Symbol Calculus for Operators of Layer Potential Type on Lipschitz Surfaces with VMO Normals, and Related Pseudodifferential Operator Calculus *

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Abstract

We show that operators of layer potential type on surfaces that are locally graphs of Lipschitz functions with gradients in vmo are equal, modulo compacts, to pseudodifferential operators (with rough symbols), for which a symbol calculus is available. We build further on the calculus of operators whose symbols have coefficients in $L^\infty \cap \text{vmo}$, and apply these results to elliptic boundary problems on domains with such boundaries, which in turn we identify with the class of Lipschitz domains with normals in vmo. This work simultaneously extends and refines classical work of Fabes, Jodeit, and Rivière, and also work of Lewis, Salvaggi, and Sisto, in the context of $\mathscr{C}^1$ surfaces.

Contents

1 Introduction

We produce a symbol calculus for a class of operators of layer potential type, of the form

$$Kf(x) = \mathrm{PV}\int_{\partial\Omega} k(x, x-y) f(y)\, d\sigma(y), \qquad x \in \partial\Omega, \tag{1.0.1}$$

in the following setting. First,

$$k \in \mathscr{C}^\infty(\mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} \setminus 0)), \tag{1.0.2}$$

with $k(x, z)$ homogeneous of degree $-n$ in z and $k(x, -z) = -k(x, z)$. Next, $\Omega \subset \mathbb{R}^{n+1}$ is a bounded Lipschitz domain, with a little extra regularity. Namely, Ω is locally the upper-graph of a function $\varphi_0 : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$\nabla\varphi_0 \in L^\infty(\mathbb{R}^n) \cap \mathrm{vmo}(\mathbb{R}^n). \tag{1.0.3}$$

We say Ω is a $\mathrm{Lip} \cap \mathrm{vmo}_1$ domain.

Since we will be dealing with a number of variants of BMO, we recall some definitions. First,

$$\mathrm{BMO}(\mathbb{R}^n) := \big\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : f^\# \in L^\infty(\mathbb{R}^n) \big\}, \tag{1.0.4}$$

where

$$f^\#(x) := \sup_{B \in \mathcal{B}(x)} \frac{1}{V(B)} \int_B |f(y) - f_B|\, dy, \tag{1.0.5}$$

with $\mathcal{B}(x) := \{B_r(x) : 0 < r < \infty\}$, $B_r(x)$ being the ball centered at x of radius r, and f_B the mean value of f on B. There are variants, giving the same space. For example, one could use cubes containing x instead of balls centered at x, and one could replace f_B in (1.0.5) by c_B, chosen to minimize the integral. We set

$$\|f\|_{\mathrm{BMO}} := \|f^\#\|_{L^\infty}. \tag{1.0.6}$$

This is not a norm, since $\|c\|_{\mathrm{BMO}} = 0$ if c is a constant; it is a seminorm. The space $\mathrm{bmo}(\mathbb{R}^n)$ is defined by

$$\mathrm{bmo}(\mathbb{R}^n) := \big\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : {}^\# f \in L^\infty(\mathbb{R}^n) \big\}, \tag{1.0.7}$$

where

$${}^\# f(x) := \sup_{B \in \mathcal{B}_1(x)} \frac{1}{V(B)} \int_B |f(y) - f_B|\, dy + \frac{1}{V(B_1(x))} \int_{B_1(x)} |f(y)|\, dy, \tag{1.0.8}$$

with $\mathcal{B}_1(x) := \{B_r(x) : 0 < r \le 1\}$. We set

$$\|f\|_{\mathrm{bmo}} := \|{}^\# f\|_{L^\infty}. \tag{1.0.9}$$

This is a norm, and $\mathrm{bmo}(\mathbb{R}^n)$ has good localization properties.

Now, $\mathrm{VMO}(\mathbb{R}^n)$ is the closure in $\mathrm{BMO}(\mathbb{R}^n)$ of $\mathrm{UC}(\mathbb{R}^n) \cap \mathrm{BMO}(\mathbb{R}^n)$, where $\mathrm{UC}(\mathbb{R}^n)$ is the space of uniformly continuous functions on $\mathbb{R}^n$, and $\mathrm{vmo}(\mathbb{R}^n)$ is the closure in $\mathrm{bmo}(\mathbb{R}^n)$ of $\mathrm{UC}(\mathbb{R}^n) \cap \mathrm{bmo}(\mathbb{R}^n)$. One can use local coordinates and partitions of unity to define $\mathrm{bmo}(M)$ and $\mathrm{vmo}(M)$ on a class of Riemannian manifolds M (cf. [37]). See also Appendix A.3 of this paper for a discussion of $\mathrm{BMO}(M)$ and $\mathrm{VMO}(M)$ on spaces M of homogeneous type. We mention that if M is compact, $\mathrm{BMO}(M)$ coincides with $\mathrm{bmo}(M)$ and $\mathrm{VMO}(M)$ coincides with $\mathrm{vmo}(M)$.

With this in mind, we mention that Ω could be an open set in a compact $(n+1)$-dimensional Riemannian manifold M, whose boundary, in local coordinates on M, is locally a graph as in

(1.0.3), and $k(x, x-y)$ in (1.0.1) could be the integral kernel of a pseudodifferential operator on M of order -1, with odd symbol.

The analysis of operators of the form (1.0.1) as bounded operators on $L^p(\partial\Omega)$ for $p \in (1, \infty)$, together with nontangential maximal function estimates for

$$\mathcal{K}f(x) = \int_{\partial\Omega} k(x, x-y) f(y) \, d\sigma(y), \qquad x \in \mathbb{R}^{n+1} \setminus \partial\Omega, \tag{1.0.10}$$

and nontangential convergence, was done for general Lipschitz domains in [5], carrying through the breakthrough initiated in [2], at least for $k = k(x-y)$.

In between [2] and [5] was another key paper, [9], which treated (1.0.1) (again with $k = k(x-y)$) when Ω has a $\mathscr{C}^1$ boundary, and gave some applications to PDE. These applications involved looking at double layer potentials

$$K_d f(x) = \mathrm{PV} \int_{\partial\Omega} \nu(x) \cdot (x-y) E(x-y) f(y) \, d\sigma(y), \qquad x \in \partial\Omega, \tag{1.0.11}$$

where $\nu(x)$ is the unit normal to $\partial\Omega$, and $E(z) = c_n |z|^{-(n+1)}$. Such an operator is of the form $K_d f(x) = \nu(x) \cdot K f(x)$, where K is as in (1.0.1), with $k(z) = zE(z)$ vector valued. In [9] it was shown that K_d is compact when Ω is a bounded domain of class $\mathscr{C}^1$. (See §3.4 of this paper for a proof that K_d is compact more generally when Ω is a bounded $\mathrm{Lip} \cap \mathrm{vmo}_1$ domain.) This compactness was applied to the Dirichlet problem for the Laplace operator on bounded $\mathscr{C}^1$ domains. In fact, if

$$\mathcal{K}_d f(x) = \int_{\partial\Omega} \nu(x) \cdot (x-y) E(x-y) f(y) \, d\sigma(y), \quad x \in \Omega, \tag{1.0.12}$$

one has

$$\mathcal{K}_d f\big|_{\partial\Omega} = (\tfrac{1}{2} I + K_d) f, \tag{1.0.13}$$

so solving the Dirichlet problem $\Delta u = 0$ on Ω, $u\big|_{\partial\Omega} = g$, as $u = \mathcal{K}_d f$ leads to solving

$$(\tfrac{1}{2} I + K_d) f = g, \tag{1.0.14}$$

and the compactness of K_d implies $\frac{1}{2} I + K_d$ is Fredholm, of index 0.

For a general bounded Lipschitz domain $\Omega \subset \mathbb{R}^{n+1}$, one continues to have (1.0.12)–(1.0.14), but K_d is typically not compact. However, it was shown in [38] that $\frac{1}{2} I + K_d$ is still Fredholm, of index 0, using Rellich identities as a tool. This led to much work on other elliptic boundary problems, including boundary problems for the Stokes system, linear elasticity systems, and the Hodge Laplacian. In [26] a program was initiated that extended the study of (1.0.1) from $k = k(x-y)$ to $k = k(x, x-y)$, a development that enabled the authors to work on Lipschitz domains in Riemannian manifolds. This led to a series of papers, including [27], and [22]. In these papers, variants of Rellich identities also played major roles.

Meanwhile, [12] established compactness of K_d in (1.0.11) when $\Omega \subset \mathbb{R}^{n+1}$ is a bounded VMO_1 domain, i.e., its boundary is locally a graph of a function φ_0 satisfying

$$\nabla \varphi_0 \in \mathrm{VMO}(\mathbb{R}^n), \tag{1.0.15}$$

which is weaker than (1.0.3). This led [14] to establish compactness of a somewhat broader class of operators, not just on VMO_1 domains, but more generally on a class of domains, introduced by

[31] and [17], called chord-arc domains with vanishing constant by those authors, but called regular SKT domains in [14]. This was applied in [14] to the Dirichlet boundary problem for the Laplace operator, on regular SKT domains in Riemannian manifolds, and also to a variety of boundary problems for other second order elliptic systems.

In these works on various elliptic boundary problems, both on Lipschitz domains and on regular SKT domains, each elliptic system seemed to need a separate treatment. This is in striking contrast to the now standard theory of regular elliptic boundary problems on smoothly bounded domains, for operators with smooth coefficients. Such cases yield operators of the form (1.0.1) that are pseudodifferential operators on $\partial\Omega$, for which a symbol calculus is effective to power the analysis. One can, for example, see the treatment of regular elliptic boundary problems in [34, Chapter 7, §12].

Our goal here is to develop a symbol calculus for operators of the form (1.0.1) in $\mathrm{Lip} \cap \mathrm{vmo}_1$ domains, and to apply this symbol calculus to the analysis of some elliptic boundary problems.

We work in local graph coordinates, in which (1.0.1) takes the form

$$Kf(x) = \mathrm{PV} \int\limits_{\mathbb{R}^n} k(\varphi(x), \varphi(x) - \varphi(y)) f(y) \Sigma(y)\, dy, \qquad x \in \mathbb{R}^n, \tag{1.0.16}$$

where $\varphi(x) = (x, \varphi_0(x))$, with $\varphi_0 : \mathbb{R}^n \to \mathbb{R}$ as in (1.0.3). In fact, we allow $\varphi_0 : \mathbb{R}^n \to \mathbb{R}^\ell$. The surface area element $d\sigma(y) = \Sigma(y)\, dy$. Our first major result is that, with $K^\#$ given by

$$K^\# f(x) = \mathrm{PV} \int\limits_{\mathbb{R}^n} k(\varphi(x), D\varphi(x)(x-y)) f(y) \Sigma(y)\, dy, \qquad x \in \mathbb{R}^n, \tag{1.0.17}$$

we have

$$K - K^\# \ \text{ compact on } \ L^p(B), \tag{1.0.18}$$

for $p \in (1, \infty)$, for any ball $B \subset \mathbb{R}^n$. Then, as we show, $K^\# f = p(x, D)(\Sigma f)$, with

$$p(x, D) \in \mathrm{OP}(L^\infty \cap \mathrm{vmo}) S^0_{\mathrm{cl}}(\mathbb{R}^n), \tag{1.0.19}$$

a class of pseudodifferential operators studied in [36] and shown to have a viable symbol calculus. Definitions and basic results are given in Appendix A.3 of this paper. The proof of (1.0.18), given in §2, makes essential use of results of [12] and further material in [14].

Since (1.0.16) and (1.0.17) are given in local graph coordinates, it is important to record how operators are related when represented in two different such coordinates, and how a symbol can be associated to such an operator, independent of the coordinate representation. These matters are handled in §3.

In connection with this, we mention the work [18], providing such an analysis on $\mathscr{C}^1$ manifolds. In particular, (1.0.18) (for $\varphi \in \mathscr{C}^1$) plays a central role there. In [18], the function $k(x, z)$ is required to be analytic in $z \in \mathbb{R}^{n+1} \setminus \{0\}$. The need for such analyticity arises from technical issues, which we can overcome here, thanks to the advances in [12] and [14]. One desirable effect of not requiring such analyticity is that our results readily allow for microlocalization. Though we do not pursue microlocal analysis on boundaries of $\mathrm{Lip} \cap \mathrm{vmo}_1$ domains here, we are pleased to advertise the potential to pursue such analysis.

The structure of the rest of this paper is as follows. Section 2 is devoted to a proof of the basic result (1.0.18). Section 3 builds on this to produce a symbol calculus, making essential use of results on operators of the form (1.0.19), recalled in an appendix. Section 4 applies these results to some boundary problems for elliptic systems on $\mathrm{Lip} \cap \mathrm{vmo}_1$ domains. These include the Dirichlet problem for a general class of second order, strongly elliptic systems, and a class of oblique derivative problems. We also produce a general result on regular boundary problems for first order elliptic systems, and show how this plays out for the Hodge-Dirac operator $d + \delta$, acting on differential forms.

A set of appendices deals with auxiliary results. The first gives material used in §2.1. The second gives a detailed analysis of just how a principal value integral like (1.0.1) works for such domains as we consider here. The third reviews material on the class of pseudodifferential operators (1.0.19). The fourth reviews matters related to $\mathrm{BMO}(M)$ and $\mathrm{VMO}(M)$ when M is a space of homogeneous type. The fifth proves that a bounded domain $\Omega \subset \mathbb{R}^{n+1}$ is locally the upper-graph of a function satisfying (1.0.3) if and only if its outward unit normal belongs to $\mathrm{VMO}(\partial\Omega)$.

2 From layer potential operators to pseudodifferential operators

The primary goal of this section is to establish the compactness of the difference between a singular integral operator K of layer potential type, as in (1.0.1) and a related operator $K^{\#}$, which belongs to the class of pseudodifferential operators $\mathrm{OP}(L^\infty \cap \mathrm{vmo})S^0_{\mathrm{cl}}$, a class that is reviewed in Appendix A.3. We proceed in stages.

2.1 General local compactness results

Below, the principal value integrals $\mathrm{PV}\int$ are understood in the sense of removing small balls centered at the singularity and passing to the limit, by letting their radii approach zero; for a more flexible view on this topic see the discussion in §A.2. We begin by recalling the following local compactness result.

Theorem 2.1. *Assume $\varphi : \mathbb{R}^n \to \mathbb{R}$ and $\psi : \mathbb{R}^n \to \mathbb{R}^m$ are two locally integrable functions satisfying*

$$\nabla\varphi \in \mathrm{vmo}(\mathbb{R}^n), \quad D\psi \in \mathrm{bmo}(\mathbb{R}^n), \tag{2.1.1}$$

and set

$$\Gamma(x,y) := \varphi(x) - \varphi(y) - \nabla\varphi(x)(x-y), \qquad x, y \in \mathbb{R}^n. \tag{2.1.2}$$

Given $F : \mathbb{R}^m \to \mathbb{R}$ smooth (of a sufficiently large order $M = M(m,n) \in \mathbb{N}$), even on $\mathbb{R}^m$, and such that

$$|F(w)| \leq C(1+|w|)^{-1} \ \ \textit{for every} \ \ w \in \mathbb{R}^m, \tag{2.1.3}$$

$$\textit{and} \ \ \partial^\alpha F \in L^1(\mathbb{R}^m) \ \ \textit{whenever} \ \ |\alpha| \leq M, \tag{2.1.4}$$

consider the principal value integral operator

$$Tf(x) := \mathrm{PV}\int_{\mathbb{R}^n} |x-y|^{-(n+1)} F\Big(\frac{\psi(x)-\psi(y)}{|x-y|}\Big)\Gamma(x,y)f(y)\,dy, \qquad x \in \mathbb{R}^n, \tag{2.1.5}$$

and the associated maximal operator

$$T_*f(x) := \sup_{\varepsilon>0}\Bigg|\int_{\substack{y\in\mathbb{R}^n\\ |x-y|>\varepsilon}} |x-y|^{-(n+1)} F\Big(\frac{\psi(x)-\psi(y)}{|x-y|}\Big)\Gamma(x,y)f(y)\,dy\Bigg|, \qquad x \in \mathbb{R}^n. \tag{2.1.6}$$

Then for each $p \in (1,\infty)$ there exists $C_{n,p} \in (0,\infty)$ such that

$$\|T_* f\|_{L^p(\mathbb{R}^n)} \leq C_{n,p}\Big(\sum_{|\alpha|\leq M} \|\partial^\alpha F\|_{L^1(\mathbb{R}^m)} + \sup_{w\in\mathbb{R}^m}\big[(1+|w|)|F(w)|\,\big]\Big)\times$$

$$\times\,\|\nabla\varphi\|_{\mathrm{BMO}(\mathbb{R}^n)}\Big(1+\|D\psi\|_{\mathrm{BMO}(\mathbb{R}^n)}\Big)^N \|f\|_{L^p(\mathbb{R}^n)}, \tag{2.1.7}$$

for every $f \in L^p(\mathbb{R}^n)$. Also, with B_R abbreviating $B(0,R) := \{x \in \mathbb{R}^n : |x| < R\}$, it follows that for each $R \in (0,\infty)$ and $p \in (1,\infty)$ the operator

$$T : L^p(B_R) \longrightarrow L^p(B_R) \quad \textit{is compact.} \tag{2.1.8}$$

This result is given in [14, Theorem 4.34, p. 2725, §4.4] and [14, Theorem 4.35, p. 2726, §4.4]. As noted there, the analysis behind it is from [12]. Of course, there is a natural analogue of Theorem 2.1 when the function φ is vector-valued (implied by the scalar case, by working componentwise). Here, the the goal is to prove the following version of Theorem 2.1.

Theorem 2.2. *Suppose $\varphi : \mathbb{R}^n \to \mathbb{R}$ and $\psi : \mathbb{R}^n \to \mathbb{R}^m$ are two locally integrable functions satisfying*

$$\nabla\varphi \in \mathrm{vmo}(\mathbb{R}^n), \quad D\psi \in L^\infty(\mathbb{R}^n), \tag{2.1.9}$$

and let the symbol $\Gamma(x,y)$ retain the same significance as in (2.1.2). Given an even real-valued function $F \in \mathscr{C}^M(\mathbb{R}^k)$ (for a sufficiently large $M \in \mathbb{N}$), along with some matrix-valued function

$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^{k\times m}, \quad A \in L^\infty(\mathbb{R}^n), \tag{2.1.10}$$

consider the principal value singular integral operator

$$T_A f(x) := \mathrm{PV}\int_{\mathbb{R}^n} |x-y|^{-(n+1)} F\Big(A(x)\frac{\psi(x)-\psi(y)}{|x-y|}\Big)\Gamma(x,y) f(y)\,dy, \qquad x \in \mathbb{R}^n. \tag{2.1.11}$$

Then for each $R \in (0,\infty)$ and $p \in (1,\infty)$ the operator

$$T_A : L^p(B_R) \longrightarrow L^p(B_R) \quad \textit{is compact.} \tag{2.1.12}$$

Once again, there is a natural analogue of Theorem 2.2 when the function φ is vector-valued (implied by the scalar case, by working componentwise).

