Gibbs Phenomena and Pinsky Phenomena for Nonlinear Schrödinger Equations

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Contents

1. Linear Schrödinger equation

Fourier transform

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx. \tag{1.1}$$

Fourier inversion formula

$$f(x) = (2\pi)^{-n/2} \int \hat{f}(\xi) e^{ix\cdot\xi}\, d\xi. \tag{1.2}$$

Heat semigroup

$$\begin{aligned} e^{t\Delta} f(x) &= (2\pi)^{-n/2} \int e^{-t|\xi|^2} \hat{f}(\xi) e^{ix\cdot\xi}\, d\xi \\ &= (4\pi t)^{-n/2} \int e^{-|y|^2/4t} f(x-y)\, dy. \end{aligned} \tag{1.3}$$

Analytic continuation yields

$$\begin{aligned} e^{it\Delta} f(x) &= (2\pi)^{-n/2} \int e^{-it|\xi|^2} \hat{f}(\xi) e^{ix\cdot\xi}\, d\xi \\ &= (4\pi it)^{-n/2} \int e^{-|y|^2/4it} f(x-y)\, dy. \end{aligned} \tag{1.4}$$

Generalization

$$\begin{aligned} L &= a_1\partial_1^2 + \cdots + a_n\partial_n^2, \quad a_j \in \mathbb{R}\setminus 0, \\ e^{itL} f(x) &= (2\pi)^{-n/2} \int e^{-itQ_a(\xi)} \hat{f}(\xi) e^{ix\cdot\xi}\, d\xi \\ &= A^{-1/2}(4\pi it)^{-n/2} \int e^{-Q_a(y)/4it} f(x-y)\, dy, \end{aligned} \tag{1.5}$$

where

$$\begin{aligned} Q^a(\xi) &= a_1\xi_1^2 + \cdots + a_n\xi_n^2, \\ Q_a(y) &= \frac{y_1^2}{a_1} + \cdots + \frac{y_n^2}{a_n}, \quad A = a_1\cdots a_n. \end{aligned}$$

Application to stationary phase method:

$$I(\lambda) = \int\limits_{\Omega \subset \mathbb{R}^n} e^{i\lambda\psi(x)} a(x)\, dx, \quad a \in C_0^\infty(\Omega). \tag{1.6}$$

One non-degenerate critical point.
Make change of variable so $\psi(x) = Q_a(x)$. Take $\lambda = 1/4t$. Then

$$I(\lambda) = (a_1 \cdots a_n)^{1/2} \Big(\frac{\pi i}{\lambda}\Big)^{n/2} e^{itL} f(0), \tag{1.7}$$

where $f \in C_0^\infty(\mathbb{R}^n)$. We have

$$e^{itL} f \longrightarrow f \ \text{ in } \ \mathcal{S}(\mathbb{R}^n), \tag{1.8}$$

as $t \to 0$.

2. Fresnel integral and "Gibbs phenomenon"

Take $n = 1,\ f = \chi_I,\ I = [-1, 1]$. We have

$$\begin{aligned} e^{it\Delta}\chi_I(x) &= (4\pi i t)^{-1/2} \int_{x-1}^{x+1} e^{-y^2/4it}\, dy \\ &= \mathrm{Fr}\Big(\frac{x+1}{\sqrt{4t}}\Big) - \mathrm{Fr}\Big(\frac{x-1}{\sqrt{4t}}\Big), \end{aligned} \tag{2.1}$$

where $\mathrm{Fr}(\lambda)$ is the Fresnel integral:

$$\begin{aligned} \mathrm{Fr}(\lambda) &= \frac{1}{\sqrt{\pi i}} \int_0^{\lambda} e^{iy^2}\, dy \\ &= \frac{1}{2} \frac{\lambda}{\sqrt{\pi i}} \int_{-1}^{1} e^{i\lambda^2 y^2}\, dy. \end{aligned} \tag{2.2}$$

Stationary phase applies at $y = 0$.
Change of variable and Fourier analysis apply at $y = \pm 1$.
Get Fourier transform, at frequency λ^2, of function with jump.
Asymptotics:

$$\mathrm{Fr}(\lambda) - \frac{1}{2}\,\mathrm{sgn}\,\lambda \sim e^{i\lambda^2} \sum_{\nu \ge 0} a_\nu \lambda^{-1-2\nu}, \tag{2.3}$$

as $|\lambda| \to \infty$.
Another formula:

$$\begin{aligned} \mathrm{Fr}(\lambda) &= \frac{1}{2} \frac{1}{\sqrt{4\pi i t}} \int_{-1}^{1} e^{iy^2/4t}\, dy \\ &= \frac{1}{2} e^{it\partial_y^2} \chi_{[-1,1]}(0), \quad \lambda = \sqrt{\frac{1}{4t}} > 0. \end{aligned} \tag{2.4}$$

