Longrightarrow P : M^{p,s}_q(\mathbb{R}^n) \rightarrow M^{p,s-m}_q(\mathbb{R}^n), \tag{2.12}$$

for $1 < q \le p < \infty,\ m, s \in \mathbb{R}$.

It follows that, for p_j and q_j related as above,

$$M^{p_1,s+|m|}_{q_1}(\mathbb{R}^n) \subset M^{p_2,s}_{q_2}(\mathbb{R}^n). \tag{2.13}$$

Another useful general result established in [T2] is:

Proposition 2.2. *Assume the Schwartz kernel* $k(x, y)$ *of* T *satisfies*

$$|k(x, y)| \le C_M |x - y|^{-n}\big(1 + |x - y|\big)^{-M} \tag{2.14}$$

for sufficiently large M*. Then, if* $1 < q \le p < \infty$,

$$T : L^q(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n) \Longrightarrow T : M^p_q(\mathbb{R}^n) \rightarrow M^p_q(\mathbb{R}^n). \tag{2.15}$$

We include another proof of Proposition 2.2 in Appendix B. Proposition 2.2 implies (2.1) and (2.12). Another application of Proposition 2.2 is the following result, noted in [T2]:

Proposition 2.3. *Given* $k \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$, *we have*

$$P \in OPS^{-k}_{1,1}(\mathbb{R}^n) \Longrightarrow D^\alpha P : M^p_q(\mathbb{R}^n) \rightarrow M^p_q(\mathbb{R}^n), \tag{2.16}$$

for $|\alpha| \le k,\ 1 < q \le p < \infty$.

That (2.16) holds follows from the fact that $D^\alpha P : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ and that

$$D^\alpha P(x, D) = \sum_{\beta+\gamma=\alpha} P_\beta(x, D) D^\gamma \in OPS^0_{1,1}, \tag{2.17}$$

so $T = D^\alpha P$ has Schwartz kernel satisfying (2.14). This fact, in conjunction with (2.1), applied to $D^\alpha \Lambda^{-k} \in OPS^0_{1,0}$, shows that, for $k \in \mathbb{Z}^+$,

$$P \in OPS^{-k}_{1,1}(\mathbb{R}^n) \Longrightarrow \Lambda^k P : M^p_q(\mathbb{R}^n) \to M^p_q(\mathbb{R}^n). \tag{2.18}$$

More generally, one can replace Λ^k in (2.18) by any $A \in OPS^k_{1,\delta},\ \delta \in [0,1)$. We can rewrite (2.18) as

$$P \in OPS^0_{1,1}(\mathbb{R}^n) \Longrightarrow \Lambda^k P \Lambda^{-k} : M^p_q(\mathbb{R}^n) \to M^p_q(\mathbb{R}^n), \tag{2.19}$$

and more generally

$$P \in OPS^0_{1,1}(\mathbb{R}^n) \Longrightarrow \Lambda^{k+i\sigma} P \Lambda^{-(k+i\sigma)} : M^p_q(\mathbb{R}^n) \to M^p_q(\mathbb{R}^n), \tag{2.20}$$

for $k \in \mathbb{Z}^+,\ \sigma \in \mathbb{R}$. For each $k \in \mathbb{Z}^+$, the family of operators has norm polynomially bounded in σ. It follows that

$$P \in OPS^0_{1,1},\ s \in [1,\infty) \Longrightarrow \Lambda^s P \Lambda^{-s} : M^p_q(\mathbb{R}^n) \to M^p_q(\mathbb{R}^n). \tag{2.21}$$

In fact, we improve Proposition 2.3 to the following.

Proposition 2.4. *Given* $s > 0,\ \delta \in (0,1]$, *and* $1 < q \le p < \infty$, *we have*

$$A \in OPS^s_{1,\delta},\ P \in OPS^{-s}_{1,1} \Longrightarrow AP : M^p_q(\mathbb{R}^n) \to M^p_q(\mathbb{R}^n). \tag{2.22}$$

Proof. It suffices to show that, for $s > 0$,

$$P \in OPS^{-s}_{1,1} \Longrightarrow \Lambda^s P : M^p_q(\mathbb{R}^n) \to M^p_q(\mathbb{R}^n), \tag{2.23}$$

and, granted (2.21), we need only consider the cases $0 < s < 1$. We want to apply Proposition 2.2 to $T = \Lambda^s P$, and we know that $T : L^q(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ under our hypotheses. Thus we need to verify that the Schwartz kernel of T satisfies (2.14). That this holds for $|x-y| \ge 1$ is easy. The fact that it holds on the region $|x-y| \le 1$ is proved in Appendix A.

Proposition 2.4 can be rephrased in the language of Morrey scales as

$$P \in OPS^m_{1,1}(\mathbb{R}^n) \Longrightarrow P : M^{p,s}_q(\mathbb{R}^n) \to M^{p,s-m}_q(\mathbb{R}^n), \quad \text{provided } s-m > 0, \tag{2.24}$$

assuming $1 < q \le p < \infty$.

If $s > n/p$ and $1 < q \le p$, then $M^{p,s}_q(\mathbb{R}^n)$ is an algebra. In fact, one can apply a general smooth nonlinear function F to (a vector-valued) $u \in M^{p,s}_q$, and obtain $F(u) \in M^{p,s}_q$, with Moser-type estimates. To see this, write $F(u)$ in terms of a paradifferential operator:

$$F(u) = M(u;x,D)u + R(u), \tag{2.25}$$

as in (1.8)–(1.9), with $R(u) \in C^\infty$, and, by (1.10),

$$u \in C^0 \Longrightarrow M(u;x,\xi) \in S^0_{1,1}. \tag{2.26}$$

Using (2.24), we obtain:

Proposition 2.5. *If $u \in M^{p,s}_q(\mathbb{R}^n)$ with $1 < q \le p$ and $s > n/p$, then, given smooth F, we have $F(u) \in M^{p,s}_q$ and*

$$\|F(u)\|_{M^{p,s}_q} \le C_F\big(\|u\|_{L^\infty}\big)\Big(1 + \|u\|_{M^{p,s}_q}\Big). \tag{2.27}$$

If also $v \in M^{p,s}_q(\mathbb{R}^n)$, then

$$\|uv\|_{M^{p,s}_q} \le C\Big[\|u\|_{L^\infty}\|v\|_{M^{p,s}_q} + \|u\|_{M^{p,s}_q}\|v\|_{L^\infty}\Big]. \tag{2.28}$$

If $s \le n/p$, such estimates fail, unless we also assume that $u \in L^\infty$. If $s = 0$, what we have in place of (2.28) is

$$v, w \in M^p_s(\mathbb{R}^n) \Longrightarrow vw \in M^{p/2}_{s/2}(\mathbb{R}^n), \tag{2.29}$$

provided $2 \le s \le p$. The following, while not sharp, will be useful in §3.

Proposition 2.6. *Let $p > 2,\ q \in (2, p],\ 0 \le \sigma \le 1$. Then*

$$v, w \in M^{p,\sigma}_q \Longrightarrow vw \in M^{p/2,\sigma}_{q/2}. \tag{2.30}$$

Proof. Say $v = \Lambda^{-\sigma} f,\ w = \Lambda^{-\sigma} g$, with $f, g \in M^p_q$. We seek an estimate of the form

$$\sup_{z\in\Omega} \big\|e^{z^2}\Lambda^z(\Lambda^{-z} f \cdot \Lambda^{-z} g)\big\|_{M^{p/2}_{q/2}} \le C\|f\|_{M^p_q}\|g\|_{M^p_q}, \tag{2.31}$$

where

$$\Omega = \{z \in \mathbb{C} : 0 \le \ \mathrm{Re}\, z \le 1\}.$$

It suffices to establish this estimate form $f, g \in \mathcal{S}(\mathbb{R}^n)$. Note that we are taking the norm of a holomorphic function, so it suffices to check $z = iy$ and $z = 1+iy,\ y \in \mathbb{R}$. We have

$$\big\|e^{-y^2}\Lambda^{iy}(\Lambda^{-iy} f \cdot \Lambda^{-iy} g)\big\|_{M^{p/2}_{q/2}} \le C'\|f \cdot g\|_{M^{p/2}_{q/2}} \le C\|f\|_{M^p_q}\|g\|_{M^p_q}, \tag{2.32}$$

by the boundedness of $\langle y\rangle^{-K}\Lambda^{iy}$ in $\mathcal{L}(M^{p/2}_{q/2})$ and in $\mathcal{L}(M^p_q)$. Similarly,

$$\begin{aligned} &\big\|e^{1-y^2}\Lambda^{1+iy}(\Lambda^{-1-iy} f \cdot \Lambda^{-1-iy} g)\big\|_{M^{p/2}_{q/2}} \\ &\le C\sum_{j=1}^n \big\|e^{-\frac{y^2}{2}}\partial_j(\Lambda^{-1-iy} f \cdot \Lambda^{-1-iy} g)\big\|_{M^{p/2}_{q/2}} \\ &\quad + C\big\|e^{-\frac{y^2}{2}}\Lambda^{-i-iy} f \cdot \Lambda^{-1-iy} g\big\|_{M^{p/2}_{q/2}}. \end{aligned} \tag{2.33}$$

Now using

$$\begin{aligned} \partial_j(\Lambda^{-1-iy} f \cdot \Lambda^{-1-iy} g) = (\partial_j\Lambda^{-1-iy} f) \cdot (\Lambda^{-1-iy} g) \\ + (\Lambda^{-1-iy} f) \cdot (\partial_j\Lambda^{-1-iy} g), \end{aligned} \tag{2.34}$$

plus $\partial_j\Lambda^{-1} \in OPS^0$, we easily bound (2.33) by $C\|f\|_{M^p_q}\|g\|_{M^p_q}$. This completes the proof.

As in the case of Sobolev spaces, we can define the notion of u belonging microlocally to a space $M^{p,s}_q$. Assume $1 < q \le p < \infty,\ s \in \mathbb{R}$. Let Γ be a closed conic subset of $T^*\mathbb{R}^n \setminus 0$. We say

$$u \in M^{p,s}_q{}_{ml}(\Gamma) \Longleftrightarrow Pu \in M^{p,s}_q(\mathbb{R}^n), \tag{2.35}$$

for some $P \in OPS^0(\mathbb{R}^n)$ which is elliptic on some conic neighborhood of Γ. There is the following variant of Rauch's lemma:

Proposition 2.7. *Assume* $u \in C^r \cap M^{p,s}_q$, *with* $r, s > 0,\ 1 < q \le p < \infty$. *If* F *is smooth, then*

$$u \in M^{p,\sigma}_q{}_{ml}(\Gamma) \Longrightarrow F(u) \in M^{p,\sigma}{}_{ml}(\Gamma), \quad \textit{provided} \ \ s \le \sigma < s + r. \tag{2.36}$$

Proof. As in (1.12)–(1.17), write, mod C^∞,

$$F(u) = M^\# u + M^b u, \quad M^\# \in OPS^0_{1,\delta}, \quad M^b \in OPS^{-r\delta}_{1,1}, \tag{2.37}$$

for any $\delta < 1$. Then $M^\# u \in M^{p,\sigma}_q{}_{ml}(\Gamma)$, by (2.1) and symbol calculus, while, by (2.24), $M^b u \in M^{p,s+r\delta}_q$. This proves (2.36).

This result can be sharpened, in a way parallel to the treatment of [Mey] for Sobolev spaces. In the decomposition $M = M^\# + M^b$, choose the second method of (1.14); then $M^b \in OPS^{-r}_{1,1}$. Furthermore, if $r > 0,\ u \in C^r \Rightarrow M^\# \in OP\mathcal{B}^r S^0_{1,1}$, where $\mathcal{B}^r S^m_{1,1}$ consists of $p(x,\xi) \in \mathcal{B}S^m_{1,1}$ satisfying the additional conditions

$$\begin{aligned} \|D^\alpha_\xi p(\cdot,\xi)\|_{C^r} &\le C_\alpha \langle\xi\rangle^{m-|\alpha|}, \\ |D^\beta_x D^\alpha_\xi p(x,\xi)| &\le C_{\alpha\beta} \langle\xi\rangle^{m-|\alpha|+\delta(|\beta|-r)}, \ \ \text{for} \ \ |\beta| > r. \end{aligned} \tag{2.38}$$

Operator calculus then yields the following.

