Vanishing Viscosity Limits for a Class of Circular Pipe Flows *

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Abstract

We consider 3D Navier-Stokes flows with no-slip boundary condition in an infinitely long pipe with circular cross section. The velocity fields we consider are independent of the variable parametrizing the axis of the pipe, and the component of the velocity normal to the axis is arranged to be circularly symmetric, though we impose no such symmetry on the component of velocity parallel to the axis. For such flows we analyze the limit as the viscosity tends to zero, including boundary layer estimates.

1 Introduction

In this paper we study a class of solutions to the 3D Navier-Stokes equations

$$\frac{\partial u^\nu}{\partial t} + \nabla_{u^\nu} u^\nu + \nabla p^\nu = \nu \Delta u^\nu + F^\nu, \quad \operatorname{div} u^\nu = 0, \tag{1.1}$$

for $u^\nu = u^\nu(t,x,z),\ p^\nu = p^\nu(t,x,z)$ with $(t,x,z) \in \mathbb{R}^+ \times \Omega$, where

$$\Omega = D \times \mathbb{R}, \quad D = \{x \in \mathbb{R}^2 : |x| < 1\}. \tag{1.2}$$

We denote the closure of D by $\overline{D}$, with boundary ∂D. We restrict attention to the following type of external force field F^ν:

$$F^\nu(t,x,z) = (0, f^\nu(t)), \tag{1.3}$$

i.e., F^ν is parallel to the z-axis, with z-component $f^\nu(t)$. We impose no-slip boundary data on the boundary, which might be rotating and translating:

$$u^\nu(t,x,z) = \Big(\frac{\alpha(t)}{2\pi} x^\perp, \beta(t)\Big), \quad |x| = 1,\ z \in \mathbb{R},\ t > 0. \tag{1.4}$$

Here $x^\perp = Jx$ where J is counterclockwise rotation by 90°. We take initial data of the following form:

$$u^\nu(0,x,z) = u_0(x) = (v_0(x), w_0(x)), \tag{1.5}$$

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where v_0 is a vector field on D and w_0 is the z-component of u_0. We require the conditions

$$\operatorname{div} u_0 = 0, \ u_0 \parallel \partial\Omega, \quad \text{i.e.,} \quad \operatorname{div} v_0 = 0, \ v_0 \parallel \partial D, \tag{1.6}$$

and we require that the vector field v_0 on D be circularly symmetric.

By definition, a vector field v_0 on D is circularly symmetric provided

$$v_0(R_\theta x) = R_\theta v_0(x), \quad \forall\, x \in D, \tag{1.7}$$

for each $\theta \in [0, 2\pi]$, where R_θ is counterclockwise rotation by θ. The general planar vector field satisfying (1.7) has the form $s_0(|x|)x^\perp + s_1(|x|)x$, with s_j scalar, but the condition $\operatorname{div} v_0 = 0$, together with the condition $v_0 \parallel \partial D$, forces $s_1 \equiv 0$, so the type of initial data we consider is characterized by

$$u_0(x) = (s_0(|x|)x^\perp, w_0(x)). \tag{1.8}$$

Another characterization of this special form for v_0 is that

$$v_0(\Phi_\omega x) = -\Phi_\omega v_0(x) \tag{1.9}$$

for all $\omega \in S^1$, where $\Phi_\omega : \mathbb{R}^2 \to \mathbb{R}^2$ is the reflection across the line generated by ω.

The fact that Ω in (1.2) is infinite makes uniqueness an issue. (We discuss this further in Appendix C.) To guarantee uniqueness, we modify the set-up by requiring the solutions to be periodic (say of period L) in z, i.e., we replace Ω in (1.2) by

$$\Omega_L = D \times (\mathbb{R}/L\mathbb{Z}). \tag{1.10}$$

In such a case, the general theory implies that (1.1), (1.4) and (1.5) has a unique strong, short-time solution, given mild regularity hypotheses on $v_0(x)$ and $w_0(x)$ (actually the solution persists globally in t, as we will see shortly) and the solution is z-translation invariant, i.e.,

$$u^\nu = (v^\nu(t,x), w^\nu(t,x)), \quad p^\nu = p^\nu(t,x). \tag{1.11}$$

Remark. While F^ν, given by (1.3), satisfies $F^\nu = \nabla(f^\nu(t)z)$, this is not the gradient of a function periodic in z.

Note that

$$\nabla_{u^\nu} u^\nu = (\nabla_{v^\nu} v^\nu, \nabla_{v^\nu} w^\nu), \quad \operatorname{div} u^\nu = \operatorname{div} v^\nu. \tag{1.12}$$

Hence, in the current setting, (1.1) is equivalent to the following system of equations on $\mathbb{R}^+ \times D$:

$$\frac{\partial v^\nu}{\partial t} + \nabla_{v^\nu} v^\nu + \nabla p^\nu = \nu \Delta v^\nu, \quad \operatorname{div} v^\nu = 0, \tag{1.13}$$

$$\frac{\partial w^\nu}{\partial t} + \nabla_{v^\nu} w^\nu = \nu \Delta w^\nu + f^\nu(t). \tag{1.14}$$

Note that (1.13) is the 2D Navier-Stokes equation for flow on D. We are imposing the boundary condition

$$v^\nu(t,x) = \frac{\alpha(t)}{2\pi} x^\perp, \quad |x| = 1, \ t > 0, \tag{1.15}$$

and the initial condition

$$v^\nu(0,x) = v_0(x) = s_0(|x|)x^\perp. \tag{1.16}$$

Meanwhile, (1.14) is a scalar equation, with boundary condition

$$w^\nu(t,x) = \beta(t), \quad |x| = 1, \ t > 0, \tag{1.17}$$

and initial condition

$$w^\nu(0,x) = w_0(x). \tag{1.18}$$

We do not require $v_0(x)$ to equal $(\alpha(0)/2\pi)x^\perp$ when $x \in \partial D$, nor do we require $w_0(x)$ to equal $\beta(0)$ when $x \in \partial D$. At this point we recall that the solvability of (1.13) for all $t \in \mathbb{R}^+$, for each $\nu > 0$, is well known, and the solvability of (1.14) for all $t \in \mathbb{R}^+$ is then relatively elementary.

Our main goal is to study the limit as $\nu \searrow 0$ of the solutions to (1.13)–(1.18), and see how $u^\nu = (v^\nu, w^\nu)$ approaches the solution of the Euler equation

$$\frac{\partial u^0}{\partial t} + \nabla_{u^0} u^0 + \nabla p^0 = F^0, \quad \operatorname{div} u^0 = 0, \tag{1.19}$$

on $\mathbb{R}^+ \times \Omega_L$, with initial condition

$$u^0(0,x,z) = (v_0(x), w_0(x)), \tag{1.20}$$

given in (1.16) and (1.18), and with boundary condition

$$u^0 \parallel \partial\Omega_L. \tag{1.21}$$

Here $F^0(t,x,z) = (0, f^0(t))$. Arguments as above give

$$u^0(t,x,z) = (v^0(t,x), w^0(t,x)), \tag{1.22}$$

where $v^0(t,x)$ and $w^0(t,x)$ solve

$$\frac{\partial v^0}{\partial t} + \nabla_{v^0} v^0 + \nabla p^0 = 0, \quad \operatorname{div} v^0 = 0, \tag{1.23}$$

$$\frac{\partial w^0}{\partial t} + \nabla_{v^0} w^0 = f^0(t). \tag{1.24}$$

Note that (1.23) is the 2D Euler equation for flows on D. We have the boundary condition

$$v^0(t,x) \parallel \partial D \ \text{ for } \ t > 0, \ x \in \partial D, \tag{1.25}$$

and initial condition

$$v^0(0,x) = v_0(x) = s_0(|x|)x^\perp. \tag{1.26}$$

As is well known, the vector field v_0 given by (1.26) is a steady solution to the Euler equation (1.23). In fact, a calculation gives

$$\nabla_{v_0} v_0 = -s_0(|x|)^2 x = -\nabla p_0(x), \tag{1.27}$$

with

$$p_0(x) = \tilde{p}_0(|x|), \quad \tilde{p}_0(r) = -\int_r^1 \rho s_0(\rho)^2 \, d\rho, \tag{1.28}$$

which proves our assertion:

$$v^0(t,x) \equiv v_0(x), \tag{1.29}$$

when $v_0(x)$ is as in (1.26). From here, we see that (1.24) becomes

$$\frac{\partial w^0}{\partial t} + \nabla_{v_0} w^0 = f^0(t). \tag{1.30}$$

The tangency condition (1.21) imposes no boundary condition for w^0. This is logical, since $\partial_t + \nabla_{v_0}$ is a vector field on $\mathbb{R} \times \overline{D}$ that is tangent to $\mathbb{R} \times \partial D$. The solution to (1.30), with initial condition

$$w^0(0, x) = w_0(x), \tag{1.31}$$

is given by

$$w^0(t, x) = w_0(\mathcal{F}_{v_0}^{-t}(x)) + \int_0^t f^0(s)\, ds, \tag{1.32}$$

where $\mathcal{F}_{v_0}^{-t} : \overline{D} \to \overline{D}$ is the backwards flow on $\overline{D}$ generated by v_0.

Now the task of analyzing how $u^\nu \to u^0$ as $\nu \searrow 0$ has two parts, namely how

$$v^\nu \longrightarrow v^0 \quad \text{as} \;\; \nu \searrow 0, \tag{1.33}$$

and how

$$w^\nu \longrightarrow w^0 \quad \text{as} \;\; \nu \searrow 0. \tag{1.34}$$

There is a literature on (1.33), including [7], [10], [1], and, recently, [5] and [6]. The first key to a successful attack on (1.33) is the following result.

Proposition 1.1 *Given that v_0 has the form (1.26), the solution v^ν to (1.13), (1.15), (1.16) is circularly symmetric for each $t > 0$, of the form*

$$v^\nu(t, x) = s^\nu(t, |x|)x^\perp, \tag{1.35}$$

and it coincides with the solution to the linear PDE

$$\frac{\partial v^\nu}{\partial t} = \nu \Delta v^\nu, \tag{1.36}$$

with boundary condition (1.15) and initial condition (1.16).

This well known result figured in the analyses in the papers cited above. A proof (using the characterization (1.9)) is recorded in Proposition 1.1 of [6]. We mention in particular that

$$\begin{aligned} \nabla_{v^\nu} v^\nu = -\nabla p^\nu, \quad p^\nu(t, x) = \tilde{p}^\nu(t, |x|), \\ \tilde{p}^\nu(t, r) = -\int_r^1 \rho s^\nu(t, \rho)^2\, d\rho. \end{aligned} \tag{1.37}$$

The structure of the rest of this paper is as follows. In §2 we recall results of [5] and [6] on the nature of the convergence $v^\nu \to v^0$ in (1.33), and give some further results, which will be of use in §3. Prior results include a variety of L^p-Sobolev space estimates, recalled in Propositions 2.3–2.5. Further results include estimates in spaces $\mathcal{V}^k(D)$ (defined in (2.33)), available thanks to [8], given in Proposition 2.6 and Corollary 2.7. New results (of crucial use in §3) include explicit boundary layer analyses, following from material in Appendix B, leading to estimates in the space $\mathcal{V}^{\infty,\infty}(D)$ (defined in (2.52)), given in Propositions 2.8–2.10.

In §3 we discuss the nature of the convergence $w^\nu \to w^0$ in (1.34). Here we apply results obtained in [8]. These results were originally directed towards a different fluid problem, involving plane parallel channel flows, but [8] found it convenient to develop the relevant singular perturbation theory on a more general level, and, thanks to the results of §2 of this paper, this development has applications to (1.34). In (3.21) we write

$$w^\nu(t,x) - w^0(t,x) = R_1(\nu,t,x) + R_2(\nu,t,x) + R_3(\nu,t,x), \tag{1.38}$$

and apply a variety of attacks on the three terms on the right, which are defined in (3.18)–(3.20). We obtain estimates on R_2 and R_3 in $L^p(D)$, for $p \in [1,\infty)$ in Proposition 3.1, and such estimates on R_1 in Proposition 3.3. These results lead to $w^\nu(t,\cdot) \to w^0(t,\cdot)$ in such L^p-norms. We obtain $H^{\sigma,q}(D)$ estimates on R_1 and R_2 in Proposition 3.5, for $q \in [2,\infty)$, $\sigma q \in [0,1)$. Propositions 3.7–3.10 yield $w^\nu(t,\cdot) \to w^0(t,\cdot)$ in the spaces $\mathcal{V}^k(D)$, leading to convergence boundedly and locally uniformly on D, established in Proposition 3.11.

Of course, Proposition 3.11 does not establish uniform convergence on $\overline{D}$. There is a boundary layer effect at work here, as there was for $v^\nu \to v$. The corresponding study of boundary layer effects for $w^\nu \to w$ is taken up in §4. Again we use the decomposition (1.38) and apply separate analyses to the three terms on the right. Results of Appendix B give an explicit boundary layer analysis of R_2. Layer potential techniques developed in [8] give an almost equally precise analysis of R_1. This leaves R_3, and as we show in (4.47)–(4.51), we can obtain at least an estimate on boundary layer thickness for this term, consistent with the boundary layer thickness results apparent for R_1 and R_2, though fine detail on the boundary layer behavior of R_3 remains a topic for further work.

In §5 we briefly focus on the special case where $w_0(x)$ in (1.5) is also circularly symmetric. In this case, the system (1.13)–(1.14) simplifies further to the linear system (5.2)–(5.3). Here we note how results of [6] and Appendix B yield definitive results on convergence $u^\nu \to u^0$ in this case, including explicit boundary layer analyses. The main message is that the lack of circular symmetry for $w_0(x)$ is the source of the difficulties (otherwise said, the most interesting phenomena) for the results discussed in §§3–4.

This paper ends with four appendices. Appendix A deals with the phenomenon of concentration of vorticity. This concentration effect was established in [6] for circularly symmetric 2D flow. There, use was made of L^1-norm bounds on the vorticity. Then [4] produced a more general result on vorticity concentration, involving however convergence in a weaker topology. We produce further variants of this result in Appendix A, and discuss problems yet to be resolved regarding L^1-vorticity estimates.

Appendix B, which has already been mentioned, studies limits as $\nu \searrow 0$ of solutions to

$$\frac{\partial u^\nu}{\partial t} = \nu \Delta u^\nu \quad \text{on} \quad \mathbb{R}^+ \times \Omega, \tag{1.39}$$

satisfying

$$u^\nu\big|_{\mathbb{R}^+\times\partial\Omega} = 0, \quad u^\nu(0,x) = f(x). \tag{1.40}$$

We take $f \in C^\infty(\overline{\Omega})$, and do not require it to vanish on the boundary. We make use of wave equation techniques to produce explicit boundary layer analyses of solutions to this equation, whose utility is manifested in §§2–5.

Appendix C discusses a class of Poisseuille flows, and places their analysis in the context of problems treated in this paper.

Finally, in Appendix D we make a close examination of a model case of layer potentials arising in §4, illustrating in particular L^1-gradient estimates applicable to R_1.

2 Nature of the convergence $v^\nu \to v^0$

As explained in the introduction, the component of the solution to (1.1)–(1.8) normal to the axis of the pipe solves the linear system

$$\frac{\partial v^\nu}{\partial t} = \nu \Delta v^\nu \quad \text{on} \;\; \mathbb{R}^+ \times D, \tag{2.1}$$

$$v^\nu(t,x) = \frac{\alpha(t)}{2\pi} x^\perp \quad \text{on} \;\; (0,\infty) \times \partial D, \tag{2.2}$$

$$v^\nu(0,x) = v_0(x) = s_0(|x|)x^\perp. \tag{2.3}$$

Here, for each $t \geq 0$, $v^\nu(t,\cdot)$ is a planar vector field on the disk $\overline{D} = \{x \in \mathbb{R}^2 : |x| \leq 1\}$, tangent to the boundary. We do not require $s_0(1)$ to be equal to $\alpha(0)/2\pi$. This non-matching is what produces the boundary layer effect. In this section we recall some results from [5] and [6] on the nature of the convergence $v^\nu \to v^0 \equiv v_0$, and produce some additional results, which will be of use in §3.