3. Pinsky phenomenon for $e^{it\Delta}$ – first look

Using spherical polar coordinates gives

$$
\begin{aligned}
e^{it\Delta}f(x) &= A_{n-1}(4\pi it)^{-n/2}\int_0^\infty \overline{f}_x(r)e^{ir^2/4t}r^{n-1}\,dr \\
&= \frac{A_{n-1}}{2}(4\pi it)^{-n/2}\int_0^\infty \overline{f}_x(s^{1/2})e^{is/4t}s^{n/2-1}\,ds,
\end{aligned}
\tag{3.1}
$$

where

$$
\overline{f}_x(r) = \text{ mean value of } f \text{ on } \partial B_r(x).
\tag{3.2}
$$

Case $n = 2$:

$$
\begin{aligned}
e^{it\Delta}f(x) &= (4it)^{-1}\int_0^\infty \overline{f}_x(s^{1/2})e^{is/4t}\,ds \\
&= \overline{f}_x(0) + \int_0^\infty \frac{\partial}{\partial s}\overline{f}_x(s^{1/2})e^{is/4t}\,ds.
\end{aligned}
\tag{3.3}
$$

Special case:

$$
\begin{aligned}
e^{it\Delta}\chi_D(0) &= 1 + \int_0^\infty \frac{\partial}{\partial s}\chi_{[0,1]}(s)e^{is/4t}\,ds \\
&= 1 - e^{i/4t},
\end{aligned}
\tag{3.4}
$$

where

$$
D = \{x \in \mathbb{R}^2 : |x| \le 1\}.
\tag{3.5}
$$

Have oscillatory divergence as $t \to 0$.
Remark. For Fourier inversion, oscillatory divergence at $n = 3$.

4. Basic asymptotics

Take $\Omega \subset \mathbb{R}^n$, smoothly bounded, and

$$f(x) = F(x)\chi_\Omega(x), \quad F \in C_0^\infty(\mathbb{R}^n). \tag{4.1}$$

Then

$$\begin{aligned} e^{it\Delta} f(x) &= (4\pi i t)^{-n/2} \int\limits_\Omega e^{i|x-y|^2/4t} f(y)\, dy \\ &= C\lambda^{n/2} \int\limits_\Omega e^{i\lambda\psi(x,y)} f(y)\, dy, \end{aligned} \tag{4.2}$$

with

$$\psi(x,y) = |x-y|^2, \quad \lambda = \frac{1}{4t}. \tag{4.3}$$

Note that

$$f \in C_0^\infty(\Omega) \Longrightarrow e^{it\Delta} f \to f \ \text{ in } \ \mathcal{S}(\mathbb{R}^n), \tag{4.4}$$

as $t \to 0$, so we can focus on the case where

$$\operatorname{supp} f \subset \overline{\Omega} \cap \mathcal{O}, \quad \mathcal{O} = B_\varepsilon(p),\ p \in \partial\Omega. \tag{4.5}$$

First take $x \in K$, a compact set disjoint from $\overline{\mathcal{O}} \cap \overline{\Omega}$.
Supress the x dependence in (4.2), and write

$$\begin{aligned} I(f,\lambda) &= \int\limits_\Omega e^{i\lambda\psi(y)} f(y)\, dy \\ &= \int\limits_{\partial\Omega} \int_0^1 e^{i\lambda\psi(s,z)} f(s,z) J(s,z)\, ds\, dS(z). \end{aligned} \tag{4.6}$$

Lemma 4.1. *If there exists a vector field X, tangent to $\partial\Omega$, such that $X\psi \neq 0$ on $\mathcal{O}$, then*

$$I(f,\lambda) = O(\lambda^{-k}), \quad \forall\, k, \tag{4.7}$$

as $|\lambda| \to \infty$.