Lemma 2.8. *If* $p(x,\xi) \in \mathcal{B}^r S^m_{1,1}$ *and* $u \in M^{p,s}_q$, *with* $1 < q \le p < \infty,\ s > 0$, *then*

$$u \in M^{p,\sigma}_q{}_{ml}(\Gamma) \Longrightarrow p(x,D)u \in M^{p,\sigma-m}_q{}_{ml}(\Gamma), \quad s \le \sigma \le s + r. \tag{2.39}$$

The proof is parallel to that in [Mey]; see also Proposition 3.4.D in [T1]. With this in hand, one can now replace the condition $s \le \sigma < s + r$ in (2.36) by $s \le \sigma \le s + r$.

3. A class of second order elliptic systems in divergence form

Here we study regularity of solutions to elliptic equations of the form

$$\sum \partial_j a_{jk}(x,u)\partial_k u + B(x,u,\nabla u) = f. \tag{3.1}$$

This can be an $M \times M$ system, with u taking values in $\mathbb{R}^M$. We assume $B(x,u,\zeta)$ is smooth in x and u, and is a quadratic form in ζ, or more generally satisfies

$$|B(x,u,\zeta)| \le C\langle\zeta\rangle^2.$$

Proposition 3.1. *Assume that a solution* u *to (3.1) satisfies*

$$\nabla u \in M^q_2, \quad \textit{for some } q > n, \ \textit{hence } u \in C^r, \tag{3.2}$$

for some $r \in (0,1)$, *and*

$$f \in M^{p,-1}_s, \tag{3.3}$$

for some $p \in (q,\infty)$, $s \in [2,p]$. *Then*

$$\nabla u \in M^p_s, \tag{3.4}$$

If $s = p$, then (3.4) is the conclusion of Proposition 2.2.I of [T1], but the hypothesis (3.2) above is weaker than the corresponding hypothesis made in [T1]. The case $f = 0$ of Proposition 3.1 is also contained in Theorem 4.1 of [Sch], when dim $\Omega = 2$.

To begin the proof of Proposition 3.1, we write

$$\sum_k a_{jk}(x,u)\partial_k u = A_j(u;x,D)u, \tag{3.5}$$

mod C^∞, with

$$A_j(u;x,\xi) \in C^r S^1_{1,0} \cap S^1_{1,1} + S^{1-r}_{1,1}, \tag{3.6}$$

as established in (3.3.23) of [T1], and hence, by (3.3.25) of [T1], given $\delta \in (0,1)$,

$$\begin{gathered} A_j(u;x,\xi) = A^\#_j(x,\xi) + A^b_j(x,\xi), \\ A^\#_j(x,\xi) \in S^1_{1,\delta}, \quad A^b_j(x,\xi) \in S^{1-r\delta}_{1,1}. \end{gathered} \tag{3.7}$$

It follows that we can write

$$\sum \partial_j a_{jk}(x,u)\partial_k u = P^\# u + P^b u, \tag{3.8}$$

with

$$P^\# = \sum \partial_j A^\#_j(x,D) \in OPS^2_{1,\delta}, \quad \text{elliptic}, \tag{3.9}$$

and

$$P^b = \sum \partial_j A^b_j(x,D). \tag{3.10}$$

By Proposition 2.4, we have

$$\Lambda^{r\delta-1} P^b \Lambda^{-1} : M^{p'}_{q'} \longrightarrow M^{p'}_{q'}, \quad 1 < q' \le p' < \infty. \tag{3.11}$$

In particular,

$$\nabla u \in M^q_2 \Longrightarrow P^b u \in M^{q,-1+r\delta}_2. \tag{3.12}$$

Now, if

$$E^\# \in OPS^{-2}_{1,\delta} \tag{3.13}$$

denotes a parametrix of $P^\#$, we have, mod C^∞,

$$u = E^\# f - E^\# B(x,u,\nabla u) - E^\# P^b u, \tag{3.14}$$

and we see that, under the hypothesis (3.2), we have some control over the last term:

$$E^{\#} P^b u \in M_2^{q,1+r\delta}. \tag{3.15}$$

Note also that, under our hypothesis on $B(x,u,\zeta)$,

$$\nabla u \in M_2^q \Longrightarrow B(x,u,\nabla u) \in M_1^{q/2}. \tag{3.16}$$

Now, by (2.12),

$$\Lambda^{-1} : M_1^{q/2} \longrightarrow M_2^{\tilde{p}}, \quad \tilde{p} = \frac{q}{2-q/n}, \quad \text{if} \ \ n < q < 2n, \tag{3.17}$$

while the range is contained in C^σ for some $\sigma > 0$ if $q > 2n$, by Morrey's Lemma, and the range is contained in BMO if $q = 2n$. Thus

$$E^{\#} B(x,u,\nabla u) \in M_2^{\tilde{p},1}, \tag{3.18}$$

with $\tilde{p} = q/(2-q/n)$ if $q < 2n$ and for all $\tilde{p} < \infty$ if $q \geq 2n$. Note that $\tilde{p} > q(1+a/n)$ if $q = n + a$. This treats the middle term on the right side of (3.14). Of course, the hypothesis (3.3) yields

$$E^{\#} f \in M_s^{p,1}, \tag{3.19}$$

which is just where we want to place u.

We can draw from (3.15) a conclusion parallel to (3.18)–(3.19), using

$$\Lambda^{-r\delta} : M_2^q \longrightarrow M_2^{\tilde{q}}, \quad \frac{1}{\tilde{q}} = \frac{1}{q} - \frac{r\delta}{n}, \tag{3.20}$$

which follows from (2.9). We then have

$$E^{\#} P^b u \in M_2^{\tilde{q},1}. \tag{3.21}$$

Having thus analyzed the three terms on the right side of (3.14), we have

$$u \in M_2^{q^{\#},1}, \quad q^{\#} = \min\,(\tilde{p}, p, \tilde{q}). \tag{3.22}$$

Iterating this argument a finite number of times, we get

$$u \in M_2^{p,1}. \tag{3.23}$$

If $s = 2$ in (3.3), our work is done.

If $s \in (2,p]$, we can proceed with an argument similar to that above. Details are omitted.

We next establish the following generalization of Proposition 3.1.

Proposition 3.2. *Assume that* $\nabla u \in M_2^q$ *for some* $q > n$, *that* u *satisfies (3.1), and that*

$$f \in M_s^{p,\tau-1}, \tag{3.24}$$

for some $p \in (q,\infty),\ s \in [2,p],\ \tau \geq 0$. *Then*

$$\nabla u \in M_s^{p,\tau}. \tag{3.25}$$

Proof. Note that Proposition 3.1 handles the case $\tau = 0$. Thus we can assume

$$u \in M_s^{p,\rho}, \tag{3.26}$$

with $\rho = 1$. We want to show that (3.26) holds with $\rho = 1+\tau$. As before, we make use of (3.14). The hypothesis (3.24) yields

$$E^{\#} f \in M_s^{p,\tau+1}, \tag{3.27}$$

which is where we want to place u. Whenever (3.26) holds, with $\rho \geq 1$, we have

$$E^{\#} P^b u \in M_s^{p,\rho+r\delta}, \tag{3.28}$$

parallel to (3.15). This is a desirable gain in regularity. It remains to examine the term $E^{\#} B(x,u,\nabla u)$ in (3.14).

To begin,

$$u \in M_s^{p,1} \Longrightarrow B(x,u,\nabla u) \in M_{s/2}^{p/2}. \tag{3.29}$$

Thus, by Proposition 2.1, for arbitrarily small $\varepsilon > 0$,

$$\Lambda^{-\mu} B(x,u,\nabla u) \in M_s^p, \quad \mu = \frac{n}{p} + \varepsilon. \tag{3.30}$$

Since $p > n$, we can take $\mu < 1$. Hence

$$u \in M_s^{p,1} \Longrightarrow E^{\#} B(x,u,\nabla u) \in M_s^{p,1+\sigma}, \quad \forall\, \sigma < 1 - \frac{n}{p}. \tag{3.31}$$

We now prove Proposition 3.2 for $0 < \tau \leq 1$. First assume $s > 2$; use Proposition 2.6 to get, for any $\beta \in (0,1]$,

$$u \in M_s^{p,1+\beta} \Longrightarrow B(x,u,\nabla u) \in M_{s/2}^{p/2,\beta}, \tag{3.32}$$

given that $B(x,u,\nabla u)$ is a quadratic form in ∇u. This time, an application of Proposition 2.1 to the analysis of $E^{\#} B(x,u,\nabla u)$ yields

$$u \in M_s^{p,1+\beta} \Longrightarrow E^{\#} B(x,u,\nabla u) \in M_s^{p,1+\beta+\sigma}, \quad \forall\, \sigma < 1 - \frac{n}{p}, \tag{3.33}$$

given $\beta \in (0,1],\ p > n$, provided $s > 2$. On the other hand, if $s = 2$, the arguments (3.27)–(3.31) yield $u \in M_s^{p,1+\beta}$ for $\beta = \min\{\tau, 1 - n/p - \varepsilon\}$, $\forall\, \varepsilon > 0$. Then, use

$$M_2^{p,1+\beta} \subset M_{2+\varepsilon}^{p,1+\beta-\delta}$$

for some small positive ε, δ, and again apply the argument above. Thus we extend the implication (3.33) to the case $s = 2$.

This is a desirable gain in regularity. Thus a finite iteration of the arguments above establishes Proposition 3.2, if $\tau \in [0, 1]$.

On the other hand, by Proposition 2.5, if $s > 2$,

$$\begin{aligned} u \in M^{p,1+\sigma},\ \sigma > \frac{n}{p} &\Longrightarrow B(x,u,\nabla u) \in M^{p,\sigma}_s \\ &\Longrightarrow E^{\#}B(x,u,\nabla u) \in M^{p,2+\sigma}_s. \end{aligned} \tag{3.34}$$

Thus, if we *have* $u \in M^{p,1+\sigma}_s$ for some $\sigma > n/p$, a finite number of iterations of this argument will yield the desired conclusion (3.26), provided $s > 2$. If $s = 2$, use

$$M^{p,1+\sigma}_2 \subset M^{p,1+\sigma-\delta}_{2+\varepsilon} \tag{3.35}$$

for small $\varepsilon > 0$, $\delta > 0$, and again apply Proposition 2.5 to get

$$u \in M^{p,1+\sigma}_2,\ \sigma > \frac{n}{p} \Longrightarrow E^{\#}B(x,u,\nabla u) \in M^{p,2+\sigma-\delta}_2, \tag{3.36}$$

and iterate.

Using this, we can establish Proposition 3.2 in the case $\tau > 1$. Indeed, in such a case, we can use the conclusion from the $\tau = 1$ case to deduce that $u \in M^{p,2}_s$. This is more than enough regularity to apply (3.34)–(3.36), so the proof is complete.

Our next goal is to derive the hypothesis (3.2) on u as a consequence of a weaker hypothesis, at least for an important special case of systems of the form (3.1).

Proposition 3.3. *Let $u \in H^1(\Omega)$ solve (3.1). Assume the very strong ellipticity condition*

$$a^{jk}_{\alpha\beta}(x,u)\zeta_{j\alpha}\zeta_{k\beta} \ge \lambda_0|\zeta|^2, \quad \lambda_0 > 0. \tag{3.37}$$

Also assume $B(x,u,\nabla u)$ is a quadratic form in ∇u. Assume furthermore that u is continuous on Ω. Then, locally, if $p > n/2$,

$$f \in M^p_2 \Longrightarrow \nabla u \in M^q_2, \quad \textit{for some } q > n. \tag{3.38}$$

Hence $u \in C^r$, for some $r > 0$.