One tool to analyze solutions to (2.1)–(2.3) is the semigroup $e^{t\Delta}$, defined by $u(t) = e^{t\Delta} f$ solving

$$\frac{\partial u}{\partial t} = \Delta u \;\; \text{on} \;\; \mathbb{R}^+ \times D, \quad u\big|_{\mathbb{R}^+ \times \partial D} = 0, \quad u(0) = f. \tag{2.4}$$

We have

$$v^\nu(t) = e^{\nu t \Delta} v_0 + \mathcal{S}^\nu \alpha, \tag{2.5}$$

where $\mathcal{S}^\nu \alpha = V^\nu$ solves

$$\begin{aligned} \frac{\partial V^\nu}{\partial t} &= \nu \Delta V^\nu, \quad V^\nu = 0 \;\; \text{for} \;\; t < 0, \\ V^\nu\big|_{\mathbb{R}^+ \times \partial D} &= \frac{\alpha(t)}{2\pi} x^\perp. \end{aligned} \tag{2.6}$$

If we set

$$\begin{aligned} C_b^\infty(\mathbb{R}) &= \{\alpha \in C^\infty(\mathbb{R}) : \alpha(t) = 0 \text{ for } t < 0\}, \\ C_b(\mathbb{R}) &= \{\alpha \in C(R) : \alpha(t) = 0 \text{ for } t < 0\}, \end{aligned} \tag{2.7}$$

we have for each $\nu > 0$,

$$\begin{aligned} \mathcal{S}^\nu : C_b^\infty(\mathbb{R}) &\longrightarrow C_b^\infty(\mathbb{R} \times \overline{D}), \\ \mathcal{S}^\nu : C_b(\mathbb{R}) &\longrightarrow C_b(\mathbb{R} \times \overline{D}), \end{aligned} \tag{2.8}$$

where the subscript b in the spaces on the right side of (2.8) also denote vanishing for $t < 0$, as it does in (2.9) below. As shown in [6], we have a continuous extension

$$\mathcal{S}^\nu : L_b^p(\mathbb{R}) \longrightarrow C([0,1], H^{-1}_{\text{loc},b}(\mathbb{R} \times \partial D)), \tag{2.9}$$

where we use polar coordinates $[0,1] \times \partial D \to \overline{D}$, $(r, e^{i\theta}) \mapsto re^{i\theta}$. In each case (2.8)–(2.9),

$$\text{Tr}(\mathcal{S}^\nu \alpha) = \frac{\alpha}{2\pi} x^\perp, \tag{2.10}$$

for each $\nu > 0$.

The behavior as $\nu \searrow 0$ of the first term of the right side of (2.5) is governed by the behavior as $t \searrow 0$ of $e^{t\Delta}$, acting on u_0. The behavior of $\mathcal{S}^\nu \alpha$ as $\nu \searrow 0$ is attacked as follows. First assume $\alpha \in C_b^\infty(\mathbb{R})$. Set

$$\widetilde{V}^\nu(t,x) = V^\nu(t,x) - \frac{\alpha(t)}{2\pi} x^\perp. \tag{2.11}$$

This solves

$$\frac{\partial \widetilde{V}^\nu}{\partial t} = \nu\Delta \widetilde{V}^\nu - \alpha'(t) f_1, \quad \widetilde{V}^\nu(0) = 0, \quad \widetilde{V}^\nu\big|_{\mathbb{R}^+\times\partial D} = 0, \tag{2.12}$$

where

$$f_1(x) = \frac{x^\perp}{2\pi}. \tag{2.13}$$

Hence, by Duhamel's formula,

$$\widetilde{V}^\nu(t) = -\int_0^t e^{\nu(t-s)\Delta} f_1\, \alpha'(s)\, ds. \tag{2.14}$$

Substitution into (2.11) gives

$$\mathcal{S}^\nu\alpha(t) = \int_0^t \big(I - e^{\nu(t-s)\Delta}\big) f_1\, \alpha'(s)\, ds. \tag{2.15}$$

A mollifier argument gives the following (Proposition 2.1 of [6]).

Proposition 2.1 *Let $\mathfrak{X}$ be a Banach space of functions on D such that $f_1 \in \mathfrak{X}$ and $\{e^{t\Delta} : t \geq 0\}$ is a strongly continuous semigroup on $\mathfrak{X}$. Then*

$$\mathcal{S}^\nu : BV_b(\mathbb{R}) \longrightarrow C_b(\mathbb{R}, \mathfrak{X}), \tag{2.16}$$

with

$$\mathcal{S}^\nu\alpha(t) = \int_{I(t)} \big(I - e^{\nu(t-s)\Delta}\big) f_1\, d\alpha(s), \tag{2.17}$$

where we can take either $I(t) = [0, t]$ or $I(t) = [0, t)$. Furthermore,

$$\mathcal{S}^\nu\alpha(t) = -\lim_{\varepsilon\to 0} \nu \int_0^{t-\varepsilon} \Delta e^{\nu(t-s)\Delta} f_1\, \alpha(s)\, ds. \tag{2.18}$$

Corollary 2.2 *In the setting of Proposition 2.1,*

$$\|\mathcal{S}^\nu\alpha(t)\|_{\mathfrak{X}} \leq \|\alpha\|_{BV([0,t])} \sup_{s\in[0,t]} \|e^{\nu s\Delta} f_1 - f_1\|_{\mathfrak{X}}. \tag{2.19}$$

Hence, if v^ν solves (2.1)–(2.3) and $v_0 \in \mathfrak{X}$, then

$$\begin{aligned} \|v^\nu(t) - v_0\|_{\mathfrak{X}} \leq\ & \|e^{\nu t\Delta} v_0 - v_0\|_{\mathfrak{X}} \\ & + \|\alpha\|_{BV([0,t])} \sup_{s\in[0,t]} \|e^{\nu s\Delta} f_1 - f_1\|_{\mathfrak{X}}. \end{aligned} \tag{2.20}$$

The following records spaces $\mathfrak{X}$ to which Proposition 2.1 applies.

Proposition 2.3 *$\{e^{t\Delta} : t \geq 0\}$ is a strongly continuous semigroup on the following spaces:*

$$L^p(D), \quad 1 \leq p < \infty, \tag{2.21}$$

more generally the L^p-Sobolev spaces

$$H^{s,p}(D), \quad 1 \leq p < \infty, \quad 0 \leq s < \frac{1}{p}. \tag{2.22}$$

Also

$$C_*(D) = \{f \in C(\overline{D}) : f|_{\partial D} = 0\}, \tag{2.23}$$

$$H_0^1(D) = \{f \in H^{1,2}(D) : f|_{\partial D} = 0\}, \tag{2.24}$$

and

$$H_0^1(D) \cap H^{\sigma,2}(D), \quad 1 \le \sigma < \frac{5}{2}. \tag{2.25}$$

See [6] for more details and references. We mention that $\{e^{t\Delta} : t \ge 0\}$ is a contraction semigroup on the spaces (2.21), and also on

$$L^\infty(D), \quad C(\overline{D}), \tag{2.26}$$

but it is not strongly continuous at $t = 0$ on the spaces (2.26).

Proposition 2.3 has obvious applications to the limiting behavior as $\nu \searrow 0$ of the first term on the right side of (2.5). As for $\mathcal{S}^\nu \alpha$, Proposition 2.3 together with the formulas (2.17) and (2.18) can be used to establish the following (Proposition 4.2 of [6]).

Proposition 2.4 *Assume $q \in (1, \infty)$ and assume*

$$0 \le \sigma < \tau < \frac{1}{q}, \quad p \in \Big[1, \frac{2}{2 - 1/q + \sigma}\Big). \tag{2.27}$$

Then

$$\mathcal{S}^\nu : L_b^{p'}(\mathbb{R}) \longrightarrow C_b(\mathbb{R}, H^{\sigma,q}(D)), \tag{2.28}$$

and

$$\|\mathcal{S}^\nu \alpha(t)\|_{H^{\sigma,q}(D)} \le C(t) \nu^{(\tau-\sigma)/2} \|\alpha\|_{L^{p'}([0,t])} \, \|f_1\|_{H^{\tau,q}(D)}, \tag{2.29}$$

provided that also

$$1 \le p < \frac{2}{2 - (\tau - \sigma)}. \tag{2.30}$$

NOTE. For a given p, there exist q, τ, σ satisfying the hypotheses above, provided $1 \le p < 2$, i.e., provided $p' > 2$.

In addition to such global convergence results as given above, there are local convergence results, which hold in stronger norms, such as the following (Proposition 7.1 of [6]).

Proposition 2.5 *Let $\mathcal{O} \subset D$ be open, $\Omega \subset \overline{\Omega} \subset \mathcal{O}$, Ω smoothly bounded. Assume*

$$v_0 \in L^2(D), \quad v_0\big|_{\mathcal{O}} \in H^k(\mathcal{O}), \quad \alpha \in L_b^1(\mathbb{R}). \tag{2.31}$$

Then, given $T_0 < \infty$,

$$\lim_{\nu \searrow 0} v^\nu(t)\big|_\Omega = v_0\big|_\Omega \quad \text{in} \quad H^k(\Omega), \tag{2.32}$$

uniformly for $t \in [0, T_0]$.

Proposition 2.5 applies in particular when $v_0 \in H^k(D)$. In such a case, we can draw a stronger conclusion, via some analysis done in [8], which will also prove useful in §3. We introduce the following spaces:

$$\mathcal{V}^k(D) = \{u \in L^2(D) : Lu \in L^2(D), \ \forall\, L \in \mathfrak{X}^k\}, \tag{2.33}$$

where

$$\mathfrak{X}^k = \text{Span}\,\{Z_1 \cdots Z_j : j \le k, \ Z_\ell \in \mathfrak{X}^1\}, \tag{2.34}$$

with

$$\mathfrak{X}^1 = \{Y \text{ smooth vector field on } \overline{D} : Y \parallel \partial D\}. \tag{2.35}$$

We note that there exists a finite family $\{Y_j : 1 \le j \le M\} \subset \mathfrak{X}^1$ that spans $\mathfrak{X}^1$ over $C^\infty(\overline{D})$. In fact, we can take $M = 3$ and

$$Y_j = (1 - r^2)\frac{\partial}{\partial x_j} \quad (j = 1, 2), \quad Y_3 = \frac{\partial}{\partial \theta}. \tag{2.36}$$

We can set

$$Y^J = Y_{j_1} \cdots Y_{j_k}, \quad |J| = k, \tag{2.37}$$

and

$$\|u\|^2_{\mathcal{V}^k} = \sum_{|J| \le k} \|Y^J u\|^2_{L^2}. \tag{2.38}$$

We mention that, for each $k \in \mathbb{Z}^+$,

$$C_0^\infty(D) \text{ is dense in } \mathcal{V}^k(D). \tag{2.39}$$

The following result is a special case of results of §3.3 of [8], to which we will return in the following section of this paper.

Proposition 2.6 *For each $k \in \mathbb{Z}^+$, $\{e^{t\Delta} : t \ge 0\}$ is a strongly continuous semigroup on $\mathcal{V}^k(D)$.*

We can then bring in Corollary 2.2 and deduce:

Corollary 2.7 *If v^ν solves (2.1)–(2.3) and $v_0 \in \mathcal{V}^k(D)$, then*

$$\begin{aligned} \|v^\nu(t) - v_0\|_{\mathcal{V}^k} \le\ & \|e^{\nu t\Delta} v_0 - v_0\|_{\mathcal{V}^k} \\ & + \|\alpha\|_{BV([0,t])} \sup_{s \in [0,t]} \|e^{\nu s\Delta} f_1 - f_1\|_{\mathcal{V}^k}, \end{aligned} \tag{2.40}$$

which tends to 0 as $\nu \searrow 0$, uniformly for $t \in [0, T_0]$, provided $\alpha \in BV([0, T_0])$.

We now describe more detailed behavior of $e^{t\Delta} v_0(x)$ in case $v_0 \in C^\infty(\overline{D})$. Parallel to Proposition 2.5, we have the interior regularity result

$$v_0 \in C^\infty(\overline{D}),\ v(t,x) = e^{t\Delta} v_0(x) \Longrightarrow v \in C^\infty([0,\infty) \times D), \tag{2.41}$$

as well as $v \in C^\infty((0,\infty) \times \overline{D})$. It remains to analyze the behavior near $t = 0$ on a neighborhood of ∂D. In [6] this was attacked via the use of layer potentials. In Appendix B of this paper we use another method, exploiting a connection with the wave equation and the method of geometrical optics. In Proposition B.1 we exhibit $e^{t\Delta} v_0(x)$ near ∂D (for $v_0 \in C^\infty(\overline{D})$) as

$$\begin{aligned} e^{t\Delta} v_0(x) = v_0(x) & + \sum_{k=1}^N \frac{t^k}{k!} \Delta^k v_0(x) \\ & - \sum_{j=0}^{2N} 2 b_j(x) (4t)^{j/2} E_j\Big(\frac{\varphi(x)}{\sqrt{4t}}\Big) + \widehat{R}_N(t,x). \end{aligned} \tag{2.42}$$

Here we have

$$b_j \in C^\infty(\overline{D}), \quad \varphi(x) = 1 - |x|, \tag{2.43}$$

and

$$E_j(y) = \frac{1}{\sqrt{\pi}} \int_y^\infty e^{-s^2} (s-y)^j \, ds \\ = \frac{e^{-y^2}}{\sqrt{\pi}} \int_0^\infty e^{-s^2-2sy} s^j \, ds. \tag{2.44}$$

The term $\widehat{R}_N$ is a remainder. Its significance is that, for each $M, k \in \mathbb{N}$ there exists N such that

$$\|\widehat{R}_N(t,\cdot)\|_{C^k(\overline{D})} \le C_{M,k} t^M, \quad t \in (0,1]. \tag{2.45}$$

Note that for each $j \ge 0$, $E_j \in C^\infty([0,\infty))$ is positive and rapidly decreasing at infinity. The boundary layer phenomenon is captured by these terms, particularly the leading term

$$-2b_0(x) E_0\Bigl(\frac{1-|x|}{\sqrt{4t}}\Bigr). \tag{2.46}$$

Note that $E_0(0) = 1/2$, and hence

$$b_0(x) = v_0(x) \ \text{ for } \ x \in \partial D. \tag{2.47}$$

These results apply to f_1, given by (2.13). In this case, $\Delta f_1 = 0$, and we get

$$e^{t\Delta} f_1(x) = f_1(x) - \sum_{j=0}^{2N} 2g_j(x) (4t)^{j/2} E_j\Bigl(\frac{1-|x|}{\sqrt{4t}}\Bigr) + \widehat{R}_N(t,x), \tag{2.48}$$

with $g_j \in C^\infty(\overline{D})$. By (2.17), we get

$$\mathcal{S}^\nu \alpha(t) = \sum_{j=0}^{2N} 2g_j(x) \int_0^t (4\nu(t-s))^{j/2} E_j\Bigl(\frac{1-|x|}{\sqrt{4\nu(t-s)}}\Bigr) \, d\alpha(s) \\ - \int_0^t \widehat{R}_N(\nu(t-s),x) \, d\alpha(s). \tag{2.49}$$

The estimate (2.45) implies

$$\Bigl\| \int_0^t \widehat{R}_N(\nu(t-s),\cdot) \, d\alpha(s) \Bigr\|_{C^k(\overline{D})} \le C_{M,k}(T) \|\alpha\|_{BV} \, \nu^M, \quad 0 < t \le T. \tag{2.50}$$

By (2.5), the results (2.42) (with t replaced by νt) and (2.49) apply to produce an asymptotic expansion for $v^\nu(t,x)$ as $\nu \searrow 0$, valid uniformly for $t \in [0,T]$.