Proof of Theorem 2.2. Fix a finite number

$$R_* > \|D\psi\|_{L^\infty(\mathbb{R}^n)} \tag{2.1.13}$$

and abbreviate $B_* := \{w \in \mathbb{R}^m : |w| < R_*\}$. Also, select a real-valued function χ satisfying

$$\begin{gathered}\chi \in \mathscr{C}^\infty(\mathbb{R}^m),\ \chi \text{ even in } \mathbb{R}^m,\ \operatorname{supp}\chi \subseteq B_*,\\ \chi(z) = 1 \ \text{ whenever } \ |z| \leq \|D\psi\|_{L^\infty(\mathbb{R}^n)}.\end{gathered} \tag{2.1.14}$$

To proceed, let $\{\vartheta_j\}_{j\in\mathbb{N}} \subset L^2(B_*)$ denote an orthonormal basis of $L^2(B_*)$ consisting of real-valued eigenfunctions of the Dirichlet Laplacian in B_* (as discussed in Appendix §A.1). For $x \in \mathbb{R}^n$, we can write in $L^2(B_*)$ and a.e. $z \in B_*$

$$F(A(x)z) = \sum_{j\in\mathbb{N}} b_j(x)\vartheta_j(z) \tag{2.1.15}$$

where, for each $j \in \mathbb{N}$, we have set

$$b_j(x) := \int_{B_*} F(A(x)z)\vartheta_j(z)\,dz, \qquad x \in \mathbb{R}^n. \tag{2.1.16}$$

To estimate the b_j's, fix $j \in \mathbb{N}$, $x \in \mathbb{R}^n$, and observe that for each $N \in \mathbb{N}$ we may write

$$\begin{aligned}
\lambda_j^N \big|b_j(x)\big| &= \Big|\int_{B_*} F(A(x)z)\big((-\Delta)^N\vartheta_j\big)(z)\,dz\Big| \\
&= \Big|\int_{B_*} (-\Delta_z)^N\big[F(A(x)z)\big]\vartheta_j(z)\,dz\Big| \\
&\leq C_N\|A\|^{2N}_{L^\infty(\mathbb{R}^n)}\Bigg\{\sup_{\substack{|w|\leq R_*\|A\|_{L^\infty(\mathbb{R}^n)} \\ |\alpha|=2N}} |(\partial^\alpha F)(w)|\Bigg\}\|\vartheta_j\|_{L^\infty(B_*)} \\
&\leq C_{A,F,R_*,N}\, j^{1/2+2/n},
\end{aligned} \tag{2.1.17}$$

by (A.1.9). In light of (A.1.8) this ultimately shows that for each $N \in \mathbb{N}$ there exists a constant $C_N \in (0,\infty)$ such that

$$\|b_j\|_{L^\infty(\mathbb{R}^n)} \leq C_N j^{-N}, \qquad \forall\, j \in \mathbb{N}. \tag{2.1.18}$$

Moving on, we note that combining (2.1.15) with its version written for $-z$ in place of z, and keeping in mind that F is even, yields

$$F\big(A(x)z\big) = \sum_{j\in\mathbb{N}} b_j(x)\widetilde{\vartheta}_j(z) \tag{2.1.19}$$

where, for each $j \in \mathbb{N}$, we have set

$$\widetilde{\vartheta}_j(z) := \frac{\vartheta_j(z) + \vartheta_j(-z)}{2}, \qquad z \in B_*. \tag{2.1.20}$$

In particular, for each $j \in \mathbb{N}$,

$$\begin{aligned}
&\widetilde{\vartheta}_j \in \mathscr{C}^\infty_{loc}(B_*) \ \text{ is even, vanishes on } \partial B_*, \\
&\text{and satisfies } \ -\Delta\widetilde{\vartheta}_j = \lambda_j\widetilde{\vartheta}_j \ \text{ in } \ B_*.
\end{aligned} \tag{2.1.21}$$

Multiplying both sides of (2.1.19) with the cut-off function χ from (2.1.14) then finally yields

$$\chi(z)F(A(x)z) = \sum_{j\in\mathbb{N}} b_j(x)F_j(z), \qquad x \in \mathbb{R}^n,\ z \in \mathbb{R}^m, \tag{2.1.22}$$

where, for each $j \in \mathbb{N}$, we have set

$$F_j(z) := \chi(z)\widetilde{\vartheta}_j(z), \qquad z \in \mathbb{R}^m, \tag{2.1.23}$$

naturally viewed as zero outside B_*. Hence, for each $j \in \mathbb{N}$,

$$F_j \in \mathscr{C}^\infty(\mathbb{R}^m) \ \text{ is an even function supported in } B_*, \tag{2.1.24}$$

and (A.1.11) implies that for every multi-index $\alpha \in \mathbb{N}_0^m$ there exists a constant $C_{m,\alpha} \in (0,\infty)$ such that

$$\|\partial^\alpha F_j\|_{L^\infty(\mathbb{R}^m)} \leq C_{m,\alpha}\, j^{1/2+2/n}. \tag{2.1.25}$$

Since

$$z = \frac{\psi(x)-\psi(y)}{|x-y|} \Longrightarrow |z| \leq \|D\psi\|_{L^\infty(\mathbb{R}^m)} \Longrightarrow \chi(z) = 1, \tag{2.1.26}$$

we deduce from (2.1.22) that

$$T_A f(x) = \sum_{j\in\mathbb{N}} b_j(x) T_j f(x), \tag{2.1.27}$$

where for each $j \in \mathbb{N}$ we have set

$$T_j f(x) := \mathrm{PV}\int_{\mathbb{R}^n} |x-y|^{-(n+1)} F_j\Big(\frac{\psi(x)-\psi(y)}{|x-y|}\Big)\Gamma(x,y) f(y)\,dy, \qquad x \in \mathbb{R}^n. \tag{2.1.28}$$

At this stage, Theorem 2.1 applies to each operator T_j. In concert, estimates (2.1.7) and (2.1.25) yield a polynomial bound in $j \in \mathbb{N}$ on the operator norms of T_j on $L^p(\mathbb{R}^n)$. Then, in the context of the expansion (2.1.27), the rapid decrease (2.1.18) implies the desired compactness on $L^p(B_R)$ for T_A, for each $R \in (0,\infty)$ and $p \in (1,\infty)$. □

It is possible to prove Theorem 2.2 using the Fourier transform in place of spectral methods, based on Dirichlet eigenfunction decompositions. We shall do so below and, in the process, derive further information about the family of truncated operators (indexed by $\varepsilon > 0$)

$$T_{A,\varepsilon} f(x) := \int_{\{y\in\mathbb{R}^n:\,|x-y|>\varepsilon\}} |x-y|^{-(n+1)} F\left(A(x)\frac{\psi(x)-\psi(y)}{|x-y|}\right)\Gamma(x,y) f(y)\,dy, \tag{2.1.29}$$

where $x \in \mathbb{R}^n$, including the pointwise a.e. existence of the associated principal value singular integral operator.

Theorem 2.3. *For each $\varepsilon > 0$ let $T_{A,\varepsilon}$ be as in* (2.1.29)*, where $\Gamma(x,y)$ is defined as in* (2.1.2) *for a function $\varphi : \mathbb{R}^n \to \mathbb{R}$ satisfying $\nabla\varphi \in \mathrm{BMO}(\mathbb{R}^n)$, $A \in L^\infty(\mathbb{R}^n)$ is a $k \times m$ matrix-valued function, $\psi : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz, and $F \in \mathscr{C}^M(\mathbb{R}^k)$ is even.*

Then, if $M = M(m,n) \in \mathbb{N}$ is large enough, there is a positive $M_0 < \infty$ such that for $1 < p < \infty$,

$$\begin{aligned}\sup_{\varepsilon>0} \|T_{A,\varepsilon} f\|_{L^p(\mathbb{R}^n)} &\leq \Big\|\sup_{\varepsilon>0} |T_{A,\varepsilon} f|\Big\|_{L^p(\mathbb{R}^n)} \\ &\leq C_0 \left(1 + \|\nabla\psi\|_{L^\infty(\mathbb{R}^n)}\right)^{M_0} \|\nabla\varphi\|_{\mathrm{BMO}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)},\end{aligned} \tag{2.1.30}$$

where the constant C_0 depends on $\|A\|_\infty$, p, n, m, k, and $\|F\|_{\mathscr{C}^M(B(0,\|A\|_\infty R_*))}$ with

$$R_* := 2(\|\nabla\psi\|_\infty + 1). \tag{2.1.31}$$

Moreover,

$$\nabla\varphi \in \mathrm{VMO}(\mathbb{R}^n) \Longrightarrow \lim_{\varepsilon\to 0^+} T_{A,\varepsilon} f(x) \ \text{ exists for a.e. } x \in \mathbb{R}^n, \quad \forall\, f \in L^p(\mathbb{R}^n). \tag{2.1.32}$$

In fact, a more general result of this nature holds. Specifically, if $B : \mathbb{R}^n \to \mathbb{R}^{m'}$ is a bi-Lipschitz function and if for each $\varepsilon > 0$ we set

$$T_{A,B,\varepsilon} f(x) := \int_{\{y\in\mathbb{R}^n:\,|B(x)-B(y)|>\varepsilon\}} |x-y|^{-(n+1)} F\left(A(x)\frac{\psi(x)-\psi(y)}{|x-y|}\right)\Gamma(x,y) f(y)\,dy, \tag{2.1.33}$$

where $x \in \mathbb{R}^n$*, then*

$$\nabla\varphi \in \mathrm{VMO}(\mathbb{R}^n) \implies \lim_{\varepsilon\to 0^+} T_{A,B,\varepsilon} f(x) \ \ \textit{exists for a.e.} \ \ x \in \mathbb{R}^n, \quad \forall\, f \in L^p(\mathbb{R}^n). \tag{2.1.34}$$

We shall prove estimate (2.1.30) by reducing it to the scalar valued case $k = m = 1$, with $A \equiv 1$, which is Theorem 1.10 in [12]. Note that given (2.1.30), for $\varphi \in \mathrm{vmo}(\mathbb{R}^n)$ one then gets local compactness (as in the statement of Theorem 2.2; cf. (2.1.12)) of the associated principal value operator, by the usual methods.

Proof of Theorem 2.3. For $z \in \mathbb{R}^m$, set $F_x(z) := F(A(x)\,z)$. Note that since $A \in L^\infty$, we have that $F_x(\cdot) \in \mathscr{C}^M$, with

$$\sup_{0\le j\le M} \|\nabla^j F_x(\cdot)\|_{L^\infty(B)} \ \text{ controlled uniformly in } x\text{, for every ball } B \subset \mathbb{R}^m. \tag{2.1.35}$$

Moreover, as before, we may suppose that

$$F_x(\cdot) \text{ is supported in the ball } B(0, R_*) \subset \mathbb{R}^m\text{, for every } x \in \mathbb{R}^n, \tag{2.1.36}$$

where R_* is as in (2.1.31). For notational convenience, we normalize F so that

$$\sup_{0\le j\le M} \|\nabla^j F(\cdot)\|_{L^\infty(B(0,\|A\|_\infty R_*))} = 1. \tag{2.1.37}$$

We may write

$$F_x(z) = c\int_{\mathbb{R}^m} \widehat{F_x}(\xi)\cos(z\cdot\xi)\,d\xi, \tag{2.1.38}$$

where $\widehat{F_x}$ is the Fourier transform of F_x, and we observe that by standard estimates for the Fourier transform, and our normalization of F from (2.1.37),

$$\operatorname{ess\,sup}_{x\in\mathbb{R}^n} |\widehat{F_x}(\xi)| \le C\,R_*^m(1+|\xi|)^{-M}. \tag{2.1.39}$$

Let $\eta \in \mathscr{C}_0^\infty(-2,2)$ be an even function, with $\eta \equiv 1$ on $[-1,1]$, and for $\xi \in \mathbb{R}^m$, $t \in \mathbb{R}$, set

$$E_\xi(t) := \cos(t)\,\eta\left(\frac{t}{(1+|\xi|)R_*}\right). \tag{2.1.40}$$

Observe that for $z \in B(0,R_*) \subset \mathbb{R}^m$, we may replace $\cos(z\cdot\xi)$ by $E_\xi(z\cdot\xi)$ in (2.1.38). In concert with (2.1.29) and (2.1.38), this permits us to write

$$\begin{aligned} T_{A,\varepsilon} f(x) &= \int_{\{y\in\mathbb{R}^n:\,|x-y|>\varepsilon\}} |x-y|^{-(n+1)} F\left(A(x)\frac{\psi(x)-\psi(y)}{|x-y|}\right)\Gamma(x,y)f(y)\,dy \\ &= c\int_{\mathbb{R}^m} \widehat{F_x}(\xi)\left\{\int_{\{y\in\mathbb{R}^n:\,|x-y|>\varepsilon\}} |x-y|^{-(n+1)}\,E_\xi\left(\xi\cdot\frac{\psi(x)-\psi(y)}{|x-y|}\right)\Gamma(x,y)f(y)\,dy\right\}d\xi \\ &= c\int_{\mathbb{R}^m} (1+|\xi|)^{M-N}\widehat{F_x}(\xi)T_{\xi,\varepsilon}f(x)\,d\xi, \end{aligned} \tag{2.1.41}$$

where

$$T_{\xi,\varepsilon} f(x) := \int_{\{y\in\mathbb{R}^n:\,|x-y|>\varepsilon\}} |x-y|^{-(n+1)}\,\widetilde{E}_\xi\left(\xi\cdot\frac{\psi(x)-\psi(y)}{|x-y|}\right)\Gamma(x,y)f(y)\,dy \tag{2.1.42}$$

and, with N a large number to be chosen later,

$$\widetilde{E}_\xi(t) := (1+|\xi|)^{N-M} E_\xi(t), \qquad \forall\, t \in \mathbb{R}. \tag{2.1.43}$$

In turn, from (2.1.41) and (2.1.39) we deduce that

$$\Big\| \sup_{\varepsilon>0} \big|T_{A,\varepsilon} f\big| \Big\|_{L^p(\mathbb{R}^n)} \leq C R_*^m \int_{\mathbb{R}^m} (1+|\xi|)^{-N} \Big\| \sup_{\varepsilon>0} \big|T_{\xi,\varepsilon} f\big| \Big\|_{L^p(\mathbb{R}^n)} \, d\xi, \tag{2.1.44}$$

We now set

$$N := M - 2, \tag{2.1.45}$$

and note that this choice ensures that for all non-negative integers j,

$$\left| \left(\frac{d}{dt}\right)^j \widetilde{E}_\xi(t) \right| \leq C_j (1+|\xi|)^{-2} \left(\frac{1}{1+|t|/((1+|\xi|)R_*)} \right)^2 \leq C_j R_*^2 (1+|t|)^{-2},$$

where the constant C_j may depend on j, but is independent of ξ. By [12, Theorem 1.10, p. 470], applied to the scalar-valued Lipschitz function $\xi \cdot \psi$, we then have that for some $M_1 < \infty$,

$$\Big\| \sup_{\varepsilon>0} \big|T_{\xi,\varepsilon} f(x)\big| \Big\|_{L^p_x(\mathbb{R}^n)} \leq C R_*^2 (1+|\xi| R_*)^{M_1} \|\nabla \varphi\|_{\mathrm{BMO}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}. \tag{2.1.46}$$

Plugging the latter estimate into (2.1.44), and finally choosing

$$M := M_1 + m + 3, \tag{2.1.47}$$

we obtain (2.1.30) thanks to (2.1.45).

Finally, there remains to consider the issue of the existence of the limits in (2.1.32) and (2.1.34). We treat in detail the former, since the argument for the latter is similar, granted our results in §A.2. To justify (2.1.32), make the standing assumption that

$$\nabla \varphi \in \mathrm{VMO}(\mathbb{R}^n), \tag{2.1.48}$$

and recall from (2.1.41), (2.1.45) that

$$T_{A,\varepsilon} f(x) = c \int_{\mathbb{R}^m} (1+|\xi|)^2 \widehat{F_x}(\xi) T_{\xi,\varepsilon} f(x) \, d\xi, \tag{2.1.49}$$

where $T_{\xi,\varepsilon} f(x)$ is as in (2.1.42). To proceed, observe that for each $f \in L^p(\mathbb{R}^n)$ there holds

$$\sup_{\varepsilon>0} \big|(1+|\xi|)^2 \widehat{F_x}(\xi) T_{\xi,\varepsilon} f(x)\big| \in L^1_\xi(\mathbb{R}^m), \quad \text{for a.e. fixed } \; x \in \mathbb{R}^n. \tag{2.1.50}$$

To see that this is the case, use Minkowski's inequality along with (2.1.39) and (2.1.46) to estimate

$$\begin{aligned}
&\left\{ \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^m} \sup_{\varepsilon>0} \big|(1+|\xi|)^2 \widehat{F_x}(\xi) T_{\xi,\varepsilon} f(x)\big| \, d\xi \Big)^p \, dx \right\}^{1/p} \\
&\qquad \leq \int_{\mathbb{R}^m} \Big\| \sup_{\varepsilon>0} \big|(1+|\xi|)^2 \widehat{F_x}(\xi) T_{\xi,\varepsilon} f(x)\big| \Big\|_{L^p_x(\mathbb{R}^n)} \, d\xi \qquad (2.1.51) \\
&\qquad \leq \int_{\mathbb{R}^m} (1+|\xi|)^2 \Big[\mathrm{esssup}_{x \in \mathbb{R}^n} |\widehat{F_x}(\xi)| \Big] \Big\| \sup_{\varepsilon>0} \big|T_{\xi,\varepsilon} f(x)\big| \Big\|_{L^p_x(\mathbb{R}^n)} \, d\xi \\
&\qquad \leq C R_*^{m+2} \|\nabla \varphi\|_{\mathrm{BMO}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)} \int_{\mathbb{R}^m} (1+|\xi|)^{2-M} (1+|\xi| R_*)^{M_1} \, d\xi < +\infty,
\end{aligned}$$

thanks to (2.1.47). With (2.1.51) in hand, the claim in (2.1.50) readily follows. Next, granted (2.1.48), we claim that for each fixed function $f \in L^p(\mathbb{R}^n)$ the following holds:

$$\text{for each fixed } \ \xi \in \mathbb{R}^m, \ \text{ the limit } \ \lim_{\varepsilon \to 0^+} T_{\xi,\varepsilon} f(x) \ \text{ exists for a.e. } \ x \in \mathbb{R}^n. \tag{2.1.52}$$

Given that we have already established (2.1.30), this may be justified along the lines of the proof of Theorem 5.11, pp. 500-501 in [12], based on Proposition A.3 and keeping in mind that VMO functions may be approximated in the BMO norm by continuous functions with compact support which, in turn, are uniformly approximable by functions in $\mathscr{C}_0^\infty$.