To begin, given $x_0 \in \Omega$, shrink Ω down to a smaller neighborhood, on which

$$|u(x) - u_0| \le E, \tag{3.39}$$

for some $u_0 \in \mathbb{R}^M$ (if (3.1) is an $M \times M$ system). We will specify E below. Write

$$\big(\partial_j a^{jk}(x,u)\partial_k u, w\big)_{L^2} = -\int \langle \nabla u, \nabla w\rangle \, dx, \tag{3.40}$$

where $a^{jk}_{\alpha\beta}(x,u)$ determines an inner product on $T^*_x \otimes \mathbb{R}^M$ for each $x \in \Omega$, in a fashion that depends on u, perhaps, but one has bounds on the set of inner products so arising. Now, if we let $\psi \in C_0^\infty(\Omega)$ and $w = \psi(x)^2(u - u_0)$, and take the inner product of (2.1) with w, we have

$$\begin{aligned} \int \psi^2 |\nabla u|^2 \, dx + 2 \int \psi (\nabla u)(\nabla \psi)(u - u_0) \, dx \\ - \int \psi^2 (u - u_0) B(x, u, \nabla u) \, dx = - \int \psi^2 f (u - u_0) \, dx. \end{aligned} \tag{3.41}$$

Hence we obtain the inequality

$$\begin{aligned} \int \psi^2 \left[|\nabla u|^2 - |u - u_0| \cdot |B(x, u, \nabla u)| - \delta^2 |\nabla u|^2 \right] dx \\ \leq \frac{1}{\delta^2} \int |\nabla \psi|^2 |u - u_0|^2 \, dx + \int \psi^2 |f| \cdot |u - u_0| \, dx, \end{aligned} \tag{3.42}$$

for any $\delta \in (0, 1)$. Now, for some $A < \infty$, we have

$$|B(x, u, \nabla u)| \leq A |\nabla u|^2. \tag{3.43}$$

Then we choose E in (3.39) so that

$$EA \leq 1 - a < 1. \tag{3.44}$$

Then take $\delta^2 = a/2$, and we have

$$\frac{a}{2} \int \psi^2 |\nabla u|^2 \, dx \leq \frac{2}{a} \int |\nabla \psi|^2 \cdot |u - u_0|^2 \, dx + \int \psi^2 |f| \cdot |u - u_0| \, dx. \tag{3.45}$$

Now, given $x \in \Omega$, for $r < \text{dist}(x, \partial\Omega)$ define $U(x, R)$ by

$$U(x, R) = R^{-n} \int_{B_R(x)} |u(y) - u_{x,R}|^2 \, dy, \tag{3.46}$$

where, as before, $u_{x,R}$ is the mean value of $u\big|_{B_R(x)}$.

Lemma 3.4. *Let $\overline{\mathcal{O}} \subset\subset \Omega$. There exist $R_0 > 0$, $\rho \in (0, 1)$, $\vartheta < 1$, and $C_0 < \infty$ such that, if $x \in \overline{\mathcal{O}}$ and $r \leq R_0$, then either*

$$U(x, r) \leq C_0 r^{2(2 - \frac{n}{p})}, \tag{3.47}$$

or

$$U(x, \rho r) \leq \vartheta U(x, r). \tag{3.48}$$

We first describe how to pick ρ, using the following; compare [Gia], pp. 91–92.

Lemma 3.5. *There is a constant* $A_0 = A_0(n, M, \lambda_1/\lambda_0)$ *such that, whenever* $b^{jk}_{\alpha\beta}$ *are constants satisfying*

$$\lambda_1|\zeta|^2 \geq \sum b^{jk}_{\alpha\beta}\zeta_{j\alpha}\zeta_{k\beta} \geq \lambda_0|\zeta|^2, \quad \lambda_0 > 0, \tag{3.49}$$

the following holds. If $u \in H^1\big(B_1(0), \mathbb{R}^M\big)$ *solves*

$$\partial_j b^{jk}_{\alpha\beta}\partial_k u^\beta = 0 \quad \text{on} \quad B_1(0), \tag{3.50}$$

then, for all $\rho \in (0, 1)$,

$$U(0, \rho) \leq A_0\rho^2 U(0, 1). \tag{3.51}$$

Proof. For $\rho \in (0, 1/2]$, we have

$$U(0, \rho) \leq \rho^{2-n} \int\limits_{B_\rho(0)} |\nabla u(y)|^2 \, dy \leq C_n\rho^2 \|\nabla u\|^2_{L^\infty(B_{1/2}(0))}. \tag{3.52}$$

On the other hand, regularity for the constant coefficient elliptic PDE (3.50) readily yields an estimate

$$\|\nabla u\|^2_{L^\infty(B_{1/2}(0))} \leq B_0\|\nabla u\|^2_{L^2(B_{3/4}(0))} \leq B_1\|u - u_{0,1}\|^2_{L^2(B_1(0))}, \tag{3.53}$$

with $B_j = B_j(n, M, \lambda_1/\lambda_0)$, from which (2.51) easily follows.

Now, to pick ρ for Lemma 3.4, we assume (3.49) holds for all frozen coefficient principal parts of (3.1), take the A_0 given by Lemma 3.5, and then pick ρ so that $A_0\rho^2 \leq 1/2$.

Having picked ρ, we proceed to prove Lemma 3.4 by contradiction. If the result is false, there exist $x_\nu \in \overline{\mathcal{O}}$, $R_\nu \to 0$, $\vartheta_\nu \to 1$, and $u_\nu \in H^1(\Omega, \mathbb{R}^M)$ solving (3.1) such that

$$U_\nu(x_\nu, R_\nu) = \varepsilon_\nu^2 > C_0 R_\nu^{2(2-n/p)} \tag{3.54}$$

and

$$U_\nu(x_\nu, \rho R_\nu) > \vartheta_\nu U_\nu(x_\nu, R_\nu). \tag{3.55}$$

The hypothesis that u is continuous implies $\varepsilon_\nu \to 0$. We want to obtain a contradiction.

We next set

$$v_\nu(x) = \varepsilon_\nu^{-1}\big[u_\nu(x_\nu + R_\nu x) - u_{\nu x_\nu, R_\nu}\big]. \tag{3.56}$$

Then v_ν solves

$$\begin{aligned} &\partial_j a^{jk}_{\alpha\beta}\big(x_\nu + R_\nu x, \varepsilon_\nu v_\nu(x) + u_{\nu x_\nu, R_\nu}\big)\partial_k v^\beta_\nu \\ &+ \varepsilon_\nu B\big(x_\nu + R_\nu x, \varepsilon_\nu v_\nu(x) + u_{\nu x_\nu, R_\nu}, \nabla v_\nu(x)\big) = \frac{R_\nu^2}{\varepsilon_\nu} f. \end{aligned} \tag{3.57}$$

Note that, by the hypothesis (3.54),

$$\frac{R_\nu^2}{\varepsilon_\nu} < \frac{1}{C_0} R_\nu^{n/p}. \tag{3.58}$$

Now set

$$V_\nu(0,r) = r^{-n} \int\limits_{B_r(0)} |v_\nu(y) - v_{\nu 0,r}|^2 \, dy. \tag{3.59}$$

Then, since $v_{\nu 0,1} = 0$, we have

$$V_\nu(0,1) = \|v_\nu\|^2_{L^2(B_1(0))} = 1, \quad V_\nu(0,\rho) > \vartheta_\nu. \tag{3.60}$$

Passing to a subsequence, we can assume that

$$v_\nu \to v \ \text{ weakly in } \ L^2\big(B_1(0), \mathbb{R}^M\big), \quad \varepsilon_\nu v_\nu \to 0 \ \text{ a.e. in } B_1(0). \tag{3.61}$$

Also

$$a^{jk}_{\alpha\beta}(x_\nu, u_{\nu x_\nu, R_\nu}) \longrightarrow b^{jk}_{\alpha\beta}, \tag{3.62}$$

an array of constants satisfying (3.49). Boundedness of $\varepsilon_\nu v_\nu(x) + u_{\nu x_\nu, R_\nu}$ plus continuity of $a^{jk}_{\alpha\beta}$ imply

$$a^{jk}_{\alpha\beta}\big(x_\nu + R_\nu x, \varepsilon_\nu v_\nu(x) + u_{\nu x_\nu, R_\nu}\big) \longrightarrow b^{jk}_{\alpha\beta} \quad \text{a.e. in } B_1(0), \tag{3.63}$$

and this is bounded convergence.

We next need to estimate the L^2-norm of ∇v_ν. Substituting $\varepsilon_\nu v_\nu\big(\frac{x - x_\nu}{R_\nu}\big) + u_{\nu x_\nu, R_\nu}$ for $u_\nu(x)$ in (3.45), and replacing u_0 by $u_{\nu x_\nu, R_\nu}$, we have

$$\begin{aligned} &\frac{a}{2} \int \psi^2 \Big|\nabla v_\nu\Big(\frac{x - x_\nu}{R_\nu}\Big)\Big|^2 \, dx \\ &\qquad \le \frac{2}{a} \int R_\nu^2 |\nabla \psi|^2 \, \Big|v_\nu\Big(\frac{x - x_\nu}{R_\nu}\Big)\Big|^2 \, dx + \frac{R_\nu^2}{\varepsilon_\nu} \int \psi^2 |f| \cdot \Big|v_\nu\Big(\frac{x - x_\nu}{R_\nu}\Big)\Big| \, dx, \end{aligned} \tag{3.64}$$

for $\psi \in C_0^\infty\big(B_{R_\nu}(x_\nu)\big)$. Actually, for this new value of u_0, the estimate (3.39) might change to $|u(x) - u_0| \le 2E$, so at this point we strengthen the hypothesis (3.44) to

$$2EA \le 1 - a < 1, \tag{3.65}$$

in order to get (3.59). Since $R_\nu^2/\varepsilon_\nu \le R_\nu^{n/p}/C_0$, we have, for $\Psi(x) = \psi(x_\nu + R_\nu x) \in C_0^\infty\big(B_1(0)\big)$,

$$\frac{a}{2} \int \Psi^2 |\nabla v_\nu|^2 \, dx \le \frac{2}{a} \int |\nabla \Psi|^2 |v_\nu|^2 \, dx + \frac{R_\nu^{n/p}}{C_0} \int \Psi^2 |F| \cdot |v_\nu| \, dx, \tag{3.66}$$

where $F(x) = f(x_\nu + R_\nu x)$.

Since $\|v_\nu\|_{L^2(B_1(0))} = 1$, if $\Psi \le 1$ we have

$$\int \Psi^2 |F| \cdot |v_\nu| \, dx \le \Big(\int_{B_1(0)} |F|^2 \, dx \Big)^{1/2} \le C_1 R_\nu^{-n/p}, \tag{3.67}$$

if $f \in M_2^p$, so we have

$$\frac{a}{2} \int \Psi^2 |\nabla v_\nu|^2 \, dx \le \frac{2}{a} \int |\nabla \Psi|^2 |v_\nu|^2 \, dx + \frac{C_1}{C_0} \|f\|_{M_2^p}. \tag{3.68}$$

This implies that v_ν is bounded in $H^1\big(B_\rho(0)\big)$ for each $\rho < 1$. Now, we can pass to a further subsequence and obtain

$$\begin{aligned} v_\nu &\longrightarrow v \;\text{ strongly in }\; L^2_{\text{loc}}\big(B_1(0)\big) \\ \nabla v_\nu &\longrightarrow \nabla v \;\text{ weakly in }\; L^2_{\text{loc}}\big(B_1(0)\big). \end{aligned} \tag{3.69}$$

Thus, we can pass to the limit in (3.57), to obtain

$$\partial_j b^{jk}_{\alpha\beta} \partial_k v^\beta = 0, \quad \text{on } B_1(0). \tag{3.70}$$

Also, by (3.60),

$$V(0,1) = \|v\|_{L^2(B_1(0))} \le 1, \quad V(0,\rho) \ge 1. \tag{3.71}$$

This contradicts Lemma 3.5, which requires $V(0,\rho) \le (1/2)V(0,1)$.