We use these asymptotic results to obtain further estimates on $e^{t\Delta} v_0$, for $v_0 \in C^\infty(\overline{D})$, which will be of use in §3. To set this up, we introduce the following generalization of $\mathcal{V}^k(D)$ in (2.33). Given $k \in \mathbb{Z}^+$, $p \in [1,\infty]$, set

$$\mathcal{V}^{k,p}(D) = \{u \in L^p(D) : Lu \in L^p(D), \ \forall \, L \in \mathfrak{X}^k\}. \tag{2.51}$$

Also set

$$\mathcal{V}^{\infty,p}(D) = \bigcap_k \mathcal{V}^{k,p}(D). \tag{2.52}$$

Proposition 2.8 *Given $v_0 \in C^\infty(\overline{D})$, we have*

$$\{e^{t\Delta} v_0 : t \ge 0\} \ \textit{ bounded in } \ \mathcal{V}^{\infty,\infty}(D). \tag{2.53}$$

Proof. The bound in $C^\infty(\overline{D}) \subset \mathcal{V}^{\infty,\infty}(D)$ for $t \ge 1$ is elementary. Also the bound in $C^\infty(\{x : |x| \le 1/2\})$ for $t \in [0,1]$ follows from (2.41). To finish, it suffices to show that

$$\Big\{ E_j\Big(\frac{1-|x|}{\sqrt{4t}}\Big) : 0 < t \le 1 \Big\} \ \text{ is bounded in } \ \mathcal{V}^{\infty,\infty}, \tag{2.54}$$

near ∂D, which follows from the assertion that

$$\Big\{ \Big(y\frac{d}{dy}\Big)^k E_j\Big(\frac{y}{\sqrt{4t}}\Big) : 0 < t \le 1 \Big\} \ \text{ is bounded in } \ L^\infty([0,1]), \ \ \forall\, j,k. \tag{2.55}$$

The identity

$$y\frac{d}{dy}E_j\Big(\frac{y}{\sqrt{4t}}\Big) = \frac{y}{\sqrt{4t}} E_j'\Big(\frac{y}{\sqrt{4t}}\Big) \tag{2.56}$$

gives (2.55) for $k = 1$, and the result for general k follows by induction. ☐

Applying this to (2.5) and (2.17), we have:

Proposition 2.9 *Given v^ν solving (2.1)–(2.3) with $v_0 \in C^\infty(\overline{D})$, $\alpha \in BV_b(\mathbb{R})$, we have*

$$\{v^\nu(t) : \nu \in (0,1]\} \ \ \textit{bounded in} \ \ \mathcal{V}^{\infty,\infty}(D), \tag{2.57}$$

uniformly for $t \in [0, T_0]$, $T_0 < \infty$.

Recall from Proposition 1.1 that solutions to (2.1)–(2.3) have the form

$$v^\nu(t,x) = \tilde{s}^\nu(t,x)x^\perp = s^\nu(t,|x|)x^\perp. \tag{2.58}$$

The following complement to Proposition 2.9 will also be useful in §3.

Proposition 2.10 *When (2.55) and (2.56) hold, we have*

$$\{\tilde{s}^\nu(t,\cdot) : \nu \in (0,1], \ t \in [0,T_0]\} \ \ \textit{bounded in} \ \ \mathcal{V}^{\infty,\infty}(D). \tag{2.59}$$

Proof. First note that

$$v^\nu(t,x) \cdot x^\perp = \tilde{s}^\nu(t,x)|x|^2, \tag{2.60}$$

which gives the desired estimate on $A_{1/2} = \{x \in \overline{D} : |x| \ge 1/2\}$:

$$\|\tilde{s}^\nu(t,\cdot)\|_{\mathcal{V}^{k,\infty}(A_{1/2})} \le C_k \|v^\nu(t,\cdot)\|_{\mathcal{V}^{k,\infty}(A_{1/2})}. \tag{2.61}$$

It remains to show that

$$\{\tilde{s}^\nu(t,\cdot)\big|_{\overline{D}_{1/2}} : \nu \in (0,1], \ t \in [0,T_0]\} \ \text{ bounded in } \ C^\infty(\overline{D}_{1/2}), \tag{2.62}$$

where $\overline{D}_{1/2} = \{x \in \mathbb{R}^2 : |x| \le 1/2\}$. To do this, note that

$$s^\nu(t,r)r = v^\nu(t,re_1)\cdot e_2, \tag{2.63}$$

where $\{e_1, e_2\}$ is the standard basis of $\mathbb{R}^2$. Hence

$$s^\nu(t,r) = \int_0^1 e_2 \cdot \nabla_{e_1} v^\nu(t, r\sigma e_1)\, d\sigma. \tag{2.64}$$

This defines $s^\nu(t,r)$ as an even function of $r \in [-1,1]$, and we have

$$\|s^\nu(t,\cdot)\|_{C^k([-1/2,1/2])} \le C_k \|v^\nu(t,\cdot)\|_{C^{k+1}(\overline{D}_{1/2})}. \tag{2.65}$$

Now $\tilde{s}^\nu$ and s^ν are related by (2.58), hence

$$\tilde{s}^\nu(t,x) = \sigma_\nu(t,|x|^2), \quad \sigma_\nu(t,\rho) = s^\nu(t,\rho^{1/2}). \tag{2.66}$$

We have

$$\|\tilde{s}^\nu(t,\cdot)\|_{C^\ell(\overline{D}_{1/2})} \le C \|\sigma_\nu(t,\cdot)\|_{C^\ell([-1/4,1/4])}. \tag{2.67}$$

Clearly

$$\|\sigma_\nu(t,\cdot)\|_{C^0([-1/4,1/4])} = \|s^\nu(t,\cdot)\|_{C^0([-1/2,1/2])}. \tag{2.68}$$

Next

$$\begin{aligned} \partial_\rho \sigma_\nu(t,\rho) &= \frac{1}{2}\rho^{-1/2}\partial_r s^\nu(t,\rho^{1/2}) \\ &= \frac{1}{2}\int_0^1 \partial_r^2 s^\nu(t,\rho^{1/2}\tau)\,d\tau, \end{aligned} \tag{2.69}$$

the latter identity using $\partial_r s^\nu(t,0) = 0$. Inductively, we obtain

$$\|\sigma_\nu(t,\cdot)\|_{C^\ell([-1/4,1/4])} \le C \|s^\nu(t,\cdot)\|_{C^{2\ell}([-1/2,1/2])}. \tag{2.70}$$

Recalling (2.65) and (2.67), we get

$$\begin{aligned} \|\tilde{s}^\nu(t,\cdot)\|_{C^k(\overline{D}_{1/2})} &\le C \|s^\nu(t,\cdot)\|_{C^{2k}([-1/2,1,2])} \\ &\le C \|v^\nu(t,\cdot)\|_{C^{2k+1}(\overline{D}_{1/2})}, \end{aligned} \tag{2.71}$$

finishing the proof of Proposition 2.10. □

We record some more estimates that follow from the arguments given above. Since

$$\int_D |x|^{-p}\,dx = 2\pi \int_0^1 r^{1-p}\,dr < \infty \quad \text{for} \quad p \in [1,2), \tag{2.72}$$

we have

$$\|\tilde{s}^\nu(t,\cdot)\|_{L^p(D)} \le C\|v^\nu(t,\cdot)\|_{L^p(D)} + C\|v^\nu(t,\cdot)\|_{L^\infty(D_{1/2})}, \quad p \in [1,2). \tag{2.73}$$

We also have

$$\|\tilde{s}^\nu(t,\cdot)\|_{L^p(D)} \le C\|v^\nu(t,\cdot)\|_{L^p(D)} + C\|v^\nu(t,\cdot)\|_{C^1(\overline{D}_{1/2})}, \quad p \in [1,\infty]. \tag{2.74}$$

3 Nature of the convergence $w^\nu \to w^0$

Recall from the introduction that the component of the solution to (1.1)–(1.8) parallel to the axis of the pipe solves the scalar equation

$$\frac{\partial w^\nu}{\partial t} = \nu \Delta w^\nu - X_\nu w^\nu + f^\nu(t) \quad \text{on} \quad \mathbb{R}^+ \times D, \tag{3.1}$$

$$w^\nu(t,x) = \beta(t) \quad \text{on} \quad (0,\infty) \times \partial D, \tag{3.2}$$

$$w^\nu(0,x) = w_0(x), \tag{3.3}$$

where

$$X_\nu g = \nabla_{v^\nu} g = s^\nu(t,|x|)\frac{\partial g}{\partial \theta}. \tag{3.4}$$

We do not require $w_0(x) = \beta(0)$ for $x \in \partial D$. We assume v^ν is given by (2.1)–(2.3), with

$$v_0 \in C^\infty(\overline{D}), \quad \alpha \in BV_b(\mathbb{R}), \tag{3.5}$$

so the results of §2 are available. In this section we show how results of §2 allow us to apply results of [8] to draw conclusions about the nature of the convergence $w^\nu \to w^0$, the solution to

$$\frac{\partial w^0}{\partial t} = -Xw^0 + f^0(t), \tag{3.6}$$

$$w^0(0,x) = w_0(x), \tag{3.7}$$

where

$$Xg = \nabla_{v_0} g = s_0(|x|)\frac{\partial g}{\partial \theta}. \tag{3.8}$$

The initial-boundary problem (3.1)–(3.3) is a singular perturbation of (3.6)–(3.8) in two ways, due both to the presence of $\nu\Delta w^\nu$ in (3.1) and to the nature of the convergence of the coefficients of X_ν to those of X. We can partially separate these two mechanisms by rewriting (3.1) as

$$\frac{\partial w^\nu}{\partial t} = (\nu\Delta - X)w^\nu + (X - X_\nu)w^\nu + f^\nu(t). \tag{3.9}$$

Also let us set

$$W^\nu(t,x) = w^\nu(t,x) - \beta(t), \tag{3.10}$$

so $W^\nu(t,x)$ solves

$$\begin{aligned} \frac{\partial W^\nu}{\partial t} &= (\nu\Delta - X)W^\nu + (X - X_\nu)W^\nu + g^\nu(t), \\ W^\nu(t,x) &= 0 \quad \text{on} \quad (0,\infty)\times\partial D, \\ W^\nu(0,x) &= W_0(x) = w_0(x) - \beta(0), \end{aligned} \tag{3.11}$$

where

$$g^\nu(t) = f^\nu(t) - \beta'(t), \tag{3.12}$$

assuming $\beta \in C_b^1(\mathbb{R})$. (We relax this requirement below.) Duhamel's formula gives

$$\begin{aligned} W^\nu(t) &= e^{t(\nu\Delta - X)}W_0 + \int_0^t e^{(t-s)(\nu\Delta - X)}(XW^\nu - X_\nu W^\nu + g^\nu(s))\,ds \\ &= e^{t(\nu\Delta - X)}W_0 + \int_0^t e^{(t-s)(\nu\Delta - X)}(s_0 - s^\nu)\frac{\partial W^\nu}{\partial \theta}\,ds \\ &\quad + \int_0^t g^\nu(s)e^{(t-s)(\nu\Delta - X)}1\,ds. \end{aligned} \tag{3.13}$$

By comparison, if we set

$$W^0(t,x) = w^0(t,x) - \beta(t), \tag{3.14}$$

which solves

$$\frac{\partial W^0}{\partial t} = -XW^0 + g^0(t), \quad W^0(0,x) = W_0(x), \tag{3.15}$$

with

$$g^0(t) = f^0(t) - \beta'(t), \tag{3.16}$$

we have

$$W^0(t) = e^{-tX} W_0 + \int_0^t g^0(s)\, ds. \tag{3.17}$$

To compare $w^\nu(t,x)$ and $w^0(t,x)$, we will separately estimate

$$R_1(\nu,t,x) = e^{t(\nu\Delta - X)} W_0 - e^{-tX} W_0, \tag{3.18}$$

$$R_2(\nu,t,x) = \int_0^t \Big[g^\nu(s) e^{(t-s)(\nu\Delta - X)} 1 - g^0(s) \Big]\, ds, \text{ and} \tag{3.19}$$

$$R_3(\nu,t,x) = \int_0^t e^{(t-s)(\nu\Delta - X)} (s_0 - s^\nu) \frac{\partial W^\nu}{\partial \theta}\, ds, \tag{3.20}$$

which fit together as follows:

$$w^\nu(t,x) - w^0(t,x) = W^\nu(t,x) - W^0(t,x) = \sum_{j=1}^3 R_j(\nu,t,x). \tag{3.21}$$

We begin with an estimate on $R_3(\nu,t,x)$. Note that

$$\frac{\partial}{\partial \theta} \text{ commutes with } X,\ X_\nu, \text{ and } \Delta. \tag{3.22}$$

Hence

$$\begin{aligned} &Z^\nu = \frac{\partial W^\nu}{\partial \theta} \Longrightarrow \\ &\frac{\partial Z^\nu}{\partial t} = (\nu\Delta - X_\nu) Z^\nu, \quad Z^\nu\big|_{\mathbb{R}^+ \times \partial D} = 0, \quad Z^\nu(0) = \frac{\partial W_0}{\partial \theta} = \frac{\partial w_0}{\partial \theta}. \end{aligned} \tag{3.23}$$

Thus the maximum principle gives

$$\Big\| \frac{\partial W^\nu}{\partial \theta}(s) \Big\|_{L^\infty(D)} \le \Big\| \frac{\partial W_0}{\partial \theta} \Big\|_{L^\infty(D)} = \Big\| \frac{\partial w_0}{\partial \theta} \Big\|_{L^\infty(D)}. \tag{3.24}$$

Since the semigroup $e^{t(\nu\Delta - X)}$ is positivity preserving, we have

$$|R_3(\nu,t,x)| \le \|\partial_\theta w_0\|_{L^\infty} \int_0^t e^{(t-s)(\nu\Delta - X)} \big| s_0(|x|) - s^\nu(s,|x|) \big|\, ds. \tag{3.25}$$

We also have, by radial symmetry,

$$e^{(t-s)(\nu\Delta - X)} |s_0 - s^\nu| = e^{\nu(t-s)\Delta} |s_0 - s^\nu|, \tag{3.26}$$

hence

$$|R_3(\nu,t,x)| \le \|\partial_\theta w_0\|_{L^\infty} \int_0^t e^{\nu(t-s)\Delta} |\tilde{s}_0 - \tilde{s}^\nu|\, ds. \tag{3.27}$$

Here, as in (2.58), $\tilde{s}^\nu(t,x) = s^\nu(t,|x|)$, and similarly $\tilde{s}_0(x) = s_0(|x|)$.

Moving on to $R_2(\nu,t,x)$, we have, as in (3.26),

$$e^{(t-s)(\nu\Delta - X)} 1 = e^{\nu(t-s)\Delta} 1, \tag{3.28}$$

and hence

$$R_2(\nu,t,x) = \int_0^t \Big[(g^\nu(s) - g^0(s)) + g^\nu(s)(e^{\nu(t-s)\Delta}1 - 1)\Big]\,ds$$
$$= \int_0^t [f^\nu(s) - f^0(s)]\,ds + \int_0^t g^\nu(s)\big(e^{\nu(t-s)\Delta}1 - 1\big)\,ds \tag{3.29}$$
$$= \int_0^t [f^\nu(s) - f^0(s)]\,ds + R_2^\#(\nu,t,x).$$

Using (3.12), we can write

$$R_2^\#(\nu,t,x) = \int_0^t f^\nu(s)(e^{\nu(t-s)\Delta}1 - 1)\,ds + \int_0^t (e^{\nu(t-s)\Delta}1 - 1)\,d\beta(s), \tag{3.30}$$

and by a mollifier argument such as described in §1 extend the validity of this identity from $\beta \in C_b^1(\mathbb{R})$ to $\beta \in BV_b(\mathbb{R})$.