In concert with the uniform integrability property (2.1.50), the existence of the limit in (2.1.52) makes it possible to use Lebesgue's Dominated Convergence Theorem in order to write that, for a.e. $x \in \mathbb{R}^n$,

$$\begin{aligned} \lim_{\varepsilon \to 0^+} T_{A,\varepsilon} f(x) &= c \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^m} (1+|\xi|)^2 \widehat{F_x}(\xi) T_{\xi,\varepsilon} f(x)\, d\xi \\ &= c \int_{\mathbb{R}^m} (1+|\xi|)^2 \widehat{F_x}(\xi) \lim_{\varepsilon \to 0^+} T_{\xi,\varepsilon} f(x)\, d\xi. \end{aligned} \tag{2.1.53}$$

This proves the claim in (2.1.32) and finishes the proof of the theorem. □

2.2 The local compactness of the remainder

Let $\varphi : \mathbb{R}^n \to \mathbb{R}^{n+\ell}$ be a Lipschitz map, of "graph" type, i.e., assume that

$$\varphi(x) = \big(x, \varphi_0(x)\big), \qquad \forall\, x \in \mathbb{R}^n, \tag{2.2.1}$$

for some

$$\varphi_0 : \mathbb{R}^n \longrightarrow \mathbb{R}^\ell \ \text{ Lipschitz}. \tag{2.2.2}$$

Note that this implies

$$|\varphi(x) - \varphi(y)| \geq |x-y|, \qquad \forall\, x, y \in \mathbb{R}^n. \tag{2.2.3}$$

Let

$$\begin{aligned} &k : \mathbb{R}^{n+\ell} \setminus \{0\} \to \mathbb{R} \ \text{ be a smooth function,} \\ &\text{positive homogeneous of degree } -n, \text{ and} \\ &\text{satisfying } k(-w) = -k(w) \text{ for all } w \in \mathbb{R}^{n+\ell} \setminus \{0\}. \end{aligned} \tag{2.2.4}$$

Then

$$\begin{aligned} Kf(x) &:= \mathrm{PV} \int_{\mathbb{R}^n} k\big(\varphi(x) - \varphi(y)\big) f(y)\, dy \\ &= \mathrm{PV} \int_{\mathbb{R}^n} |x-y|^{-n} k\Big(\frac{\varphi(x) - \varphi(y)}{|x-y|}\Big) f(y)\, dy, \qquad x \in \mathbb{R}^n, \end{aligned} \tag{2.2.5}$$

defines a bounded operator on $L^p(\mathbb{R}^n)$, for each $p \in (1, \infty)$. We aim to establish a finer structure when $\varphi \in \mathscr{C}^1(\mathbb{R}^n)$ or, more generally, when the Jacobian $D\varphi$ of φ satisfies

$$D\varphi \in L^\infty(\mathbb{R}^n) \cap \mathrm{vmo}(\mathbb{R}^n). \tag{2.2.6}$$

Namely, we set

$$R := K - K_0, \tag{2.2.7}$$

with

$$K_0 f(x) := \mathrm{PV}\int_{\mathbb{R}^n} k\big(D\varphi(x)(x-y)\big) f(y)\,dy, \qquad x\in\mathbb{R}^n. \tag{2.2.8}$$

Note that (2.2.3) implies $|D\varphi(x)z| \ge |z|$ for all $z\in\mathbb{R}^n$. We have

$$\begin{gathered}\varphi\in\mathscr{C}^1(\mathbb{R}^n) \Longrightarrow K_0\in \mathrm{OP}\mathscr{C}^0 S^0_{\mathrm{cl}},\\ D\varphi\in L^\infty(\mathbb{R}^n)\cap \mathrm{vmo}(\mathbb{R}^n) \Longrightarrow K_0\in \mathrm{OP}(L^\infty\cap\mathrm{vmo})S^0_{\mathrm{cl}}.\end{gathered} \tag{2.2.9}$$

The latter class is studied in [36, Chapter 1, § 11] and, for reader's convenience, useful background material on this topic is presented in §A.3. See Theorem 2.6 for a derivation of the second part of (2.2.9), in a more general setting. As for the "remainder" R in (2.2.7) we have

$$Rf(x) = \mathrm{PV}\int_{\mathbb{R}^n} r(x,y) f(y)\,dy, \qquad x\in\mathbb{R}^n, \tag{2.2.10}$$

where

$$r(x,y) := k\big(\varphi(x)-\varphi(y)\big) - k\big(D\varphi(x)(x-y)\big) = \int_0^1 r_\tau(x,y)\,d\tau, \tag{2.2.11}$$

with

$$\begin{gathered} r_\tau(x,y) := (\nabla k)\big(\varphi(x)-\varphi(y)+\tau\Gamma(x,y)\big)\cdot\Gamma(x,y),\\ \Gamma(x,y) := \varphi(x)-\varphi(y) - D\varphi(x)(x-y).\end{gathered} \tag{2.2.12}$$

The following is our first major result.

Theorem 2.4. *Let φ be as in* (2.2.1)-(2.2.2)*, suppose k is as in* (2.2.4)*, and define R as in* (2.2.7)*, where K, K_0 are as in* (2.2.5) *and* (2.2.8)*, respectively. Finally, assume that* (2.2.6) *holds. Then for each ball $B\subset\mathbb{R}^n$ and $p\in(1,\infty)$, the operator*

$$R : L^p(B) \longrightarrow L^p(B) \ \ \text{is compact.} \tag{2.2.13}$$

In the case when $\varphi\in\mathscr{C}^1(\mathbb{R}^n)$ and $D\varphi$ has a modulus of continuity satisfying a Dini condition, the compactness result (2.2.13) is straightforward. See [36, Chapter 3, § 4].

Proof of Theorem 2.4. Note that

$$R = \int_0^1 R_\tau\,d\tau, \tag{2.2.14}$$

interpreted as a Bochner integral, with

$$R_\tau f(x) := \mathrm{PV}\int_{\mathbb{R}^n} r_\tau(x,y) f(y)\,dy, \qquad x\in\mathbb{R}^n, \tag{2.2.15}$$

and the integral kernel $r_\tau(x,y)$ as in (2.2.12). Given this, and bearing in mind that the collection of compact operators on $L^p(B)$ is a closed linear subspace of $\mathcal{L}\big(L^p(B), L^p(B)\big)$, it suffices to show that each operator R_τ has the compactness property (2.2.13).

With this goal in mind, for each $\tau\in[0,1]$ observe that the operator R_τ has the form

$$R_\tau f(x) = \mathrm{PV}\int_{\mathbb{R}^n} |x-y|^{-(n+1)} F\Big(\frac{D\varphi(x)(x-y)+\tau\Gamma(x,y)}{|x-y|}\Big)\Gamma(x,y) f(y)\,dy, \tag{2.2.16}$$

with $\Gamma(x,y)$ as in (2.2.12) and $F := \nabla k$. Note that the argument of F in (2.2.23) is

$$D\varphi(x)(x-y) + \tau\Gamma(x,y) = \big(x-y, D\varphi_0(x)(x-y) + \tau\Gamma_0(x,y)\big), \tag{2.2.17}$$

with φ_0 as in (2.2.1)-(2.2.2) and $\Gamma_0(x,y)$ as in (2.2.12), but with φ replaced by φ_0. In particular, there exists a constant $C \in (1,\infty)$ such that

$$1 \le \frac{|D\varphi(x)(x-y) + \tau\Gamma(x,y)|}{|x-y|} \le C, \tag{2.2.18}$$

for all $x, y \in \mathbb{R}^n$ and all $\tau \in [0,1]$. As such, we can alter the function $F(w)$ at will off the set $\{w \in \mathbb{R}^{n+\ell} : 1 \le |w| \le C\}$, and arrange that

$$F \in \mathscr{C}_0^\infty(\mathbb{R}^{n+\ell}), \tag{2.2.19}$$

while keeping F even.

Moving on, observe that another way of looking at the argument of F in (2.2.23) is to write

$$\begin{aligned}
D\varphi(x)&(x-y) + \tau\Gamma(x,y) \\
&= \tau\big(\varphi(x) - \varphi(y)\big) + (1-\tau)D\varphi(x)(x-y) \\
&= \big[\tau\varphi(x) + (1-\tau)D\varphi(x)x\big] - \big[\tau\varphi(y) + (1-\tau)D\varphi(x)y\big] \\
&= A_\tau(x)\big(\psi(x) - \psi(y)\big),
\end{aligned} \tag{2.2.20}$$

with

$$A_\tau(x) := \begin{pmatrix}\tau I & (1-\tau)D\varphi(x)\end{pmatrix}, \tag{2.2.21}$$

and

$$\psi(x) := \begin{pmatrix}\varphi(x) \\ x\end{pmatrix}, \qquad \psi : \mathbb{R}^n \longrightarrow \mathbb{R}^{2n+\ell}. \tag{2.2.22}$$

The bottom line is that for each $\tau \in [0,1]$ we have

$$R_\tau f(x) = \mathrm{PV}\int_{\mathbb{R}^n} |x-y|^{-(n+1)} F\Big(A_\tau(x)\frac{\psi(x)-\psi(y)}{|x-y|}\Big)\Gamma(x,y)f(y)\,dy, \qquad x \in \mathbb{R}^n, \tag{2.2.23}$$

where A_τ, ψ are as in (2.2.21)-(2.2.22) and we can assume F is even and satisfies (2.2.19). Granted this, Theorem 2.2 applies and yields that each R_τ has the compactness property (2.2.13). $\square$

2.3 A variable coefficient version of the local compactness theorem

Here the goal is to work out a variable coefficient version of Theorem 2.4, by treating the following class of operators. Let $k \in \mathscr{C}^\infty\big(\mathbb{R}^{n+\ell} \times (\mathbb{R}^{n+\ell} \setminus 0)\big)$. Suppose $k(w,z)$ is odd in z and homogeneous of degree $-n$ in z. In addition, assume bounds

$$|D_w^\alpha D_z^\beta k(w,z)| \le C_{\alpha\beta}|z|^{-n-|\beta|} \tag{2.3.1}$$

We take $\varphi : \mathbb{R}^n \to \mathbb{R}^{n+\ell}$ as in (2.2.1)-(2.2.2), (2.2.6), and consider

$$Kf(x) := \mathrm{PV}\int_{\mathbb{R}^n} k\big(\varphi(x), \varphi(x) - \varphi(y)\big)f(y)\,dy, \qquad x \in \mathbb{R}^n. \tag{2.3.2}$$

To analyze this type of singular integral operator with variable coefficient kernel, it is convenient to expand

$$k(w,z)=\sum_j a_j(w)\Omega_{n,j}(z), \tag{2.3.3}$$

where, starting with orthonormal, real-valued, spherical harmonics Ω_j on S^{n-1}, we have set

$$\Omega_{n,j}(z):=\Omega_j\Big(\frac{z}{|z|}\Big)\,|z|^{-n}, \qquad z\in\mathbb{R}^n\setminus\{0\}, \tag{2.3.4}$$

and where the coefficient functions a_j are given by

$$a_j(w):=\int\limits_{S^{n-1}} k(w,z)\Omega_j(z)\,dz. \tag{2.3.5}$$

We can arrange that all the functions $\Omega_{n,j}(z)$ in (2.3.3) are odd. There is a polynomial bound in j on the $\mathscr{C}^m$ norm of $\Omega_{n,j}\big|_{S^{n-1}}$, for each $m\in\mathbb{N}$, and the coefficients a_j are rapidly decreasing in $\mathscr{C}^m$ norm, for each $m\in\mathbb{N}$. We have

$$K=\sum_j K_j, \tag{2.3.6}$$

where, for each j,

$$K_jf(x):=a_j(\varphi(x))\,\mathrm{PV}\int\limits_{\mathbb{R}^n}\Omega_{n,j}\big(\varphi(x)-\varphi(y)\big)f(y)\,dy, \qquad x\in\mathbb{R}^n. \tag{2.3.7}$$

The series (2.3.6) converges rapidly in L^p-operator norm, for each $p\in(1,\infty)$.

Let us compare K with $K^{\#}$, defined as

$$K^{\#}f(x):=\mathrm{PV}\int\limits_{\mathbb{R}^n}k\big(\varphi(x),D\varphi(x)(x-y)\big)f(y)\,dy, \qquad x\in\mathbb{R}^n. \tag{2.3.8}$$

This time (2.3.3) yields

$$K^{\#}=\sum_j K_j^{\#}, \tag{2.3.9}$$

with $K_j^{\#}$ given by

$$K_j^{\#}f(x):=a_j(\varphi(x))\,\mathrm{PV}\int\limits_{\mathbb{R}^n}\Omega_{n,j}\big(D\varphi(x)(x-y)\big)f(y)\,dy, \qquad x\in\mathbb{R}^n. \tag{2.3.10}$$

We claim that the series (2.3.9) is rapidly convergent in L^p-operator norm for each $p\in(1,\infty)$. Indeed, Theorem 2.4 directly implies that, for each j,

$$K_j-K_j^{\#}\ \text{ is compact on }\ L^p(B), \tag{2.3.11}$$

for each ball $B\subset\mathbb{R}^n$, and each $p\in(1,\infty)$. The operator norm convergence of (2.3.6) and (2.3.9) then yield the following variable coefficient counterpart to Theorem 2.4.

Theorem 2.5. *Given K as in (2.3.3) and $K^{\#}$ as in (2.3.8),*

$$K-K^{\#}\ \textit{ is compact on }\ L^p(B). \tag{2.3.12}$$

Moving on, we propose to further analyze (2.3.8) and show that (again, see the discussion in §A.3 for relevant definitions)

$$K^{\#} \in \mathrm{OP}(L^\infty \cap \mathrm{vmo})S^0_{\mathrm{cl}}. \tag{2.3.13}$$

To this end, it is convenient to write

$$k(w, Az) = \sum_j b_j(w, A)\Omega_{n,j}(z), \tag{2.3.14}$$

for $A : \mathbb{R}^n \to \mathbb{R}^{n+\ell}$ of the form

$$A = \begin{pmatrix} I \\ A_0 \end{pmatrix}, \tag{2.3.15}$$

with

$$b_j(w, A) := \int_{S^{n-1}} k(w, Az)\Omega_j(z)\, d\sigma(z). \tag{2.3.16}$$

Again, we can arrange that only odd functions $\Omega_{n,j}$ arise in (2.3.14). As A_0 runs over a compact subset of $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^\ell)$, the space of linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^\ell$. we have uniform rapid decay of $b_j(w, A)$ and each of its derivatives. We have the following conclusion.

Theorem 2.6. *The operator $K^{\#}$ defined by* (2.3.8) *satisfies*

$$K^{\#} f(x) = \sum_j b_j\big(\varphi(x), D\varphi(x)\big)\, \mathrm{PV} \int_{\mathbb{R}^n} \Omega_{n,j}(x - y) f(y)\, dy, \quad x \in \mathbb{R}^n, \tag{2.3.17}$$

hence

$$K^{\#} f(x) = p(x, D) f(x), \qquad x \in \mathbb{R}^n, \tag{2.3.18}$$

with

$$p(x, \xi) := \sum_j b_j\big(\varphi(x), D\varphi(x)\big)\widehat{\Omega}_{n,j}(\xi). \tag{2.3.19}$$

Consequently,

$$p \in (L^\infty \cap \mathrm{vmo})S^0_{\mathrm{cl}} \tag{2.3.20}$$

and (2.3.13) *follows.*

3 Symbol calculus

Our goals here are to associate symbols to the operators studied in Section 2 and to examine how these operators behave under coordinate changes.

3.1 Principal symbols

Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded Lip$\cap$ vmo$_1$ domain, so $\partial\Omega$ is locally a graph of the form (2.2.1)-(2.2.2), (2.2.6), with $\ell = 1$. Let $\partial^*\Omega$ denote the subset of $\partial\Omega$ of the form $\varphi(x)$ such that x is an L^p-Lebesgue point of $D\varphi$ with $p > n$ (so in particular φ is differentiable at x). Then we set

$$T_{\varphi(x)}\partial^*\Omega := \big\{D\varphi(x)v : v \in \mathbb{R}^n\big\}, \qquad \text{whenever } \varphi(x) \in \partial^*\Omega. \tag{3.1.1}$$

In this fashion, we can talk about the tangent bundle and cotangent bundle over $\partial^*\Omega$,

$$T\partial^*\Omega, \qquad T^*\partial^*\Omega, \tag{3.1.2}$$

in the latter case, the fiber $T^*_{\varphi(x)}\partial^*\Omega$ being the dual space to (3.1.1).