Proof. $e^{i\lambda\psi} = (i\lambda X\psi)^{-1} X e^{i\lambda\psi} = L(\lambda) e^{i\lambda\psi}$, so

$$I(f,\lambda) = \int\limits_\Omega e^{i\lambda\psi(y)} (L(\lambda)^t)^k f(y)\, dy = O(\lambda^{-k}). \tag{4.8}$$

If Lemma 4.1 does not apply, then we can assume

$$\partial_s \psi(s,z) \neq 0 \quad \text{on} \quad \mathcal{O}. \tag{4.9}$$

Then elementary Fourier analysis gives

$$\begin{aligned} &\int_0^1 e^{i\lambda\psi(s,z)} f(s,z) J(s,z)\, ds \\ &\sim e^{i\lambda\psi(0,z)} \sum_{k\geq 0} a_k(z) \lambda^{-1-k}. \end{aligned} \tag{4.10}$$

Plugging into (4.6) and restoring x-dependence, we get

$$\begin{aligned} e^{it\Delta} f(x) \sim &\sum_{k\geq 0} \lambda^{n/2-1-k} e^{i\lambda|x|^2} \\ &\times \int_{\partial\Omega} e^{i\lambda(|z|^2 - 2x\cdot z)} a_k(x,z) dS(z), \end{aligned} \tag{4.11}$$

as

$$t = \frac{1}{4\lambda} \to 0.$$

Valid uniformly for $x \in K$, compact set disjoint from supp f.

The analysis of (4.11) splits into several cases.

A. Non-caustic region $\mathcal{C}_0$

$\mathcal{C}_0 =$ set of x such that $\psi_x(z) = \psi(x,z)$, as a function of $x \in \partial\Omega$, has only non-degenerate critical points, say at $p_\ell(x) \in \partial\Omega$. Stationary phase method yields

$$\begin{aligned} &\int_{\partial\Omega} e^{i\lambda\psi(x,z)} a_k(x,z)\, dS(z) \\ &\sim \lambda^{-(n-1)/2} \sum_\ell e^{i\lambda\psi(x,p_\ell(x))} \sum_{m\geq 0} a_{k\ell m}(x) \lambda^{-m}. \end{aligned} \tag{4.12}$$

Hence, on $\mathcal{C}_0$,

$$e^{it\Delta} f(x) \sim \sum_\ell e^{i\psi(x,p_\ell(x))/4t} \sum_{k\geq 0} b_{k\ell}(x) t^{k+1/2}. \tag{4.13}$$

Leading term is $O(t^{1/2})$. (Recall $x \neq \operatorname{supp} f$.)

B. Fold set, $\mathcal{C}_1$

$$e^{it\Delta}f(x) \sim e^{i\theta(x)/4t}\Big[Ai\Big(\frac{\rho(x)}{(4t)^{2/3}}\Big)\sum_{k\geq 0} b_{0k}(x)t^{k+1/3} + Ai'\Big(\frac{\rho(x)}{(4t)^{2/3}}\Big)\sum_{k\geq 0} b_{1k}(x)t^{k+2/3}\Big]. \tag{4.14}$$

Here $\rho(x) = 0$ on $\mathcal{C}_1$, and $Ai(s)$ is the Airy function:

$$\begin{aligned} Ai(s) &\sim \frac{1}{2\sqrt{\pi}} s^{-1/4} e^{-(2/3)s^{3/2}}, \\ Ai(-s) &\sim \frac{1}{\sqrt{\pi}} s^{-1/4} \cos\Big(\frac{2}{3}s^{3/2} - \frac{\pi}{4}\Big), \end{aligned} \tag{4.15}$$

as $s \to +\infty$.
Leading term is $O(t^{1/3})$.

C. Higher order stable caustics

Leading term $O(t^{1/4})$ for a cusp,

$O(t^{1/(k+2)})$ for other stable caustics.

D. Perfect focus caustics. $\Omega = B_a$.
Then (4.11) applies, with $|z|^2 = a^2$ on ∂B_a.
On $B_b \subset\subset B_a$, one has (given $f \equiv 0$ on B_b)

$$e^{it\Delta}f(x) \sim e^{i(|x|^2+a^2)/4t}\sum_{k\geq 0}\hat{\alpha}_k\Big(x, \frac{x}{2t}\Big)t^{-n/2+1+k}, \tag{4.16}$$

where

$$\hat{\alpha}_k(x,\xi) = \int_{\partial B_a} e^{-i\xi\cdot z} a_k(x,z)\, dS(z). \tag{4.17}$$