We record some L^p-estimates on R_3 and R_2. Since $e^{t\Delta}$ is a contraction semigroup on $L^p(D)$, (3.27) yields

$$\begin{aligned}\|R_3(\nu,t,\cdot)\|_{L^p} &\le \|\partial_\theta w_0\|_{L^\infty} \int_0^t \|\tilde{s}_0 - \tilde{s}^\nu(s,\cdot)\|_{L^p}\,ds \\ &\le \|\partial_\theta w_0\|_{L^\infty} \sup_{s\in[0,t]} \|\tilde{s}_0(\cdot) - \tilde{s}^\nu(s,\cdot)\|_{L^p}\cdot t.\end{aligned} \tag{3.31}$$

Meanwhile, (3.30) yields

$$\|R_2^\#(\nu,t,\cdot)\|_{L^p} \le \Big(\|f^\nu\|_{L^1([0,t])} + \|\beta\|_{BV([0,t])}\Big)\cdot \sup_{s\in[0,t]} \|1 - e^{\nu(t-s)\Delta}1\|_{L^p}. \tag{3.32}$$

The arguments yielding (2.73)–(2.74) also yield

$$\begin{aligned}&\|\tilde{s}_0(\cdot) - \tilde{s}^\nu(t,\cdot)\|_{L^p(D)} \\ &\le C\|v^\nu(t,\cdot) - v_0(\cdot)\|_{L^p(D)} + C\|v^\nu(t,\cdot) - v_0(\cdot)\|_{L^\infty(D_{1/2})}, \quad p\in[1,2),\end{aligned} \tag{3.33}$$

and

$$\begin{aligned}&\|\tilde{s}_0(\cdot) - \tilde{s}^\nu(t,\cdot)\|_{L^p(D)} \\ &\le C\|v^\nu(t,\cdot) - v_0(\cdot)\|_{L^p(D)} + C\|v^\nu(t,\cdot) - v_0(\cdot)\|_{C^1(D_{1/2})}, \quad p\in[2,\infty).\end{aligned} \tag{3.34}$$

Results of §2 guarantee that these quantities tend to 0 as $\nu \searrow 0$, uniformly in $t\in[0,T_0]$, under hypotheses weaker than (3.5). Results of Appendix B give

$$\|1 - e^{\nu t\Delta}1\|_{L^p(D)} \le C(T_0)\nu^{1/2p}, \tag{3.35}$$

for $t\in[0,T_0]$. We have:

Proposition 3.1 *Assume that (3.5) holds. Also assume* $w_0 \in C^1(\overline{D})$, $\beta \in BV_b(\mathbb{R})$, $f^0, f^\nu \in L_b^1(\mathbb{R})$ *and* $\|f^\nu - f^0\|_{L^1([0,T])} \to 0$, *for each* $T<\infty$. *Then, for each* $p\in[1,\infty)$,

$$\|R_2(\nu,t,\cdot)\|_{L^p(D)} + \|R_3(\nu,t,\cdot)\|_{L^p(D)} \to 0 \quad \textit{as} \quad \nu\searrow 0, \tag{3.36}$$

uniformly in $t\in[0,T]$.

We turn our attention to $R_1(\nu, t, x)$, given by (3.18), i.e., to the nature of the convergence

$$e^{t(\nu\Delta - X)} W_0 \longrightarrow e^{-tX} W_0, \tag{3.37}$$

as $\nu \searrow 0$. Chapter 3 of [8] was devoted to such an analysis, in a more general setting, in which $\overline{D}$ is replaced by a general compact Riemannian manifold with boundary $\overline{\mathcal{O}}$, with Laplace-Beltrami operator Δ, and X is taken to be a smooth vector field on $\overline{\mathcal{O}}$, tangent to the boundary $\partial\mathcal{O}$ and satisfying $\operatorname{div} X = 0$. We recall some of these results.

Let us set

$$U^\nu(t, x) = e^{t(\nu\Delta - X)} U_0(x). \tag{3.38}$$

The identity

$$U^\nu(t) = e^{-tX} U_0 + \nu \int_0^t e^{-(t-s)X} \Delta U^\nu(s)\, ds \tag{3.39}$$

is useful once one has the following (Lemma 3.1.2 of [8]).

Lemma 3.2 *There exists $K \in (0, \infty)$, independent of $\nu \in (0, 1]$, such that, if $U_0 \in \mathcal{D}(\Delta^2)$,*

$$\|\Delta U^\nu(t)\|_{L^2}^2 \le e^{2Kt} \|\Delta U_0\|_{L^2}^2. \tag{3.40}$$

This is proven by estimating

$$\frac{d}{dt} \|\Delta U^\nu(t)\|_{L^2}^2 = 2 \operatorname{Re} (\Delta \partial_t U^\nu, \Delta U^\nu)_{L^2} = \cdots . \tag{3.41}$$

The proof exploits the identity

$$\mathcal{D}((\nu\Delta - X)^2) = \mathcal{D}(\Delta^2). \tag{3.42}$$

With this in hand, one proceeds to Proposition 3.1.3 of [8]:

Proposition 3.3 *Given $p \in [1, \infty)$,*

$$e^{t(\nu\Delta - X)} W_0 \longrightarrow e^{-tX} W_0 \quad \textit{as} \quad \nu \to 0, \tag{3.43}$$

in L^p-norm, for all $W_0 \in L^p(D)$.

Ingredients in the proof include the contraction property on $L^p(D)$ of $e^{t(\nu\Delta - X)}$, the validity of (3.43) on a dense subspace of $L^2(D)$, via Lemma 3.2, to get (3.43) for $p \in [1, 2]$, and then use of duality to get (3.43) for $p > 2$, first weak*, then, via uniform convexity, in norm.

Putting together Propositions 3.1 and 3.3, we have L^p estimates on

$$\begin{aligned} w^\nu(t, x) - w^0(t, x) &= W^\nu(t, x) - W^0(t, x) \\ &= R_1(\nu, t, x) + R_2(\nu, t, x) + R_3(\nu, t, x). \end{aligned} \tag{3.44}$$

Proposition 3.4 *Under the hypotheses of Proposition 3.1, as $\nu \searrow 0$,*

$$W^\nu(t, \cdot) \longrightarrow W^0(t, \cdot), \quad \textit{and hence} \tag{3.45}$$

$$w^\nu(t, \cdot) \longrightarrow w^0(t, \cdot), \tag{3.46}$$

in norm, in $L^p(D)$, for each $p \in [1, \infty)$.

We now discuss convergence in stronger topologies, starting with $R_1(\nu, t, x)$. In (3.1.19) of [8], we obtain

$$\|e^{t(\nu\Delta - X)} W_0\|_{H^s(D)} \le C e^{Kt} \|W_0\|_{H^s(D)}, \quad s \in \Big[0, \frac{1}{2}\Big), \tag{3.47}$$

with C and K independent of $\nu \in (0, 1]$. On the other hand (as mentioned above), $e^{t(\nu\Delta - X)}$ is a contraction semigroup on $L^p(D)$ for each p. Interpolation with (3.47) gives

$$\|e^{t(\nu\Delta - X)} W_0\|_{H^{\sigma,q}(D)} \le C e^{Kt} \|W_0\|_{H^{\sigma,q}(D)}, \tag{3.48}$$

provided

$$q \in [2, \infty), \quad \sigma q \in [0, 1). \tag{3.49}$$

Using this leads to Proposition 3.1.4 of [8]:

Proposition 3.5 *Given (3.49), then*

$$W_0 \in H^{\sigma,q}(D) \Longrightarrow \lim_{\nu \searrow 0} e^{t(\nu\Delta - X)} W_0 = e^{-tX} W_0, \tag{3.50}$$

in $H^{\sigma,q}$-norm, uniformly in $t \in [0, T_0]$.

A similar (but slightly more elementary) analysis of $R_2(\nu, t, x)$ via (3.29)–(3.30) gives, under the hypotheses of Proposition 3.1,

$$\lim_{\nu \searrow 0} R_2(\nu, t, \cdot) = 0, \quad \text{in} \quad H^{\sigma,q}\text{-norm}, \tag{3.51}$$

for σ and q as in (3.49).

We move on to estimates in the spaces $\mathcal{V}^k(D)$, defined in (2.33)–(2.38). The following result (which extends Proposition 2.6) is Proposition 3.3.3 of [8].

Proposition 3.6 *For each $k \in \mathbb{Z}^+$, $\nu > 0$, $e^{t(\nu\Delta - X)}$ is a strongly continuous semigroup on $\mathcal{V}^k(D)$, and, with B_k independent of $\nu \in (0, 1]$,*

$$\|e^{t(\nu\Delta - X)} W_0\|_{\mathcal{V}^k} \le e^{tB_k} \|W_0\|_{\mathcal{V}^k}. \tag{3.52}$$

The proof involves estimating a weighted sum of terms

$$\frac{d}{dt} \|Y^J U^\nu(t)\|_{L^2}^2, \quad |J| \le k, \tag{3.53}$$

and takes about 4 pages in [8]. From here, we get Proposition 3.3.4 of [8]:

Proposition 3.7 *In the setting of Proposition 3.6,*

$$W_0 \in \mathcal{V}^k(D) \Longrightarrow \lim_{\nu \searrow 0} e^{t(\nu\Delta - X)} W_0 \to e^{-tX} W_0, \tag{3.54}$$

in norm, in $\mathcal{V}^k(D)$.

We make some comments about the proof of this result. The boundedness result (3.52) plus the L^2-convergence from (3.43) imply convergence in (3.54), weak* in $\mathcal{V}^k(D)$. To get norm convergence, one argues further. It suffices to get norm convergence on a dense subspace, e.g.,

$$C_0^\infty(D) \subset \mathcal{V}^{2k}(D) \subset \mathcal{V}^k(D). \tag{3.55}$$

Appendix A of [8] establishes the complex interpolation property

$$\mathcal{V}^k(D) = [L^2(D), \mathcal{V}^{2k}(D)]_{1/2}. \tag{3.56}$$

Hence, for $f \in \mathcal{V}^{2k}(D)$,

$$\begin{aligned} &\|e^{t(\nu\Delta - X)}f - e^{-tX}f\|_{\mathcal{V}^k} \\ &\le \|e^{t(\nu\Delta - X)}f - e^{-tX}f\|_{L^2}^{1/2} \cdot \|e^{t(\nu\Delta - X)}f - e^{-tX}f\|_{\mathcal{V}^{2k}}^{1/2}. \end{aligned} \tag{3.57}$$

The first factor on the right side of (3.57) tends to 0 as $\nu \searrow 0$, by Proposition 3.3, and the last factor is uniformly bounded as $\nu \searrow 0$ by (3.52), with k replaced by $2k$.

We next discuss convergence of (3.45) in the spaces $\mathcal{V}^k(D)$. For simplicity, we assume here that

$$g^\nu,\ g^0 \equiv 0, \tag{3.58}$$

so W^ν is given by

$$\frac{\partial W^\nu}{\partial t} = (\nu\Delta - X_\nu)W^\nu, \quad W^\nu\big|_{\mathbb{R}^+ \times \partial D} = 0, \quad W^\nu(0,x) = W_0(x), \tag{3.59}$$

and W^0 by

$$\frac{\partial W^0}{\partial t} = -XW^0, \quad W^0(0,x) = W_0(x), \quad \text{i.e.,}\ \ W^0(t) = e^{-tX}W_0. \tag{3.60}$$

(Treating the general case simply involves one more use of Duhamel's formula.) To set up the analysis, Chapter 4 of [8] defined a class $\widehat{\mathfrak{X}}^1$ of vector fields on D, depending on t and ν, as follows. Recall $\mathfrak{X}^1$, given by (2.35), and let $\{Y_j\}$ be a finite spanning set, as in (2.36). We say $Z_\nu \in \widehat{\mathfrak{X}}^1$ provided we can write

$$Z_\nu = \sum_j A_j^\nu(t,x)Y_j, \tag{3.61}$$

with

$$\{A_j^\nu(t,\cdot) : \nu \in (0,1],\ t \in [0,T_0]\} \ \text{ bounded in } \ \mathcal{V}^{\infty,\infty}(D), \tag{3.62}$$

for each $T_0 \in (0,\infty)$. In the current case of interest,

$$X_\nu = \tilde{s}^\nu(t,x)\frac{\partial}{\partial\theta}, \tag{3.63}$$

Proposition 2.10 gives

$$X_\nu \in \widehat{\mathfrak{X}}^1. \tag{3.64}$$

We also set

$$\widehat{\mathfrak{X}}^k = \operatorname{Span}\{Z_\nu Y^J : Z_\nu \in \widehat{\mathfrak{X}}^1,\ Y^J \in \mathfrak{X}^{k-1}\}. \tag{3.65}$$

The following results are established in Chapter 4 of [8]:

$$\begin{aligned} Z_\nu \in \widehat{\mathfrak{X}}^1,\ Y \in \mathfrak{X}^1 &\Longrightarrow [Z_\nu, Y] \in \widehat{\mathfrak{X}}^1, \\ P_\nu \in \widehat{\mathfrak{X}}^k,\ Y^I \in \mathfrak{X}^\ell &\Longrightarrow Y^I P_\nu \in \widehat{\mathfrak{X}}^{k+\ell}, \end{aligned} \tag{3.66}$$

and play a role in the demonstration of the next result (Proposition 4.1.5 of [8]).

Proposition 3.8 *Assume $W_0 \in \mathcal{V}^k(D)$. Given that X_ν satisfies (3.64), there is a unique solution W^ν to (3.59), satisfying*

$$W^\nu \in C([0,\infty), \mathcal{V}^k(D)) \cap C^\infty((0,\infty) \times \overline{D}), \tag{3.67}$$

and we have

$$\|W^\nu(t)\|_{\mathcal{V}^k} \le e^{tB_k} \|W_0\|_{\mathcal{V}^k}, \tag{3.68}$$

with B_k independent of $\nu \in (0,1]$.

The proof involves estimating a weighted sum of terms

$$\frac{d}{dt} \|Y^J W^\nu(t)\|_{L^2}^2, \quad |J| \le k, \tag{3.69}$$

and is a bit more elaborate than the proof of Proposition 3.6, bringing in the results of (3.66).

The uniform bounds (3.68) plus the L^p-norm convergence (3.45), with $p = 2$, imply the following, as shown in Proposition 4.2.1 of [8]:

Proposition 3.9 *Retain the hypotheses of Proposition 3.1 (which imply (3.64), and assume (3.58). Assume $W_0 \in \mathcal{V}^k(D)$. Then, as $\nu \searrow 0$,*

$$W^\nu(t) \longrightarrow e^{-tX} W_0, \tag{3.70}$$

weak* in $\mathcal{V}^k(D)$.