Let $k(w,z)$ be smooth on $\mathbb{R}^{n+1}\times(\mathbb{R}^{n+1}\setminus 0)$, odd in z, and homogeneous of degree $-n$ in z. Consider

$$Kf(x) := \mathrm{PV}\int\limits_{\partial\Omega} k(x,x-y)f(y)\,d\sigma(y), \quad K : L^p(\partial\Omega)\to L^p(\partial\Omega),\ p\in(1,\infty). \tag{3.1.3}$$

In a local coordinate system described above,

$$Kf(x) = \mathrm{PV}\int\limits_{\mathcal{O}} k\bigl(\varphi(x),\varphi(x)-\varphi(y)\bigr)f(y)\Sigma(y)\,dy, \tag{3.1.4}$$

with $\mathcal{O}\subset\mathbb{R}^n$ and $d\sigma(y)=\Sigma(y)\,dy$. Note that $\Sigma\in L^\infty\cap\mathrm{vmo}$. As we have seen in §2.3,

$$K = p(x,D) \ \text{ mod compact}, \tag{3.1.5}$$

with $p(x,\xi)\in(L^\infty\cap\mathrm{vmo})S^0_{\mathrm{cl}}$, odd and homogeneous of degree 0 in ξ. We want to associate to K a principal symbol σ_K, defined on $T^*\partial^*\Omega$. We propose

$$\sigma_K(\varphi(x),\xi) := p\bigl(x, D\varphi(x)^\top\xi\bigr), \tag{3.1.6}$$

for $x\in\mathcal{O}$, $\varphi(x)\in\partial^*\Omega$, with p as in (3.1.5). If $\partial\Omega$ is smooth, this coincides with the classical transformation formula for the symbol of a pseudodifferential operator. Now $K=K^\#$ mod compact, with $K^\#$ given by (2.3.8), with a factor of $\Sigma(y)$ thrown in. This factor can be changed to $\Sigma(x)$, mod compact, so we can take

$$p(x,D)f(x) = \mathrm{PV}\int k(\varphi(x), D\varphi(x)(x-y))\Sigma(x)f(y)\,dy. \tag{3.1.7}$$

The standard formula connecting a pseudodifferential operator and its symbol yields

$$p(x,\zeta) = \int_{\mathbb{R}^n} k\bigl(\varphi(x), D\varphi(x)z\bigr)e^{-iz\cdot\zeta}\Sigma(x)\,dz, \tag{3.1.8}$$

so (compare (3.2.22)–(3.2.23))

$$\begin{aligned} p\bigl(x, D\varphi(x)^\top\xi\bigr) &= \int_{\mathbb{R}^n} k\bigl(\varphi(x), D\varphi(x)z\bigr)e^{-iD\varphi(x)z\cdot\xi}\Sigma(x)\,dz \\ &= \int\limits_{T_{\varphi(x)}\partial^*\Omega} k\bigl(\varphi(x), z^0\bigr)e^{-iz^0\cdot\xi}\,dz^0, \end{aligned} \tag{3.1.9}$$

since the area element of $\partial\Omega$ at $w\in\partial^*\Omega$ coincides with that of $T_w\partial^*\Omega$. Hence

$$\sigma_K(w,\xi) = \int\limits_{T_w\partial^*\Omega} k(w,z^0)e^{-iz^0\cdot\xi}\,dz^0, \qquad w\in\partial^*\Omega. \tag{3.1.10}$$

This last formula is independent of the choice of local coordinates on $\partial\Omega$. In case $\partial\Omega$ is smooth, (3.1.10) is the standard formula. We note that $T^*_w\partial^*\Omega$ inherits an inner product, hence a volume form, as a linear subspace of $\mathbb{R}^{n+1}$, and $dz^0 = \Sigma(x)\, dz$, when $w = \varphi(x)$.

Suppose K is an $\ell \times \ell$ system of singular integral operators. We say K is *elliptic on* $\partial\Omega$ if there exists a constant $C > 0$ such that

$$\|\sigma_K(w,\xi)v\| \geq C\|v\|, \quad \forall\, v \in \mathbb{C}^\ell, \ \text{ for } \ \sigma\text{-a.e. } \ w \in \partial^*\Omega. \tag{3.1.11}$$

In such a case, by (3.1.6), the operator $p(x,D) \in \mathrm{OP}(L^\infty \cap \mathrm{vmo})S^0_{\mathrm{cl}}$ associated to K in a local graph coordinate system, is elliptic, i.e., its symbol $p(x,\xi)$ satisfies the analogue of (3.1.11). We can hence prove the following.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded* $\mathrm{Lip} \cap \mathrm{vmo}_1$ *domain. If K is an $\ell \times \ell$ elliptic system of singular integral operators of the form* (3.1.3) *and satisfies the ellipticity condition* (3.1.11)*, then*

$$K : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \ \ \textit{is Fredholm}, \ \forall\, p \in (1,\infty). \tag{3.1.12}$$

Moreover, the index of K in (3.1.12) *is independent of $p \in (1,\infty)$, and the following regularity result holds:*

$$\textit{if } 1 < p < q < \infty \ \textit{ and } f \in L^p(\partial\Omega),\ Kf \in L^q(\partial\Omega) \Longrightarrow f \in L^q(\partial\Omega). \tag{3.1.13}$$

Proof. Let $\{\mathcal{O}_j\}_j$ be an open cover of $\partial\Omega$ on which we have graph coordinates. (We also identify each $\mathcal{O}_j$ with an open subset of $\mathbb{R}^n$.) Let $\{\psi_j\}_j$ be a Lipschitz partition of unity on $\partial\Omega$ subordinate to this cover. Let $\varphi_j \in \mathrm{Lip}(\mathcal{O}_j)$ have compact support and satisfy $\varphi_j \equiv 1$ on a neighborhood of $\mathrm{supp}\, \psi_j$. Then

$$K = \sum_j KM_{\psi_j} = \sum_j M_{\varphi_j}KM_{\psi_j}, \quad \text{mod compacts}, \tag{3.1.14}$$

where, generally speaking, $M_\psi f := \psi f$. Now we have (cf. (3.1.5))

$$M_{\varphi_j}KM_{\psi_j} = M_{\varphi_j}p_j(x,D)M_{\psi_j}, \quad \text{mod compacts}, \tag{3.1.15}$$

with $p_j(x,D) \in \mathrm{OP}(L^\infty \cap \mathrm{vmo})S^0_{\mathrm{cl}}$, elliptic. We have a parametrix $e_j(x,D) \in \mathrm{OP}(L^\infty \cap \mathrm{vmo})S^0_{\mathrm{cl}}$, satisfying

$$M_{\varphi_i}e_i(x,D)M_{\psi_i}M_{\varphi_j}KM_{\psi_j} = M_{\psi_i\psi_j}, \quad \text{mod compacts}. \tag{3.1.16}$$

Set

$$E := \sum_i M_{\varphi_i}e_i(x,D)M_{\psi_i}. \tag{3.1.17}$$

Then

$$\begin{aligned} EK &= \sum_{i,j} M_{\varphi_i}e_i(x,D)M_{\psi_i}M_{\varphi_j}KM_{\psi_j}, \quad \text{mod compacts} \\ &= \sum_{i,j} M_{\psi_i\psi_j}, \quad \text{mod compacts} \\ &= I, \quad \text{mod compacts}. \end{aligned} \tag{3.1.18}$$

Similarly, E is a right Fredholm inverse of K, and we have (3.1.12).

Going further, for each $p \in (1, \infty)$ let $\iota_p(K)$ denote the index of K on $L^p(\partial\Omega)$. Then, if $1 < p < q < \infty$ and $\mathcal{N}_p$ denotes the null space of K on $L^p(\partial\Omega)$, $\mathcal{N}'_p$ that of K^* on $L^{p'}(\partial\Omega)$, we have

$$\mathcal{N}_q \subset \mathcal{N}_p, \quad \mathcal{N}'_p \subset \mathcal{N}'_q, \quad \text{hence} \;\; \iota_p(K) \geq \iota_q(K). \tag{3.1.19}$$

The same type of argument applies to E, yielding $\iota_p(E) \geq \iota_q(E)$, hence

$$\iota_p(K) = \iota_q(K), \tag{3.1.20}$$

as wanted. Note that, together with (3.1.19), this actually forces

$$\mathcal{N}_q = \mathcal{N}_p \quad \text{and} \quad \mathcal{N}'_p = \mathcal{N}'_q. \tag{3.1.21}$$

Finally, for (3.1.13), if $f \in L^p(\partial\Omega)$ and $Kf = g \in L^q(\partial\Omega)$, then g annihilates $\mathcal{N}'_p$. Since $\mathcal{N}'_q = \mathcal{N}'_p$, g annihilates $\mathcal{N}'_q$, so $g = K\tilde{f}$ for some $\tilde{f} \in L^q(\partial\Omega)$. Given $p < q$, we have $f - \tilde{f} \in \mathcal{N}_p$. Hence $f - \tilde{f} \in \mathcal{N}_q$, and thus $f \in L^q(\partial\Omega)$, as asserted in (3.1.13). □

3.2 Transformations of operators under coordinate changes

Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a bi-Lipschitz map, so there exist $a, b \in (0, \infty)$ such that

$$a|x - y| \leq |\varphi(x) - \varphi(y)| \leq b|x - y|, \qquad \forall\, x, y \in \mathbb{R}^n. \tag{3.2.1}$$

In addition, we assume

$$D\varphi \in \mathrm{vmo}(\mathbb{R}^n). \tag{3.2.2}$$

Given

$$k \in \mathscr{C}^\infty(\mathbb{R}^n \setminus 0), \;\; \text{homogeneous of degree } \; -n, \quad k(-z) = -k(z), \tag{3.2.3}$$

we set

$$Kf(x) := \mathrm{PV} \int_{\mathbb{R}^n} k(x - y) f(y)\, dy, \qquad x \in \mathbb{R}^n. \tag{3.2.4}$$

Let us also set

$$K_\varphi f(x) := \mathrm{PV} \int_{\mathbb{R}^n} k\big(\varphi(x) - \varphi(y)\big) f(y)\, dy, \qquad x \in \mathbb{R}^n. \tag{3.2.5}$$

As in the past, we let M_χ denote the operator of pointwise multiplication by χ.

Definition 3.1. *Say that $\varphi \in \mathfrak{T}(\mathbb{R}^n)$ provided that* (3.2.1)–(3.2.2) *hold and, in addition, whenever* (3.2.3) *holds, then the singular integral operator K_φ associated with φ as in* (3.2.5) *may be decomposed*

$$K_\varphi f(x) = \mathrm{PV} \int_{\mathbb{R}^n} k\big(D\varphi(x)(x - y)\big) f(y)\, dy + R_\varphi f(x), \qquad x \in \mathbb{R}^n, \tag{3.2.6}$$

for a remainder with the property that for each cut-off function $\chi \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ one has

$$M_\chi R_\varphi M_\chi : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n) \;\; \textit{compact}, \;\; \forall\, p \in (1, \infty). \tag{3.2.7}$$

By Theorem 2.6, the principal value integral on the right-hand side of (3.2.6) defines an operator

$$\widetilde{K}_\varphi \in \mathrm{OP}(L^\infty \cap \mathrm{vmo}) S^0_{\mathrm{cl}}, \tag{3.2.8}$$

which is bounded on $L^p(\mathbb{R}^n)$ for each $p \in (1, \infty)$.

The following is a variant of Theorem 2.4, proven by the same sort of arguments.

Theorem 3.2. *Assume φ satisfies* (3.2.1)–(3.2.2). *Assume also that there exists $\kappa > 0$ such that, for all $\tau \in [0,1]$,*

$$\Big|\tau[\varphi(x)-\varphi(y)]+(1-\tau)D\varphi(x)(x-y)\Big| \geq \kappa|x-y|, \qquad \forall\, x,y \in \mathbb{R}^n. \tag{3.2.9}$$

Then $\varphi \in \mathfrak{T}(\mathbb{R}^n)$.

In fact, given a function $\chi \in \mathscr{C}_0^\infty(\mathbb{R}^n)$, one has (3.2.7) *provided the estimate in* (3.2.9) *holds for all points $x, y \in \text{supp}\chi$.*

Note the similarity of (3.2.9) and (2.2.18). In this connection, if $\Sigma \subset \mathbb{R}^{n+\ell}$ is an n-dimensional graph over $\mathbb{R}^n$, as introduced in §2.2, and if it is also represented as a graph over a nearby n-dimensional linear space V, then one gets a bi-Lipschitz map from $\mathbb{R}^n$ to $V \equiv \mathbb{R}^n$, satisfying (3.2.9). In such a way, one can represent Σ as a $\text{Lip} \cap \text{vmo}_1$ *manifold*, whose transition maps satisfy the conditions of Theorem 3.2. See the next section for more on this.

We proceed to a variable coefficient version of (3.2.3)–(3.2.7). Take k measurable on $\mathbb{R}^n \times \mathbb{R}^n$, satisfying

$$k(x,z) \text{ homogeneous of degree } -n \text{ in } z, \quad k(x,-z) = -k(x,z). \tag{3.2.10}$$

Assume $k(x,z)$ is smooth in $z \neq 0$, and that for each multiindex α there exists a finite constant $C_\alpha > 0$ such that

$$\|\partial_z^\alpha k(\cdot,z)\|_{L^\infty \cap \text{vmo}} \leq C_\alpha |z|^{-n-|\alpha|}, \tag{3.2.11}$$

where, for $f \in L^\infty(\mathbb{R}^n)$,

$$\|f\|_{L^\infty\cap\text{vmo}} := \begin{cases} \|f\|_{L^\infty} & \text{if } f \in \text{vmo}, \\ \infty & \text{if } f \notin \text{vmo}. \end{cases} \tag{3.2.12}$$

Then we can write

$$k(x,z) = \sum_{j\geq 0} k_j(x)|z|^{-n}\Omega_j\Big(\frac{z}{|z|}\Big), \tag{3.2.13}$$

where $\{\Omega_j\}_j$ is an orthonormal set of spherical harmonics on S^{n-1}, all odd, and for each $j \in \mathbb{N}$ we have

$$\|k_j\|_{L^\infty\cap\text{vmo}} \leq C_N \langle j\rangle^{-N}, \quad \text{for every } N \in \mathbb{N}. \tag{3.2.14}$$

In place of (3.2.4)–(3.2.6), we take

$$Kf(x) := \text{PV}\int_{\mathbb{R}^n} k(x,x-y)f(y)\,dy, \qquad x \in \mathbb{R}^n, \tag{3.2.15}$$

$$K_\varphi f(x) := \text{PV}\int_{\mathbb{R}^n} k\big(\varphi(x),\varphi(x)-\varphi(y)\big)f(y)\,dy, \qquad x \in \mathbb{R}^n, \tag{3.2.16}$$

and write

$$K_\varphi f(x) = \text{PV}\int_{\mathbb{R}^n} k\big(\varphi(x), D\varphi(x)(x-y)\big)f(y)\,dy + R_\varphi f(x), \qquad x \in \mathbb{R}^n. \tag{3.2.17}$$

Using (3.2.13)–(3.2.14), we can write these as rapidly convergent series, and deduce that

$$\varphi \in \mathfrak{T}(\mathbb{R}^n) \Longrightarrow M_\chi R_\varphi M_\chi : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n) \text{ compact}, \forall\, p \in (1,\infty), \tag{3.2.18}$$

whenever $\chi \in \mathscr{C}_0^\infty(\mathbb{R}^n)$. Implementing this for (3.2.16) involves using the following result.

Lemma 3.3. *The function spaces* $\mathrm{bmo}(\mathbb{R}^n)$ *and* $\mathrm{vmo}(\mathbb{R}^n)$ *are invariant under* $u \mapsto u \circ \varphi$*, provided* $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ *is a bi-Lipschitz map.*

Proof. This has the same proof as Proposition A.15 (cf. also [37, Proposition 3.3] and [1, Theorem 2, p. 516]). □

As in (3.2.8), the integral on the right side of (3.2.17) defines an operator

$$\widetilde{K}_\varphi \in \mathrm{OP}(L^\infty \cap \mathrm{vmo})S^0_{\mathrm{cl}}. \tag{3.2.19}$$

We use these results to analyze how an operator $P = p(x, D) \in \mathrm{OP}(L^\infty \cap \mathrm{vmo})S^0_{\mathrm{cl}}$ transforms under a map $\varphi \in \mathfrak{T}(\mathbb{R}^n)$. In more detail, given $P : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, set

$$P_\varphi g(x) := Pf(\varphi(x)), \qquad f \in L^p(\mathbb{R}^n), \quad g(x) = f(\varphi(x)). \tag{3.2.20}$$

Our hypothesis (3.2.1) implies $\|g\|_{L^p} \approx \|f\|_{L^p}$, so $P_\varphi : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$. We claim that $P_\varphi \in \mathrm{OP}(L^\infty \cap \mathrm{vmo})S^0_{\mathrm{cl}}$, at least modulo an operator with the compactness property (3.2.18). Furthermore, we obtain a formula for its principal symbol.

We take $p(x, \xi)$ to be homogeneous of degree 0 in ξ. To start, we assume

$$p(x, \xi) = -p(x, -\xi). \tag{3.2.21}$$

Now

$$Pf(x) = \mathrm{PV}\int_{\mathbb{R}^n} k(x, x-y) f(y)\, dy, \qquad x \in \mathbb{R}^n, \tag{3.2.22}$$

with

$$k(x, z) = (2\pi)^{-n} \int_{\mathbb{R}^n} p(x, \xi) e^{iz\cdot\xi}\, d\xi, \tag{3.2.23}$$

so

$$p(x, \xi) = \int_{\mathbb{R}^n} k(x, z) e^{-iz\cdot\xi}\, dz. \tag{3.2.24}$$

Note that

$$p(x, \xi) = \sum_{j\geq 0} p_j(x)\Omega_j\Big(\frac{\xi}{|\xi|}\Big), \tag{3.2.25}$$

with $\{\Omega_j\}_j$ as in (3.2.13) (again, all odd), and

$$\|p_j\|_{L^\infty\cap\mathrm{vmo}} \leq C_N \langle j\rangle^{-N}, \qquad \forall\, N \in \mathbb{N}. \tag{3.2.26}$$

It follows that $k(x, z)$ satisfies (3.2.10)–(3.2.11). Hence (3.2.15)–(3.2.19) apply. Consequently, with $J_\varphi(y) := |\det D\varphi(y)|$,

$$\begin{aligned}
P_\varphi g(x) &= Pf(\varphi(x)) && (3.2.27)\\
&= \mathrm{PV}\int_{\mathbb{R}^n} k\big(\varphi(x), \varphi(x) - y'\big) f(y')\, dy' && (3.2.28)\\
&= \mathrm{PV}\int_{\mathbb{R}^n} k\big(\varphi(x), \varphi(x) - \varphi(y)\big) f(\varphi(y)) J_\varphi(y)\, dy && (3.2.29)\\
&= \mathrm{PV}\int_{\mathbb{R}^n} k\big(\varphi(x), \varphi(x) - \varphi(y)\big) g(y) J_\varphi(y)\, dy. && (3.2.30)
\end{aligned}$$

Applying (3.2.15)–(3.2.18), we have

$$P_\varphi g(x) = \mathrm{PV} \int_{\mathbb{R}^n} k\bigl(\varphi(x), D\varphi(x)(x-y)\bigr) g(y) J_\varphi(y)\, dy + R_{1\varphi}, \tag{3.2.31}$$

where $R_{1\varphi}$ has the compactness property (3.2.18). Also, $J_\varphi \in L^\infty \cap \mathrm{vmo}$, so we can use the commutator estimate from [6] to replace $J_\varphi(y)$ by $J_\varphi(x)$ in (3.2.31), replacing $R_{1\varphi}$ by $R_{2\varphi}$, also satisfying (3.2.18). Consequently, we have

$$P_\varphi g(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} p_\varphi(x,\xi) e^{i(x-y)\cdot\xi} g(y)\, dy\, d\xi + R_{2\varphi}, \tag{3.2.32}$$

and

$$(2\pi)^{-n} \int_{\mathbb{R}^n} p_\varphi(x,\xi') e^{iz\cdot\xi'}\, d\xi' = J_\varphi(x) k\bigl(\varphi(x), D\varphi(x) z\bigr). \tag{3.2.33}$$

Taking $\xi' = D\varphi(x)^\top \xi$ gives $d\xi' = J_\varphi(x)\, d\xi$. We have cancellation of the factors $J_\varphi(x)$, hence

$$(2\pi)^{-n} \int_{\mathbb{R}^n} p_\varphi(x, D\varphi(x)^\top \xi) e^{i\nabla\varphi(x) z\cdot\xi}\, d\xi = k\bigl(\varphi(x), D\varphi(x) z\bigr). \tag{3.2.34}$$

Hence, with

$$\sigma(x,\xi) = p_\varphi(x, D\varphi(x)^\top \xi), \quad z' = D\varphi(x) z, \tag{3.2.35}$$

we have

$$(2\pi)^{-n} \int_{\mathbb{R}^n} \sigma(x,\xi) e^{iz'\cdot\xi}\, d\xi = k\bigl(\varphi(x), z'\bigr), \tag{3.2.36}$$

so

$$\sigma(x,\xi) = \int_{\mathbb{R}^n} k\bigl(\varphi(x), z'\bigr) e^{-iz'\cdot\xi}\, dz'. \tag{3.2.37}$$

Comparison with (3.2.24) yields the formula

$$p_\varphi\bigl(x, D\varphi(x)^\top \xi\bigr) = p(\varphi(x), \xi). \tag{3.2.38}$$

This has been derived for $p(x,\xi)$ satisfying (3.2.21). We now address the general case.