Now that we have Lemma 3.4, the proof of Proposition 3.3 is easily completed. From (3.47)–(3.48) we have

$$U(x,r) \le C r^{2\alpha} \tag{3.72}$$

for some $\alpha > 0$. In other words

$$\int_{B_r(x)} \big| u(y) - u_{x,r} \big|^2 \, dy \le C r^{n+2\alpha}, \tag{3.73}$$

uniformly for $x \in \overline{\mathcal{O}} \subset\subset \Omega$. This in itself implies $u \in C^\alpha(\overline{\mathcal{O}})$. Furthermore, by (3.45), we have

$$\int_{B_r(x)} |\nabla u|^2 \, dy \le C r^{n-2(1-\alpha)}, \tag{3.74}$$

which implies

$$\nabla u \big|_{\overline{\mathcal{O}}} \in M_2^q, \quad q = \frac{n}{1-\alpha}. \tag{3.75}$$

Thus Proposition 3.3 is proved.

We can extend Proposition 3.3 to the following result, which interfaces most conveniently with Propositions 3.1–3.2.

Proposition 3.6. *Under the hypotheses of Proposition 3.3, if* $p > n$,

$$f \in M_2^{p,-1} \Longrightarrow u \in M_2^{q,1}, \quad \textit{for some} \;\; q > n. \tag{3.76}$$

Proof. Writing $f = \sum \partial_j g_j,\ g_j \in M_2^p$, we replace the right side of (3.41) by (the sum over j of)

$$-\int \psi^2(\partial_j g_j)(u-u_0)\, dx = \int \psi^2 g_j(\partial_j u)\, dx + 2\int \psi(\partial_j\psi) g_j (u-u_0)\, dx. \tag{3.77}$$

Thus, in place of (3.42), we have the inequality

$$\begin{aligned} &\int \psi^2 \bigl[|\nabla u|^2 - |u-u_0|\cdot|B(x,u,\nabla u)| - 2\delta^2|\nabla u|^2\bigr]\, dx \\ &\le \frac{1}{\delta^2}\int \bigl\{|\nabla\psi|^2|u-u_0|^2 + \psi^2|g|^2\bigr\}\, dx + 2\int |\psi|\cdot|\nabla\psi|\cdot|g|\cdot|u-u_0|\, dx, \end{aligned} \tag{3.78}$$

where $|g|^2 = \sum |g_j|^2$. The estimates (3.43)–(3.75) proceed essentially as before, with a few minor changes, resulting from replacing the estimate for $F(x) = f(x_\nu + R_\nu x)$ by the following estimate for $G_j(x) = g_j(x_\nu + R_\nu x)$:

$$\Bigl(\int_{B_1(0)} |G_j|^2\, dx\Bigr)^{1/2} \le C_1' R^{-n/p}, \tag{3.79}$$

if $g_j \in M_2^p$. Details are left to the reader.

Combining Propositions 3.2 and 3.6, we have:

Proposition 3.7. *Assume* $u \in H^1(\Omega) \cap C(\Omega)$ *solves (3.1), that the very strong ellipticity condition (3.37) holds, and that* $B(x,u,\nabla u)$ *is a quadratic form in* ∇u. *If* $p > n,\ \tau \ge 0,\ 2 \le s \le p$, *then*

$$f \in M_s^{p,\tau-1} \Longrightarrow u \in M_s^{p,\tau+1}. \tag{3.80}$$

4. Connections with Ricci curvature bounds

Consider a Riemannian metric g_{jk} defined on the unit ball $B_1 \subset \mathbb{R}^n$. We will work under the following hypotheses:
(i) For some constants $a_j \in (0,\infty)$, *there are estimates*

$$0 < a_0 I \le \bigl(g_{jk}(x)\bigr) \le a_1 I. \tag{4.1}$$

(ii) The coordinates $x_1, \dots, x_n$ *are harmonic, i.e.,*

$$\Delta x_\ell = 0. \tag{4.2}$$

Here, Δ is the Laplace operator determined by the metric g_{jk}. In general,

$$\Delta v = g^{jk}\partial_j\partial_k v - \lambda^\ell \partial_\ell v, \quad \lambda^\ell = g^{jk}\Gamma^\ell{}_{jk}. \tag{4.3}$$

Note that $\Delta x_\ell = \lambda^\ell$, so the coordinates are harmonic if and only if $\lambda^\ell = 0$. Thus, in harmonic coordinates,

$$\Delta v = g^{jk}\partial_j\partial_k v. \tag{4.4}$$

We will also assume some bounds on the Ricci tensor, and desire to see how this influences the regularity of g_{jk} in these coordinates. Generally, the Ricci tensor is given by

$$\begin{aligned} \mathrm{Ric}_{jk} &= \frac{1}{2} g^{\ell m}\big[-\partial_\ell\partial_m g_{jk} - \partial_j\partial_k g_{\ell m} + \partial_k\partial_m g_{\ell j} + \partial_\ell\partial_j g_{km}\big] + M_{jk}(g,\nabla g) \\ &= -\frac{1}{2} g^{\ell m}\partial_\ell\partial_m g_{jk} + \frac{1}{2} g_{j\ell}\partial_k\lambda^\ell + \frac{1}{2} g_{k\ell}\partial_j\lambda^\ell + H_{jk}(g,\nabla g), \end{aligned} \tag{4.5}$$

with λ^ℓ as in (4.3). In harmonic coordinates, we obtain

$$\sum \partial_j g^{jk}(x)\partial_k g_{\ell m} + Q_{\ell m}(g,\nabla g) = \mathrm{Ric}_{\ell m}, \tag{4.6}$$

and $Q_{\ell m}(g,\nabla g)$ is a quadratic form in ∇g, with coefficients which are smooth functions of g, as long as (4.1) holds. Also, when (4.1) holds, the equation (4.6) is elliptic, of the form (3.1). Thus Proposition 3.7 directly implies the following.

Proposition 4.1. *Assume the metric tensor satisfies hypotheses (i) and (ii). Also assume that*

$$g_{jk} \in H^1(B_1) \cap C(B_1), \tag{4.7}$$

and

$$Ric_{\ell m} \in M^{p,r-1}_s, \tag{4.8}$$

for some $p \in (n,\infty),\ 2 \le s \le p,\ r \ge 0$. *Then, on the ball* $B_{9/10}$,

$$g_{jk} \in M^{p,r+1}_s. \tag{4.9}$$

Geometrical consequences of estimates on the Ricci tensor can be found in [An], [AC], and references given in these papers.

5. Commutator estimates on Morrey scales

In this section we establish a number of commutator estimates, starting with the following variant of an estimate of T. Kato and G. Ponce [KP]:

Theorem 5.1. *If $P \in OP\mathcal{B}S^m_{1,1}$ and $m > 0$, then*

$$\|P(fu) - fPu\|_{\mathcal{M}^{p,s}_q} \le C\|f\|_{\mathrm{Lip}^1}\|u\|_{\mathcal{M}^{p,m-1+s}_q} + C\|f\|_{\mathcal{M}^{p,m+s}_q}\|u\|_{L^\infty}, \tag{5.1}$$

provided $s \ge 0,\ 1 < q \le p < \infty$.

Proof. We start with

$$\begin{aligned} f(Pu) &= T_f Pu + T_{Pu} f + R(f, Pu), \\ P(fu) &= PT_f u + PT_u f + PR(f,u). \end{aligned} \tag{5.2}$$

As shown in Proposition 4.2 of [AT], possibly replacing the '4' in (1.19) by a larger number, we have $[T_f, P] \in OP\mathcal{B}S^{m-1}_{1,1}$ when $f \in \mathrm{Lip}^1(\mathbb{R}^n)$. Hence

$$\|[T_f, P]u\|_{\mathcal{M}^{p,s}_q} \le C\|f\|_{\mathrm{Lip}^1}\|u\|_{\mathcal{M}^{p,m-1+s}_q}. \tag{5.3}$$

Next, $u \in L^\infty \Rightarrow T_u \in OP\mathcal{B}S^0_{1,1}$, so

$$\|PT_u f\|_{\mathcal{M}^{p,s}_q} \le C\|u\|_{L^\infty}\|f\|_{\mathcal{M}^{p,m+s}_q}. \tag{5.4}$$

Furthermore,

$$u \in L^\infty \Longrightarrow Pu \in C^{-m}_* \Longrightarrow T_{Pu} \in OP\mathcal{B}S^m_{1,1}, \quad \text{if} \ \ m > 0, \tag{5.5}$$

so

$$\|T_{Pu} f\|_{\mathcal{M}^{p,s}_q} \le C\|u\|_{L^\infty}\|f\|_{\mathcal{M}^{p,m+s}_q}. \tag{5.6}$$

It remains to estimate $R(f, Pu)$ and $PR(f,u)$.

First, we mention that R_f, given by $R_f u = R(f,u)$, is a Calderon-Zygmund operator, for any $f \in \mathrm{bmo}$, satisfying

$$\|R_f u\|_{L^q} \le C_q\|f\|_{\mathrm{BMO}}\|u\|_{L^q}, \quad 1 < q < \infty, \tag{5.7}$$

and with Schwartz kernel K_f satisfying (cf. Lemma 3.5.E of [T1])

$$|K_f(x,y)| \le C\|f\|_{C^0_*}|x-y|^{-n}, \tag{5.8}$$

as well as

$$|\nabla_{x,y} K_f(x,y)| \le C\|f\|_{C^0_*}|x-y|^{-n-1}. \tag{5.9}$$

As shown in Appendix B, (5.7)–(5.8) lead to $\mathcal{M}^p_q$ boundedness. Since $\mathrm{bmo} \subset \mathrm{BMO} \cap C^0_*$, we have

$$\|R(f,u)\|_{\mathcal{M}^p_q} \le C_{pq}\|f\|_{\mathrm{bmo}}\|u\|_{\mathcal{M}^p_q}, \tag{5.10}$$

for $1 < q \le p < \infty$. Now we establish a variant of Proposition 3.5.D in [T1]:

Lemma 5.2. *Let $\mathfrak{h}^{r,\infty}$ denote the bmo-Sobolev space, which has the property that*

$$P \in OPS^r_{1,0} \Longrightarrow P : \mathfrak{h}^{r,\infty} \to bmo. \tag{5.11}$$

Then

$$\|R(f,u)\|_{\mathcal{M}^{p,s}_q} \le C_{pqrs} \|f\|_{\mathfrak{h}^{r,\infty}} \|u\|_{\mathcal{M}^{p,s-r}_q}, \quad s \ge 0,\ 1 < q \le p < \infty. \tag{5.12}$$

Proof. First we treat the case $s = 0$. Decompose f into $\sum_{\ell=1}^{20} f_\ell$, via operators in $OPS^0_{1,0}$, so that

$$\text{supp } \hat{f} \subset \bigcup \Big\{ 2^k \le |\xi| \le 2^{k+2} : k = \ell \text{ mod } 20 \Big\}.$$

Similarly decompose u. (We needn't worry about pieces left over with spectrum contained in, say, $|\xi| \le 3$.) It suffices to estimate such $R(f_\ell, u_m)$. In such a case, we can find

$$F_\ell = Q_+ f_\ell \in \text{bmo}, \quad V_m = Q_- u_m, \quad Q_\pm \in OPS^{\pm r}_{1,0} \tag{5.13}$$

such that, for each k,

$$\psi^a_k(D) f_\ell = 2^{-kr} \psi^a_k(D) F_\ell, \quad \psi_k(D) u_m = 2^{kr} \psi_k(D) V_m. \tag{5.14}$$

Here, $\{\psi_k\}$ is a Littlewood-Paley partition of unity and $\psi^a_k(\xi) = \sum_{\ell=k-5}^{k+5} \psi_\ell(\xi)$, so that

$$R(f,u) = \sum_k \big(\psi^a_k(D) f\big) \cdot \psi_k(D) u. \tag{5.15}$$

Hence

$$R(f,u) = R(F_\ell, V_m), \tag{5.16}$$

so the $s = 0$ case of (5.12) follows from the estimate

$$\|V_m\|_{\mathcal{M}^p_q} \le C \|u_m\|_{\mathcal{M}^{p,-r}_q}, \tag{5.17}$$

plus (5.10).

