# The "Oscillator-Dirac" Operator, and Related Operators on $\mathbb{H}^n$

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Abstract. We have several goals: to construct an interesting first order differential operator on $\mathbb{R}^n$ of index one, to construct a related invertible pseudodifferential operator on $\mathbb{R}^n$, and to construct related differential and pseudodifferential operators on the Heisenberg group $\mathbb{H}^n$, including a first order operator that is hypoelliptic with loss of 1/2 derivative.

## 1. The Oscillator-Dirac Operator

To construct the Oscillator-Dirac operator, we begin as follows. Let $\{A_j, B_j : 1 \le j \le n\}$ define the representation of the Clifford algebra $\mathrm{Cl}_{2n}$ on $V = \Lambda^* \mathbb{C}^{2n}$, given by

$$A_j \varphi = e_j \wedge \varphi + \iota_{e_j} \varphi, \quad B_j \varphi = e_{n+j} \wedge \varphi + \iota_{e_{n+j}} \varphi, \quad 1 \le j \le n, \tag{1.1}$$

where $\{e_j : 1 \le j \le 2n\}$ is the standard basis of $\mathbb{C}^n$. We have

$$A_j^* = A_j, \quad B_j^* = B_j, \tag{1.2}$$

and

$$A_j A_k + A_k A_j = B_j B_k + B_k B_j = 2\delta_{jk} I, \quad A_j B_k + B_k A_j = 0. \tag{1.3}$$

Also define unbounded self-adjoint operators $D_j$ and $X_j$ on $L^2(\mathbb{R}^n)$ by

$$D_j u(x) = \frac{1}{i} \frac{\partial u}{\partial x_j}, \quad X_j u(x) = x_j u(x), \tag{1.4}$$

and note the commutation relations

$$[D_j, D_k] = 0, \quad [X_j, X_k] = 0, \quad [D_j, X_k] = -i\delta_{jk} I. \tag{1.5}$$

Now we define the following operator on $V$-valued functions on $\mathbb{R}^n$:

$$\mathcal{D} = \sum_{j=1}^{n} \big( A_j D_j + B_j X_j \big). \tag{1.6}$$

This is an elliptic operator in the class denoted $OP\mathcal{H}^1_b$ in (2.13) of [T], Chapter 2. Note that $A_j$ and $B_j$ map $\Lambda^e\mathbb{C}^{2n}$ to $\Lambda^o\mathbb{C}^{2n}$ and vice-versa, so we can write $\mathcal{D}$ in block form as

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}_1 \\ \mathcal{D}_0 & 0 \end{pmatrix}, \quad \text{on functions with values in } \ \Lambda^e\mathbb{C}^{2n} \oplus \Lambda^o\mathbb{C}^{2n}. \tag{1.7}$$

Note that $\mathcal{D}$ is formally self-adjoint, hence actually self-adjoint, with domain $H^1(\mathbb{R}^n, V)$, where we set

$$H^k(\mathbb{R}^n, V) = \{u \in L^2(\mathbb{R}^n, V) : x^\alpha \partial_x^\beta u \in L^2(\mathbb{R}^n, V), \text{ for } |\alpha| + |\beta| \le k\}. \tag{1.8}$$

In particular, $\mathcal{D}_1 = \mathcal{D}_0^*$ and $\mathcal{D}_0 = \mathcal{D}_1^*$, and

$$\mathcal{D}^2 = \begin{pmatrix} \mathcal{D}_0^*\mathcal{D}_0 & \\ & \mathcal{D}_1^*\mathcal{D}_1 \end{pmatrix}. \tag{1.9}$$

We next use (1.6) to produce a useful formula for $\mathcal{D}^2$. We have

$$\begin{aligned} \mathcal{D}^2 &= \sum_{j,k} \big(A_j D_j + B_j X_j\big)\big(A_k D_k + B_k X_k\big) \\ &= \sum_{j,k} \big(A_j A_k D_j D_k + B_j B_k X_j X_k + A_j B_k D_j X_k + B_j A_k X_j D_k\big). \end{aligned} \tag{1.10}$$