The same argument used to go from weak* convergence to $\mathcal{V}^k$-norm convergence in Proposition 3.7, involving (3.55)–(3.57), works here, yielding the following improvement of Proposition 3.9, proved as in Proposition 4.2.4 of [8]:

Proposition 3.10 *In the setting of Proposition 3.9, we have convergence in (3.70), in $\mathcal{V}^k$-norm.*

We conclude this section with some complementary results. First there is the contraction property:

$$\|W^\nu(t,\cdot)\|_{L^p} \le \|W_0\|_{L^p}, \quad 1 \le p \le \infty. \tag{3.71}$$

Next, if also $W_0 \in \mathcal{V}^k(D)$ with $k > 1$, the result (3.70) implies

$$W^\nu(t,x) \longrightarrow e^{-tX} W_0, \ \text{ locally uniformly in } \ D. \tag{3.72}$$

In particular,

$$W_0 \in C^\infty(\overline{D}) \Longrightarrow W^\nu(t) \to e^{-tX} W_0, \ \text{ boundedly and locally uniformly on } \ D. \tag{3.73}$$

Combining (3.71) and (3.73) and using standard approximation arguments yields:

Proposition 3.11 *In the setting of Proposition 3.9,*

$$W_0 \in C(\overline{D}) \Longrightarrow W^\nu(t) \to e^{-tX} W_0, \ \textit{ boundedly and locally uniformly on } \ D. \tag{3.74}$$

4 Further boundary layer estimates

In §3 we have seen various spaces in which

$$w^\nu(t,x) - w^0(t,x) \longrightarrow 0 \tag{4.1}$$

as $\nu \searrow 0$. Here we take a closer look at the boundary layers that form and prevent (4.1) from holding in sup norm. We work with the decomposition (3.21), i.e.,

$$w^\nu(t,x) - w^0(t,x) = R_1(\nu,t,x) + R_2(\nu,t,x) + R_3(\nu,t,x), \tag{4.2}$$

where, we recall,

$$\begin{aligned} R_1(\nu,t,x) &= e^{t(\nu\Delta - X)}W_0 - e^{-tX}W_0, &(4.3)\\ R_2(\nu,t,x) &= \int_0^t \big[f^\nu(s) - f^0(s)\big]\,ds &(4.4)\\ &\quad + \int_0^t f^\nu(s)\big(e^{\nu(t-s)\Delta}1 - 1\big)\,ds &(4.5)\\ &\quad + \int_0^t \big(e^{\nu(t-s)\Delta}1 - 1\big)\,d\beta(s), &(4.6)\\ R_3(\nu,t,x) &= \int_0^t e^{(t-s)(\nu\Delta - X)}(s_0 - s^\nu)\frac{\partial W^\nu}{\partial\theta}\,ds. &(4.7) \end{aligned}$$

We also recall that $W_0(x) = w_0(x) - \beta(0)$ and $W^\nu(t,x)$ is given by (3.10)–(3.11). Note therefore that

$$\frac{\partial W^\nu}{\partial\theta} = \frac{\partial w^\nu}{\partial\theta}. \tag{4.8}$$

Of the three terms on the right side of (4.2), $R_2(\nu,t,x)$ is the easiest to analyze precisely. Proposition B.1 applied to $f \equiv 1$ gives

$$\begin{aligned} &e^{\nu(t-s)\Delta}1 - 1\\ &= -\sum_{j=0}^{2N} 2b_j(x)\big(4\nu(t-s)\big)^{j/2}E_j\Big(\frac{\varphi(x)}{\sqrt{4\nu(t-s)}}\Big) + \widehat{R}_N(\nu(t-s),x), \end{aligned} \tag{4.9}$$

where, for each $M,k \in \mathbb{N}$, there exists N such that

$$\|\widehat{R}_N(\nu(t-s),\cdot)\|_{C^k(\overline{D})} \le C_{M,k}\big(\nu(t-s)\big)^M, \quad 0 \le s \le t \le T_0. \tag{4.10}$$

Thus, for example, the term (4.6) has the form

$$\begin{aligned} &-\sum_{j=0}^{2N} 2b_j(x)\int_0^t \big(4\nu(t-s)\big)^{j/2}E_j\Big(\frac{\varphi(x)}{\sqrt{4\nu(t-s)}}\Big)\,d\beta(s)\\ &+ \int_0^t \widehat{R}_N(\nu(t-s),x)\,d\beta(s). \end{aligned} \tag{4.11}$$

We recall that

$$\varphi(x) = 1 - |x|, \quad b_0\big|_{\partial D} = 1, \tag{4.12}$$

and that E_j is given by (B.39); in particular,

$$E_0(y) = \frac{1}{\sqrt{\pi}} \int_y^\infty e^{-s^2}\, ds. \tag{4.13}$$

Thus the principal term in (4.11) is

$$-2b_0(x) \int_0^t E_0\Big(\frac{1-|x|}{\sqrt{4\nu(t-s)}}\Big)\, d\beta(s), \tag{4.14}$$

and for $j \geq 1$ the jth term in (4.11) is $\leq C_j \nu^{j/2} t^{j/2}$, uniformly in $x \in \overline{D}$.

The term (4.5) has a similar form as (4.6), and of course the term on the right side of (4.4) is completely elementary. We summarize.

Proposition 4.1 *We have*

$$\begin{aligned} R_2(\nu,t,x) = -2b_0(x) &\int_0^t E_0\Big(\frac{1-|x|}{\sqrt{4\nu(t-s)}}\Big)\big(d\beta(s) + f^\nu(s)\, ds\big) \\ &+ \int_0^t \big[f^\nu(s) - f^0(s)\big]\, ds + O(\nu^{1/2}t^{1/2}), \end{aligned} \tag{4.15}$$

uniformly in $x \in \overline{D},\ t \in [0,T_0],\ \nu \in (0,1]$.

We turn to $R_1(\nu,t,x)$, given by (4.3). We assume $W_0 \in C^\infty(\overline{D})$. Results here are as precise as those for R_2, but somewhat more complicated. For the analysis, we use results from §§3.6–3.7 of [8]. The attack combines the use of layer potentials and semiclassical analysis. Before starting this attack, we first deal with the fact that the equation

$$\frac{\partial W_\nu}{\partial t} = \nu\Delta W_\nu - XW_\nu, \quad W_\nu\big|_{\mathbb{R}^+\times\partial D} = 0, \quad W_\nu(0,x) = W_0(x) \tag{4.16}$$

for

$$W_\nu(t) = e^{t(\nu\Delta - X)}W_0 \tag{4.17}$$

does not fit the pattern typically encountered in semiclassical analysis. One could regard (4.16) as semiclassical (with $\nu = \hbar^2$) if X were zero order (which it is not) or if the vector field X were accompanied by a factor $\hbar = \nu^{1/2}$. As it is, (4.16) is a more singular perturbation than that. The first step is to ameliorate this by considering

$$v_\nu(t,x) = e^{tX}e^{t(\nu\Delta - X)}W_0(x), \tag{4.18}$$

which solves

$$\frac{\partial v_\nu}{\partial t} = \nu L(t)v_\nu \ \text{ on } \ \mathbb{R}^+\times D, \quad v_\nu\big|_{\mathbb{R}^+\times\partial D} = 0, \quad v_\nu(0) = W_0, \tag{4.19}$$

with

$$L(t) = e^{tX}\Delta e^{-tX}, \tag{4.20}$$

a t-dependent family of second order, strongly elliptic operators with smooth coefficients on $\overline{D}$. (The functions v_ν and u_ν, used below, are not to be confused with v^ν and u^ν from previous sections. Nor should W_ν be confused with W^ν.) The solutions to (4.16) and (4.20) are related by the simple transformation

$$W_\nu(t) = e^{-tX}v_\nu(t). \tag{4.21}$$

We aim to express the solution to (4.19) as a sum of a "free space" solution and a layer-potential correction. To construct the free space solution, we put $\overline{D}$ in a box in $\mathbb{R}^2$ and identify opposite edges, so $\overline{D}$ is a domain with boundary in a compact manifold without boundary, say $M = \mathbb{T}^2 = \mathbb{R}^2/(4\mathbb{Z}^2)$. We extend X from $\overline{D}$ to a smooth vector field on M, and of course Δ extends to the Laplace operator on M, with its standard flat metric tensor. Then $L(t)$ in (4.20) is a well defined t-dependent family of second order strongly elliptic operators on M. Also extend $W_0 \in C^\infty(\overline{D})$ to $\widetilde{W}_0 \in C^\infty(M)$. The free space solution is then $V_\nu(t)$, given by

$$\frac{\partial V_\nu}{\partial t} = \nu L(t) V_\nu \quad \text{on} \quad \mathbb{R}^+ \times M, \quad V_\nu(0,x) = \widetilde{W}_0(x). \tag{4.22}$$

The solution to (4.19) then has the form

$$v_\nu(t,x) = V_\nu(t,x) - u_\nu(t,x), \quad t \ge 0, \ x \in D, \tag{4.23}$$

where $u_\nu(t,x)$ satisfies

$$\begin{aligned} \frac{\partial u_\nu}{\partial t} &= \nu L(t) u_\nu \quad \text{on} \quad \mathbb{R} \times D, \\ u_\nu &= g_\nu = \chi_{\mathbb{R}^+}(t) V_\nu(t,x), \quad x \in \partial D, \\ u_\nu &= 0 \quad \text{on} \quad (-\infty, 0) \times D. \end{aligned} \tag{4.24}$$

The method of layer potentials is brought to bear to solve (4.24). This method involves the use of functions $H(\nu,s,t,x,y)$, defined as follows. First, the solution to (4.22) is given by

$$V_\nu(t,x) = \int_M \widetilde{W}_0(y) H(\nu,0,t,x,y)\, dV(y), \tag{4.25}$$

where $dV(y) = dy$ is the standard area element on $M = \mathbb{T}^2$. More generally, for $0 \le s \le t$,

$$V_\nu(t,x) = \int_M V_\nu(s,y) H(\nu,s,t,x,y)\, dV_s(y), \tag{4.26}$$

where $dV_s(y) = \sqrt{g(s,y)}\, dy$ is the pull-back of dV via the flow generated by X, i.e., the Riemannian area element for g_s, the pull-back via this flow of the standard flat metric tensor on M.

It is not hard to analyze V_ν as a smooth function on $[0,\infty) \times M$, depending smoothly on $\nu \in [0,1]$, given $\widetilde{W}_0 \in C^\infty(M)$. Details can be found in §3.5 of [8]. Going from here to a layer potential analysis of u_ν in (4.24) requires an accurate approximation to the integral kernel $H(\nu,s,t,x,y)$. This was carried out in §3.6 of [8], in the more general context where D is a smoothly bounded domain in a compact n-dimensional Riemannian manifold M, via techniques of semiclassical analysis. We summarize the results. We have

$$H(\nu,s,t,x,y) = g(s,y)^{-1/2} K(\nu,s,t,x,x-y), \tag{4.27}$$

where $K(\nu,s,t,x,x-y)$ has the form

$$K(\nu,s,t,x,z) = \sum_{j=0}^{N} K_j(\nu,s,t,x,z) + R_N(\nu,s,t,x,z), \tag{4.28}$$

where R_N is increasingly negligible for large N (cf. [8], Proposition 3.6.6), and the principal term $K_0(\nu,s,t,x,z)$ is given (with $n = 2$) by

$$\begin{aligned} &K_0(\nu,s,t,x,z) \\ &= \big(4\pi\nu(t-s)\big)^{-n/2} \det \mathcal{G}(s,t,x)^{1/2} e^{-\mathcal{G}(s,t,x)z\cdot z/4\nu(t-s)}. \end{aligned} \tag{4.29}$$

Here $\mathcal{G}(s,t,x)$ is a smooth positive-definite $n \times n$ (i.e., 2×2) matrix valued function of $(s,t,x) \in [0,\infty) \times [0,\infty) \times M$, whose construction involves a transport equation; cf. (3.6.79) of [8]. For $j \geq 1$, formulas for $K_j(\nu,s,t,x,z)$ are somewhat more elaborate variants of (4.29). They are given in (3.6.90) and (3.6.93) of [8]. The main point to take from these formulas is that, for $j \geq 1$, $K_j(\nu,s,t,x,z)$ is smaller and smoother than $K_0(\nu,s,t,x,z)$, which has a δ-function type singularity in the limit $\nu \searrow 0$. In addition, these terms get progressively smaller and smoother as j increases.

With these results in hand, we bring in the method of layer potentials to treat (4.24), following §3.7 of [8]. The double layer potential is given by

$$\mathcal{D}_\nu h(t,x) = \nu \int_0^t \int_{\partial D} h(s,y) \frac{\partial H}{\partial n_{s,y}}(\nu,s,t,x,y)\, dS_s(y)\, ds. \tag{4.30}$$

Here dS_s is the arc length on ∂D induced by the metric tensor g_s, and $\partial/\partial n_{s,y}$ is the outward unit normal to ∂D at $y \in \partial D$, determined by this metric tensor. The boundary trace relation for $\mathcal{D}_\nu$ is

$$\mathcal{D}_\nu h\big|_{\mathbb{R}\times\partial D} = \Big(\frac{1}{2}I + \nu N_\nu\Big)h, \tag{4.31}$$

for supp $h \subset \mathbb{R}^+ \times \partial D$, where

$$N_\nu h(t,x) = \int_0^t \int_{\partial D} h(s,y) \frac{\partial H}{\partial n_{s,y}}(\nu,s,t,x,y)\, dS_s(y)\, ds. \tag{4.32}$$

Thus the solution to (4.24) has the form

$$u_\nu(t,x) = \mathcal{D}_\nu h_\nu(t,x), \tag{4.33}$$

provided h_ν solves

$$\Big(\frac{1}{2}I + \nu N_\nu\Big)h_\nu = g_\nu, \tag{4.34}$$

with g_ν given in (4.24).

Solvability of (4.34), on any given $I = [0,T_0]$, for $\nu > 0$ small enough, is achieved as follows. From (4.27)–(4.29) and related results on K_j, one has

$$\|\nu N_\nu h\|_{L^\infty(I\times\partial D)} \leq C(I)\nu^{1/2}\|h\|_{L^\infty(I\times\partial D)}. \tag{4.35}$$

Cf. [8], (3.7.27). Hence, as long as $\nu^{1/2} \leq 1/2C(I)$, if $g_\nu \in L^\infty(I \times \partial D)$, the equation (4.34) is solved by

$$\begin{aligned} h_\nu &= 2(I + 2\nu N_\nu)^{-1} g_\nu \\ &= 2(I - 2\nu N_\nu + 4\nu^2 N_\nu^2 - \cdots) g_\nu. \end{aligned} \tag{4.36}$$

We can take some finite sum of the series in (4.36) and have a rather small remainder. In particular,

$$\|h_\nu - 2g_\nu\|_{L^\infty(I\times\partial D)} \leq C(I)\nu^{1/2}\|g_\nu\|_{L^\infty(I\times\partial D)}. \tag{4.37}$$

Since, by (4.33),

$$u_\nu = 2\mathcal{D}_\nu g_\nu + \mathcal{D}_\nu(h_\nu - 2g_\nu), \tag{4.38}$$

it is useful to know that

$$\|\mathcal{D}_\nu h\|_{L^\infty(I\times D)} \leq C\|h\|_{L^\infty(I\times\partial D)}, \tag{4.39}$$

with C independent of $\nu \in (0,1]$; cf. (3.7.34) of [8]. Hence

$$\begin{aligned} \|u_\nu - 2\mathcal{D}_\nu g_\nu\|_{L^\infty(I\times D)} &\le C(I)\nu^{1/2}\|g_\nu\|_{L^\infty(I\times\partial D)} \\ &\le C'(I)\nu^{1/2}\|\widetilde{W}_0\|_{L^\infty(M)}, \end{aligned} \tag{4.40}$$

the latter inequality by (4.22)–(4.24) and the maximum principle.

The estimate (4.40) implies that $2\mathcal{D}_\nu g_\nu$ is a good enough approximation to u_ν to resolve the boundary layer behavior of u_ν, hence, via (4.23), that of v_ν, and hence, by (4.21), the boundary layer behavior of $W_\nu = e^{t(\nu\Delta - X)}W_0$, given by (4.16)–(4.17). Thus we resolve the boundary layer behavior of $R_1(\nu,t,x)$, at least to leading order. Taking more terms in the series in (4.36) leads to higher order approximation.

As for approximating u_ν within $O(\nu^{1/2})$ in sup norm, one can do this with a simplification of $2\mathcal{D}_\nu g_\nu$, namely $2\mathcal{D}_\nu^0 g_\nu$, where

$$\mathcal{D}_\nu^0 h(t,x) = \nu \int_0^t \int_{\partial D} h(s,y)\frac{\partial H_0}{\partial n_{s,y}}(\nu,s,t,x,y)\,dS_s(y)\,ds, \tag{4.41}$$

where, in place of (4.27)–(4.28), we take

$$H_0(\nu,s,t,x,y) = g(s,y)^{-1}K_0(\nu,s,t,x,x-y), \tag{4.42}$$

again with K_0 as in (4.29). We have, via estimates on K_j for $j \ge 1$,

$$\|\mathcal{D}_\nu h - \mathcal{D}_\nu^0 h\|_{L^\infty(I\times D)} \le C(I)\nu^{1/2}\|h\|_{L^\infty(I\times\partial D)}. \tag{4.43}$$

Further estimates on V_ν in (4.22)–(4.24) yield, for $\delta > 0$,

$$\|v_\nu - (W_0 - 2\mathcal{D}_\nu^0 W_0^b)\|_{L^\infty(I\times D)} \le C(I)\nu^{1/2}\|W_0\|_{C^{1+\delta}(\overline{D})}, \tag{4.44}$$

where

$$W_0^b = \chi_{\mathbb{R}^+}(t)W_0\big|_{\partial D}. \tag{4.45}$$

Cf. [8], Proposition 3.7.4. Recalling (4.16)–(4.18), we reach the following conclusion.