$\hat{\alpha}_k \in C^\infty$, and stationary phase method gives

$$\hat{\alpha}_k(x,\xi) \sim \sum_{\pm} e^{\pm ia|\xi|}\sum_{\ell\geq 0}\alpha^{\pm}_{k\ell}\Big(x, a\frac{\xi}{|\xi|}\Big)|\xi|^{-(n-1)/2-\ell}, \tag{4.18}$$

as $|\xi| \to \infty$. In particular (if $f \equiv 0$ near 0),

$$e^{it\Delta}f(0) \sim e^{ia^2/4t}\sum_{k\geq 0}\hat{\alpha}_k(0,0)t^{-n/2+1+k}. \tag{4.19}$$

Note that

$$n = 2 \Rightarrow -\frac{n}{2} + 1 = 0, \quad n = 3 \Rightarrow -\frac{n}{2} + 1 = -\frac{1}{2}. \tag{4.16}$$

Recall from §3 that

$$\text{For } n = 2, \quad e^{it\Delta}\chi_D(0) = 1 - e^{i/4t}. \tag{4.17}$$

Proposition 4.2. *In case $n \geq 3$,*

$$\sup_{|t|\leq 1} \|e^{it\Delta}\chi_B\|_{L^p} < \infty \Longleftrightarrow 2 \leq p \leq \frac{2n}{n-2}. \tag{4.18}$$

5. Gibbs phenomenon

Given $f = F\chi_\Omega$, supported in a small neighborhood $\mathcal{O}$ of $p \in \partial\Omega$, it remains to investigate $e^{it\Delta} f(x)$ for $x \in \mathcal{O}$.
New approach:

$$e^{it\Delta} f(x) = (4\pi i t)^{-1/2} \int_{-\infty}^{\infty} e^{is^2/4t} u(s,x)\, ds, \tag{5.1}$$

where

$$u(s,x) = \cos s\sqrt{-\Delta}\, f(x), \tag{5.2}$$

solution to the wave equation. Special case of

$$\varphi(\sqrt{-\Delta}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\varphi}(s)\, \cos s\sqrt{-\Delta}\, ds, \tag{5.3}$$

for even φ. Take $\varphi(\lambda) = \varphi_t(\lambda) = e^{-it\lambda^2}$.
Equivalent to (5.1) is

$$e^{it\Delta} f(x) = e^{it\partial_s^2} u(s,x)\Big|_{s=0}. \tag{5.4}$$

Progressing wave expansion for $u(s,x)$. Set

$$\begin{aligned} \psi(x) &= \operatorname{dist}(x, \partial\Omega), \qquad x \in \mathcal{O} \cap \Omega, \\ &\quad - \operatorname{dist}(x, \partial\Omega), \quad x \in \mathcal{O} \setminus \Omega. \end{aligned} \tag{5.5}$$

Then $u(s,x) \sim$

$$\begin{aligned} &A_0(s,x)\chi_+(\psi(x) - s) + \sum_{j\ge 1} A_j(s,x)\chi_+^j(\psi(x) - s) \\ &+ A_0(-s,x)\chi_+(\psi(x) + s) + \sum_{j\ge 1} A_j(-s,x)\chi_+^j(\psi(x) + s), \end{aligned} \tag{5.6}$$

where

$$A_j \in C^\infty(\mathbb{R} \times \mathcal{O}), \quad \chi_+(s) = \chi_{\mathbb{R}^+}(s), \quad \chi_j^+(s) = s^j \chi_+(s). \tag{5.7}$$

Now,

$$\begin{aligned} &e^{it\partial_s^2} A_0(s,x)\chi_+(\psi(x) - s) \\ &= A_0(\psi(x), x)\Big[\mathrm{Fr}\Big(\frac{\psi(x) - s}{\sqrt{4t}}\Big) + \frac{1}{2}\Big] + \text{ lower order terms.} \end{aligned} \tag{5.8}$$

Proposition 5.1. *Take* $f = F\chi_\Omega,\ F \in C_0^\infty(\mathcal{O})$.
For $x \in \mathcal{O},\ t \in (-1,1)$,

$$e^{it\Delta} f(x) = 2A_0(\psi(x),x)\Big[\mathrm{Fr}\Big(\frac{\psi(x)}{\sqrt{4t}}\Big) + \frac{1}{2}\Big] + R(t,x), \tag{5.9}$$

with $a_0 \in C^\infty(\mathbb{R}\times\mathcal{O})$ *and, as* $t \to 0$,

$$R(t,x) \to f(x) - 2A_0(\psi(x),x)\chi_\Omega(x), \quad \textit{uniformly on } \ \mathcal{O}. \tag{5.10}$$

Note. By (5.6),

$$x \in \partial\Omega \Rightarrow 2A_0(\psi(x),x) = 2A_0(0,x) = f(x). \tag{5.11}$$

Hence the right side of (5.10) is piecewise smooth and Lipschitz continuous.