Theorem 3.4. *Assume* $\varphi \in \mathfrak{T}(\mathbb{R}^n)$. *Given* $P \in \mathrm{OP}(L^\infty \cap \mathrm{vmo}) S^0_{\mathrm{cl}}$, *with principal symbol* $p(x,\xi)$, *and* P_φ *defined by* (3.2.20), *one can decompose*

$$P_\varphi = p_\varphi(x,D) + R_\varphi, \tag{3.2.39}$$

with R_φ *satisfying* (3.2.18) *and* $p_\varphi(x,D) \in \mathrm{OP}(L^\infty \cap \mathrm{vmo}) S^0_{\mathrm{cl}}$ *satisfying* (3.2.38).

Proof. We have this when $p(x,\xi)$ satisfies (3.2.21). It remains to treat the case $p(x,-\xi) = p(x,\xi)$. For this, we can write

$$\begin{aligned} p(x,D) &= \sum_{j=1}^n q_j(x,D) s_j(x,D), \\ s_j(x,\xi) &= \frac{\xi_j}{|\xi|}, \quad q_j(x,\xi) = p(x,\xi) \frac{\xi_j}{|\xi|}. \end{aligned} \tag{3.2.40}$$

The previous analysis holds for the factors $q_j(x,D)$ and $s_j(x,D)$, and our conclusion follows by basic operator calculus for $\mathrm{OP}(L^\infty \cap \mathrm{vmo}) S^0_{\mathrm{cl}}$. □

3.3 Admissible coordinate changes on a $\mathrm{Lip} \cap \mathrm{vmo}_1$ surface

Let $\varphi : \mathbb{R}^n \to \mathbb{R}^{n+\ell}$ have the form $\varphi(x) = (x, \varphi_0(x))$, with $D\varphi_0(x) \in L^\infty(\mathbb{R}^n) \cap \mathrm{vmo}(\mathbb{R}^n)$, as in §2.2. Thus φ maps $\mathbb{R}^n$ onto an n-dimensional surface Σ. Let $V \subset \mathbb{R}^{n+\ell}$ be an n-dimensional linear space. If V is not too far from $\mathbb{R}^n$ (depending on $\|D\varphi_0\|_{L^\infty}$), then Σ is also a graph over V, and we have the coordinate change map

$$\psi : \mathbb{R}^n \longrightarrow V, \quad \psi(x) = Q\varphi(x), \tag{3.3.1}$$

where $Q : \mathbb{R}^{n+\ell} \to V$ is the orthogonal projection. Consequently,

$$\psi(x) = Q\begin{pmatrix} x \\ \varphi_0(x) \end{pmatrix}, \quad D\psi(x)v = Q\begin{pmatrix} v \\ D\varphi_0(x)v \end{pmatrix}. \tag{3.3.2}$$

Consequently,

$$\begin{aligned} &\tau[\psi(x) - \psi(y)] + (1-\tau)D\psi(x)(x-y) \\ &\qquad = Q\begin{pmatrix} x - y \\ \tau[\varphi_0(x) - \varphi_0(y)] + (1-\tau)D\varphi_0(x)(x-y) \end{pmatrix}. \end{aligned} \tag{3.3.3}$$

Recall that the condition for Theorem 3.2 to apply is that (3.3.3) has norm $\geq \kappa|x-y|$, for some $\kappa > 0$, for $x, y \in \mathbb{R}^n$, $\tau \in [0,1]$. We see that the norm of (3.3.3) is

$$\geq \|Q(x-y)\| - \gamma(x,y), \tag{3.3.4}$$

where, with Q_0 denoting the orthogonal projection of $\mathbb{R}^{n+\ell}$ onto $\mathbb{R}^n$,

$$\begin{aligned} \gamma(x,y) &= \|Q(I-Q_0)(\tau[\varphi_0(x) - \varphi_0(y)] + (1-\tau)D\varphi_0(x)(x-y))\| \\ &\leq \|D\varphi_0\|_{L^\infty} \|Q(I-Q_0)\| \cdot |x-y|. \end{aligned} \tag{3.3.5}$$

Since $Q(x-y) = (x-y) + (I-Q)Q_0(x-y)$, we deduce that the norm of (3.3.3) is

$$\geq \Big(1 - \|(I-Q)Q_0\| - \|(I-Q_0)Q\| \cdot \|D\varphi_0\|_{L^\infty}\Big)|x-y|. \tag{3.3.6}$$

Consequently, Theorem 3.2 applies as long as

$$\|(I-Q)Q_0\| + \|(I-Q_0)Q\| \cdot \|D\varphi_0\|_{L^\infty} < 1. \tag{3.3.7}$$

This in turn holds provided

$$\|Q - Q_0\| < (1 + \|D\varphi_0\|_{L^\infty})^{-1}. \tag{3.3.8}$$

We have the following conclusion.

Proposition 3.5. *Let $\psi : \mathbb{R}^n \to V$ be as constructed in* (3.3.1)*. Assume* (3.3.8) *holds, where Q and Q_0 are the orthogonal projections of $\mathbb{R}^{n+\ell}$ onto V and $\mathbb{R}^n$, respectively. Take a linear isomorphism $J : V \to \mathbb{R}^n$. Then $J \circ \psi$ belongs to $\mathfrak{T}(\mathbb{R}^n)$.*

3.4 Remark on double layer potentials

Assume that a kernel

$$
\begin{aligned}
&E : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R} \text{ be a smooth function,} \\
&\text{positive homogeneous of degree } -(n+1), \text{ and} \\
&\text{satisfying } E(-X) = E(X) \text{ for all } X \in \mathbb{R}^{n+1} \setminus \{0\}
\end{aligned}
\tag{3.4.1}
$$

has been given. Also, let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded $\mathrm{Lip} \cap \mathrm{vmo}_1$ domain, and consider the singular integral operator of the type

$$
Kf(X) := \mathrm{PV} \int_{\partial\Omega} \langle \nu(X), X - Y \rangle E(X - Y) f(Y)\, d\sigma(Y), \qquad X \in \partial\Omega, \tag{3.4.2}
$$

where ν and σ are, respectively, the outward unit normal and surface measure on $\partial\Omega$. To study this, focus on a local version of (3.4.2) of the following sort. Let

$$
\varphi_0 : \mathcal{O} \longrightarrow \mathbb{R} \ \text{ Lipschitz, with } \ \nabla\varphi_0 \in \mathrm{vmo} \tag{3.4.3}
$$

where $\mathcal{O} \subset \mathbb{R}^n$ is open, be such that its graph is contained in $\partial\Omega$, and define the Lipschitz map $\varphi : \mathcal{O} \to \mathbb{R}^{n+1}$ by setting

$$
\varphi(x) := \big(x, \varphi_0(x)\big), \qquad \forall\, x \in \mathcal{O}. \tag{3.4.4}
$$

Then in these local coordinates, K takes the form

$$
K_\varphi f(x) = \mathrm{PV} \int_{\mathcal{O}} \big\langle (\nabla\varphi_0(x), -1), \varphi(x) - \varphi(y) \big\rangle E\big(\varphi(x) - \varphi(y)\big) f(y)\, dy. \tag{3.4.5}
$$

Its "sharp" form, obtained by replacing $\varphi(x) - \varphi(y)$ with $D\varphi(x)(x - y)$ is then

$$
\begin{aligned}
K_\varphi^{\#} f(x) &:= \mathrm{PV} \int_{\mathcal{O}} \big\langle (\nabla\varphi_0(x), -1), D\varphi(x)(x - y) \big\rangle E\big(D\varphi(x)(x - y)\big) f(y)\, dy \\
&= \mathrm{PV} \int_{\mathcal{O}} \big\langle D\varphi(x)^\top (\nabla\varphi_0(x), -1), x - y \big\rangle E\big(D\varphi(x)(x - y)\big) f(y)\, dy \\
&= 0,
\end{aligned}
\tag{3.4.6}
$$

since

$$
D\varphi(x) = \begin{pmatrix} I_{n\times n} \\ \nabla\varphi_0(x) \end{pmatrix} \Rightarrow D\varphi(x)^\top (\nabla\varphi_0(x), -1) = \begin{pmatrix} I_{n\times n} & \nabla\varphi_0(x) \end{pmatrix} \begin{pmatrix} \nabla\varphi_0(x)^\top \\ -1 \end{pmatrix} = 0. \tag{3.4.7}
$$

In concert with our local compactness result, according to which $K_\varphi - K_\varphi^{\#}$ is compact on L^p for each $p \in (1, \infty)$, this ultimately gives that

$$
\begin{gathered}
\text{if } \Omega \subset \mathbb{R}^{n+1} \text{ is a bounded } \mathrm{Lip} \cap \mathrm{vmo}_1 \text{ domain and } E \text{ is as in (3.4.1)} \\
\text{then } K \text{ from (3.4.2) is compact on } L^p(\partial\Omega), \text{ for each } p \in (1, \infty).
\end{gathered}
\tag{3.4.8}
$$

Of course, the above result contains as a particular case the fact (which is a key result in the work of Fabes, Jodeit, and Riviére in [9]) that the principal-value harmonic double layer operator

$$
Kf(X) := \lim_{\varepsilon \to 0^+} \frac{1}{\omega_n} \int_{\substack{Y \in \partial\Omega \\ |X - Y| > \varepsilon}} \frac{\langle \nu(Y), Y - X \rangle}{|X - Y|^{n+1}} f(Y)\, d\sigma(Y), \qquad X \in \partial\Omega, \tag{3.4.9}
$$

is compact on $L^p(\partial\Omega)$, for each $p \in (1, \infty)$, if $\Omega \subset \mathbb{R}^{n+1}$ is a bounded $\mathscr{C}^1$ domain.

3.5 Cauchy integrals and their symbols

Given $\ell \in \mathbb{N}$, we let $M(\ell, \mathbb{C})$ denote the collection of $\ell \times \ell$ matrices with complex entries. Let $\mathcal{D}$ be a first order elliptic $\ell \times \ell$ system of differential operators on $\mathbb{R}^{n+1}$,

$$\mathcal{D}u(x) = \sum_j A_j \partial_j u, \quad A_j \in M(\ell, \mathbb{C}). \tag{3.5.1}$$

Thus $\sigma_{\mathcal{D}}(\zeta) = i \sum_j A_j \zeta_j$ is invertible for each nonzero $\zeta \in \mathbb{R}^{n+1}$, and $\mathcal{D}$ has a fundamental solution

$$k(z) = (2\pi)^{-(n+1)} \int\limits_{\mathbb{R}^{n+1}} E(\zeta) e^{iz\cdot\zeta} \, d\zeta, \quad E(\zeta) = \sigma_{\mathcal{D}}(\zeta)^{-1}, \tag{3.5.2}$$

odd and homogeneous of degree $-n$ in z. If $\Omega \subset \mathbb{R}^{n+1}$ is a bounded UR (acronym for uniformly rectifiable) domain, we can form

$$\mathcal{B}f(x) = \int\limits_{\partial\Omega} k(x-y) f(y) \, d\sigma(y), \qquad x \in \Omega, \tag{3.5.3}$$

with nontangential limits (cf. (4.1.3))

$$\Big(\mathcal{B}f\Big|_{\partial\Omega}^{\text{n.t.}}\Big)(z) := \lim_{\Gamma_\kappa(x) \ni z \to x} \mathcal{B}f(z) = \frac{1}{2i} \sigma_{\mathcal{D}}(\nu(x))^{-1} f(x) + Bf(x), \tag{3.5.4}$$

for σ-a.e. $x \in \partial\Omega$, where $\Gamma_\kappa(x) \subset \Omega$ is a region of nontangential approach to $x \in \partial\Omega$ (cf. (4.1.2)), and

$$Bf(x) := \mathrm{PV} \int\limits_{\partial\Omega} k(x-y) f(y) \, d\sigma(y), \quad x \in \partial\Omega. \tag{3.5.5}$$

One is hence motivated to consider the "Cauchy integral"

$$\mathcal{C}_{\mathcal{D}} f(x) = i \int\limits_{\partial\Omega} k(x-y) \sigma_{\mathcal{D}}(\nu(y)) f(y) \, d\sigma(y), \qquad x \in \Omega, \tag{3.5.6}$$

with nontangential limits

$$\mathcal{C}_{\mathcal{D}} f\Big|_{\partial\Omega}^{\text{n.t.}}(x) = \tfrac{1}{2} f(x) + C_{\mathcal{D}} f(x), \tag{3.5.7}$$

for σ-a.e. $x \in \partial\Omega$, where

$$C_{\mathcal{D}} f(x) := i\mathrm{PV} \int\limits_{\partial\Omega} k(x-y) \sigma_{\mathcal{D}}(\nu(y)) f(y) \, d\sigma(y), \quad x \in \partial\Omega. \tag{3.5.8}$$

As shown in [23], a reproducing formula yields

$$P_{\mathcal{D}} = \tfrac{1}{2} I + C_{\mathcal{D}} \Longrightarrow P_{\mathcal{D}}^2 = P_{\mathcal{D}}. \tag{3.5.9}$$

This is studied in [23] in the setting of UR domains (and also for variable coefficient situations, which for simplicity we do not take up here in detail). The operator $P_{\mathcal{D}}$ is a Calderón projector.

Here, we take Ω to be a $\mathrm{Lip} \cap \mathrm{vmo}_1$ domain and analyze the principal symbol of $P_{\mathcal{D}}$, as a projection-valued function on $T^*\partial^*\Omega \setminus 0$. To start, we recall from (3.1.10) that, for B in (3.5.5),

$$\sigma_B(w, \xi) = \int\limits_{T_w \partial^* \Omega} k(z^0) e^{-iz^0 \cdot \xi} \, dz^0, \qquad w \in \partial^* \Omega. \tag{3.5.10}$$

Plugging in (3.5.2) and using basic Fourier analysis, we obtain

$$\sigma_B(w, \xi) = \frac{1}{2\pi} \mathrm{PV} \int_{-\infty}^{\infty} E\big(\xi + s\nu(w)\big) \, ds. \tag{3.5.11}$$

We then have

$$\sigma_{C_{\mathcal{D}}}(w, \xi) = \frac{i}{2\pi} \mathrm{PV} \int_{-\infty}^{\infty} \sigma_{\mathcal{D}}\big(\xi + is\nu(w)\big)^{-1} \sigma_{\mathcal{D}}(\nu(w)) \, ds. \tag{3.5.12}$$

Now $\sigma_{\mathcal{D}}\big(\xi + s\nu(w)\big) = \sigma_{\mathcal{D}}(\xi) + s\sigma_{\mathcal{D}}(\nu(w))$, so

$$\sigma_{\mathcal{D}}\big(\xi + s\nu(w)\big)^{-1} \sigma_{\mathcal{D}}(\nu(w)) = \Big(M(w, \xi) + sI\Big)^{-1}, \tag{3.5.13}$$

with

$$M(w, \xi) = \sigma_{\mathcal{D}}(\nu(w))^{-1} \sigma_{\mathcal{D}}(\xi). \tag{3.5.14}$$

The invertibility of $\sigma_{\mathcal{D}}(\xi + s\nu(w))$ and of $\sigma_{\mathcal{D}}(\nu(w))$ imply that

$$\mathrm{Spec}\, M(w, \xi) \cap \mathbb{R} = \emptyset. \tag{3.5.15}$$

We have

$$\sigma_{C_{\mathcal{D}}}(w, \xi) = \frac{i}{2\pi} \mathrm{PV} \int_{-\infty}^{\infty} \Big(sI + M(w, \xi)\Big)^{-1} \, ds. \tag{3.5.16}$$

Lemma 3.6. *Assume $A \in M(\ell, \mathbb{C})$ and $\mathrm{Spec}\, A \cap \mathbb{R} = \emptyset$. Then*

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} (s - A)^{-1} e^{i\varepsilon s} \, ds = \begin{cases} e^{i\varepsilon A} P_+(A) & \text{if } \varepsilon > 0, \\ -e^{i\varepsilon A} P_-(A) & \text{if } \varepsilon < 0, \end{cases} \tag{3.5.17}$$

where $P_+(A)$ is the projection of $\mathbb{C}^\ell$ onto the linear span of the generalized eigenvectors of A associated to eigenvalues in $\mathrm{Spec}A$ *with positive imaginary part, annihilating those associated to eigenvectors with negative imaginary part, and $P_-(A) = I - P_+(A)$. Hence*

$$\frac{1}{2\pi i} \mathrm{PV} \int_{-\infty}^{\infty} (s - A)^{-1} \, ds = P_+(A) - \tfrac{1}{2} I. \tag{3.5.18}$$

Proof. In case $\varepsilon > 0$, the left-hand side of (3.5.17) is equal to

$$\lim_{R \to \infty} \frac{1}{2\pi i} \int\limits_{\partial D_R^+} (s - A)^{-1} \, ds, \tag{3.5.19}$$

where $D_R := \{s \in \mathbb{C} : |s| < R\}$ and $D_R^+ := D_R \cap \{s \in \mathbb{C} : \Im s > 0\}$. This path integral stabilizes when $R > \|A\|$, and the desired conclusion in this case follows from the Riesz functional calculus. The treatment of the case when $\varepsilon < 0$ is similar. Then (3.5.18) follows readily from (3.5.17). $\square$

We apply Lemma 3.6 to (3.5.16) with $A := -M(w,\xi)$. Making use of the identity $P_+(-M) = P_-(M)$, we have the following conclusion.