Proposition 4.2 *Assuming $v_0,\ w_0 \in C^\infty(\overline{D})$,*

$$\|R_1(\nu,\cdot,\cdot) + 2e^{-tX}\mathcal{D}_\nu^0 W_0^b\|_{L^\infty(I\times D)} \le C(I)\nu^{1/2}\|W_0\|_{C^{1+\delta}(\overline{D})}. \tag{4.46}$$

It remains to analyze $R_3(\nu,t,x)$. Since W^ν occurs on the right side of (4.7), we do not have as precise an analysis of R_3 as we got for R_1 and R_2, but we are able to show the following.

Proposition 4.3 *In the setting of Proposition 4.2,*

$$R_3(\nu,t,x) \longrightarrow 0, \tag{4.47}$$

as long as

$$\frac{1-|x|}{\sqrt{\nu t}} \longrightarrow \infty. \tag{4.48}$$

We get this from the estimate (3.27), i.e.,

$$|R_3(\nu,t,x)| \le \|\partial_\theta w_0\|_{L^\infty} \int_0^t e^{\nu(t-s)\Delta} |\tilde{s}_0 - \tilde{s}^\nu(s)|\, ds, \tag{4.49}$$

together with the fact that, uniformly on $t \in (0, T_0]$, $\nu \in (0, 1]$,

$$|\tilde{s}_0(x) - \tilde{s}^\nu(t,x)| \le \psi\Big(\frac{1-|x|}{\sqrt{\nu t}}\Big) + C\nu, \tag{4.50}$$

where $\psi(\lambda) \to 0$ as $\lambda \to \infty$. Such an estimate on

$$|v_0(x) - v^\nu(t,x)| \tag{4.51}$$

follows from (2.42)–(2.49), and then the estimate (4.50) follows by the arguments involving (2.60)–(2.71).

5 Completely circularly symmetric pipe flows

If we impose not only the circular symmetry hypothesis (1.26) on $v_0(x)$ but also the following circular symmetry hypothesis on $w_0(x)$,

$$w_0(R_\theta x) = w_0(x), \quad \forall\, x \in D,\ \theta \in [0, 2\pi], \tag{5.1}$$

which implies $w_0(x)$ is a function of $|x|$, it follows that $w^\nu(t,x)$ satisfies such circular symmetry for all $t > 0$, and the system (1.13)–(1.14) simplifies to

$$\frac{\partial v^\nu}{\partial t} = \nu \Delta v^\nu, \quad \operatorname{div} v^\nu = 0, \tag{5.2}$$

$$\frac{\partial w^\nu}{\partial t} = \nu \Delta w^\nu + f^\nu(t). \tag{5.3}$$

We continue to have the boundary conditions (1.15) and (1.17), i.e.,

$$v^\nu(t,x) = \frac{\alpha(t)}{2\pi} x^\perp, \quad w^\nu(t,x) = \beta(t), \quad |x| = 1,\ t > 0. \tag{5.4}$$

Of course, (5.2) is identical to (1.36), and we have nothing further beyond the material of §2 to say about that. The simplification occurs in (5.3).

Let us suppose to start that

$$f^\nu,\ \beta \in C_b^\infty(\mathbb{R}), \tag{5.5}$$

i.e., f^ν and β are C^∞ on $\mathbb{R}$ and vanish on $(-\infty, 0]$. Setting

$$W^\nu(t,x) = w^\nu(t,x) - \beta(t) \tag{5.6}$$

yields

$$\begin{aligned} \frac{\partial W^\nu}{\partial t} &= \nu \Delta W^\nu + f^\nu(t) - \beta'(t), \\ W^\nu(0,x) = w_0(x), &\quad W^\nu\big|_{\mathbb{R}^+ \times \partial D} = 0, \end{aligned} \tag{5.7}$$

and an application of Duhamel's formula gives

$$W^\nu(t,x) = e^{\nu t \Delta} w_0(x) + \int_0^t [f^\nu(s) - \beta'(s)] e^{\nu(t-s)\Delta} 1(x)\, ds, \tag{5.8}$$

hence

$$w^\nu(t,x)=e^{\nu t\Delta}w_0(x)+\beta(t)+\int_0^t [f^\nu(s)-\beta'(s)]e^{\nu(t-s)\Delta}1(x)\,ds. \tag{5.9}$$

Alternatively,

$$w^\nu(t,x)=e^{\nu t\Delta}w_0(x)+\int_0^t f^\nu(s)e^{\nu(t-s)\Delta}1(x)\,ds+\int\limits_{I(t)}\Bigl(I-e^{\nu(t-s)\Delta}\Bigr)1(x)\,d\beta(s), \tag{5.10}$$

where $I(t)=[0,t]$.

Recall we are interested in the nature of the convergence of $w^\nu(t,x)$ to $w^0(t,x)$, given in this case by

$$\frac{\partial w^0}{\partial t}=f^0(t),\quad w^0(0,x)=w_0(x), \tag{5.11}$$

i.e.,

$$w^0(t,x)=w_0(x)+\int_0^t f^0(s)\,ds. \tag{5.12}$$

Reasoning from [6], recalled in §2 of this paper, gives the following.

Proposition 5.1 *Let X be a Banach space of (real valued) functions on D with the properties that $1,w_0\in X$ and that $\{e^{t\Delta}:t\ge 0\}$ is a strongly continuous semigroup on X. Take $T\in(0,\infty)$ and assume*

$$\beta\in BV([0,T]), \tag{5.13}$$

$$f^\nu\in L^1([0,T]),\quad f^\nu\to f^0 \ \text{ in } \ L^1([0,T]). \tag{5.14}$$

Then $w^\nu(t,x)$, given by (5.10), satisfies

$$w^\nu(t,\cdot)\longrightarrow w^0 \ \text{ in norm, in } \ X,\quad \text{as } \ \nu\searrow 0, \tag{5.15}$$

uniformly in $t\in[0,T]$.

In case $w_0\in C^\infty(\overline{D})$, results of Appendix B yield precise asymptotic expansions for $w^\nu(t,x)$ in (5.10), analogous to the expansion for $v^\nu(t,x)$ described by (2.42)–(2.50). There is no need to repeat such formulas, but we do point out the following consequence of Corollary B.2, which is relevant for vorticity concentration.

Proposition 5.2 *In the setting of Proposition 5.1, if (5.13)–(5.14) hold, and if $w_0\in C^\infty(\overline{D})$, then*

$$\|\nabla w^\nu(t,\cdot)\|_{L^1(D)}\le C_{w_0}+C\|f^\nu\|_{L^1([0,T])}+C\|\beta\|_{BV([0,T])}, \tag{5.16}$$

with constants independent of $\nu\in(0,1],\ t\in[0,T]$.

A Variant of J. Kelliher's concentration calculation

In [6] it was shown that for a class of circularly symmetric planar vector fields u^ν on $D=\{x\in\mathbb{R}^2: |x|<1\}$, solving the Navier-Stokes equations, converging to u, solving the Euler equations, one has

$$\operatorname{rot} u^\nu\longrightarrow \operatorname{rot} u-(u\cdot\tau)\sigma, \tag{A.1}$$

weak* in the space $\mathcal{M}(\overline{D})$ of finite Borel measures on $\overline{D}$, where τ is the unit tangent to ∂D and σ is arclength measure on ∂D. In [4] such a limit was shown to hold in a much more general context (including more general domains Ω), provided one weakens the convergence to convergence in $H^1(\Omega)'$. That result included concentration of vorticity on $\partial\Omega$ for domains $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary. In that treatment, one sees that the concentration of vorticity phenomenon is actually a consequence of a general concentration of gradient phenomenon, for scalar fields. Here, we present another perspective on this calculation, with the goal of allowing $\partial\Omega$ to be quite rough.

Let $\Omega \subset \mathbb{R}^n$ be an open set. We assume u^ν and u are real valued functions on Ω satisfying

$$u^\nu \in H_0^1(\Omega), \quad u = \tilde{u}\big|_\Omega, \quad \tilde{u} \in H^1(\mathbb{R}^n), \tag{A.2}$$

and that

$$u^\nu \longrightarrow u \;\text{ in }\; L^2(\Omega), \;\text{ weakly.} \tag{A.3}$$

We can regard u^ν as an element of $H_0^1(\mathbb{R}^n)$, equal to 0 on $\mathbb{R}^n \setminus \Omega$. Then (A.3) implies

$$u^\nu \longrightarrow \chi_\Omega \tilde{u} \;\text{ in }\; L^2(\mathbb{R}^n), \;\text{ weakly.} \tag{A.4}$$

Standard distribution theory implies

$$\nabla u^\nu \longrightarrow \nabla(\chi_\Omega \tilde{u}) \;\text{ in }\; H^{-1}(\mathbb{R}^n), \;\text{ weakly.} \tag{A.5}$$

Now we claim that

$$\nabla(\chi_\Omega \tilde{u}) = \chi_\Omega \nabla \tilde{u} + \tilde{u} \nabla \chi_\Omega. \tag{A.6}$$

To see this, let $v_\varepsilon = \psi_\varepsilon * \tilde{u} \to \tilde{u}$ be a family of mollifications of $\tilde{u}$, $v_\varepsilon \in C^\infty(\mathbb{R}^n)$. It is elementary that

$$\nabla(\chi_\Omega v_\varepsilon) = \chi_\Omega \nabla v_\varepsilon + v_\varepsilon \nabla \chi_\Omega. \tag{A.7}$$

In the limit as $\varepsilon \to 0$, $\chi_\Omega v_\varepsilon \to \chi_\Omega \tilde{u}$ in $L^2(\mathbb{R}^n)$, so $\nabla(\chi_\Omega v_\varepsilon) \to \nabla(\chi_\Omega \tilde{u})$ in $H^{-1}(\mathbb{R}^n)$. Meanwhile $\nabla v_\varepsilon \to \nabla \tilde{u}$ in $L^2(\mathbb{R}^n)$, so $\chi_\Omega \nabla v_\varepsilon \to \chi_\Omega \nabla \tilde{u}$ in $L^2(\mathbb{R}^n)$. Furthermore, $v_\varepsilon \to \tilde{u}$ in $H^1(\mathbb{R}^n)$, while $\nabla \chi_\Omega \in H^{-1}_{loc}(\mathbb{R}^n)$, so $v_\varepsilon \nabla \chi_\Omega \to \tilde{u} \nabla \chi_\Omega$ in $\mathcal{D}'(\mathbb{R}^n)$. Thus (A.6) follows in the limit from (A.7). Note that the last part of this demonstration just has $\tilde{u} \nabla \chi_\Omega \in \mathcal{D}'(\mathbb{R}^n)$, but now that we have (A.6) we also have

$$\tilde{u} \nabla \chi_\Omega \in H^{-1}(\mathbb{R}^n). \tag{A.8}$$

From (A.5) and (A.6) we have, weakly in $H^{-1}(\mathbb{R}^n)$,

$$\begin{aligned} \nabla u^\nu &\longrightarrow \chi_\Omega \nabla \tilde{u} + \tilde{u} \nabla \chi_\Omega \\ &= \nabla u + \tilde{u} \nabla \chi_\Omega. \end{aligned} \tag{A.9}$$

In case Ω has a mildly regular boundary

$$\nabla \chi_\Omega = -n\,\sigma, \tag{A.10}$$

where n is the unit outward pointing normal and σ is surface area on $\partial\Omega$.

We expand on this last point. In geometric measure theory, one says Ω is a domain of locally finite perimeter provided $\nabla \chi_\Omega$ is a locally finite $\mathbb{R}^n$-valued measure. It is a result of E. DeGiorgi that in such a case, (A.10) holds, where n is the "measure-theoretic unit normal," and

$$\sigma = \mathcal{H}^{n-1} \lfloor \partial_* \Omega, \tag{A.11}$$

where $\mathcal{H}^{n-1}$ is $(n-1)$-dimensional Hausdorff measure and $\partial_*\Omega \subset \partial\Omega$ is the measure-theoretic boundary of Ω. When $\partial\Omega$ is smooth, or even Lipschitz, $\partial_*\Omega = \partial\Omega$. When $\partial\Omega$ is locally the graph of a continuous function with gradient in L^1, it is known that $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$. In rougher cases, $\partial\Omega \setminus \partial_*\Omega$ might be quite large.

Putting these results together, we have the following extension of (A.1).

Proposition A.1 *Let $\Omega \subset \mathbb{R}^n$ be an open set with locally finite perimeter. Under the hypotheses (A.2)–(A.3), we have*

$$\nabla u^\nu \longrightarrow \nabla u - (\tilde{u}n)\sigma, \tag{A.12}$$

weakly in $H^{-1}(\mathbb{R}^n)$. Here, $\nabla u \in L^2(\Omega)$ is extended by 0 on $\mathbb{R}^n \setminus \Omega$.

Vector analogues can be deduced from Proposition A.1 via linear algebra, or established directly. For example, suppose $n = 3$ and u, u^ν are vector fields on $\Omega \subset \mathbb{R}^3$, satisfying (A.2)–(A.3). We have analogues of (A.4)–(A.5) for $\operatorname{curl} u^\nu$, with (A.6) replaced by

$$\operatorname{curl}(\chi_\Omega \tilde{u}) = \chi_\Omega \operatorname{curl} \tilde{u} + \nabla \chi_\Omega \times \tilde{u}. \tag{A.13}$$

Hence in place of (A.12), we have

$$\operatorname{curl} u^\nu \longrightarrow \operatorname{curl} u - (n \times \tilde{u})\sigma, \tag{A.14}$$

weakly in $H^{-1}(\mathbb{R}^3)$.

We mention that the first part of (A.2) applies directly to fluids with zero velocity on $\partial\Omega$. In case there is a nonzero velocity specified on $\partial\Omega$, the appropriate replacement for the first part of (A.2) would be

$$u^\nu - \varphi \in H_0^1(\Omega), \quad \varphi = \tilde{\varphi}\big|_\Omega, \quad \tilde{\varphi} \in H^1(\mathbb{R}^n). \tag{A.15}$$

We keep the rest of (A.2) and (A.3). Then the previous argument applies directly to $u^\nu - \varphi$, to give, in the setting of Proposition A.1,

$$\nabla u^\nu \longrightarrow \nabla u - (\tilde{u} - \tilde{\varphi})n\sigma, \tag{A.16}$$

and similarly, for vector fields on $\Omega \subset \mathbb{R}^3$, one replaces (A.14) by

$$\operatorname{curl} u^\nu \longrightarrow \operatorname{curl} u - n \times (\tilde{u} - \tilde{\varphi})\sigma. \tag{A.17}$$

We make some further comments on the vorticity of a t-dependent family of vector fields on Ω of the form (1.11), i.e.,

$$u^\nu(t, x, z) = (v^\nu(t, x), w^\nu(t, x)), \tag{A.18}$$

where $x \in D$, $v = v_1 i + v_2 j$ is a planar vector field, and w is the z-component of u. Then

$$\begin{aligned} \operatorname{curl} u^\nu &= \det \begin{pmatrix} i & j & k \\ \partial_{x_1} & \partial_{x_2} & \partial_z \\ v_1^\nu & v_2^\nu & w^\nu \end{pmatrix} \\ &= (\partial_{x_2} w^\nu) i - (\partial_{x_1} w^\nu) j + (\partial_{x_1} v_2^\nu - \partial_{x_2} v_1^\nu) k, \end{aligned} \tag{A.19}$$

hence

$$\operatorname{curl} u^\nu = (\nabla^\perp w^\nu, \operatorname{rot} v^\nu). \tag{A.20}$$

Now [6] obtained uniform L^1 bounds on $\operatorname{rot} v^\nu$, given such a bound on the initial data $v_0(x)$, which led to the weak* convergence in the space $\mathcal{M}(\overline{D})$ in (A.1) (with a slight change of notation, namely v^ν and v in place of u^ν and u). It would be interesting to know whether we also have L^1-gradient bounds on w^ν in (A.20), given such bounds on $w_0(x)$, as well as appropriate bounds on $v_0(x)$.