6. Nonlinear Schrödinger equations

$$u_t = i\Delta u + F(u) \text{ on } I \times \mathbb{R}^n, \quad u(0) = f. \tag{6.1}$$

Assume $F : \mathbb{R}^2 \to \mathbb{R}^2$ is smooth and

$$F(0) = 0, \quad DF(0) = 0. \tag{6.2}$$

Hence

$$\begin{gathered} |u| \le A \Rightarrow |F(u)| \le C(A),\ C(A)|u|,\ C(A)|u|^2, \\ |DF(u)| \le C(A),\ C(A)|u|. \end{gathered} \tag{6.3}$$

Assume

$$\begin{aligned} &f \in L^2(\mathbb{R}^n) \text{ if } n = 1, \\ &f \in H^{\sigma,2}(\mathbb{R}^n),\ \sigma > \frac{n}{2} - 1 \text{ if } n \ge 2, \end{aligned} \tag{6.4}$$

and

$$\|e^{it\Delta} f\|_{L^\infty} \le A, \quad |t| \le T_1, \tag{6.5}$$

for some $T_1 > 0$. Rewrite (6.1) as

$$u(t) = e^{it\Delta} f + \int_0^t e^{i(t-s)\Delta} F(u(s))\, ds. \tag{6.6}$$

Write

$$u(t) = u_0(t) + v(t), \quad u_0(t) = e^{it\Delta} f. \tag{6.7}$$

Get equation for v:

$$v(t) = \int_0^t e^{i(t-s)\Delta} F(u_0(s) + v(s))\, ds = \Psi v(t). \tag{6.8}$$

Want to solve this, estimate v, and estimate the remainder w in

$$u(t) = u_0(t) + \int_0^t e^{i(t-s)\Delta} F(u_0(s))\, ds + w(t). \tag{6.9}$$

Theorem 6.1. *For some $T_0 > 0$, we have a solution*

$$u \in C([0,T_0], H^{\sigma,2}(\mathbb{R}^n)) \cap L^\infty([0,T_0]\times\mathbb{R}^n), \tag{6.10}$$

with $\sigma > n/2-1$ ($\sigma = 0$ if $n=1$). Also, for $n = 1$,

$$\begin{aligned} \|v(t)\|_{L^2} \le Ct, \quad \|v(t)\|_{L^\infty} \le Ct^{1/2}, \\ \|w(t)\|_{L^2} \le Ct^2, \quad \|w(t)\|_{L^\infty} \le Ct^{3/2}. \end{aligned} \tag{6.11}$$

For $n \ge 2$, $\delta > 0$,

$$\begin{aligned} &\|v(t)\|_{H^{\sigma,2}} \le Ct, \quad \|v(t)\|_{L^\infty} \le C_\delta t^{1+\sigma-n/2-\delta}, \\ &\|w(t)\|_{H^{\sigma,2}} \le C_\delta t^{2+\sigma-n/2-\delta}, \\ &\|w(t)\|_{L^\infty} \le C_\delta t^{2(1+\sigma-n/2-\delta)}, \end{aligned} \tag{6.12}$$

provided also $\sigma < n/2$.

Tools:

Dispersive estimates.

$$\|e^{it\Delta}f\|_{L^\infty} \le Ct^{-n/2}\|f\|_{L^1}, \quad \|e^{it\Delta}f\|_{L^2} = \|f\|_{L^2}. \tag{6.13}$$

Interpolation gives

$$\|e^{it\Delta}f\|_{L^p} \le Ct^{-n(1/2-1/p)}\|f\|_{L^{p'}}, \quad 2 \le p \le \infty. \tag{6.14}$$

Moser estimates

$$F(0)=0,\ \|u\|_{L^\infty} \le A \Rightarrow \|F(u)\|_{H^{s,p}} \le C_{sp}(A)\|u\|_{H^{s,p}}. \tag{6.15}$$

Christ-Weinstein estimates

$$\|uv\|_{H^{\sigma,a}} \le C\|u\|_{L^q}\|v\|_{H^{\sigma,r}} + C\|u\|_{H^{\sigma,r}}\|v\|_{L^q}, \tag{6.16}$$

with $\sigma > 0$ and

$$\frac{1}{a} = \frac{1}{q} + \frac{1}{r}, \quad r \in (1,\infty),\ q \in (1,\infty],\ a \in (1,\infty). \tag{6.17}$$