Proposition 3.7. *The operator $C_{\mathcal{D}}$ and the associated Calderón projector, derived from the Cauchy integral* (3.5.6) *via* (3.5.7)–(3.5.9)*, have symbols given by*

$$\begin{aligned}\sigma_{C_{\mathcal{D}}}(w,\xi) &= -\Big(P_-(M(w,\xi)) - \tfrac{1}{2}I\Big)\\ &= \tfrac{1}{2}I - P_-(M(w,\xi)),\end{aligned} \tag{3.5.20}$$

and

$$\sigma_{P_{\mathcal{D}}}(w,\xi) = P_+(M(w,\xi)), \tag{3.5.21}$$

respectively, with $M(w,\xi)$ as in (3.5.14) *and $P_+(A)$ as described in Lemma 3.6.*

Remark 3.1. *Extensions of the results in this section to variable coefficient operators (acting between vector bundles) and to domains on manifolds can be worked out using the formalism developed in [23], [24].*

4 Applications to elliptic boundary problems

Here we apply the results of Sections 2–3 to several classes of elliptic boundary problems, including the Dirichlet problem for general strongly elliptic, second order systems and general regular boundary problems for first order elliptic systems of differential operators.

4.1 Single layers and boundary problems for elliptic systems

Let M be a smooth, compact, $(n+1)$-dimensional manifold, equipped with a Riemannian metric tensor

$$g = \sum_{j,k} g_{jk}\, dx_j \otimes dx_k, \qquad \text{with} \;\; g_{jk} \in \mathscr{C}^2. \tag{4.1.1}$$

Also, consider a $\mathrm{Lip} \cap \mathrm{vmo}_1$ domain $\Omega \subset M$ (cf. the discussion in the last part of §A.5). Having some fixed $\kappa \in (0,\infty)$, for each $x \in \partial\Omega$, define the nontangential approach region with vertex at x by setting

$$\Gamma_\kappa(x) := \big\{ y \in \Omega :\, \mathrm{dist}(x,y) < (1+\kappa)\,\mathrm{dist}\,(y,\partial\Omega)\big\}. \tag{4.1.2}$$

Next, given an arbitrary $u : \Omega \to \mathbb{C}$, define its nontangential maximal function and its pointwise nontangential boundary trace at $x \in \partial\Omega$, respectively, as

$$\big(\mathcal{N}_\kappa u\big)(x) := \sup\big\{|u(y)| :\, y \in \Gamma_\kappa(x)\big\}, \qquad \Big(u\Big|_{\partial\Omega}^{\mathrm{n.t.}}\Big)(x) := \lim_{\Gamma_\kappa(x)\ni y \to x} u(y), \tag{4.1.3}$$

whenever the limit exists. The parameter κ plays a somewhat secondary role in the proceedings, since for any $\kappa_1, \kappa_2 \in (0,\infty)$ and $p \in (0,\infty)$ there exists $C = C(\kappa_1,\kappa_2,p) \in (1,\infty)$ with the property that

$$C^{-1}\big\|\mathcal{N}_{\kappa_1} u\big\|_{L^p(\partial\Omega)} \leq \big\|\mathcal{N}_{\kappa_2} u\big\|_{L^p(\partial\Omega)} \leq C\big\|\mathcal{N}_{\kappa_1} u\big\|_{L^p(\partial\Omega)}, \tag{4.1.4}$$

for each $u : \Omega \to \mathbb{C}$. Given this, we will simplify notation and write $\mathcal{N}$ in place of $\mathcal{N}_\kappa$.

Moving on, let L be a second-order, strongly elliptic, $k \times k$ system of differential operators on M. Assume that, locally,

$$Lu = \sum_{i,j} \partial_j A^{ij}(x) \partial_j u + \sum_j B^j(x) \partial_j u + V(x)u, \tag{4.1.5}$$

where

$$A^{ij} \in \mathscr{C}^2, \quad B^j \in \mathscr{C}^1, \quad V \in L^\infty. \tag{4.1.6}$$

Also, suppose

$$L : H^{1,p}(M) \longrightarrow H^{-1,p}(M) \text{ is an isomorphism, for } 1 < p < \infty. \tag{4.1.7}$$

We want to solve the Dirichlet boundary problem

$$Lu = 0 \quad \text{on} \quad \Omega, \qquad u\Big|_{\partial\Omega}^{\text{n.t.}} = f \in L^p(\partial\Omega), \qquad \mathcal{N}u \in L^p(\partial\Omega), \tag{4.1.8}$$

via the layer potential method. To this end, let E denote the Schwartz kernel of L^{-1}, so that

$$L^{-1}v(x) = \int_M E(x,y)v(y)\, d\text{Vol}(y), \qquad x \in M, \tag{4.1.9}$$

where dVol stands for the volume element on M. Then, with σ denoting the surface measure on $\partial\Omega$, define the single layer potential operator and its boundary version by

$$\begin{gathered} \mathcal{S}g(x) := \int_{\partial\Omega} E(x,y)g(y)\, d\sigma(y), \qquad x \in M \setminus \partial\Omega, \\ \text{and} \quad Sg := \mathcal{S}g\Big|_{\partial\Omega}^{\text{n.t.}} \quad \text{on} \quad \partial\Omega. \end{gathered} \tag{4.1.10}$$

We want to solve (4.1.8) in the form

$$u = \mathcal{S}g, \qquad \text{where } g \text{ is chosen so that } \; Sg = f. \tag{4.1.11}$$

As such, if $H^{s,p}(\partial\Omega)$, with $1 < p < \infty$ and $-1 \leq s \leq 1$, denotes the L^p-based scale of Sobolev spaces of fractional order s on $\partial\Omega$, we would like to show

$$S : H^{-1,p}(\partial\Omega) \longrightarrow L^p(\partial\Omega) \text{ is Fredholm, of index 0.} \tag{4.1.12}$$

Since the adjoint of S is the single layer associated with L^* (which continues to be a second-order, strongly elliptic, $k \times k$ system of differential operators on M), this is further equivalent (with $q := p'$ the Hölder conjugate exponent of p) to the condition that

$$S : L^q(\partial\Omega) \longrightarrow H^{1,q}(\partial\Omega) \text{ is Fredholm, of index 0.} \tag{4.1.13}$$

Such a result was established, for q close to 2, when Ω is a Lipschitz domain, in Chapter 3 of [22]. The argument made use of a Rellich type identity. In the scalar case the result was established (in the setting of regular SKT domains) in [14, Section 6.4], and applied in §7.1 of that paper to the Dirichlet problem. In case $\partial\Omega$ is smooth, it is standard that $S \in \text{OP}S^{-1}(\partial\Omega)$ and it is strongly elliptic, from which (4.1.12) and (4.1.13) follow. Here is what we propose.

Proposition 4.1. *Let Ω be a* $\text{Lip} \cap \text{vmo}_1$ *domain and let L be a second-order, strongly elliptic, $k \times k$ system of differential operators on M, as in* (4.1.5)-(4.1.6)*, and satisfying* (4.1.7)*. Then* (4.1.12) *holds for all $p \in (1,\infty)$ and* (4.1.13) *holds for all $q \in (1,\infty)$.*

Proof. We start with the proof of (4.1.13). Pick $L^\infty \cap \text{vmo}$ vector fields X_j, $1 \le j \le N$, tangent to $\partial\Omega$, such that

$$\sum_{j=1}^{N} |X_j(x)| \ge A > 0 \quad \text{for a.e.} \quad x \in \partial\Omega. \tag{4.1.14}$$

Then set $\nabla_T f := \{X_j f : 1 \le j \le N\}$. We have $\nabla_T S : L^q(\partial\Omega) \to L^q(\partial\Omega)$ for all $q \in (1, \infty)$. Theorem 2.4 (or rather its standard "variable coefficient" extension) implies

$$\nabla_T S = k_0(x, D) + R, \qquad k_0(x, D) \in \mathrm{OP}(L^\infty \cap \text{vmo}) S^0_{\text{cl}}, \tag{4.1.15}$$

with R compact on $L^q(\partial\Omega)$. At this point we make the following

`Claim.` *We have the (overdetermined) ellipticity property*

$$\|k_0(x, \xi) v\| \ge A_0 \|v\|, \qquad A_0 > 0. \tag{4.1.16}$$

Assuming for now this claim (whose proof will be provided later), we obtain that

$$k_0^*(x, D) k_0(x, D) \in \mathrm{OP}(L^\infty \cap \text{vmo}) S^0_{\text{cl}} \qquad \text{(mod compacts)} \tag{4.1.17}$$

is a (determined) elliptic operator, so it has a parametrix $Q \in \mathrm{OP}(L^\infty \cap \text{vmo}) S^0_{\text{cl}}$ (cf. §A.3). Hence,

$$Q k_0^*(x, D) \nabla_T S = I + R_1, \quad \text{with } R_1 \text{ compact on } L^q(\partial\Omega). \tag{4.1.18}$$

This implies that

$$S : L^q(\partial\Omega) \longrightarrow H^{-1,q}(\partial\Omega) \quad \text{is semi-Fredholm}, \tag{4.1.19}$$

namely, it has closed range and finite dimensional null-space.

To complete the argument, we take a continuous family L_τ of second order, strongly elliptic operators on M, $\tau \in [0, 1]$, such that $L_1 = L$ and L_0 is *scalar*. This gives a norm continuous family

$$S_\tau : L^q(\partial\Omega) \longrightarrow H^{1,q}(\partial\Omega), \quad \text{all semi-Fredholm}. \tag{4.1.20}$$

We know that S_0 is Fredholm of index 0. Hence so are all the operators S_τ in (4.1.20). This gives (4.1.13) which, by duality, also yields (4.1.12).

Now we return to the proof of the claim made in (4.1.16). That is, we shall establish the (overdetermined) ellipticity of $k_0(x, D) \in \mathrm{OP}(L^\infty \cap \text{vmo}) S^0_{\text{cl}}$, arising in (4.1.15) (which is equal modulo a compact operator to $\nabla_T S$). To begin, we discuss the smooth case. If $\partial\Omega$ is smooth, and L is strongly elliptic of second order with smooth coefficients, then actually $S \in \mathrm{OP}S^{-1}(\partial\Omega)$, and this operator is strongly elliptic. In fact, given $(x, \xi) \in T^*\partial\Omega \setminus 0$, and with $\nu \in T^*_x\partial\Omega$ the outward unit conormal to $\partial\Omega$, we have

$$\sigma_S(x, \xi) = C_n \int_{-\infty}^{+\infty} \sigma_E(x, \xi + t\nu)\, dt = C_n \int_{-\infty}^{+\infty} \sigma_L(x, \xi + t\nu)^{-1}\, dt. \tag{4.1.21}$$

This is seen as in [34, (11.11)-(11.12) in Chapter 7, Vol. 2], where we take $m = -2$, $x_n = 0$. Strong ellipticity of S then follows from (4.1.21), keeping in mind the strong ellipticity of L. Specifically, note that $\sigma_S(x, \xi)$ is positive homogeneous of degree -1 in ξ and the integrals in (4.1.21) are absolutely convergent since $|\sigma_L(x,\xi+t\nu(x))^{-1}| \le C(|\xi|^2+t^2)^{-1}$. Keeping this in mind, for any section η and any $0 \ne \xi \in T^*_x\partial\Omega \subset T^*_xM$, we may estimate

$$\begin{aligned}\big\langle -\sigma_S(x,\xi)\eta,\eta\big\rangle_x &= C_n\int_{-\infty}^{+\infty}\big\langle -\sigma_L(x,\xi+t\nu(x))^{-1}\eta,\eta\big\rangle\,dt\\ &\ge C|\eta|^2\int_{-\infty}^{+\infty}(|\xi|^2+t^2)^{-1}\,dt\\ &\ge C|\eta|^2|\xi|^{-1},\end{aligned}\tag{4.1.22}$$

for some $C>0$. This yields the strong ellipticity of S. Next, since $\sigma_{X_jS}=\sigma_{X_j}\sigma_S$, the ellipticity of $\nabla_T S$ is an immediate consequence of what we have just proved and (4.1.14).

To tackle the case when Ω is a $\mathrm{Lip}\cap\mathrm{vmo}_1$ domain, we take local graph coordinates $\varphi(x)=(x,\varphi_0(x))$, and arrange that the vector fields $\{X_j\}_{1\le j\le N}$ include those associated with coordinate differentiation. The integral kernel $E(x,y)$ has the form

$$E(x,y)=E_0(x,x-y)+r(x,y),\tag{4.1.23}$$

where $E_0(x,z)$ is smooth on $\{z\ne 0\}$ and homogeneous of degree $-(n-1)$ in z (note that dim $\partial\Omega=n$), and $r(x,y)$ has lower order. See the analysis in [22]. Locally, the operator S has the form

$$Sg(x)=\int_{\mathbb{R}^n}E_0\big(\varphi(x),\varphi(x)-\varphi(y)\big)g(y)\Sigma(y)\,dy+Rg(x),\qquad x\in\mathbb{R}^n,\tag{4.1.24}$$

where $d\sigma(y)=\Sigma(y)\,dy$ and R denotes the integral operator with kernel $r(x,y)$. Hence, for each $j\in\{1,...,n\}$,

$$\partial_jSg(x)=\mathrm{PV}\int_{\mathbb{R}^n}\partial_j\varphi(x)\cdot\nabla_2E_0\big(\varphi(x),\varphi(x)-\varphi(y)\big)g(y)\Sigma(y)\,dy+R_jg(x),\quad x\in\mathbb{R}^n,\tag{4.1.25}$$

where here and below R_j will denote (perhaps different) operators that are compact on L^p, for $1<p<\infty$. Theorem 2.4 (or rather its natural "variable coefficient" extension from §2.3) gives

$$\partial_jSg(x)=\mathrm{PV}\int_{\mathbb{R}^n}\partial_j\varphi(x)\cdot\nabla_2E_0\big(\varphi(x),D\varphi(x)(x-y)\big)g(y)\Sigma(y)\,dy+R_jg(x),\quad x\in\mathbb{R}^n,\tag{4.1.26}$$

i.e.,

$$\partial_jSg(x)=T_j(x,D)(\Sigma g)(x)+R_jg(x),\qquad x\in\mathbb{R}^n,\tag{4.1.27}$$

where $T_j(x,D)(\Sigma g)(x)$ is given by the principal value integral in (4.1.26). We therefore have $T_j(x,D)\in\mathrm{OP}(L^\infty\cap\mathrm{vmo})S^0_{\mathrm{cl}}$, with symbol

$$T_j(x,\xi)=\int_{\mathbb{R}^n}e^{-iz\cdot\xi}\,\partial_j\varphi(x)\cdot\nabla_2E_0\big(\varphi(x),D\varphi(x)z\big)\,dz.\tag{4.1.28}$$

Given that L is a $k\times k$ system, $T_j(x,\xi)$ is a $k\times k$ matrix, i.e., $T_j(x,\xi)\in M(k,\mathbb{C})$, for $\xi\ne0$, and a.e. x. We need to show that there exists $C>0$ such that, for all $\xi\ne0$ and $v\in\mathbb{C}^k$,

$$\sum_j\|T_j(x,\xi)v\|\ge C\|v\|,\quad\text{for a.e. }\ x.\tag{4.1.29}$$

Recall that φ has the form (1.1), so $D\varphi(x) : \mathbb{R}^n \to \mathbb{R}^{n+1}$ has the form

$$D\varphi(x) = \begin{pmatrix} I \\ D\varphi_0(x) \end{pmatrix}, \qquad D\varphi_0(x) : \mathbb{R}^n \to \mathbb{R}, \tag{4.1.30}$$

for a.e. $x \in \mathbb{R}^n$. Freezing coefficients at a point where φ is differentiable, we can rephrase our task as follows. Let $L_0(\zeta)$ be a matrix in $M(k,\mathbb{C})$ whose entries are polynomials in $\zeta \in \mathbb{R}^{n+1}$, homogeneous of degree 2, and which is positive definite for each $\zeta \neq 0$. For $\zeta \neq 0$ set $E_0(\zeta) := L_0(\zeta)^{-1}$. In addition, consider a linear mapping $A : \mathbb{R}^n \to \mathbb{R}^{n+1}$ of the form

$$A = \begin{pmatrix} I \\ A_0 \end{pmatrix}, \qquad A_0 : \mathbb{R}^n \to \mathbb{R}. \tag{4.1.31}$$

Let A_0 run over a compact set in $\mathcal{L}(\mathbb{R}^n,\mathbb{R})$. Also let L_0 and $E_0 = L_0^{-1}$ run over compact sets of symbols. Take

$$T_j(\xi) := \int\limits_{\mathbb{R}^n} e^{-iz\cdot\xi}\, Ae_j \cdot \nabla E_0(Az)\, dz, \tag{4.1.32}$$

where $\{e_j\}_{1\leq j\leq n}$ denotes the standard orthonormal basis of $\mathbb{R}^n$. We need to prove that there exists a finite constant $C > 0$ such that, for all $v \in \mathbb{C}^k$ and $\xi \neq 0$,

$$\sum_j \|T_j(\xi)v\| \geq C\|v\|, \tag{4.1.33}$$

uniformly in A_0, L_0, E_0. Note that this is equivalent to the ellipticity of $\nabla_T S$ in case $\varphi(x) = Ax$, so $\partial\Omega$ is a hyperplane in $\mathbb{R}^{n+1}$. In this case, the previous analysis applies, since $S \in \mathrm{OP}S^{-1}(\partial\Omega)$ is strongly elliptic, and (4.1.33) follows.