So far, we have

$$R_f : \mathcal{M}^{p,-r}_q \longrightarrow \mathcal{M}^p_q, \quad \text{for} \quad f \in \mathfrak{h}^{r,\infty}, \tag{5.18}$$

under the hypothesis (5.11). Next, we claim $R_f : \mathcal{M}^{p,1-r}_q \to \mathcal{M}^{p,1}_q$, for $f \in \mathfrak{h}^{r,\infty}$. This follows from

$$\partial_j(R_f u) = R_{(\partial_j f)} u + R_f(\partial_j u), \tag{5.19}$$

plus the fact that, if $f \in \mathfrak{h}^{r,\infty}$, then $P \in OPS^{r-1}_{1,0} \Rightarrow P(\partial_j f) \in \text{bmo}$, and hence the argument above shows that $R_{(\partial_j f)} : \mathcal{M}^{p,1-r}_q \to \mathcal{M}^p_q$. Once we have (5.19), then by induction we obtain

$$R_f : \mathcal{M}^{p,j-r}_q \longrightarrow \mathcal{M}^{p,j}_q, \quad j = 0, 1, 2, \dots \tag{5.20}$$

for $f \in \mathfrak{h}^{r,\infty}$, and then (5.12) follows by interpolation.

Our application of Lemma 5.2 to the estimation of $R(f, Pu)$ and $PR(f, u)$ in (5.2) is the following:

$$\|R(f,u)\|_{\mathcal{M}^{p,\sigma}_q} \le C\|f\|_{\mathrm{Lip}^1}\|w\|_{\mathcal{M}^{p,\sigma-1}_q}, \quad \sigma \ge 0,\ 1 < q \le p < \infty. \tag{5.21}$$

Hence, given $P \in OP\mathcal{B}S^m_{1,1}$, we have, taking $\sigma = s$,

$$\|R(f,Pu)\|_{\mathcal{M}^{p,s}_q} \le C\|f\|_{\mathrm{Lip}^1}\|u\|_{\mathcal{M}^{p,m-1+s}_q}, \quad s \ge 0, \tag{5.22}$$

and, taking $\sigma = s + m$,

$$\|PR(f,u)\|_{\mathcal{M}^{p,s}_q} \le C\|f\|_{\mathrm{Lip}^1}\|u\|_{\mathcal{M}^{p,m-1+s}_q}, \quad s + m \ge 0. \tag{5.23}$$

The proof of Theorem 5.1 is complete.

We next establish a commutator result along the lines of Lemma 1.13 in [Be]. Set $M_b u = bu$.

Proposition 5.3. *Let $1 < p \le q < \infty$; consider*

$$v \in M^{p,s}_q(\mathbb{R}^n) \cap M^{p,r}_q{}_{ml}(\Gamma), \quad b \in M^{p,s+1}_q(\mathbb{R}^n) \cap M^{p,r+1}_q{}_{ml}(\Gamma). \tag{5.24}$$

Assume

$$\frac{n}{p} < s \le r < 2s - \frac{n}{p}. \tag{5.25}$$

Then

$$P \in OPS^1_{1,0} \Longrightarrow [P, M_b]v \in M^{p,s}_q(\mathbb{R}^n) \cap M^{p,r}_q{}_{ml}(\Gamma). \tag{5.26}$$

Proof. Write

$$M_b v = T_b v + T_v b + R_b v. \tag{5.27}$$

Then, as in (5.2),

$$[P, M_b]v = [P, T_b]v + PT_v b - T_{Pv} b + PR_b v - R_b P v. \tag{5.28}$$

The hypotheses imply $b \in C^{s+1-n/p}_*$, hence, if $s > n/p$, the $OP\mathcal{B}S^m_{1,1}$ calculus gives

$$T_b \in OP\mathcal{B}^{\sigma+1}S^0_{1,1}, \quad [P, T_b] \in OP\mathcal{B}^{\sigma}S^0_{1,1}, \quad \sigma = s - \frac{n}{p}, \tag{5.29}$$

where $\mathcal{B}^\sigma S^m_{1,1}$ is the subspace of $\mathcal{B}S^m_{1,1}$ defined by (2.38).

Also, $v \in C^\sigma_*$, hence

$$T_v \in OP\mathcal{B}^\sigma S^0_{1,1}, \quad T_{Pv} \in OP\mathcal{B}^\sigma S^1_{1,1}. \tag{5.30}$$

Now (5.29) plus the hypothesis (5.24) on v gives

$$(5.31)\qquad [P,T_b]v \in M^{p,s}_q(\mathbb{R}^n)\cap M^{p,r}_q{}_{ml}(\Gamma),\quad r<s+\sigma.$$

Also, (5.30) implies that, for $r<s+\sigma$,

$$(5.32)\qquad \begin{aligned} T_v b &\in M^{p,s+1}_q(\mathbb{R}^n)\cap M^{p,r+1}_q{}_{ml}(\Gamma),\\ T_{Pv} &\in M^{p,s}_q(\mathbb{R}^n)\cap M^{p,r}_q{}_{ml}(\Gamma).\end{aligned}$$

Finally, we have

$$(5.33)\qquad R_b\in OPS^{-\sigma-1}_{1,1},\ \text{ hence }\ PR_b v,\, R_bPv\in M^{p,s+\sigma}_q(\mathbb{R}^n).$$

It follows from (5.31)–(5.33) that

$$(5.34)\qquad [P,M_b]v\in M^{p,s}_q(\mathbb{R}^n)\cap M^{p,r}_q{}_{ml}(\Gamma)+M^{p,2s-n/p}_q(\mathbb{R}^n),$$

which gives (5.25).

The next result was proven for $P\in OPS^0_{cl}$ in [DR1], following the seminal L^p estimate of [CRW]. This estimate will be useful in §6. We sketch a proof of an extension given in [AT].

Proposition 5.4. *If $P\in OP\mathcal{B}S^0_{1,1}$, $f\in bmo(\mathbb{R}^n)$, and $1<q\le p<\infty$, then*

$$(5.35)\qquad \|fPu-P(fu)\|_{M^q_p}\le C\|f\|_{\mathrm{bmo}}\|u\|_{M^p_q}.$$

Sketch of proof. As before, we use (5.2). We have (5.10), and similarly

$$(5.36)\qquad \|T_uf\|_{M^p_q}\le C\|f\|_{\mathrm{bmo}}\|u\|_{M^p_q}.$$

Hence

$$(5.37)\qquad \begin{aligned}&\|T_{Pu}f\|_{M^p_q}+\|PT_uf\|_{M^p_q}+\|R(f,Pu)\|_{M^p_q}+\|PR(f,u)\|_{M^p_q}\\ &\le C\|f\|_{\mathrm{bmo}}\|u\|_{M^p_q}.\end{aligned}$$

On the other hand, $\mathrm{bmo}(\mathbb{R}^n)\subset C^0_*(\mathbb{R}^n)$, and, as shown in [AT],

$$(5.38)\qquad f\in C^0_*(\mathbb{R}^n)\Longrightarrow [T_f,P]\in OP\mathcal{B}S^0_{1,1},$$

so

$$(5.39)\qquad \|[T_f,P]u\|_{M^p_q}\le C\|f\|_{\mathrm{bmo}}\|u\|_{M^p_q}.$$

This gives (5.35).

6. Operators with vmo coefficients

Consider a symbol

$$p(x,\xi) \in \mathrm{bmo}S^m_{1,0} \tag{6.1}$$

such that

$$p(x,r\xi) = r^m p(x,\xi), \quad r \geq 1,\ |\xi| \geq 1. \tag{6.2}$$

Thus, if $\{w_j : j \geq 1\}$ is an orthonormal basis of $L^2(S^{n-1})$ consisting of eigenfunctions of the Laplace operator Δ_S on S^{n-1}, we can write

$$p(x,\xi) = p_0(x,\xi) + \sum_j f_j(x) w_j\Bigl(\frac{\xi}{|\xi|}\Bigr)|\xi|^m\bigl(1-\varphi(\xi)\bigr), \tag{6.3}$$

where $\varphi \in C_0^\infty(\mathbb{R}^n),\ \varphi(\xi) = 1$ for $|\xi| \leq 1/2$, and $p_0(x,\xi)$ is supported on $|\xi| \leq 1$. Furthermore,

$$\|f_j\|_{\mathrm{bmo}} \leq C_N \langle j\rangle^{-N}. \tag{6.4}$$

Write

$$p_j(x,\xi) = f_j(x) w_j\Bigl(\frac{\xi}{|\xi|}\Bigr)|\xi|^m\bigl(1-\varphi(\xi)\bigr) = f_j(x) a_{jm}(\xi), \tag{6.5}$$

so we have

$$p(x,\xi) = \sum_{j\geq 0} p_j(x,\xi). \tag{6.6}$$

The operator $p_0(x,D)$ has a simple analysis. One can write

$$p_0(x,\xi) = \sum_\ell p_\ell(x) e^{i\ell\cdot\xi}\varphi(\xi/2), \tag{6.7}$$

with

$$\|p_\ell\|_{\mathrm{bmo}} \leq C'_N \langle \ell\rangle^{-N}. \tag{6.8}$$

Thus

$$p_0(x,D)u = \sum p_\ell(x)\psi_\ell(D)u = \sum p_\ell(x)\hat{\psi}_\ell * u, \tag{6.9}$$

where

$$\hat{\psi}_\ell(x) = C\hat{\varphi}(2x+\ell). \tag{6.10}$$

Hence

$$p_0(x,D) : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n) \cap \mathrm{bmo}(\mathbb{R}^n), \qquad \forall\, p \in [1,\infty). \tag{6.11}$$

We now establish some commutator estimates. First, suppose

$$B = b(x,D) \in OPS^0_{1,\delta},\ \ 0 \le \delta < 1. \tag{6.12}$$

Take $m = 0$ above, and use the notation $a_j(\xi)$ instead of $a_{j0}(\xi)$. Then

$$\begin{aligned} [B, p_j(x,D)]u &= [B, M_{f_j} a_j(D)]u \\ &= f_j(x)[B, a_j(D)]u + [B, M_{f_j}]a_j(D)u. \end{aligned} \tag{6.13}$$

Since

$$C_j = [B, a_j(D)] \in OPS^{-(1-\delta)}_{1,\delta}, \tag{6.14}$$

and there are polynomial bounds (in j) on the relevant seminorms of the symbols, we have

$$\|f_j[B, a_j(D)]u\|_{M^p_q} \le C_{N,K}\langle j\rangle^{-N}\|u\|_{M^p_q}, \quad 1 < q \le p < \infty, \tag{6.15}$$

given supp $u \subset K$, compact. Also, by Proposition 5.4,

$$\|[B, M_{f_j}]v\|_{M^p_q} \le C\|f_j\|_{\text{bmo}}\|v\|_{M^p_q}, \tag{6.16}$$

so we have

$$\|[B, p_j(x,D)]u\|_{M^p_q} \le C_N\langle j\rangle^{-N}\|u\|_{M^p_q}, \tag{6.17}$$

for $j \ge 1$. Summing over j, we have:

Proposition 6.1. *If $p(x,\xi) \in bmoS^0_{cl}$ and $B \in OPS^0_{1,\delta},\ \delta < 1$, then, for $K \subset \mathbb{R}^n$ compact,*

$$[B, p(x,D)] : M^p_q(K) \longrightarrow M^p_q, \quad 1 < q \le p < \infty. \tag{6.18}$$

If $p(x,\xi) \in vmoS^0_{cl}$, this comutator is compact.