7. Case $n = 1$

Assume $\|f\|_{L^2} \le A,\ \|e^{it\Delta} f\|_{L^\infty} \le A$. Set

$$X = \{v \in C([0, T_0], L^2) : \|v(t)\|_{L^2} \le A,\ \|v(t)\|_{L^\infty} \le A\}. \tag{7.1}$$

Claim $\exists\, T_0 > 0$ such that

$$\Psi : X \longrightarrow X \tag{7.2}$$

is a contraction map, where (with $u_0(t) = e^{it\Delta} f$)

$$\Psi v(t) = \int_0^t e^{i(t-s)\Delta} F(u_0(s) + v(s))\, ds. \tag{7.3}$$

Estimates (given $v \in X$):

$$\begin{aligned} \|\Psi v(t)\|_{L^2} &\le \int_0^t \|F(u_0(s) + v(s))\|_{L^2}\, ds \\ &\le C(A) \int_0^t \|u_0(s) + v(s)\|_{L^2}\, ds \\ &\le C'(A) t, \end{aligned} \tag{7.4}$$

and

$$\begin{aligned} \|\Psi v(t)\|_{L^\infty} &\le C \int_0^t |t-s|^{-1/2} \|F(u_0(s) + v(s))\|_{L^1}\, ds \\ &\le C \int_0^t |t-s|^{-1/2} \|u_0(s) + v(s)\|_{L^2}^2\, ds \\ &\le C(A) t^{1/2}. \end{aligned} \tag{7.5}$$

Now (7.4)–(7.5) $\Rightarrow \Psi : X \to X$ if T_0 is small.
Contraction estimates use

$$\begin{aligned} &\Psi v(t) - \Psi v_0(t) \\ &= \int_0^t e^{i(s-t)\Delta} [F(u_0(s) + v(s)) - F(u_0(s) + v_0(s))]\, ds \\ &= \int_0^t e^{i(t-s)\Delta} [R(s)(v(s) - v_0(s))]\, ds, \end{aligned} \tag{7.6}$$

where

$$R(s) = \int_0^1 DF(u_0(s) + \tau v(s) + (1-\tau) v_0(s))\, d\tau. \tag{7.7}$$

Recall

$$|u| \le A \Rightarrow |DF(u)| \le C(A) \text{ and } |DF(u)| \le C(A)|u|. \tag{7.8}$$

Contraction estimates (given $v,\ v_0 \in X$):

$$\begin{aligned} &\|\Psi v(t) - \Psi v_0(t)\|_{L^2} \\ &\le \int_0^t \|R(s)\|_{L^\infty} \|v(s) - v_0(s)\|_{L^2}\, ds \\ &\le C(A)t \sup_{0\le s\le t} \|v(s) - v_0(s)\|_{L^2}, \end{aligned} \tag{7.9}$$

and

$$\begin{aligned} &\|\Psi v(t) - \Psi v_0(t)\|_{L^\infty} \\ &\le C \int_0^t |t-s|^{-1/2} \|R(s)(v(s) - v_0(s))\|_{L^1}\, ds \\ &\le C(A) \int_0^t |t-s|^{-1/2} \|u_0(s) + \tau v(s) + (1-\tau)v_0(s)\|_{L^2} \\ &\qquad\qquad \times \|v(s) - v_0(s)\|_{L^2}\, ds \\ &\le C(A)t^{1/2} \sup_{0\le s\le t} \|v(s) - v_0(s)\|_{L^2}. \end{aligned} \tag{7.10}$$

Hence, for $T_0 > 0$ small, Ψ is a contraction in (7.2).
This gives existence in Theorem 6.1.
The solution $v(t) = \Psi v(t)$ thus has bounds (7.4)–(7.5).
This gives the first line of the estimates in (6.11).
Note that $w(t)$ in (6.9) is given by

$$w(t) = \Psi v(t) - \Psi v_0(t), \quad \text{with} \;\; v_0 \equiv 0. \tag{7.11}$$

Hence (7.9)–(7.10) imply the second line of estimates in (6.11).