The identities (1.3) and (1.5) yield

$$\sum_{j,k} A_j A_k D_j D_k = \sum_j D_j^2, \quad \sum_{j,k} B_j B_k X_j X_k = \sum_j X_j^2, \tag{1.11}$$

and

$$\sum_{j \ne k} \big(A_j B_k D_j X_k + B_j A_k X_j D_k\big) = 0. \tag{1.12}$$

Furthermore,

$$\sum_j \big(A_j B_j D_j X_j + B_j A_j X_j D_j\big) = -i \sum_j A_j B_j, \tag{1.13}$$

so we have

$$\mathcal{D}^2 = H - i \sum_j A_j B_j, \tag{1.14}$$

where $H$ is the quantum harmonic oscillator,

$$Hu = (-\Delta + |x|^2)u. \tag{1.15}$$

As is well known,

$$\operatorname{Spec} H = \{n + 2k : k = 0, 1, 2, \dots\}, \tag{1.16}$$

and the $n$-eigenspace of $H$, acting on $V$-valued functions, consists of

$$\varphi\, e^{-|x|^2/2}, \quad \varphi \in V. \tag{1.17}$$

The operators $iA_jB_j$ are each self-adjoint on $L^2(\mathbb{R}^n, V)$, and they commute with $H$ and with each other. We take a closer look at these operators. Note that

$$A_jB_j\varphi = e_j \wedge e_{n+j} \wedge \varphi + \iota_{e_j}\iota_{e_{n+j}}\varphi. \tag{1.18}$$

If $\psi$ is a monomial that does not contain $e_j$ or $e_{n+1}$, then

$$\begin{gathered} A_jB_j\psi = e_j \wedge e_{n+j} \wedge \psi, \\ A_jB_j(e_j \wedge \psi) = A_jB_j(e_{n+j} \wedge \psi) = 0, \\ A_jB_j(e_j \wedge e_{n+j} \wedge \psi) = -\psi. \end{gathered} \tag{1.19}$$

Thus

$$\operatorname{Spec} A_jB_j = \{0, \pm i\}, \tag{1.20}$$

and the $-i$-eigenspace of $A_jB_j$ is

$$\{(1 + ie_j \wedge e_{n+j}) \wedge \psi : \psi \in \operatorname{Ker} \iota_{e_j}\iota_{e_{n+j}}\}. \tag{1.21}$$

It follows that

$$\operatorname{Spec} \sum_j A_jB_j \subset \{-ni, \dots, -i, 0, i, \dots, ni\}, \tag{1.22}$$

and the $-ni$-eigenspace of this operator is the intersection of the spaces (1.21), over $1 \le j \le n$.

We deduce that the null space of

$$\mathcal{D}^2 : H^2(\mathbb{R}^n, V) \longrightarrow L^2(\mathbb{R}^n, V) \tag{1.23}$$

is one-dimensional, spanned by

$$\psi\, e^{-|x|^2}, \quad \psi = (1 + ie_1 \wedge e_{n+1}) \wedge \cdots \wedge (1 + ie_n \wedge e_{2n}). \tag{1.24}$$

In turn we have that

$$\mathcal{D}_0, \mathcal{D}_1 : H^1(\mathbb{R}^n, \Lambda^{eo}) \longrightarrow L^2(\mathbb{R}^n, \Lambda^{oe}) \tag{1.25}$$

are Fredholm, with

$$\text{Dim Ker}\,\mathcal{D}_0 = 1, \quad \text{Dim Ker}\,\mathcal{D}_1 = 0, \tag{1.26}$$

and hence

$$\text{Index}\,\mathcal{D}_0 = 1, \quad \text{Index}\,\mathcal{D}_1 = -1. \tag{1.27}$$

Furthermore, we see that if $\pi_0$ is the orthogonal projection of $L^2(\mathbb{R}^n, \Lambda^e)$ onto $\text{Ker}\,\mathcal{D}_0$, then

$$\mathcal{E} = \begin{pmatrix} \pi_0 & \mathcal{D}_1 \\ \mathcal{D}_0 & 0 \end{pmatrix} : H^1(\mathbb{R}^n, V) \xrightarrow{\approx} L^2(\mathbb{R}^n, V), \tag{1.28}$$