Next, we consider the commutator $[M_g, p(x,D)]$. We have

$$[M_g, p(x,D)] = [M_g, p_0(x,D)] + \sum_{j\ge1}[M_g, p_j(x,D)]. \tag{6.19}$$

Clearly, for $g \in$ bmo,

$$[M_g, p_0(x,D)] : M^p_q(K) \longrightarrow M^p_q, \quad 1 < q \le p < \infty. \tag{6.20}$$

Next, for $j \ge 1$,

$$[M_g, p_j(x,D)] = M_{f_j}[M_g, a_j(D)]. \tag{6.21}$$

Thus, applying Proposition 5.4 to the commutator $[M_g, a_j(D)]$, we have

$$\|[M_g, p_j(x,D)]u\|_{M^p_q} \le C\|f_j\|_{L^\infty}\|g\|_{\text{bmo}}\|u\|_{M^p_q}, \tag{6.22}$$

and, if $g \in$ vmo, this commutator is compact. Summing, we have:

Proposition 6.2. *If $p(x,\xi)\in L^\infty S^0_{cl}$ and $g\in bmo$, then*

$$[M_g, p(x,D)] : M^p_q(K) \longrightarrow M^p_q, \quad 1<q\le p<\infty. \tag{6.23}$$

If also $g\in vmo$ and $p(x,\xi)$ is supported on $x\in K$ compact, then this commutator is compact.

For the spaces L^p, this result was proved in [CFL]; see Theorem 2.11 there. Furthermore, weighted L^p estimates are obtained in Theorem 2.1 of [DR2], and the Morrey space estimate (6.23) is contained in Theorems 2.2–2.3 of [DR2].

Now suppose $q(x,\xi)$ has the form

$$\begin{gathered} q(x,\xi)=\sum_{j\ge 0} q_j(x,\xi), \quad q_0(x,\xi)=\sum_\ell q_\ell(x)\psi_\ell(\xi), \\ q_j(x,\xi)=g_j(x)a_j(\xi),\ j\ge 1, \end{gathered} \tag{6.24}$$

with ψ_ℓ as in (6.9)–(6.10). Then

$$[q(x,D),p(x,D)]=[q_0(x,D),p(x,D)]+\sum_{j\ge 1}[q_j(x,D),p(x,D)]. \tag{6.25}$$

Clearly, if $p(x,\xi)$ and $q(x,\xi)$ have compact x-support, then

$$p(x,\xi),q(x,\xi)\in L^\infty S^0_{cl} \Longrightarrow [q_0(x,D),p(x,D)] \ \text{ compact on } \ M^p_q. \tag{6.26}$$

Next,

$$[q_j(x,D),p(x,D)]=M_{g_j}[a_j(D),p(x,D)]+a_j(D)[M_{g_j},p(x,D)]. \tag{6.27}$$

Now we have, for some $M<\infty$,

$$\|M_{g_j}[a_j(D),p(x,D)]u\|_{M^p_q}\le C\|g_j\|_{L^\infty}\langle j\rangle^M\|u\|_{M^p_q}, \tag{6.28}$$

if $p(x,\xi)\in \text{bmo}S^0_{cl}$, and compactness if $p(x,\xi)\in \text{vmo}S^0_{cl}$. Also, we have

$$\|a_j(D)[M_{g_j},p(x,D)]u\|_{M^p_q}\le C\langle j\rangle^M\|g\|_{\text{bmo}}\|u\|_{M^p_q}, \tag{6.29}$$

if $p(x,\xi)\in L^\infty S^0_{cl}$, and compactness if $g_j\in \text{vmo}$. This proves:

Proposition 6.3. *Assume $p(x,\xi),q(x,\xi)\in (L^\infty\cap vmo)S^0_{cl}$, with compact x-support. Then*

$$[p(x,D),q(x,D)] \ \textit{ is compact on } \ M^p_q(\mathbb{R}^n), \quad 1<p<\infty. \tag{6.30}$$

In fact, we have the following result, which is more precise than Proposition 6.3.

Theorem 6.4. *Assume* $p(x,\xi)$, $q(x,\xi) \in (L^\infty \cap \mathrm{vmo})S^0_{cl}$, *with compact support. Then*

$$p(x,D)q(x,D) = a(x,D) + K, \quad a(x,\xi) = p(x,\xi)q(x,\xi), \tag{6.31}$$

where K *is compact on* M^p_q.

Proof. The argument is similar to that given above. We have

$$\begin{aligned} p(x,D)q(x,D) = p_0(x,D)q(x,D) &+ p(x,D)q_0(x,D) \\ &+ \sum_{j,k} M_{f_j} a_j(D) M_{g_k}\, a_k(D), \end{aligned} \tag{6.32}$$

and the first two terms on the right are compact. The double sum is equal to

$$\sum_{j,k} M_{f_j g_k} a_j(D) a_k(D) + \sum_{j,k} M_{f_j} [a_j(D), M_{g_k}] a_k(D). \tag{6.33}$$

The first sum in (6.33) differs from $a(x,D)$ by a compact operator, and the second sum is equal to

$$\sum_j M_{f_j} [a_j(D), \tilde{q}(x,D)], \tag{6.34}$$

where $\tilde{q}(x,\xi) = q(x,\xi) - q_0(x,\xi)$. The estimate (6.28) (with the roles of $p(x,\xi)$ and $q(x,\xi)$ reversed) shows this is a norm convergent sum of compact operators, so (6.31) is proven.

It is known that $L^\infty \cap \mathrm{vmo}$ is a closed linear subspace of $L^\infty(\mathbb{R}^n)$, and also an algebra. We will sketch a proof, shown to the author by Pascal Auscher, of these two facts.

First, assume $f_j \in L^\infty \cap \mathrm{vmo}$, $f_{j\nu} \in \mathcal{B}^\infty$ (given by (1.22)). Assume $f_{j\nu} \to f_j$ in bmo-norm, and $f_j \to f$ in L^∞-norm. Then

$$\|f - f_{j\nu}\|_{\mathrm{bmo}} \le \|f - f_j\|_{L^\infty} + \|f_j - f_{j\nu}\|_{\mathrm{bmo}},$$

which implies $f \in \mathrm{vmo}$.

Next, if $f, g \in L^\infty \cap \mathrm{vmo}$ and B is some ball of radius r, then

$$r^{-n} \int_B |fg - f_B g_B|\, dx \le \|f\|_{L^\infty} r^{-n} \int_B |g - g_B|\, dx + \|g\|_{L^\infty} r^{-n} \int_B |f - f_B|\, dx,$$

which implies $fg \in \mathrm{vmo}$.

It is a general fact that, if $\mathfrak{A}$ is a C^*-algebra and $\mathfrak{B}$ a closed $*$-subalgebra od $\mathfrak{A}$, containing the identity element, and if $f \in \mathfrak{B}$, then f is invertible in $\mathfrak{B}$ if and only if f is invertible in $\mathfrak{A}$. To see this, consider $h = f^* f$ and expand $(h + 1 - z)^{-1}$ in a power series about $z = 0$.

As a consequence, if $p(x,\xi) \in (L^\infty \cap \mathrm{vmo})S^0_{cl}$ is elliptic, i.e., $|p(x,\xi)^{-1}| \le C_1$ for $|\xi| \le C_2$, then $p(x,\xi)^{-1}(1 - \varphi(\xi)) \in (L^\infty \cap \mathrm{vmo})S^0_{cl}$, where $\varphi(\xi)$ is an appropriate cut-off.

We next consider a "parametrix" for an elliptic PDE with vmo coefficients. Consider an operator of the form

$$Lu = \sum a_{jk}(x)\partial_j\partial_k u. \tag{6.35}$$

Assume

$$a_{jk} \in L^\infty \cap \mathrm{vmo}, \quad A^{-1}|\xi|^2 \le \sum a_{jk}(x)\xi_j\xi_k \le A|\xi|^2, \tag{6.36}$$

for some $A \in (0,\infty)$. Then form

$$B(x,\xi) = -\Bigl(\sum a_{jk}(x)\xi_j\xi_k\Bigr)^{-1}\bigl(1-\varphi(\xi)\bigr) \in (L^\infty \cap \mathrm{vmo})S^{-2}_{cl}, \tag{6.37}$$

where $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\varphi(\xi) = 1$ for $|\xi| \le 1$. Thus

$$B_{jk}(x,\xi) = B(x,D)\partial_j\partial_k \in OP(L^\infty \cap \mathrm{vmo})S^0_{cl}. \tag{6.38}$$

The following result arose in [CFL], and was also used in [DR2]:

Lemma 6.5. *If $u \in H^{2,p}$ has compact support, then*

$$\partial_j\partial_k u = B_{jk}(x,D)Lu + \sum_{\ell,m}[M_{a_{\ell m}}, B_{jk}(x,D)]\partial_\ell\partial_m u + R_{jk}u, \tag{6.39}$$

where

$$R_{jk}u = \varphi(D)\partial_j\partial_k u \in C^\infty. \tag{6.40}$$

Proof. The right side of (6.39), with $R_{jk}u$ omitted, is equal to

$$\begin{aligned} &\sum_{\ell,m} a_{\ell m}(x)B_{jk}(x,D)\partial_\ell\partial_m u \\ &= \int \frac{\sum\limits_{\ell,m} a_{\ell m}(x)\xi_\ell\xi_m}{\sum\limits_{\mu,\nu} a_{\mu\nu}(x)\xi_\mu\xi_\nu}\bigl(1-\varphi(\xi)\bigr)\xi_j\xi_k\widehat{u}(\xi)e^{ix\cdot\xi}\,d\xi, \end{aligned} \tag{6.41}$$

and the fraction is equal to 1.

We now examine Fredholm properties of L. For simplicity, let us suppose u is defined on the torus $\mathbb{T}^n$. Set

$$E = (1-\Delta)^{-1}B(x,D)(1-\Delta). \tag{6.42}$$

We have, under the standing assumption $1 < q \le p < \infty$,

$$E : M^p_q \longrightarrow M^{p,2}_q. \tag{6.43}$$

Proposition 6.6. *Under the hypotheses (6.35)–(6.36), E is a two-sided Fredholm inverse of L.*

Proof. If we sum (6.39) over $j = k = 1, \dots, n$, we get

$$\Delta u = B(x,D)\Delta L u + \sum_{\ell,m} [M_{a_{\ell m}}, B(x,D)\Delta]\partial_\ell \partial_m u + \varphi(D)\Delta u, \tag{6.44}$$

hence, for $u \in M^{p,2}_q$,

$$u = ELu + (1-\Delta)^{-1}\mathcal{K}(1-\Delta)u = ELu + \widetilde{\mathcal{K}}u, \tag{6.45}$$

with

$$\mathcal{K} : M^p_q \longrightarrow M^p_q \ \text{ compact}, \tag{6.46}$$

as a consequence of Proposition 6.2, hence

$$\widetilde{\mathcal{K}} : M^{p,2}_q \longrightarrow M^{p,2}_q \ \text{ compact}. \tag{6.47}$$

Thus E is a left Fredholm inverse of L.

On the other hand,

$$LE = L(1-\Delta)^{-1}B(x,D)(1-\Delta) = P(x,D)Q(x,D), \tag{6.48}$$

with

$$P(x,\xi) = -\sum a_{jk}(x)\xi_j\xi_k\langle\xi\rangle^{-2}, \quad Q(x,\xi) = P(x,\xi)^{-1}(1-\varphi(\xi)), \tag{6.49}$$

both symbols belonging to $(L^\infty \cap \text{vmo})S^0_{cl}$. Thus Theorem 6.4 implies $P(x,D)Q(x,D) = I + \mathcal{K}_2$, with $\mathcal{K}_2$ compact. Hence E is a two-sided Fredholm inverse of L.