8. Case $n \geq 2$

Assume $\|f\|_{H^{\sigma,2}} \leq A,\ \|e^{it\Delta}f\|_{L^\infty} \leq A$. Set

$$X = \{v \in C([0,T_0], H^{\sigma,2}) \\ : \|v(t)\|_{H^{\sigma,2}} \leq A,\ \|v(t)\|_{L^\infty} \leq A\}. \tag{8.1}$$

Constraints on σ derived below.
Claim there exists $T_0 > 0$ such that

$$\Psi : X \longrightarrow X \tag{8.2}$$

is a contraction map, where (with $u_0(t) = e^{it\Delta}f$)

$$\Psi v(t) = \int_0^t e^{i(t-s)\Delta} F(u_0(s) + v(s))\, ds. \tag{8.3}$$

Estimates (given $v \in X$):

$$\begin{aligned} \|\Psi v(t)\|_{H^{\sigma,2}} &\leq \int_0^t \|F(u_0(s)+v(s))\|_{H^{\sigma,2}}\, ds \\ &\leq C(A)\int_0^t \|u_0(s)+v(s)\|_{H^{\sigma,2}}\, ds \\ &\leq C'(A)t. \end{aligned} \tag{8.4}$$

The second line follows from a Moser estimate.
Next, pick a, b such that

$$\sigma b > n, \quad b > 2, \quad a = b'. \tag{8.5}$$

Then $\|\Psi v(t)\|_{L^\infty} \leq$

$$\begin{aligned} &C\|\Psi v(t)\|_{H^{\sigma,b}} \\ &\leq C\int_0^t |t-s|^{-n(1/2-1/b)}\|F(u_0(s)+v(s))\|_{H^{\sigma,a}}\, ds. \end{aligned} \tag{8.6}$$

For this to be useful, we need

$$n\Big(\frac{1}{2} - \frac{1}{b}\Big) < 1, \text{ i.e., } b < \frac{2n}{n-2} = \frac{n}{n/2-1}. \tag{8.7}$$

Say

$$b = \frac{n}{\sigma - \delta}, \quad \delta << \sigma. \tag{8.8}$$

Need

$$\frac{n}{2} - 1 < \sigma < \frac{n}{2}. \tag{8.8A}$$

Then (8.5) and (8.7) hold.
To estimate right side of (8.6), use

$$F(u) = R(u,0)u, \quad R(u,0) = \int_0^t DF(\tau u)\,d\tau. \tag{8.9}$$

Then $R(0,0) = 0$ and (by Moser estimates)

$$\begin{aligned} &\|u\|_{L^\infty},\ \|u\|_{H^{\sigma,2}} \le A \\ &\Longrightarrow \|R(u,0)\|_{H^{\sigma,2}},\ \|R(u,0)\|_{L^\infty} \le C(A). \end{aligned} \tag{8.10}$$

Use Christ-Weinstein estimate to get

$$\begin{aligned} &\|R(u,0)u\|_{H^{\sigma,a}} \\ &\le C\|R(u,0)\|_{L^q}\|u\|_{H^{\sigma,r}} + C\|R(u,0)\|_{H^{\sigma,r}}\|u\|_{L^q}, \end{aligned} \tag{8.11}$$

with

$$\frac{1}{a} = \frac{1}{q} + \frac{1}{r}, \quad r \in (1,\infty),\ q \in (1,\infty]. \tag{8.12}$$

Take

$$r = 2, \quad \text{so} \ \ \frac{1}{q} = \frac{1}{2} - \frac{1}{b}. \tag{8.13}$$

(Recall $b = a'$.) By (8.8)–(8.8A),

$$q = \frac{2n}{n - 2\sigma + 2\delta}, \quad \text{so} \ \ 2 < q < \infty. \tag{8.14}$$

Hence $\|g\|_{L^q} \le \|g\|_{L^2} + \|g\|_{L^\infty}$, so

$$\begin{aligned} \|u\|_{L^\infty},\ \|u\|_{H^{\sigma,2}} &\le A \\ \Rightarrow \|F(u)\|_{H^{\sigma,a}} &\le C\|R(u,0)\|_{L^q}\|u\|_{H^{\sigma,2}} \\ &\quad + C\|R(u,0)\|_{H^{\sigma,2}}\|u\|_{L^q} \\ &\le C(A). \end{aligned} \tag{8.15}$$

Thus (8.6) yields, for $v \in X$,

$$\begin{aligned} \|\Psi v(t)\|_{L^\infty} &\le C(A)t^{1-n(1/2-1/b)} \\ &= C(A)t^{1-n/q} \\ &= C(A)t^{1+\sigma-n/2-\delta}. \end{aligned} \tag{8.16}$$

Now (8.4) and (8.16) $\Rightarrow \Psi : X \to X$ if T_0 is small,
provided σ satisfies (8.8A).