In this regard, the following comments are in order. In the setting of (4.2) we have

$$w^\nu(t, x) - w^0(t, x) = R_1(\nu, t, x) + R_2(\nu, t, x) + R_3(\nu, t, x) \tag{A.21}$$

(denoting the limit here by w^0 rather than w). An L^1-gradient bound on R_2 follows from the representation (4.4)–(4.8) and results of Appendix B, via Corollary B.2. An L^1-gradient bound on R_1 follows from the representation (4.3) and the layer potential attack described in §4, together with layer potential estimates of the sort discussed in Appendix D. This leaves R_3 to analyze. The authors hope to return to this point in future work.

B Purely diffusive boundary layers

Here we give some rather explicit formulas for the asymptotic behavior as $\nu \searrow 0$ of solutions to

$$\frac{\partial u^\nu}{\partial t} = \nu \Delta u^\nu \quad \text{on} \quad \mathbb{R}^+ \times \Omega, \tag{B.1}$$

satisfying

$$u^\nu\big|_{\mathbb{R}^+ \times \partial\Omega} = 0, \quad u^\nu(0,x) = f(x), \tag{B.2}$$

where $\overline{\Omega}$ is a compact Riemannian manifold with smooth boundary and Laplace-Beltrami operator Δ, and

$$f \in C^\infty(\overline{\Omega}). \tag{B.3}$$

Note that the solution to (B.1)–(B.2) is

$$u^\nu(t,x) = e^{\nu t \Delta} f(x) = u(\nu t, x), \tag{B.4}$$

where $u = u^\nu$ with $\nu = 1$. Thus the small ν analysis of (B.1)–(B.3) is just the small t analysis of $e^{t\Delta} f$, for $f \in C^\infty(\overline{\Omega})$. Here Δ is the self-adjoint extension of the Laplace-Beltrami operator with domain $\mathcal{D}(\Delta) = H^2(\Omega) \cap H^1_0(\Omega)$.

We assume (without loss of generality) that Ω is a smoothly bounded open subset of M, a compact Riemannian manifold without boundary. Let L denote the Laplace-Beltrami operator on M, and assume

$$f = \tilde{f}\big|_\Omega, \quad \tilde{f} \in C^\infty(M). \tag{B.5}$$

Note that, for $x \in \Omega,\ t > 0$,

$$e^{t\Delta} f(x) = e^{tL} \tilde{f}(x) - U(t,x), \tag{B.6}$$

where $U(t,x)$ satisfies

$$\begin{aligned} (\partial_t - \Delta) U = 0 \quad &\text{on} \quad \mathbb{R} \times \Omega, \\ U(t,x) = 0 \quad \text{for} \quad t < 0, \quad &U(t,\cdot)\big|_{\partial\Omega} = \chi_{\mathbb{R}^+}(t) e^{tL} \tilde{f}\big|_{\partial\Omega}. \end{aligned} \tag{B.7}$$

Standard hypoellipticity results give $U \in C^\infty(\mathbb{R} \times \Omega)$, hence, for $t \in (0,1],\ \overline{\mathcal{O}} \subset\subset \Omega,\ k, N \in \mathbb{N}$,

$$\|U(t,\cdot)\|_{C^k(\overline{\mathcal{O}})} \le C_{N,k} t^N. \tag{B.8}$$

On the other hand, given $\tilde{f} \in C^\infty(M)$, the nature of the convergence of $e^{tL}\tilde{f}$ to $\tilde{f}$ is elementary and well known. From the fact that $e^{tL}\tilde{f} \in C^\infty([0,\infty) \times M)$ it follows that, for each $k, N \in \mathbb{N}$,

$$e^{tL}\tilde{f}(x) = \tilde{f}(x) + tL\tilde{f}(x) + \cdots + \frac{t^N}{N!} L^N \tilde{f}(x) + R_N(x), \tag{B.9}$$

with

$$\|R_N\|_{C^k(M)} \le C_{k,N} t^N, \quad 0 < t \le 1. \tag{B.10}$$

What remains is to analyze the precise nature of the boundary layer that forms for $U(t,x)$ as $t \searrow 0$, preventing the uniform convergence to 0 on $\overline{\Omega}$. One way to attack this problem is via the method of layer potentials. This was used in [6], and extended in [8] to study the more complicated problem in which (B.1) is replaced by $\partial u^\nu / \partial t = \nu \Delta u^\nu + X u^\nu$, where X is a smooth vector field on $\overline{\Omega}$, tangent to $\partial\Omega$. (These results are recalled in §4.) Here we bring in another method, based on a wave equation approach.

This approach starts with the identity

$$e^{t\Delta}f(x)=\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}e^{-s^2/4t}\cos s\sqrt{-\Delta}\,f(x)\,ds, \tag{B.11}$$

where $v(s,x)=\cos s\sqrt{-\Delta}\,f(x)$ solves the wave equation

$$\begin{aligned}(\partial_t^2-\Delta)v=0 \quad &\text{on } \ \mathbb{R}\times\Omega,\\ v\big|_{\mathbb{R}\times\partial\Omega}=0, \quad &v(0,x)=f(x),\ \ \partial_s v(0,x)=0.\end{aligned} \tag{B.12}$$

The identity (B.11) follows from the Fourier inversion formula and the spectral theorem (cf. [9], Chapter 8, and [2]). More generally than (B.11), one has

$$\varphi(\sqrt{-\Delta})f=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{\varphi}(s)\cos s\sqrt{-\Delta}\,f\,ds, \tag{B.13}$$

valid for even $\varphi\in\mathcal{S}(\mathbb{R})$, where $\hat{\varphi}(s)=(2\pi)^{-1/2}\int\varphi(\lambda)e^{-i\lambda s}\,d\lambda$. Taking $\varphi(\lambda)=e^{-t\lambda^2}$ yields (B.11).

Parallel to (B.6), we have, for $x\in\Omega,\ s\ge 0$,

$$\cos s\sqrt{-\Delta}\,f(x)=\cos s\sqrt{-L}\,\tilde{f}(x)-V(s,x), \tag{B.14}$$

where $V(s,x)$ solves

$$\begin{aligned}(\partial_s^2-\Delta)V=0 \quad \text{on } \ \mathbb{R}\times\Omega, \quad V(s,x)=0 \ \text{ for } \ s<0,\\ V(s,\cdot)\big|_{\partial\Omega}=g(s,\cdot)=\chi_{\mathbb{R}^+}(s)\cos s\sqrt{-L}\,\tilde{f}\big|_{\partial\Omega}.\end{aligned} \tag{B.15}$$

Also, parallel to (B.11),

$$e^{tL}\tilde{f}(x)=\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}e^{-s^2/4t}\cos s\sqrt{-L}\,\tilde{f}(x)\,ds. \tag{B.16}$$

Together (B.11), (B.14) and (B.16) and the evenness in s of $e^{-s^2/4t}$ yield, for $t>0,\ x\in\Omega$,

$$e^{t\Delta}f(x)=e^{tL}\tilde{f}(x)-\frac{2}{\sqrt{4\pi t}}\int_0^{\infty}e^{-s^2/4t}V(s,x)\,ds, \tag{B.17}$$

hence, by (B.6),

$$U(t,x)=\frac{1}{\sqrt{\pi t}}\int_0^{\infty}e^{-s^2/4t}V(s,x)\,ds. \tag{B.18}$$

We aim to analyze $V(s,x)$ and use this analysis in (B.18). The first step is to localize this analysis to small s. Given $a>0$, pick an even function $\psi_1\in C_0^\infty(\mathbb{R})$ such that $\psi_1(s)=1$ for $|s|\le a,\ 0$ for $|s|\ge 2a$, and set $\psi_2(s)=1-\psi_1(s)$. We have

$$\begin{aligned}U(t,x)&=U_1(t,x)+U_2(t,x),\\ U_j(t,x)&=\frac{1}{\sqrt{\pi t}}\int_0^{\infty}\psi_j(s)e^{-s^2/4t}V(s,x)\,ds.\end{aligned} \tag{B.19}$$

In turn

$$e^{t\Delta}=\Phi_1^t(\sqrt{-\Delta})+\Phi_2^t(\sqrt{-\Delta}), \tag{B.20}$$

where

$$\Phi_j^t(\sqrt{-\Delta})=\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}\psi_j(s)e^{-s^2/4t}\cos s\sqrt{-\Delta}\,ds. \tag{B.21}$$

We see that for $t \in (0,1]$, $k, N \in \mathbb{N}$,

$$\Phi_2^t(\lambda) \le C_{k,N} t^N (1+|\lambda|)^{-k}, \tag{B.22}$$

and hence

$$\|\Phi_2^t(\sqrt{-\Delta})f\|_{H^k(\Omega)} \le C_{k,N} t^N \|f\|_{L^2(\Omega)}. \tag{B.23}$$

Similarly

$$e^{tL} = \Phi_1^t(\sqrt{-L}) + \Phi_2^t(\sqrt{-L}), \tag{B.24}$$

with similar estimates, including

$$\|\Phi_2^t(\sqrt{-L})\tilde{f}\|_{H^k(M)} \le C_{k,N} t^N \|\tilde{f}\|_{L^2(M)}. \tag{B.25}$$

Consequently we have

$$\begin{gathered} U_2(t,x) = \Phi_2^t(\sqrt{-L})\tilde{f}(x) - \Phi_2^t(\sqrt{-\Delta})f(x), \\ \|U_2(t,\cdot)\|_{H^k(\Omega)} \le C_{k,N} t^N \Big(\|f\|_{L^2(\Omega)} + \|\tilde{f}\|_{L^2(M)} \Big). \end{gathered} \tag{B.26}$$

Thus the boundary layer behavior of $U(t,x)$ is completely captured by $U_1(t,x)$. Hence we need a further analysis of $V(s,x)$ only for $s \in [0,2a]$, where $a > 0$ can be taken as small as desired.

Note that in (B.15), g, defined on $\mathbb{R} \times \partial\Omega$, is supported in $\{s \ge 0\}$ and piecewise smooth, with a simple jump across $\{s = 0\}$. Finite propagation speed assures that for $s \ge 0$, $x \in \Omega$,

$$V(s,x) = 0 \quad \text{for} \ \ \varphi(x) > s, \tag{B.27}$$

where

$$\varphi(x) = \text{dist}\,(x, \partial\Omega). \tag{B.28}$$

Let us pick $a > 0$ so small that

$$\overline{\mathcal{C}} = \{x \in \overline{\Omega} : \varphi(x) \le 2a\} \Longrightarrow \varphi \in C^\infty(\overline{\mathcal{C}}), \tag{B.29}$$

and use this value of a to pick ψ_1 and ψ_2 in (B.19). Then, for $s \in [0,2a]$, $V(s,x)$ is given by a progressing wave expansion of the form

$$V(s,x) \sim \sum_{j \ge 0} a_j(s,x)(s - \varphi(x))_+^j, \tag{B.30}$$

with coefficients $a_j \in C^\infty([0,2a] \times \overline{\Omega})$, determined by certain transport equations. See [9], Chapter 6, §6. The meaning of (B.30) is that for each $N \in \mathbb{N}$,

$$V(s,x) = \sum_{j=0}^{N} a_j(s,x)(s - \varphi(x))_+^j + R_N(s,x), \tag{B.31}$$

where

$$R_N(s,x) = 0 \ \ \text{for} \ \ \varphi(x) > s, \quad R_N \in C^N([0,2a] \times \overline{\Omega}). \tag{B.32}$$

Writing

$$a_0(s,x) = a_0(\varphi(x),x) + \tilde{a}_1(s,x)(s - \varphi(x)), \tag{B.33}$$

we can shift the latter term onto the $j = 1$ term in (B.31). Continuing this process, we have

$$V(s,x) = \sum_{j=0}^{N} b_j(x)(s-\varphi(x))_+^j + R_N(s,x), \tag{B.34}$$

(with slightly altered R_N, still satisfying (B.32)), valid on $[0,2a] \times \overline{\Omega}$, with $b_j \in C^\infty(\overline{\Omega})$. Inserting this into the formula for $U_1(t,x)$ given by (B.19), we have

$$\begin{aligned} U_1(t,x) = &\sum_{j=0}^{N} \frac{b_j(x)}{\sqrt{\pi t}} \int_0^\infty e^{-s^2/4t}(s-\varphi(x))_+^j \psi_1(s)\,ds \\ &+ \int_0^\infty e^{-s^2/4t} R_N(s,x)\psi_1(s)\,ds. \end{aligned} \tag{B.35}$$

Recall the partition of unity $1 = \psi_1(s) + \psi_2(s)$, specified below (B.18). Elementary estimates show that

$$\int_0^\infty e^{-s^2/4t}(s-\varphi(x))_+^j \psi_2(s)\,ds \tag{B.36}$$

is rapidly decreasing as $t \searrow 0$, together with all x-derivatives, so the sum over $0 \le j \le N$ in (B.35) has the identical asymptotic behavior as $t \searrow 0$ as does

$$\begin{aligned} &\sum_{j=0}^{N} b_j(x) W_j(t,x), \\ &W_j(t,x) = \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-s^2/4t}(s-\varphi(x))_+^j\,ds. \end{aligned} \tag{B.37}$$

A change of variable gives

$$W_j(t,x) = 2(4t)^{j/2} E_j\Big(\frac{\varphi(x)}{\sqrt{4t}}\Big), \tag{B.38}$$

where

$$\begin{aligned} E_j(y) &= \frac{1}{\sqrt{\pi}} \int_y^\infty e^{-s^2}(s-y)^j\,ds \\ &= \frac{e^{-y^2}}{\sqrt{\pi}} \int_0^\infty e^{-s^2-2sy} s^j\,ds. \end{aligned} \tag{B.39}$$

Using (B.32), one easily bounds the last integral in (B.35) by $CW_N(t,x)$. Consequently

$$U_1(t,x) = \sum_{j=0}^{N} 2b_j(x)(4t)^{j/2} E_j\Big(\frac{\varphi(x)}{\sqrt{4t}}\Big) + \widetilde{R}_N(t,x), \tag{B.40}$$

with

$$\|\widetilde{R}_N(t,\cdot)\|_{C^0(\overline{\Omega})} \le Ct^{N/2}. \tag{B.41}$$

Similar arguments give estimates $\|\widetilde{R}_N(t,\cdot)\|_{C^k(\overline{\Omega})} \le Ct^{M/2}$, for each $k, M \in \mathbb{N}$, if N is large enough.

Putting together (B.6), (B.9), (B.18), (B.19), (B.26), and (B.40), we obtain our main result:

Proposition B.1 *Given $f \in C^\infty(\overline{\Omega})$,*

$$
\begin{aligned}
e^{t\Delta} f(x) = f(x) &+ \sum_{k=1}^{N} \frac{t^k}{k!} \Delta^k f(x) \\
&- \sum_{j=0}^{2N} 2b_j(x)(4t)^{j/2} E_j\Big(\frac{\varphi(x)}{\sqrt{4t}}\Big) + \widehat{R}_N(t,x),
\end{aligned}
\tag{B.42}
$$

where $b_j \in C^\infty(\overline{\Omega})$ are as in (B.34), and, for each $M, k \in \mathbb{N}$, there exists N such that

$$
\|\widehat{R}_N(t,\cdot)\|_{C^k(\overline{\Omega})} \le C_{M,k} t^M, \quad t \in (0,1]. \tag{B.43}
$$

Remark. It follows readily from (B.15) and (B.34) that $b_j|_{\partial\Omega} = 0$ when j is odd. Also $b_0|_{\partial\Omega} = f|_{\partial\Omega}$, and $E_0(0) = 1/2$.

The following corollary, which follows by inspection of (B.42), is relevant for vorticity concentration.

Corollary B.2 *Given $f \in C^\infty(\overline{\Omega})$, we have*

$$
\|\nabla e^{t\Delta} f\|_{L^1(\Omega)} \le C_f, \quad \forall\, t \in (0,\infty). \tag{B.44}
$$

Remark. Such a uniform bound does not hold in any L^p-space with $p > 1$, unless $f|_{\partial\Omega} = 0$.