7. Morrey-space estimates for wave equations

Proposition 7.1. *Assume n is odd. Let $w(t,x)$ solve the Cauchy problem*

$$(\partial_t^2 - \Delta)w = 0, \quad w(0) = f, \quad w_t(0) = 0, \tag{7.1}$$

on $\mathbb{R} \times \mathbb{R}^n$. If $f \in L^\infty(\mathbb{R}^n)$, then, for $z \in \mathbb{R}^n$, $\rho \in (0,1]$,

$$\|w(t,\cdot)\|_{L^2(B_\rho(z))} \le C\langle t\rangle \|f\|_{L^\infty}\rho^{1/2}. \tag{7.2}$$

Proof. By the strong Huygens principle, the value of $w(t,x)$ for $x \in B_\rho(z)$ is unaffected if f is replaced by

$$\begin{aligned} f^\#(x) = f(x) \quad &\text{if } \ |t| - 2\rho \le |x-z| \le |t| + 2\rho, \\ 0 \quad &\text{otherwise}. \end{aligned} \tag{7.3}$$

Clearly

$$\|f^{\#}\|_{L^2(\mathbb{R}^n)} \le C\langle t\rangle \rho^{1/2} \|f\|_{L^\infty}, \tag{7.4}$$

so

$$w^{\#}(t) = \cos t\sqrt{-\Delta}\, f^{\#} \Longrightarrow \|w^{\#}(t)\|_{L^2} \le C\langle t\rangle \rho^{1/2}\|f\|_{L^\infty}. \tag{7.5}$$

Since $w(t,x) = w^{\#}(t,x)$ for $x \in B_\rho(z)$, we have (7.2).

Note that Proposition 7.1 can be stated in terms of a Morrey space:

$$\cos t\sqrt{-\Delta} : L^\infty(\mathbb{R}^n) \longrightarrow M_2^{2n/(n-1)}(\mathbb{R}^n). \tag{7.6}$$

More generally, we can replace L^∞ by $L^p,\ 2 \le p \le \infty$, obtaining

$$\|w(t,\cdot)\|_{L^2(B_\rho(z))} \le C(t)\|f\|_{L^p}\, \rho^{1/2-1/p}, \tag{7.7}$$

so, for $p \ge 2, n$ odd,

$$\cos t\sqrt{-\Delta} : L^p(\mathbb{R}^n) \longrightarrow M_2^q(\mathbb{R}^n), \quad q = \frac{2n}{n-1+2/p}. \tag{7.8}$$

We now extend Proposition 7.1 to the case where Δ is the Laplace operator on a complete Riemannian manifold M, with bounded geometry, whose dimension n is odd. In such a case, there exists $\tau \in (0,\infty]$ such that the solution to (7.1) can be written

$$w(t) = R'(t)f, \tag{7.9}$$

and, for $|t| < \tau$,

$$R(t) = R_0(t) + B(t), \tag{7.10}$$

where $R_0(t)$ and $B(t)$ have the following properties. First, the Schwartz kernel of $R_0(t)$ is supported on the "light cone" $\{(t,x,y) : \mathrm{dist}(x,y) = |t|\}$. Next, for $|t| < \tau,\ B(t)$ is a family of FIOs of order -2, and $B'(t)$ is a family of FIOs of order -1, having the mapping properties

$$B(t) : H^s(M) \to H^{s+2}(M), \quad B'(t) : H^s(M) \to H^{s+1}(M). \tag{7.11}$$

We can now prove the following extension of Proposition 7.1.

Proposition 7.2. *Let $w(t,z)$ solve the Cauchy problem (7.1) on $\mathbb{R}\times M$, where M is a complete Riemannian manifold with bounded geometry whose dimension n is odd. There exists $\tau \in (0,\infty]$ such that, if $p \in [2,\infty]$ and $f \in L^p(M)$, then, for $|t| < \tau,\ z \in M$, and $\rho \in (0,1]$, we have*

$$\|w(t,\cdot)\|_{L^2(B_\rho(z))} \le C(t)\|f\|_{L^p}\, \rho^{1/2-1/p}. \tag{7.12}$$

Proof. Defining $f^{\#}$ as in (7.3), but using $\mathrm{dist}(x,z)$, we see that

$$R_0'(t)f = R_0'(t)f^{\#}, \quad \text{for } x \in B_\rho(z), \tag{7.13}$$

and since $R_0'(t) : L^2(M) \to L^2(M)$, we obtain

$$\|R_0'(t)f\|_{L^2(B_\rho(z))} \le C(t)\|f^\#\|_{L^2} \le C(t)\|f\|_{L^p}\,\rho^{1/2-1/p}. \tag{7.14}$$

Meanwhile, by finite propagation speed and (7.11) we have

$$\|B'(t)f\|_{H^1(B_1(z))} \le C(t)\|f\|_{L^p}, \tag{7.15}$$

and (7.12) follows from (7.14)–(7.15), since

$$H^1(B_1(z)) \subset L^{2n/(n-2)}(B_1(z)) \subset M_2^{2n/(n-2)}(B_1(z)). \tag{7.16}$$

A statement equivalent to (7.12) is

$$\|w(t)\|_{M_2^q} \le C(t)\|f\|_{L^p}, \quad q = \frac{2n}{n-1+2/p}. \tag{7.17}$$

8. Conormal spaces and variants

We now define a class of spaces that includes "conormal spaces." Let $\mathcal{M}$ be a collection of vector fields in $\mathbb{R}^n$ (which may or may not be smooth everywhere). If $J = (j_\ell, \dots, j_1)$, we set $X^J = X_{j_\ell} \cdots X_{j_1}$, and we set $|J| = \ell$. By convention, $X^\emptyset u = u$. Assume $1 < q \le p < \infty,\ k \in \mathbb{Z}^+,\ s \in \mathbb{R}$. We say $u \in N^k(M_q^{p,s}, \mathcal{M})$ if $u \in M_q^{p,s}$ and

$$X_{j_\nu} \in \mathcal{M},\ |J| \le k \Longrightarrow X^J u \in M_q^{p,s}. \tag{8.1}$$

Important special cases include the following. Suppose $\Sigma \subset \mathbb{R}^n$ is a smooth submanifold and $\mathcal{M}$ consists of all smooth vector fields (well behaved at infinity) which are tangent to Σ. We denote the space defined above by $N^k(M_q^{p,s}, \Sigma)$ in this case. Compare the definition of $N^{k,s}(\Sigma)$ in [Be], p. 52. As another example, $\mathcal{F}$ could be a smooth foliation of $\mathbb{R}^n$ (by submanifolds of dimension d), and $\mathcal{M}$ could consist of all smooth vector fields tangent to $\mathcal{F}$. We denote the resulting space by $N^k(M_q^{p,s}, \mathcal{F})$. An example of a collection $\mathcal{M}$ of vector fields X_j which are smooth on $\mathbb{R}^n \setminus 0$ is given in [Be], p. 119.

Let f be smooth; we want to estimate $X^J f(u)$, for $u \in N^k(M_q^{p,s}, \mathcal{M})$. Repeated application of the chain rule gives

$$X^J f(u) = \sum_{I_1+\cdots+I_\ell=J} C_I (X^{I_1}u)\cdots(X^{I_\ell}u)\, f^{(\ell)}(u). \tag{8.2}$$

If we set

$$g(u, X^{I_1}u, \dots, X^{I_\ell}u) = (X^{I_1}u)\cdots(X^{I_\ell}u) f^{(\ell)}(u), \tag{8.3}$$

then we know that, for $s > 0$,

$$\begin{aligned} &\|g(u, v_{I_1}, \dots, v_{I_\ell})\|_{M_q^{p,s}} \\ &\quad \le C(\|u\|_{L^\infty}, \|v_{I_1}\|_{L^\infty}, \dots, \|v_{I_\ell}\|_{L^\infty}) \\ &\qquad \cdot \big(1 + \|u\|_{M_q^{p,s}} + \|v_{I_1}\|_{M_q^{p,s}} + \cdots + \|v_{I_\ell}\|_{M_q^{p,s}}\big). \end{aligned} \tag{8.4}$$

Hence we have the estimate:

$$\|X^J f(u)\|_{M_q^{p,s}} \le C(\|X^I u\|_{L^\infty} : I \le J) \cdot \Big(1 + \sum_{I \le J} \|X^I u\|_{M_q^{p,s}}\Big). \tag{8.5}$$

While we have briefly allowed the possibility that $\mathcal{M}$ contains nonsmooth vector fields, we will henceforth assume that all the vector fields in $\mathcal{M}$ are smooth, with coefficients that are bounded on $\mathbb{R}^n$, together with all their derivatives. We will also adopt the standing assumption that

$$1 < q \le p < \infty. \tag{8.6}$$

Proposition 8.1. *Given* $P \in OPS^m_{1,0}$*,* $s, m \in \mathbb{R}$*,*

$$P : N^k(M_q^{p,s}, \mathcal{M}) \longrightarrow N^k(M_q^{p,s-m}, \mathcal{M}). \tag{8.7}$$

Proof. For any $X \in \mathcal{M}$, we have

$$XPu = PXu + P_X u, \tag{8.8}$$

where $P_X = [X, P] \in OPS^m_{1,0}$; in fact, if $X = \sum a_j(x)\partial/\partial x_j$,

$$P_X(x,\xi) = X \cdot P(x,\xi) + P_X^b(x,\xi), \tag{8.9}$$

where

$$\begin{aligned} X \cdot P(x,\xi) &= \sum a_j(x) \partial_{x_j} P(x,\xi), \\ P_X^b(x,\xi) &\sim \sum_{|\alpha| \ge 1} \frac{i^{|\alpha|}}{\alpha!} P^{(\alpha)}(x,\xi) \partial_x^\alpha X(x,\xi). \end{aligned} \tag{8.10}$$

Inductively, we obtain

$$X^J P u = \sum_{I \le J} P_{J\setminus I} X^I u, \quad P_{J\setminus I} \in OPS^m_{1,0}. \tag{8.11}$$

Given this, (8.7) follows from (2.4).

In fact, using (8.8)–(8.9), we can say more. Let us say that

$$p(x,\xi) \in (\mathcal{M}^k) S^m_{1,\delta} \tag{8.12}$$

provided $p(x,\xi) \in S^m_{1,\delta}$ and

$$X^J \cdot p(x,\xi) \in S^m_{1,\delta}, \quad \forall\ |J| \le k. \tag{8.13}$$

Similarly define $(\mathcal{M}^k)\mathcal{B}S^m_{1,1}$ to consist of $p(x,\xi) \in \mathcal{B}S^m_{1,1}$ such that $X^J \cdot p(x,\xi) \in \mathcal{B}S^m_{1,1}$ whenever $|J| \le k$. Replacing the use of (2.4) by that of (2.12) and (2.24), and noting that (8.9)–(8.10) is valid even for $P \in OPS^m_{1,1}$, we have:

Proposition 8.2. *Given $P \in OP(\mathcal{M}^k)\mathcal{B}S^m_{1,1}$, the property (8.7) holds. Furthermore, given $P \in OP(\mathcal{M}^k)S^m_{1,1}$, (8.7) holds provided $s - m > 0$.*

Note that we can substitute other spaces for $M^{p,s}_q$ in (8.1), producing such spaces as $N^k(C^s_*, \mathcal{M})$, for which we have analogues of Propositions 8.1–8.2.

Next, given a smooth function F, write

$$F(u) = M_F(u; x, D)u + F(u_0), \tag{8.14}$$

as in (1.8)–(1.9).