Similar arguments yield contraction estimates. If $v, v_0 \in X$,

$$\begin{aligned}&\|\Psi v(t) - \Psi v_0(t)\|_{H^{\sigma,2}} \\ &\le C(A)t \sup_{0\le s\le t} \Big(\|v(s) - v_0(s)\|_{H^{\sigma,2}} + \|v(s) - v_0(s)\|_{L^\infty}\Big),\end{aligned} \tag{8.17}$$

and

$$\begin{aligned}&\|\Psi v(t) - \Psi v_0(t)\|_{L^\infty} \\ &\le C(A)t^{1-n(1/2-1/b)} \\ &\quad \times \sup_{0\le s\le t} \Big(\|v(s) - v_0(s)\|_{H^{\sigma,2}} + \|v(s) - v_0(s)\|_{L^\infty}\Big) \\ &\le C(A)t^{1+\sigma-n/2-\delta} \\ &\quad \sup_{0\le s\le t} \Big(\|v(s) - v_0(s)\|_{H^{\sigma,2}} + \|v(s) - v_0(s)\|_{L^\infty}\Big).\end{aligned} \tag{8.18}$$

Hence, for $T_0 > 0$ small, Ψ is a contraction in (8.2).

This gives existence in Theorem 6.1.
The function $v(t) = \Psi v(t)$ then satisfies bounds in (8.4) and (8.16).
This gives the first line of estimates in (6.12).

Note that $w(t)$ in (6.9) is given by

$$w(t) = \Psi v(t) - \Psi v_0(t), \quad \text{with} \ \ v_0 \equiv 0. \tag{8.19}$$

Hence (8.17)–(8.18) imply the second and third lines of estimates
in (6.12).

9. Some results of Ben Dodson (JFA, 2010)

$$iu_t = -\Delta u + \sigma |u|^2 u, \quad \text{on} \quad I \times \mathbb{R}^3, \quad \sigma = \pm 1. \tag{9.1}$$

$$u(0) = \chi_B, \quad B \subset \mathbb{R}^3 \;\; \text{ball.} \tag{9.2}$$

Theorem 9.1. *Have local solution*

$$u(t) = e^{it\Delta}\chi_B + v(t), \tag{9.3}$$

$$v(t) \in L^\infty([0, T_0], H^1(\mathbb{R}^3)), \tag{9.4}$$

$$\|v(t)\|_{H^1} \le Ct^\delta (\textit{for some } \delta > 0). \tag{9.5}$$

Theorem 9.2. *For $\sigma = +1$ (defocusing), have global solution (9.3), with*

$$v \in L^\infty(\mathbb{R}^+, H^1(\mathbb{R}^3)). \tag{9.6}$$

Linear results $+$ (9.4) $\Rightarrow$ Gibbs phenomenon results
$+$ (9.5) $\Rightarrow$ Pinsky phenomenon results

Theorem 9.3. *Theorem 9.2 extends to*

$$iu_t = -\Delta u + |u|^\alpha u, \quad \textit{on} \quad \mathbb{R}^+ \times \mathbb{R}^n, \tag{9.7}$$

provided

$$\frac{2}{n} < \alpha < \frac{3}{n-2}, \quad \alpha \le 2. \tag{9.8}$$

Part of strategy: show that if (9.8) holds,

$$\Psi : \mathfrak{X} \longrightarrow \mathfrak{X} \tag{9.9}$$

is a contraction, for $T_0 > 0$ small, where Ψ is as in previous sections, and, with $I = [0, T_0]$,

$$\mathfrak{X} = \{u \in S^1(I \times \mathbb{R}^n) : \|u\|_{S^1(I\times\mathbb{R}^n)} \le B\}. \tag{9.10}$$

Here,

$$\|u\|_{S^1(I\times\mathbb{R}^n)} = \|u\|_{S^0(I\times\mathbb{R}^n)} + \|\nabla_x u\|_{S^0(I\times\mathbb{R}^n)}, \tag{9.11}$$

where

$$\|u\|_{S^0(I\times\mathbb{R}^n)} = \sup_{(p,q)\in\mathcal{A}_n} \|u\|_{L^p(I,L^q(\mathbb{R}^n))}, \tag{9.12}$$

and $\mathcal{A}_n$ is the set of “admissible pairs”:

$$(p,q)\in\mathcal{A}_n \Leftrightarrow \frac{2}{p} = n\Big(\frac{1}{2}-\frac{1}{q}\Big), \quad \text{and} \quad 2\le p\le\infty, \tag{9.13}$$

provided $n\geq 3$.

$$\text{Example:} \quad (p,q) = (\infty,2).$$

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