This finishes the proof of the claim made in (4.1.16) which, in turn, completes the proof of Proposition 4.1. □

We next note a regularity result, under the assumption that Ω is a $\mathrm{Lip} \cap \mathrm{vmo}_1$ domain. Let us temporarily denote

$$S_{s,p} = S : H^{s,p}(\partial\Omega) \longrightarrow H^{s+1,p}(\partial\Omega), \qquad s \in \{0, -1\}, \tag{4.1.34}$$

with adjoint

$$S^*_{s,p} = S^* : H^{-1-s,q}(\partial\Omega) \longrightarrow H^{-s,q}(\partial\Omega), \qquad q = p'. \tag{4.1.35}$$

Clearly the null-spaces $\mathrm{Ker}(S_{s,p})$ and $\mathrm{Ker}(S^*_{s,p})$ of these operators satisfy

$$\mathrm{Ker}(S_{0,p}) \subset \mathrm{Ker}(S_{-1,p}), \quad \mathrm{Ker}(S^*_{-1,p}) \subset \mathrm{Ker}(S^*_{0,p}), \tag{4.1.36}$$

so the vanishing index property established in Proposition 4.1 forces

$$\mathrm{Ker}(S_{0,p}) = \mathrm{Ker}(S_{-1,p}) \quad \text{and} \quad \mathrm{Ker}(S^*_{-1,p}) = \mathrm{Ker}(S^*_{0,p}). \tag{4.1.37}$$

Also,

$$1 < p < \tilde{p} < \infty \implies \mathrm{Ker}(S_{0,\tilde{p}}) = \mathrm{Ker}(S_{0,p}), \quad \mathrm{Ker}(S^*_{0,p}) = \mathrm{Ker}(S^*_{0,\tilde{p}}), \tag{4.1.38}$$

and, again, the aforementioned vanishing index property implies

$$\mathrm{Ker}(S_{0,p}) = \mathrm{Ker}(S_{0,\tilde{p}}). \tag{4.1.39}$$

Collectively, (4.1.37) and (4.1.39) prove the following regularity result:

Proposition 4.2. *Assume that Ω is a $\mathrm{Lip}\cap\mathrm{vmo}_1$ domain in M and suppose L is a second-order, strongly elliptic, system of differential operators on M as in* (4.1.5)-(4.1.6)*, and satisfying* (4.1.7)*. Then, given $f\in H^{-1,p}(\partial\Omega)$ for some $p\in(1,\infty)$, one has*

$$Sf=0 \implies f\in\bigcap_{1<q<\infty}L^q(\partial\Omega). \tag{4.1.40}$$

Recall that standard Lipschitz theory (cf. [22]) gives

$$f\in L^p(\partial\Omega) \ \text{ with } \ p\in(1,\infty) \ \text{ and } \ u:=\mathcal{S}f \implies \begin{cases} Lu=0 \ \text{ on } \ M\setminus\partial\Omega,\\ \mathcal{N}u,\,\mathcal{N}(\nabla u)\in L^p(\partial\Omega),\\ u\big|_{\partial\Omega}^{\text{n.t.}}=Sf,\end{cases} \tag{4.1.41}$$

and

$$f\in H^{-1,p}(\partial\Omega) \ \text{ with } \ p\in(1,\infty) \ \text{ and } \ u:=\mathcal{S}f \implies \begin{cases} Lu=0 \ \text{ on } \ M\setminus\partial\Omega,\\ \mathcal{N}u\in L^p(\partial\Omega),\\ u\big|_{\partial\Omega}^{\text{n.t.}}=Sf.\end{cases} \tag{4.1.42}$$

In addition, we single out the following additional properties. Let $H^{s,p}(\Omega)$, with $s\in\mathbb{R}$ and $p\in(1,\infty)$ stand for the L^p-based Sobolev space of fractional smoothness s in Ω. Also, let $\mathrm{Tr}:H^{1,2}(\Omega)\to H^{1/2,2}(\partial\Omega)$ denote the boundary trace operator in the sense of Sobolev spaces, and set $H^{1,2}_0(\Omega):=\mathrm{Ker}\,\mathrm{Tr}$. Then

$$f\in L^2(\partial\Omega) \implies u:=\mathcal{S}f\in H^1(\Omega),\ \mathrm{Tr}\,u=u\big|_{\partial\Omega}^{\text{n.t.}}=Sf. \tag{4.1.43}$$

These considerations are relevant in the context of the following well-posedness result.

Theorem 4.3. *Suppose $\Omega\subset M$ is a $\mathrm{Lip}\cap\mathrm{vmo}_1$ domain and suppose L is a second-order, strongly elliptic, system of differential operators on M as in* (4.1.5)-(4.1.6)*, and satisfying* (4.1.7)*. Set*

$$\Omega_+:=\Omega,\qquad \Omega_-:=M\setminus\overline{\Omega}, \tag{4.1.44}$$

and assume that the following nondegeneracy conditions hold:

$$\begin{aligned} &u\in H^{1,2}_0(\Omega_+),\ Lu=0 \ \textit{ in } \ \Omega_+ \implies u=0 \ \textit{ in } \ \Omega_+,\\ &u\in H^{1,2}_0(\Omega_-),\ Lu=0 \ \textit{ in } \ \Omega_- \implies u=0 \ \textit{ in } \ \Omega_-.\end{aligned} \tag{4.1.45}$$

Then

$$\begin{aligned} &S:H^{-1,p}(\partial\Omega)\longrightarrow L^p(\partial\Omega) \ \textit{ is invertible for each } p\in(1,\infty),\\ &S:L^p(\partial\Omega)\longrightarrow H^{1,p}(\partial\Omega) \ \textit{ is invertible for each } p\in(1,\infty).\end{aligned} \tag{4.1.46}$$

In particular, the Dirichlet problem

$$Lu=0 \ \textit{ in } \ \Omega,\quad u\big|_{\partial\Omega}^{\text{n.t.}}=f\in L^p(\partial\Omega),\quad \mathcal{N}u\in L^p(\partial\Omega), \tag{4.1.47}$$

is well-posed, and its unique solution is given by $u=\mathcal{S}(S^{-1}f)$, where $S^{-1}f\in H^{-1,p}(\partial\Omega)$.

Furthermore, the Regularity problem

$$Lu=0 \ \textit{ in } \ \Omega,\quad u\big|_{\partial\Omega}^{\text{n.t.}}=f\in H^{1,p}(\partial\Omega),\quad \mathcal{N}u,\,\mathcal{N}(\nabla u)\in L^p(\partial\Omega), \tag{4.1.48}$$

is well-posed, and its unique solution is given by $u=\mathcal{S}(S^{-1}f)$, where $S^{-1}f\in L^p(\partial\Omega)$.

It is worth pointing out that the nondegeneracy conditions in (4.1.45) hold, in particular, if system in question is of the form

$$L = \mathfrak{D}^*\mathfrak{D} \tag{4.1.49}$$

where

$$\mathfrak{D} \text{ is a first-order system with the unique continuation property,} \tag{4.1.50}$$

in the sense that if $u \in H^{1,2}(M)$ is such that $\mathfrak{D}u = 0$ on M and u vanishes on some nonempty open subset of M then necessary $u = 0$ everywhere on M. As a consequence, Theorem 4.3 applies to the Laplace-Beltrami operator on a Riemannian manifold, in which scenario the present well-posedness results complement those in [26].

Proof of Theorem 4.3. In a first stage we shall show that

$$f \in L^2(\partial\Omega) \text{ and } Sf = 0 \Longrightarrow f = 0. \tag{4.1.51}$$

Suppose f is as in the left-hand side of (4.1.51) and set $u := \mathcal{S}f$ in $M \setminus \partial\Omega$. In light of (4.1.43), the hypothesis (4.1.45) then yields $u = 0$ both in Ω_+ and in Ω_-. Recall that L is as in (4.1.5)-(4.1.6) and set (with $\nu = (\nu_i)_i$ denoting the outward unit conormal to Ω)

$$\Xi_\pm f := \sum_{i,j} \nu_i A^{ij}(\partial_j \mathcal{S}f)\Big|^{\text{n.t.}}_{\partial\Omega_\pm}. \tag{4.1.52}$$

Then, on the one hand, the jump-formulas from [22, Theorem 2.9, p. 21] yield

$$\Xi_\pm f = \big(\mp \tfrac{1}{2}I + K^*\big)f, \tag{4.1.53}$$

where K^* is a principal value singular integral operator on $\partial\Omega$ and I is the identity. As such, we have the jump relation

$$f = \Xi_- f - \Xi_+ f. \tag{4.1.54}$$

On the other hand, clearly $u = \mathcal{S}f = 0$ on $\Omega_+ \cup \Omega_-$ implies $\Xi_\pm f = 0$. We conclude that $f = 0$, finishing the proof of (4.1.51).

In turn, (4.1.51), Proposition 4.2, and Proposition 4.1 imply that for each $p \in (1, \infty)$, the operator S is an isomorphism in (4.1.12) and (4.1.13). This proves the claims in (4.1.46). With these in hand, the fact that the Dirichlet and Regularity boundary value problems (4.1.47)-(4.1.48) may be solved in the form $u = \mathcal{S}(S^{-1}f)$ follows from (4.1.41)-(4.1.42).

Turning to the uniqueness part, it suffices to show that any solution u of the homogeneous version of the Dirichlet problem (4.1.47) vanishes identically in Ω. To this end, we introduce the Green function

$$G(x,y) := \Gamma(x,y) - \mathcal{S}\left[S^{-1}\left(E(x,\cdot)\big|_{\partial\Omega}\right)\right](y), \qquad (x,y) \in \Omega \times \Omega \setminus \text{diag}, \tag{4.1.55}$$

where the intervening single layer potential operators are associated with L^*. Note that for each fixed $x \in \Omega$ the function $E(x,\cdot)\big|_{\partial\Omega}$ belongs to $H^{1,q}(\partial\Omega)$ for any $q \in (1,\infty)$. Thus, on account of (4.1.46) we see that $G(x,y)$ is well-defined. To proceed, consider a sequence of Lipschitz subdomains Ω_j of Ω so that $\Omega_j \nearrow \Omega$ as $j \to \infty$ as in [26, Appendix A]; in particular, their Lipschitz character is controlled uniformly in j. Let G_j stand for the Green function corresponding to Ω_j. By construction, $G_j(x,\cdot)|_{\partial\Omega_j} = 0$ and we claim that for each $q \in (1,\infty)$ there exists a constant $C_q \in (0,\infty)$ with the property that

$$\sup_{j\in\mathbb{N}} \big\|\mathcal{N}_j\big(\nabla_2 G_j(x,\cdot)\big)\big\|_{L^q(\partial\Omega_j)} \leq C_q. \tag{4.1.56}$$

This follows from the fact that if S_j denotes the single layer constructed in relation to $\partial\Omega_j$, then for each $q \in (1,\infty)$ the operator norm of $S_j^{-1} : H^{1,q}(\partial\Omega_j) \to L^q(\partial\Omega_j)$ is uniformly bounded in j. In turn, this is seen from (4.1.18) and reasoning by contradiction.

For each $j \in \mathbb{N}$ let σ_j denote the surface measure on $\partial\Omega_j$. Integrations by parts against these Green functions give that if u solves the homogeneous version of the Dirichlet problem (4.1.47) and if $x \in \Omega$ is an arbitrary fixed point, then for $j \in \mathbb{N}$ sufficiently large we have

$$\begin{aligned} |u(x)| &= \left| \int_{\Omega_j} \big\langle \big(L^* G_j(x,\cdot)\big)(y), u(y) \big\rangle \, d\mathrm{Vol}(y) \right| \\ &= \int_{\partial\Omega_j} O\big(|u| \cdot |\nabla_2 G_j(x,\cdot)|\big) \, d\sigma_j \\ &\le C \|u\|_{L^p(\partial\Omega_j)}, \end{aligned} \tag{4.1.57}$$

where the last step utilizes Hölder's inequality and (4.1.56). Because $\|u\|_{L^p(\partial\Omega_j)} \to 0$ by Lebesgue's Dominated Convergence Theorem (and the manner in which $\Omega_j \nearrow \Omega$ as $j \to \infty$), we ultimately obtain $u(x) = 0$. Given that $x \in \Omega$ was arbitrary, the desired uniqueness statement follows.

We wish to note that for the proof of uniqueness we could have avoided using the approximating family $\Omega_j \nearrow \Omega$ and, instead, worked directly with the Green function for L^*, constructed as in (4.1.55), by reasoning as in the proof of [14, Theorem 7.2, p. 2831] as carried out in Step 3 on pp. 2832–2837 of [14]. $\square$

In the last part of this section we discuss the Poisson problem for strongly elliptic systems with data in Sobolev-Besov spaces in Lipschitz domains with normal in vmo. Throughout, retain the setting of Theorem 4.3. For starters, from (4.1.46) and complex interpolation we deduce, with the help of [10, Lemma 8.4], that

$$S : H^{s-1,p}(\partial\Omega) \longrightarrow H^{s,p}(\partial\Omega) \text{ is invertible for each } p \in (1,\infty) \text{ and } s \in [0,1]. \tag{4.1.58}$$

With $B_s^{p,q}(\partial\Omega)$, for $p, q \in (0,\infty]$ and $0 \neq s \in (-1,1)$, denoting the scale of Besov spaces on $\partial\Omega$, real interpolation then also gives that

$$S : B_{s-1}^{p,q}(\partial\Omega) \longrightarrow B_s^{p,q}(\partial\Omega) \text{ is invertible for } p \in (1,\infty),\ q \in (0,\infty], \text{ and } s \in (0,1). \tag{4.1.59}$$

Furthermore, the action of the single layer potential operator $\mathcal{S}$ on Sobolev-Besov spaces on Lipschitz domains has been studied in [27]. The emphasis in [27] is on the Hodge-Laplacian but the approach there (which utilizes size estimates for the integral kernel and its derivatives) is general enough to work in the present setting. Indeed, the mapping properties from [27, Lemmas 7.27.3] are directly applicable here. They imply that if $B_s^{p,q}(\Omega)$, for $p, q \in (0,\infty]$ and $s \in \mathbb{R}$, stands for the scale of Besov spaces in Ω, the single layer operator induces well-defined and bounded linear mappings in the following contexts:

$$\mathcal{S} : B_{-s}^{p,p}(\partial\Omega) \longrightarrow B_{1+\frac{1}{p}-s}^{p,p}(\Omega), \text{ for } 1 \le p \le \infty \text{ and } 0 < s < 1, \tag{4.1.60}$$

$$\mathcal{S} : B_{-s}^{p,p}(\partial\Omega) \longrightarrow H^{1+\frac{1}{p}-s,p}(\Omega), \text{ for } 1 < p < \infty \text{ and } 0 < s < 1. \tag{4.1.61}$$

$$\mathcal{S} : H^{-s,p}(\partial\Omega) \longrightarrow B_{1-s+1/p}^{p,\max\{p,2\}}(\Omega), \text{ for } 1 < p < \infty,\ 0 \le s \le 1. \tag{4.1.62}$$

Theorem 4.4. *Suppose $\Omega \subset M$ is a* $\mathrm{Lip} \cap \mathrm{vmo}_1$ *domain and suppose L is a second-order, strongly elliptic, system of differential operators on M as in* (4.1.5)-(4.1.6)*, and satisfying* (4.1.7) *and* (4.1.45)*. In addition assume that L^*, the adjoint of L, also satisfies the nondegeneracy conditions in* (4.1.45)*.*

Then for any $p \in (1, \infty)$ and any $s \in (0, 1)$ the Poisson problem with a Dirichlet boundary condition

$$\begin{cases} Lu = f \in H^{s+\frac{1}{p}-2,p}(\Omega), \\ \operatorname{Tr} u = g \in B^{p,p}_s(\partial\Omega), \\ u \in H^{s+\frac{1}{p},p}(\Omega), \end{cases} \tag{4.1.63}$$

has a unique solution.

Proof. Extend the given $f \in H^{s+\frac{1}{p}-2,p}(\Omega)$ to some $\widetilde{f} \in H^{s+\frac{1}{p}-2,p}(M)$ then consider

$$v := \big(L^{-1}\widetilde{f}\,\big)\big|_{\Omega} \in H^{s+\frac{1}{p},p}(\Omega). \tag{4.1.64}$$

In particular, $h := \operatorname{Tr} v \in B^{p,p}_s(\partial\Omega)$ and a solution u of the boundary value problem (4.1.63) is given by

$$u := v - \mathcal{S}\big(S^{-1}(h-g)\big) \ \text{ in } \ \Omega, \tag{4.1.65}$$

with S^{-1} the inverse of the operator in (4.1.59) (with $q = p$), and $\mathcal{S}$ considered as in (4.1.61).

There remains to prove uniqueness. The existence result just established may be interpreted (taking $g = 0$) as the statement that

$$L : H_0^{s+\frac{1}{p},p}(\Omega) \longrightarrow H^{s+\frac{1}{p}-2,p}(\Omega) \ \text{ is surjective, for each } p \in (1,\infty) \text{ and } s \in (0,1), \tag{4.1.66}$$

in the class of operators L described in the statement. Since the class in question is stable under taking adjoints, writing (4.1.66) for L^* then taking adjoints yields (after adjusting notation) that

$$L : H_0^{s+\frac{1}{p},p}(\Omega) \longrightarrow H^{s+\frac{1}{p}-2,p}(\Omega) \ \text{ is injective, for each } p \in (1,\infty) \text{ and } s \in (0,1). \tag{4.1.67}$$

With this in hand, the fact that any null-solution of (4.1.63) necessarily vanishes identically in Ω readily follows. This completes the proof of the theorem. $\square$

4.2 Oblique derivative problems

To start, let $\Omega \subset \mathbb{R}^n$ be a bounded, regular SKT domain, so its unit normal field ν belongs to $\mathrm{vmo}(\partial\Omega)$. We have tangential vector fields

$$\partial_{\tau_{jk}} = \nu_k \partial_j - \nu_j \partial_k, \qquad 1 \le j, k \le n \tag{4.2.1}$$

(see [14, Section 3.6]).