Proposition 8.3. *If $u \in N^k(C^r_*, \mathcal{M}),\ r > 0$, then*

$$M_F(u; x, \xi) \in (\mathcal{M}^k)\mathcal{A}^r_* S^m_{1,1}. \tag{8.15}$$

Here, $\mathcal{A}^r_* S^m_{1,\delta} \subset S^m_{1,\delta}$ consists of symbols satisfying

$$\|D_\xi p(\cdot, \xi)\|_{C^s_*} \le C_s \langle \xi \rangle^{m - |\alpha| + \delta(s-r)}, \quad s \ge r. \tag{8.16}$$

Proof. Using (1.9), we need to estimate

$$m_\ell(x) = \int_0^1 F'(u_{\ell,\tau})\, d\tau, \quad u_{\ell,\tau} = \Psi_\ell(D)u + \tau \psi_{\ell+1}(D)u. \tag{8.17}$$

The analogue of (8.5), with $M^{p,s}_q$ replaced by C^s_*, is

$$\|X^J F'(u_{\ell,\tau})\|_{C^s_*} \le C(\|X^I u_{\ell,\tau}\|_{L^\infty} : I \le J) \cdot \Bigl(1 + \sum_{I \le J} \|X^I u_{\ell,\tau}\|_{C^s_*}\Bigr). \tag{8.18}$$

To proceed, we use the following:

Lemma 8.4. *If $u \in N^k(C^r_*, \mathcal{M}),\ r > 0$, then*

$$\|X^I \Psi_\ell(D)u\|_{C^s_*} \le C_{s,I} \cdot 2^{\ell(s-r)}, \quad s \ge r,\ |I| \le k. \tag{8.19}$$

We will establish this after using it to prove the proposition. In fact, we now deduce from (8.18) that

$$\|X^J m_\ell\|_{C^s_*} \le C_{J,s} \cdot 2^{\ell(s-r)}, \quad s \ge r,\ |J| \le k. \tag{8.20}$$

Since $\psi_{\ell+1}(\xi)$ in (1.9) is supported on $\langle \xi \rangle \sim 2^\ell$, we have (8.16), and Proposition 8.3 is established, modulo a proof of Lemma 8.4.

To prove Lemma 8.4, we can treat $X^I \Psi_\ell(D)$ as in (8.8)–(8.11), obtaining

$$X^I \Psi_\ell(D) = \sum_{K \le I} \psi_{I \setminus K, \ell}(x, D) X^K. \tag{8.21}$$

Furthermore,

$$\psi_{I\setminus K,\ell}(x, D) - \psi_{I \setminus K, \ell}(x, D)\Psi_{\ell+3}(D) \tag{8.22}$$

is bounded in $OPS^{-\infty}_{1,0}$. Since $X^K u \in C^r_*$ for $|K| \le k$, (8.19) follows from:

$$v \in C^r_* \Longrightarrow \|\Psi_{\ell+3}(D)v\|_{C^s_*} \le C_s \cdot 2^{\ell(s-r)}, \quad s \ge r, \tag{8.23}$$

which is elementary.

A. A Schwartz kernel estimate

Here we want to prove that, if $p(x,\xi) \in S^0_{1,1}(\mathbb{R}^n)$, then the operator product

$$\Lambda^s p(x,D)\Lambda^{-s} = P_s \tag{A.1}$$

has Schwartz kernel K_s satisfying

$$|K_s(x,y)| \le C_s|x-y|^{-n}, \tag{A.2}$$

for $0 < s < 1$. Note that P_s defines a bounded linear operator on $H^{\sigma,p}(\mathbb{R}^n)$ for all $\sigma > -s$, $p \in (1,\infty)$. However, P_s is perhaps not an element of $OPS^0_{1,1}(\mathbb{R}^n)$. Of course, (A.2) clearly holds for $s = 0$, as a consequence of the implication

$$|D^\alpha_\xi A(\xi)| \le C_\alpha|\xi|^{\tau-|\alpha|} \Longrightarrow |\widehat{A}(x)| \le C_\tau|x|^{-n-\tau}, \quad \tau > -n. \tag{A.3}$$

Recall that Λ^s is Fourier multiplication by $\langle\xi\rangle^s$. It will be convenient for the dilation argument we intend to apply, to replace Λ^s by λ^s, Fourier multiplication by $|\xi|^s$. We will show that

$$\widetilde{P}_s = \lambda^s p(x,D)\lambda^{-s} \tag{A.4}$$

has Schwartz kernel $\widetilde{K}_s$ satisfying

$$|\widetilde{K}_s(x,y)| \le C_s|x-y|^{-n}. \tag{A.5}$$

It is clear that

$$\widetilde{P}_s - P_s : H^{\sigma,p}_{\text{comp}}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n), \quad \sigma > -s,\ 1 < p < \infty, \tag{A.6}$$

provided $0 < s < n$, and hence (A.5) readily implies (A.2).

To prove (A.5), we will examine

$$\vartheta_r K_s(x,y) = K_s(rx,ry), \tag{A.7}$$

which is the Schwartz kernel of

$$\widetilde{P}_{sr} = r^{-n}\delta_r\widetilde{P}_s\delta_r^{-1}; \quad \delta_r f(x) = f(rx). \tag{A.8}$$

We will show that, for $r \in (0,1]$,

$$|\vartheta_r K_s(x,y)| \le Cr^{-n} \text{ on } \Omega = \{(x,y) : 1 \le |x-y| \le 2\}, \tag{A.9}$$

which implies (A.5) for $|x-y|\le 1$, hence (A.2) for $|x-y|\le 1$. It is relatively easy to show that $K_s(x,y)$ is rapidly decreasing as $|x-y|\to\infty$, so this will suffice. Now, since $\lambda^s\delta_r^{-1} = r^{-s}\delta_r^{-1}\lambda^s$, we have

$$\vartheta_r K_s(x,y) = r^{-n}\widetilde{K}_{s,r}(x,y) \tag{A.10}$$

where

$$\widetilde{K}_{s,r}(x,y) = \text{ Schwartz kernel of } \lambda^s\delta_r p(x,D)\delta_r^{-1}\lambda^{-s}, \tag{A.11}$$

or, setting

$$\kappa_r(x,y) = \text{ Schwartz kernel of } p_r(x,D) = \delta_r p(x,D)\delta_r^{-1}, \tag{A.12}$$

we have

$$\widetilde{K}_{s,r}(x,y) = \lambda_x^s\lambda_y^{-s}\kappa_r(x,y), \tag{A.13}$$

and we want to show that

$$|\lambda_x^s\lambda_y^{-s}\kappa_r(x,y)| \le C_s \text{ on } \Omega, \quad 0<r\le 1, \tag{A.14}$$

with C_s independent of r.

Note that the symbol of $p_r(x,D)$ is

$$p_r(x,\xi) = p(rx,\xi/r), \tag{A.15}$$

which satisfies

$$\begin{aligned} |D_x^\beta D_\xi^\alpha p_r(x,\xi)| &\le C_{\alpha\beta} r^{|\beta|-|\alpha|}\langle\xi/r\rangle^{|\beta|-|\alpha|} \\ &= C_{\alpha\beta}\big(r^2+|\xi|^2\big)^{(|\beta|-|\alpha|)/2} \\ &\le C_{\alpha\beta}\langle\xi\rangle^{|\beta|}|\xi|^{-|\alpha|}, \quad \text{for } 0<r\le 1. \end{aligned} \tag{A.16}$$

Hence, by (A.3),

$$|\kappa_r(x,y)| \le C|x-y|^{-n}, \tag{A.17}$$

with C independent of $r\in(0,1]$.

Similarly, $\lambda_y^{-s}\kappa_r(x,y)$ is the Schwartz kernel of $q_r(x,D) = p_r(x,D)\lambda^{-s}$, with symbol

$$q_r(x,\xi) = p(rx,\xi/r)|\xi|^{-s} \tag{A.18}$$

satisfying

$$|D_x^\beta D_\xi^\alpha q_r(x,\xi)| \le C_{\alpha\beta}\langle\xi\rangle^{|\beta|}|\xi|^{-s-|\alpha|}, \tag{A.19}$$

which implies

$$|\lambda_y^{-s}\kappa_r(x,y)| \le C_s|x-y|^{s-n}, \tag{A.20}$$

and more generally

$$|D_x^\beta D_y^\gamma \lambda_y^{-s}\kappa_r(x,y)| \le C_{s\alpha\beta}|x-y|^{s-n-|\beta|-|\gamma|}, \tag{A.21}$$

provided $s<n$. The estimate (A.14) is a simple consequence of this.

B. Another proof of Proposition 2.2

Here we include a self-contained proof of:

Proposition B.1. *Assume the Schwartz kernel $k(x,y)$ of T satisfies*

$$|k(x,y)| \le C|x-y|^{-n}\big(1+|x-y|\big)^{-M} \tag{B.1}$$

for some $M>0$. Then, if $1<q\le p<\infty$,

$$T: L^q(\mathbb{R}^n) \to L^q(\mathbb{R}^n) \Longrightarrow T: M^p_q(\mathbb{R}^n) \to M^p_q(\mathbb{R}^n). \tag{B.2}$$

Proof. Let $f \in M^p_q(\mathbb{R}^n)$. Pick $z \in \mathbb{R}^n,\ r\in(0,1]$, and write

$$f = f_0 + \sum_{2^j r \le 1} g_j + h, \tag{B.3}$$

where

$$f_0 = \chi_{B_{2r}(z)} f, \quad g_j = \chi_{A_{rj}} f, \quad A_{rj} = \{x : |x-z| \in [2^j r, 2^{j+1} r]\}, \tag{B.4}$$

and $j \ge 1$ in the sum. We want to estimate Tf on $B_r(z)$. Clearly

$$\|Tf_0\|_{L^q(\mathbb{R}^n)} \le C\|f_0\|_{L^q(\mathbb{R}^n)} \le Cr^a, \quad a = \frac{n}{q} - \frac{n}{p}, \tag{B.5}$$

and the estimate (B.1) for $|x-y|\ge 1$ implies

$$\|Th\|_{L^\infty(B_{1/2}(z))} \le C\|h\|_{M^p_q(\mathbb{R}^n)}. \tag{B.6}$$

It remains to estimate $\sum Tg_j$ on $B_r(z)$. To do this, write

$$\chi_{B_r(z)} Tg_j = T_j(\chi_{A_{rj}} f), \tag{B.7}$$

where T_j has integral kernel

$$k_j(x,y) = \chi_{B_r(z)} k(x,y) \chi_{A_{rj}}(y). \tag{B.8}$$

Now, using (B.1) for $|x-y| \le 1$, we have

$$\begin{aligned} \int |k_j(x,y)|\,dx &= \int_{B_r(z)} |k(x,y)|\chi_{A_{rj}}(y)\,dx \\ &\le C(2^j r)^{-n} \cdot \ \text{vol}\ B_r(z) \\ &\le C2^{-jn}, \end{aligned} \tag{B.9}$$

and

$$\begin{aligned} \int |k_j(x,y)|\,dy &= \int_{A_{rj}} \chi_{B_r(z)} |k(x,y)|\,dy \\ &\le C(2^j r)^{-n} \cdot \ \text{vol}\ A_{rj} \\ &\le C. \end{aligned} \tag{B.10}$$

Hence, if $f \in M^p_q(\mathbb{R}^n)$,

(B.11)
$$\begin{aligned}\|T_j(\chi_{A_{rj}}f)\|_{L^q} &\le C2^{-jn/q}\|\chi_{A_{rj}}f\|_{L^q} \\ &\le C2^{-jn/q}(2^j r)^a \\ &\le C2^{-jn/p}r^a,\end{aligned}$$

so, if $p < \infty$,

(B.12)
$$\sum \|Tg_j\|_{L^q(B_r(z))} \le C\Big(\sum_{j\ge 1} 2^{-jn/p}\Big) r^a \le C' r^a,$$

as desired. This completes the proof.

If $f \in \mathcal{M}^p_q(\mathbb{R}^n)$, one can replace (B.3) by

$$f = f_0 + \sum_{j=1}^{\infty} g_j$$

and repeat the estimates above, obtaining:

Proposition B.2. *Assume the Schwartz kernel $k(x,y)$ of T satisfies*

(B.13)
$$|k(x,y)| \le C|x-y|^{-n}.$$

Then, if $1 < q \le p < \infty$,

(B.14)
$$T : L^q(\mathbb{R}^n) \to L^q(\mathbb{R}^n) \Longrightarrow T : \mathcal{M}^p_q(\mathbb{R}^n) \to \mathcal{M}^p_q(\mathbb{R}^n).$$

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