and, by Proposition 2.8 of [T], Chapter 2,

$$\mathcal{E}^{-1} \in OP\mathcal{H}_b^{-1}. \tag{1.29}$$

## 2. Related operators on the Heisenberg group

We can define some first and second order operators on the Heisenberg group $\mathbb{H}^n$, associated with the operators discussed in §1. We take $(t, q, p) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ to be coordinates on $\mathbb{H}^n$, such that the group law is given by (1.1) in [T], Chapter 2, and set

$$T = \frac{\partial}{\partial t}, \quad L_j = \frac{\partial}{\partial q_j} - \frac{p_j}{2}\frac{\partial}{\partial t}, \quad M_j = \frac{\partial}{\partial p_j} + \frac{q_j}{2}\frac{\partial}{\partial t}, \tag{2.1}$$

so

$$[L_j, M_j] = -[M_j, L_j] = T, \tag{2.2}$$

all other commutators being zero. Parallel to (1.3), we set

$$\mathcal{K} = \sum_j \bigl(A_j M_j + B_j L_j\bigr), \tag{2.3}$$

a first order differential operator on $\mathbb{H}^n$. In the notation of Definition 2.1 on [T], Chapter 2, we have

$$\mathcal{K} \in OP\Psi_0^1. \tag{2.4}$$

Furthermore, symbols $\sigma_{\mathcal{K}}(\pm 1)(X,D)$ associated to $\mathcal{K}$ as in §1, Chapter 2 of [T], have the following nature:

$$\sigma_{\mathcal{K}}(\pm 1)(X,D) = \mathcal{D}^{\pm}, \tag{2.5}$$

with $\mathcal{D}^{+} = \mathcal{D}$, given by (1.6), and

$$\mathcal{D}^{-} = \sum_{j=1}^{n} (A_j D_j - B_j X_j) = \begin{pmatrix} 0 & \mathcal{D}_1^{-} \\ \mathcal{D}_0^{-} & 0 \end{pmatrix}. \tag{2.6}$$

Calculations analogous to (1.10)–(1.14) give

$$(\mathcal{D}^{-})^2 = H + i \sum_j A_j B_j, \tag{2.7}$$

whose null space is spanned by

$$\psi^{-}\, e^{-|x|^2/2}, \quad \psi^{-} = (1 - ie_1 \wedge e_{n+1}) \wedge \cdots \wedge (1 - ie_n \wedge e_{2n}), \tag{2.8}$$

so if $\pi_0^{-}$ is the orthogonal projection of $L^2(\mathbb{R}^n, \Lambda^e)$ onto $\operatorname{Ker} \mathcal{D}_0^{-}$,

$$\mathcal{E}^{-} = \begin{pmatrix} \pi_0^{-} & \mathcal{D}_1^{-} \\ \mathcal{D}_0^{-} & 0 \end{pmatrix} : H^1(\mathbb{R}^n, V) \stackrel{\approx}{\longrightarrow} L^2(\mathbb{R}^n, V), \tag{2.9}$$

and

$$(\mathcal{E}^{-})^{-1} \in OP\mathcal{H}_b^{-1}. \tag{2.10}$$

Furthermore, we can define

$$\mathcal{S} \in OP\Psi_0^{1,\infty}, \quad \sigma_{\mathcal{S}}(\pm 1)(X,D) = \begin{pmatrix} \pi_0^{\pm} & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.11}$$

where $\pi_0^{+} = \pi_0$. The class $OP\Psi_0^{1,\infty}$ is as in Definition 2.9 of [T], Chapter 2. Then

$$\mathcal{K}_b = \mathcal{K} + \mathcal{S} \in OP\Psi_0^1 \tag{2.12}$$

has inverse

$$\mathcal{K}_b^{-1} \in OP\Psi_0^{-1} \subset OPS_{1/2,1/2}^{-1/2}. \tag{2.13}$$

Thus the first order operator $\mathcal{K}_b$ is hypoelliptic, with loss of 1/2 derivative.

REMARK. The operator $\mathcal{D}$ in (1.6) is vaguely similar to $\mathcal{D}_{+}$ in (113) of [E].

## References

[E] C. Epstein, Subelliptic Spin$_{\mathbb{C}}$ Dirac Operators, II, Preprint, 2005.
[T] M. Taylor, Noncommutative Microlocal Analysis (Revised Version), 1999 update of Memoir #313, AMS, 1984.