C Poiseuille flow in a circular pipe

Given $\alpha \in \mathbb{R} \setminus 0$, the velocity field

$$
u_0(x,z) = \alpha(0, 1 - |x|^2) \tag{C.1}
$$

is a well known example of a steady solution to the Navier-Stokes system

$$
\begin{aligned}
\frac{\partial u^\nu}{\partial t} + \nabla_{u^\nu} u^\nu + \nabla p^\nu = \nu\Delta u^\nu + F^\nu, &\quad \operatorname{div} u^\nu = 0, \\
u^\nu\big|_{\mathbb{R}^+ \times \partial D} = 0, &\quad u^\nu(0,x,z) = u_0(x,z),
\end{aligned}
\tag{C.2}
$$

on the infinite circular pipe

$$
\Omega = D \times \mathbb{R}, \quad D = \{x \in \mathbb{R}^2 : |x| < 1\}, \tag{C.3}
$$

an example of Poiseuille flow in a circular pipe (cf. [3], §3.1). In such a case, $\partial_t u^\nu = 0$ and $\nabla_{u^\nu} u^\nu = 0$. It is common to say that this flow is driven along the pipe by a uniform pressure gradient.

There are two ways to complete the description of how $u^\nu \equiv u_0$ solves (C.2). One is to set

$$
p^\nu(t,x,z) = -4\nu\alpha z, \quad F^\nu(t,x,z) = 0. \tag{C.4}
$$

The other is to set

$$
p^\nu(t,x,z) = 0, \quad F^\nu(t,x,z) = (0, 4\nu\alpha). \tag{C.5}
$$

Here we point out that this flow fits into the framework of our paper, in the setting of (C.5), but not in the setting of (C.4). Indeed, our analysis imposed the condition of periodicity in z on all quantities, and hence passed to the quotient $D \times (\mathbb{R}/L\mathbb{Z})$, consequently obtaining solutions independent of z. However, $p^\nu(t,x,z) = -4\nu\alpha z$ is not periodic in z, and (C.4) is not well defined on Ω_L, while (C.5) is well defined. Physically, it is appropriate to understand this flow, "driven by a uniform pressure gradient," as driven by an external force. This favors the use of (C.5) over (C.4).

In fact, if we set $F^\nu \equiv 0$ in (C.2) and solve this, with initial data given by (C.1), as per the set-up in §1, we get, not a steady solution, but a solution $u^\nu(t,x,z)$ that decays to 0 as $t \nearrow \infty$. This is physically reasonable, since the energy dissipation due to $\nu\Delta u^\nu$ is not offset by energy input from an external force. We record what solution does arise.

The unique solution to (C.1)–(C.2) on $\mathbb{R}^+ \times \Omega_L$, with $F^\nu \equiv 0$, has the form

$$u^\nu(t,x,z) = (0, w^\nu(t,x)), \tag{C.6}$$

where w^ν solves (1.14) with $v^\nu \equiv 0$ and $f^\nu \equiv 0$, i.e.,

$$\frac{\partial w^\nu}{\partial t} = \nu\Delta w^\nu, \quad \text{on } \mathbb{R}^+ \times D. \tag{C.7}$$

The initial and boundary conditions are

$$w^\nu(0,x) = w_0(x) = \alpha(1-|x|^2), \quad w^\nu\big|_{\mathbb{R}^+\times\partial D} = 0. \tag{C.8}$$

In other words,

$$w^\nu(t,x) = e^{\nu t\Delta} w_0(x). \tag{C.9}$$

D Analysis of a model layer potential

Recall the layer potential $\mathcal{D}^0_\nu$ introduced in (5.36), and tied to the analysis of R_1 in (4.46). To illustrate how applying $\mathcal{D}^0_\nu$ manifests a boundary layer, we do some explicit calculations with a toy model $\widetilde{\mathcal{D}}_\nu$, in which $g(s,y) \equiv 1$, $\mathcal{G}(s,t,x) \equiv I$, $dS_s(y) \equiv dy_1$, and the disk $\overline{D}$ is replaced by the upper half space $\overline{U} = \{(x_1,x_2) : x_2 \ge 0\}$, so ∂D is replaced by $\partial U = \mathbb{R}$. We have $\partial/\partial n_{s,y} = -\partial/\partial y_2$, and hence

$$\begin{aligned} &2\widetilde{\mathcal{D}}_\nu W_0(t,x_1,x_2) \\ &\quad = \frac{2\nu}{\pi}\int_0^t\int_{-\infty}^\infty W_0(y_1)\frac{2x_2}{(4\nu)^2(t-s)^2} e^{-[(x_1-y_1)^2+x_2^2]/4\nu(t-s)}\,dy_1\,ds. \end{aligned} \tag{D.1}$$

We can integrate out ds, using

$$\begin{aligned} \int_0^t \frac{1}{(t-s)^2} e^{-A/(t-s)}\,ds &= \int_0^t \frac{1}{s^2} e^{-A/s}\,ds \\ &= \int_{1/t}^\infty e^{-A\tau}\,d\tau \\ &= \frac{e^{-A/t}}{A}. \end{aligned} \tag{D.2}$$

We get

$$2\widetilde{\mathcal{D}}_\nu W_0(t,x_1,x_2) = \frac{1}{\pi}\int_{-\infty}^\infty W_0(y_1)\frac{x_2}{(x_1-y_1)^2+x_2^2} e^{-[(x_1-y_1)^2+x_2^2]/4\nu t}\,dy_1. \tag{D.3}$$

It is clear that, if for example $W_0 \in C_0^\infty(\mathbb{R})$,

$$2\widetilde{\mathcal{D}}_\nu W_0(t, x_1, x_2) \longrightarrow W(x_1), \quad \text{as} \;\; x_2 \searrow 0, \tag{D.4}$$

uniformly in x_1, for each t, $\nu > 0$, in fact uniformly for νt in a compact subset of $(0, \infty)$. On the other hand,

$$2\widetilde{\mathcal{D}}_\nu W_0(t, x_1, x_2) \longrightarrow 0, \quad \text{as} \;\; \nu \searrow 0, \tag{D.5}$$

uniformly in x_1, for each t, $x_2 > 0$, and more generally as long as

$$\frac{x_2^2}{\nu t} \longrightarrow \infty. \tag{D.6}$$

For further insight into the behavior of $\widetilde{\mathcal{D}}_\nu$, we rewrite (D.3) in Fourier integral form:

$$2\widetilde{\mathcal{D}}_\nu W_0(t, x_1, x_2) = \frac{1}{\sqrt{2\pi}} e^{-x_2^2/4\nu t} \int_{-\infty}^{\infty} e^{ix_1 \cdot \xi} \widehat{W_0}(\xi) A_{x_2} * B_{\nu t}(\xi)\, d\xi, \tag{D.7}$$

where

$$A_{x_2}(\xi) = e^{-x_2|\xi|}, \quad B_{\nu t}(\xi) = \sqrt{\frac{\nu t}{\pi}} e^{-\nu t \xi^2}, \tag{D.8}$$

and $A_{x_2} * B_{\nu t}(\xi)$ is the convolution. We introduce the variable

$$\mu = \frac{x_2^2}{4\nu t}, \tag{D.9}$$

and write this convolution as follows:

$$\begin{aligned} A_{x_2} * B_{\nu t}(\xi) &= \sqrt{\frac{\nu t}{\pi}} \int_{-\infty}^{\infty} e^{-x_2|\xi - \zeta|} e^{-\nu t \zeta^2}\, d\zeta \\ &= \frac{x_2}{\sqrt{4\pi\mu}} \int_{-\infty}^{\infty} e^{-x_2|\xi-\zeta|} e^{-x_2^2 \zeta^2/4\mu}\, d\zeta \\ &= \frac{1}{\sqrt{4\pi\mu}} \int_{-\infty}^{\infty} e^{-|x_2\xi - \tau|} e^{-\tau^2/4\mu}\, d\tau. \end{aligned} \tag{D.10}$$

Hence

$$A_{x_2} * B_{\nu t}(\xi) = \left(e^{\mu \partial_\tau^2} \Omega\right)(x_2 \xi), \quad \Omega(\tau) = e^{-|\tau|}. \tag{D.11}$$

The behavior of $\mathcal{D}_\nu^0 W_0(t, x)$ is similar to that of the model just described, though the formulas are a bit more complicated.

We next use (D.7) to derive some L^1-gradient bounds on $\widetilde{\mathcal{D}}_\nu W_0$. To start, note that

$$\sup_\xi |A_{x_2}(\xi)| = 1, \quad \int |B_{\nu t}(\xi)|\, d\xi = 1, \tag{D.12}$$

so clearly if

$$\widetilde{\mathcal{D}}_\nu(t) W_0(x_1, x_2) = \widetilde{\mathcal{D}}_\nu W_0(t, x_1, x_2), \tag{D.13}$$

we have

$$\widetilde{\mathcal{D}}_\nu(t) : L^2(\mathbb{R}) \longrightarrow L^\infty_{x_2}(\mathbb{R}^+, L^2_{x_1}(\mathbb{R})), \tag{D.14}$$

with operator norm ≤ 1, a bound that is independent of $t \in [0, \infty)$, $\nu \in (0, 1]$. Our goal here is to show that, given $T_0 \in (0, \infty)$,

$$\nabla \widetilde{\mathcal{D}}_\nu(t) : H^{1,2}(\mathbb{R}) \longrightarrow L^1_{x_2}([0, 1], L^2_{x_1}(\mathbb{R})), \tag{D.15}$$

with operator norm bounded independent of $t \in [0,T_0],\ \nu \in (0,1]$. We mention that $L^1_{x_2}$ cannot be replaced by $L^p_{x_2}$ for any $p > 1$. We expect that (D.15) extends to more general estimates, such as, for $\mathcal{D}^0_\nu(t)h(x) = \mathcal{D}^0_\nu h(t,x)$, as in (4.41),

$$\nabla \mathcal{D}^0_\nu(t) : H^{1,2}(\partial D) \longrightarrow L^1(D), \tag{D.16}$$

with operator norm bound independent of $\nu \in (0,1],\ t \in [0,T_0]$, and that similar estimates hold for $\nabla \mathcal{D}_\nu$, with $\mathcal{D}_\nu$ as in (4.30).

To begin the proof of (D.15), we note that ∂_{x_1} commutes with $\widetilde{\mathcal{D}}_\nu(t)$, so (D.14) readily yields

$$\partial_{x_1}\widetilde{\mathcal{D}}_\nu(t) : H^{1,2}(\mathbb{R}) \longrightarrow L^\infty_{x_2}(\mathbb{R}^+, L^2_{x_1}(\mathbb{R})), \tag{D.17}$$

with operator norm bounded independent of $\nu \in (0,1],\ t \in \mathbb{R}^+$. Thus it remains to estimate $\partial_{x_2}\widetilde{\mathcal{D}}_\nu(t)$. Let us write

$$2\widetilde{\mathcal{D}}_\nu(t)W_0(x_1,x_2) = e^{-x_2^2/4\nu t}\mathcal{E}_\nu(t)W_0(x_1,x_2), \tag{D.18}$$

where

$$\mathcal{E}_\nu(t)W_0(x_1,x_2) = \frac{1}{\sqrt{2\pi}}\int e^{ix_1\cdot\xi}\widehat{W_0}(\xi)A_{x_2} * B_{\nu t}(\xi)\, d\xi. \tag{D.19}$$

We have

$$\begin{aligned} 2\partial_{x_2}\widetilde{\mathcal{D}}_\nu(t)W_0(x_1,x_2) = &-\frac{x_2}{2\nu t}e^{-x_2^2/4\nu t}\mathcal{E}_\nu(t)W_0(x_1,x_2) \\ &+ e^{-x_2^2/4\nu t}\partial_{x_2}\mathcal{E}_\nu(t)W_0(x_1,x_2). \end{aligned} \tag{D.20}$$

The desired estimate on the first term on the right side of (D.20) follows from the fact that, just as in (D.14),

$$\mathcal{E}_\nu(t) : L^2(\mathbb{R}) \longrightarrow L^\infty_{x_2}(\mathbb{R}^+, L^2_{x_1}(\mathbb{R})), \tag{D.21}$$

with operator norm ≤ 1, for all $t \in \mathbb{R}^+,\ \nu > 0$, together with the fact that $(x_2/2\nu t)e^{-x_2^2/4\nu t}$ has $L^1(\mathbb{R}^+)$-norm equal to 1. Our next goal is to show that

$$\partial_{x_2}\mathcal{E}_\nu(t) : H^{1,2}(\mathbb{R}) \longrightarrow L^1_{x_2}(\mathbb{R}^+, L^2_{x_1}(\mathbb{R})), \tag{D.22}$$

with operator norm bound independent of $\nu \in (0,1],\ t \in [0,T_0]$. Note that

$$\partial_{x_2}\mathcal{E}_\nu(t)W_0(x_1,x_2) = \frac{1}{\sqrt{2\pi}}\int e^{ix_1\cdot\xi}\widehat{W_0}(\xi)G_{\nu t,x_2}(\xi)\, d\xi, \tag{D.23}$$

where

$$G_{\nu t,x_2}(\xi) = E_{x_2} * B_{\nu t}(\xi), \tag{D.24}$$

with $B_{\nu t}(\xi)$ as in (D.8) and

$$E_{x_2}(\xi) = -|\xi|e^{-x_2|\xi|}. \tag{D.25}$$

We can also write

$$E_{x_2}(\xi) = -\frac{1}{x_2}F_{x_2}(\xi), \quad F_{x_2}(\xi) = x_2|\xi|e^{-x_2|\xi|}, \tag{D.26}$$

so

$$G_{\nu t,x_2}(\xi) = -\frac{1}{x_2}F_{x_2} * B_{\nu t}(\xi). \tag{D.27}$$

Since $G_{\nu t,x_2}$ depends on ν and t only via νt, we may as well set $t = 1$ and produce estimates on $G_{\nu,x_2} = -(1/x_2)F_{x_2} * B_\nu$.

Taking $\nu \in (0, \nu_0]$, we first estimate $G_{\nu,x_2}(\xi)$ for $0 < x_2 \le \sqrt{\nu}$. An examination of (D.16)–(D.17) yields

$$\begin{aligned} 0 < x_2 \le \sqrt{\nu} \Rightarrow |G_{\nu,x_2}(\xi)| &\le \frac{C}{x_2}\Big(x_2|\xi| + \frac{x_2}{\sqrt{\nu}}\Big) \\ &= C\Big(|\xi| + \frac{1}{\sqrt{\nu}}\Big). \end{aligned} \tag{D.28}$$

Next, for $k \in \mathbb{Z}^+ = \{0, 1, 2, \dots\}$, we have

$$\begin{aligned} 2^k\sqrt{\nu} \le x_2 \le 2^{k+1}\sqrt{\nu} &\Rightarrow \frac{2^k}{x_2} \le \frac{1}{\sqrt{\nu}} \le \frac{2^{k+1}}{x_2} \\ &\Rightarrow |G_{\nu,x_2}(\xi)| \le \frac{C}{2^k x_2}. \end{aligned} \tag{D.29}$$

From (D.28) we have

$$\int_0^{\sqrt{\nu}} \|\partial_{x_2}\mathcal{E}_\nu(t)W_0(\cdot, x_2)\|_{L^2(\mathbb{R})}\, dx_2 \le C\|W_0\|_{L^2(\mathbb{R})} + C\sqrt{\nu}\|W_0\|_{H^{1,2}(\mathbb{R})}. \tag{D.30}$$

Meanwhile, (D.29) yields

$$\int_{2^k\sqrt{\nu}}^{2^{k+1}\sqrt{\nu}} \|\partial_{x_2}\mathcal{E}_\nu(t)W_0(\cdot, x_2)\|_{L^2(\mathbb{R})}\, dx_2 \le \frac{C}{2^k}\|W_0\|_{L^2(\mathbb{R})}. \tag{D.31}$$

Hence

$$\int_0^{\infty} \|\partial_{x_2}\mathcal{E}_\nu(t)W_0(\cdot, x_2)\|_{L^2(\mathbb{R})}\, dx_2 \le C\|W_0\|_{L^2(\mathbb{R})} + C\sqrt{\nu}\|W_0\|_{H^{1,2}(\mathbb{R})}. \tag{D.32}$$

This implies the desired estimate on (D.22).

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