Let ξ_{jk}, $1 \le j, k \le n$, be real-valued functions on $\partial\Omega$, and define the tangential vector field

$$X := \sum_{j,k=1}^{n} \xi_{jk} \partial_{\tau_{jk}}. \tag{4.2.2}$$

Assume that for each $j, k \in \{1, ..., n\}$ we have

$$\xi_{jk}\nu_j, \ \xi_{jk}\nu_k \in \mathrm{vmo}(\partial\Omega) \cap L^\infty(\partial\Omega). \tag{4.2.3}$$

Given $p \in (1, \infty)$, the goal here is to study the oblique derivative problem

$$\Delta u = 0 \text{ on } \Omega, \quad (\partial_\nu + X)u = f \text{ on } \partial\Omega, \quad \mathcal{N}u, \mathcal{N}(\nabla u) \in L^p(\partial\Omega), \tag{4.2.4}$$

where $f \in L^p(\partial\Omega)$ is given. Above, $\partial_\nu u$ and Xu are understood, respectively, as

$$\partial_\nu u := \sum_{j=1}^n \nu_j \Big((\partial_j u)\Big|_{\partial\Omega}^{\text{n.t.}} \Big) \quad \text{and} \quad Xu := \sum_{j,k=1}^n \xi_{jk} \partial_{\tau_{jk}} \Big(u\Big|_{\partial\Omega}^{\text{n.t.}} \Big). \tag{4.2.5}$$

We look for a solution of (4.2.4) in the form

$$u := \mathcal{S}g \quad \text{in} \quad \Omega, \tag{4.2.6}$$

where $g \in L^p(\partial\Omega)$ is yet to be determined and $\mathcal{S}$ is the harmonic single layer potential operator associated with Ω. That is,

$$\mathcal{S}g(x) := \int_{\partial\Omega} E(x-y) g(y)\, d\sigma(y), \qquad x \in \Omega, \tag{4.2.7}$$

with E denoting the standard fundamental solution for the Laplacian in $\mathbb{R}^n$, i.e.,

$$E(x) := \begin{cases} \dfrac{1}{\omega_{n-1}(2-n)} |x|^{2-n}, & \text{if } n \geq 3, \\ \dfrac{1}{2\pi} \ln|x|, & \text{if } n = 2, \end{cases} \qquad \forall\, x \in \mathbb{R}^n \setminus \{0\}, \tag{4.2.8}$$

where ω_{n-1} is the surface measure of the unit sphere S^{n-1} in $\mathbb{R}^n$. As shown in [14, Section 4],

$$\partial_\nu \mathcal{S}g \Big|_{\partial\Omega}^{\text{n.t.}} = \big(-\tfrac{1}{2} I + K^* \big) g, \tag{4.2.9}$$

where

$$K^* : L^p(\partial\Omega) \to L^p(\partial\Omega) \text{ is compact for every } p \in (1, \infty). \tag{4.2.10}$$

Meanwhile,

$$X(\mathcal{S}g) = Cg := \sum_{j,k} (A_{jk} g - B_{jk} g) \quad \text{on} \quad \partial\Omega, \tag{4.2.11}$$

where

$$A_{jk} g(x) := \text{PV} \int_{\partial\Omega} a_{jk}(x) \partial_j E(x-y) g(y)\, d\sigma(y), \qquad x \in \partial\Omega, \tag{4.2.12}$$

and

$$B_{jk} g(x) := \text{PV} \int_{\partial\Omega} b_{jk}(x) \partial_k E(x-y) g(y)\, d\sigma(y), \qquad x \in \partial\Omega, \tag{4.2.13}$$

with

$$a_{jk}(x) := \xi_{jk}(x) \nu_k(x), \qquad b_{jk}(x) := \xi_{jk}(x) \nu_j(x). \tag{4.2.14}$$

The following provides a key to the study of (4.2.4).

Lemma 4.5. *If $\Omega \subset \mathbb{R}^n$ is a bounded regular* SKT *domain and* (4.2.3) *holds, then*

$$A_{jk} + A_{jk}^* \quad \textit{and} \quad B_{jk} + B_{jk}^* \quad \textit{are compact on} \quad L^p(\partial\Omega), \quad \forall\, p \in (1, \infty). \tag{4.2.15}$$

Proof. For each $j \in \{1, ..., n\}$,

$$F_j g(x) := \mathrm{PV} \int_{\partial\Omega} \partial_j E(x-y) g(y)\, d\sigma(y), \qquad x \in \partial\Omega, \tag{4.2.16}$$

defines an operator of Calderón-Zygmund type that is bounded on $L^p(\partial\Omega)$ for all $p \in (1, \infty)$, since Ω is a UR domain. Then

$$A_{jk} + A^*_{jk} = [a_{jk}, F_j], \qquad B_{jk} + B^*_{jk} = [b_{jk}, F_k], \tag{4.2.17}$$

so (4.2.15) follows from a general commutator estimate of Coifman-Rochberg-Weiss type (cf. [14, Section 2.4]), since $a_{jk},\, b_{jk} \in \mathrm{vmo}(\partial\Omega)$. $\square$

In light of (4.2.9) and (4.2.11), solving the oblique derivative boundary value problem (4.2.4) via the single layer representation (4.2.6) is equivalent to finding a function $g \in L^p(\partial\Omega)$ satisfying

$$\big(-\tfrac{1}{2} I + C + K^* \big) g = f. \tag{4.2.18}$$

In this regard, the following Fredholmness result is particularly relevant.

Proposition 4.6. *If Ω is bounded regular* SKT *domain in $\mathbb{R}^n$ and if* (4.2.3) *holds, then*

$$-\tfrac{1}{2} I + C + K^* : L^2(\partial\Omega) \to L^2(\partial\Omega) \ \text{ is Fredholm, of index 0.} \tag{4.2.19}$$

Proof. By Lemma 4.5, we can write

$$\begin{gathered} C + K^* = \widetilde{C} + K_2 \ \text{ where } \ \widetilde{C}^* := -\widetilde{C} \ \text{ and} \\ K_2 \ \text{ is a compact operator on } \ L^p(\partial\Omega), \quad \forall\, p \in (1, \infty) \end{gathered} \tag{4.2.20}$$

Then, for $g \in L^2(\partial\Omega)$,

$$\mathrm{Re}\Big(\big(-\tfrac{1}{2} I + \widetilde{C} \big) g \,,\, g \Big) = -\tfrac{1}{2} \|g\|^2_{L^2(\partial\Omega)} \tag{4.2.21}$$

which, in turn, shows that

$$-\tfrac{1}{2} I + \widetilde{C} \ \text{ is invertible on } \ L^2(\partial\Omega). \tag{4.2.22}$$

Since the operator in (4.2.19) is a compact perturbation of this, the desired conclusion follows. $\square$

Corollary 4.7. *In the setting of Proposition 4.6, there exists $\varepsilon > 0$ such that*

$$-\tfrac{1}{2} I + C + K^* : L^p(\partial\Omega) \to L^p(\partial\Omega) \quad \text{ is Fredholm, of index } 0 \tag{4.2.23}$$

whenever $|p - 2| < \varepsilon$.

Proof. That

$$-\tfrac{1}{2} I + \widetilde{C} : L^p(\partial\Omega) \to L^p(\partial\Omega) \quad \text{ is invertible} \tag{4.2.24}$$

for p close to 2 follows from (4.2.22) and the stability results in [33] (cf. also [15]). Meanwhile, the operator in (4.2.23) is a compact perturbation of that in (4.2.24) for all $p \in (1, \infty)$. $\square$

In the context of Corollary 4.7, one wonders whether (4.2.23) holds for all $p \in (1, \infty)$. We show that it does hold if Ω is a bounded $\mathrm{Lip} \cap \mathrm{vmo}_1$ domain in $\mathbb{R}^n$.

Proposition 4.8. *If Ω is a bounded* $\mathrm{Lip} \cap \mathrm{vmo}_1$ *domain in $\mathbb{R}^n$ and if* (4.2.3) *holds, then the Fredholmness result* (4.2.23) *is true for all $p \in (1, \infty)$.*

Proof. For starters, we note that since (4.2.3) and (4.2.14) imply that $a_{jk}, b_{jk} \in \mathrm{vmo}(\partial\Omega)$, it follows from Lemma A.16 that $a_{jk} \circ \phi$, $b_{jk} \circ \phi \in \mathrm{vmo}(U)$, whenever $\phi : U \to \partial\Omega$ is a coordinate chart for $\partial\Omega$ (in the sense of Definition A.1). Keeping this in mind it follows that, in the present setting, the operator C defined by (4.2.11) belongs to $\mathrm{OP}(L^\infty \cap \mathrm{vmo})S^0_{\mathrm{cl}}$, and (4.2.20) implies that its principal symbol is purely imaginary. Hence, for each $s \in \mathbb{R}$, $F_s := -\frac{1}{2}I + sC$ is an elliptic operator in $\mathrm{OP}(L^\infty \cap \mathrm{vmo})S^0_{\mathrm{cl}}$. Thus, these operators F_s are all Fredholm on $L^p(\partial\Omega)$, and all have index independent of s. Clearly, F_0 has index zero, hence so does F_1, and the desired conclusion follows. □

We are now ready to state of main Fredholm solvability result for the oblique derivative problem. This builds on the earlier work of Calderón [3]. Other extensions in the Euclidean setting are in [16], [28]; cf. also [24] for some recent refinements in the two dimensional setting. For Lipschitz domains on manifolds see [26].

Theorem 4.9. *Let Ω is a bounded $\mathrm{Lip} \cap \mathrm{vmo}_1$ domain in $\mathbb{R}^n$ with outward unit normal ν. Assume that (4.2.3) holds and defined the tangential vector field X as in (4.2.2). Finally, fix $p \in (1,\infty)$ arbitrary.*

Then for any boundary datum $f \in L^p(\partial\Omega)$ satisfying finitely many (necessary) linear conditions the oblique derivative problem (4.2.4) has a solution. Moreover, such a solution is unique modulo a finite-dimensional linear space, whose dimension coincides with the number of linearly independent constraints required for the boundary data.

Hence, the oblique derivative problem (4.2.4) is Fredholm solvable, with index zero.

Proof. Fatou results in Lipschitz domains give that

$$\begin{aligned} &\Delta u = 0 \ \text{ on } \ \Omega \ \text{ and } \ \mathcal{N}u, \mathcal{N}(\nabla u) \in L^p(\partial\Omega) \\ &\text{imply that } \ u\Big|_{\partial\Omega}^{\mathrm{n.t.}} \ \text{ exists and belongs to } \ H^{1,p}(\partial\Omega). \end{aligned} \tag{4.2.25}$$

Going further, from (4.2.25) and the well-posedness of the L^p Regularity problem for the Laplacian in bounded $\mathrm{Lip} \cap \mathrm{vmo}_1$ domains established in Theorem 4.3 it follows that

$$\begin{aligned} &\text{any function } u \text{ satisfying } \ \Delta u = 0 \ \text{ on } \ \Omega \ \text{ and } \ \mathcal{N}u, \mathcal{N}(\nabla u) \in L^p(\partial\Omega) \\ &\text{is of the form } \ u = \mathcal{S}g \ \text{ in } \ \Omega, \ \text{ for some (unique) function } \ g \in L^p(\partial\Omega). \end{aligned} \tag{4.2.26}$$

In turn, from (4.2.26) we deduce that if the boundary datum $f \in L^p(\partial\Omega)$ is such that the oblique derivative problem (4.2.4) has a solution u, then there exists a (unique) function $g \in L^p(\partial\Omega)$ with the property that

$$f = (\partial_\nu + X)u = (\partial_\nu + X)(\mathcal{S}g) = \big(-\tfrac{1}{2}I + C + K^*\big)g. \tag{4.2.27}$$

This analysis shows that the oblique derivative problem (4.2.4) is solvable precisely for boundary data f belonging to the image of the operator $-\frac{1}{2}I + C + K^*$ on $L^p(\partial\Omega)$. By Proposition 4.8, this is a closed subspace of $L^p(\partial\Omega)$, of finite codimension. The above analysis also shows that the space of null-solutions for the oblique derivative problem (4.2.4) is isomorphic to the kernel of the operator $-\frac{1}{2}I + C + K^*$ on $L^p(\partial\Omega)$. Again, by Proposition 4.8, this is a finite dimensional subspace of $L^p(\partial\Omega)$. Moreover, since the operator in question has index zero, we conclude that number of the (necessary) linear conditions which the boundary data must satisfy coincide with dimension of the space of null-solutions. Hence, the problem in question is Fredholm solvable, with index zero. □

4.3 Regular boundary problems for first order elliptic systems

Suppose $\Omega \subset M$ a $\mathrm{Lip} \cap \mathrm{vmo}_1$ domain, and let $\mathcal{D}$ be a first order elliptic differential operator on M. It is permissible that $\mathcal{D}$ acts on sections of a vector bundle $E \to M$. In local coordinates, assume that

$$\mathcal{D}u(x) = \sum_j A_j(x)\partial_j u(x) + B(x)u(x), \quad \text{where } \ A_j \in \mathscr{C}^2, \ B \in \mathscr{C}^1. \tag{4.3.1}$$

As in §3.5 (cf. especially Remark 3.1), we associate to $\mathcal{D}$ a Cauchy integral $\mathcal{C}_{\mathcal{D}}$ and a projection $P_{\mathcal{D}}$, which is an element of $\mathrm{OP}(L^\infty \cap \mathrm{vmo})S^0_{\mathrm{cl}}$ in local graph coordinates.

When Ω is smooth, there is a well-established theory of regular boundary problems associated to $\mathcal{D}$ (though sometimes regular boundary conditions do not exist). We want to investigate the situation where $\Omega \subset M$ is a $\mathrm{Lip} \cap \mathrm{vmo}_1$ domain.

Let $F \to \partial^*\Omega$ be an $L^\infty \cap \mathrm{vmo}$ vector bundle, of rank k, so F is locally trivializable, to $\mathbb{C}^k \times \mathcal{O}$, with transition matrices in $L^\infty \cap \mathrm{vmo}$. Let

$$B : L^p(\partial\Omega, E) \longrightarrow L^p(\partial\Omega, F) \tag{4.3.2}$$

be an operator that, in local graph coordinates and local trivializations of E and F, satisfies

$$B \in \mathrm{OP}(L^\infty \cap \mathrm{vmo})S^0_{\mathrm{cl}}. \tag{4.3.3}$$

We can use analogues of (3.1.6)–(3.1.10) to define

$$\sigma_B(x,\xi) : E_x \longrightarrow F_x, \tag{4.3.4}$$

for almost all $(x,\xi) \in T^*\partial^*\Omega \setminus 0$. Extending the setup used when $\partial\Omega$ is smooth, we propose the following criterion for regularity:

$$\sigma_B(x,\xi) : \sigma_{P_{\mathcal{D}}}(x,\xi)E_x \longrightarrow F_x, \ \text{isomorphism, for a.e. } \ (x,\xi) \in T^*\partial^*\Omega \setminus 0, \tag{4.3.5}$$

and there exists $C > 0$ such that, for almost all $(x,\xi) \in T^*\partial^*\Omega \setminus 0$,

$$v \in E_x, \ \sigma_{P_{\mathcal{D}}} v = v \Longrightarrow \|\sigma_B(x,\xi)v\| \geq C\|v\|. \tag{4.3.6}$$

Note that (4.3.5)–(4.3.6) is equivalent to (4.3.6) alone plus

$$\dim \sigma_{P_{\mathcal{D}}}(x,\xi)E_x = \dim F_x. \tag{4.3.7}$$

Also, $\sigma_{P_{\mathcal{D}}}(x,-\xi) = I - \sigma_{P_{\mathcal{D}}}(x,\xi)$, so if $\dim \partial\Omega \geq 2$, the left side of (4.3.7) is equal to $(1/2)\dim E_x$.

Here is our basic Fredholm result.

Proposition 4.10. *Assume $\Omega \subset M$ is a $\mathrm{Lip} \cap \mathrm{vmo}_1$ domain, and suppose $\mathcal{D} : E \to E$ is a first order elliptic differential operator, as in* (4.3.1)*. Under the hypotheses* (4.3.5)–(4.3.6)*, the operator*

$$B : P_{\mathcal{D}} L^p(\partial\Omega, E) \longrightarrow L^p(\partial\Omega, F) \ \textit{ is Fredholm}, \tag{4.3.8}$$

for each $p \in (1,\infty)$.

Proof. The hypotheses imply that $\sigma_{BP_{\mathcal{D}}}(x,\xi) : E_x \to F_x$ is surjective for almost every (x,ξ), and furthermore

$$BP_{\mathcal{D}}P_{\mathcal{D}}^*B^* \in \mathrm{OP}(L^\infty \cap \mathrm{vmo})S^0_{\mathrm{cl}} \text{ is elliptic.} \tag{4.3.9}$$

Hence B has a right Fredholm inverse, so B in (4.3.8) has closed range, of finite codimension. Also,

$$f \in P_{\mathcal{D}}L^p(\partial\Omega, E), \quad Bf = 0 \tag{4.3.10}$$

is equivalent to

$$\begin{pmatrix} B \\ I - P_{\mathcal{D}} \end{pmatrix} f = 0, \quad f \in L^p(\partial\Omega, E), \tag{4.3.11}$$

and the operator on the left side of (4.3.11) (call it Q) is an element of $\mathrm{OP}(L^\infty \cap \mathrm{vmo})S^0_{\mathrm{cl}}$ (mod compacts) with symbol $\sigma_Q(x,\xi)$ injective, and furthermore

$$Q^*Q \in \mathrm{OP}(L^\infty \cap \mathrm{vmo})S^0_{\mathrm{cl}} \text{ is elliptic.} \tag{4.3.12}$$

Thus Q has a left Fredholm inverse, so its null space in $L^p(\partial\Omega, E)$ is finite dimensional. This proves (4.3.8). $\square$

Theorem 4.11. *Under the hypotheses of Proposition 4.10, the boundary problem*

$$\begin{cases} \mathcal{D}u = 0 \ \text{ on } \Omega, \\ \mathcal{N}u \in L^p(\partial\Omega), \\ Bu = f \in L^p(\partial\Omega, F), \end{cases} \tag{4.3.13}$$

is Fredholm solvable, for each $p \in (1,\infty)$.

Proof. To restate the result, consider

$$\mathcal{H}^p(\Omega, \mathcal{D}) := \big\{u \in \mathscr{C}^1(\Omega, E) : \mathcal{D}u = 0 \,\mathrm{on}\, \Omega,\ \mathcal{N}u \in L^p(\partial\Omega)\big\}. \tag{4.3.14}$$

In [24], a Fatou type lemma is established showing that each $u \in \mathcal{H}^p(\Omega, \mathcal{D})$ has a boundary trace, provided Ω is a regular SKT domain. From there, results in [23, §3.1] (see also [24]) imply that the boundary trace yields an isomorphism

$$\tau : \mathcal{H}^p(\Omega, \mathcal{D}) \xrightarrow{\approx} P_{\mathcal{D}}L^p(\partial\Omega, E), \tag{4.3.15}$$

for $p \in (1,\infty)$. The assertion of Theorem 4.11 is that if B satisfies the hypotheses of Proposition 4.10, then

$$B \circ \tau : \mathcal{H}^p(\Omega, \mathcal{D}) \longrightarrow L^p(\partial\Omega, F) \text{ is Fredholm.} \tag{4.3.16}$$

In light of (4.3.15), the result (4.3.16) is equivalent to (4.3.8). $\square$

As we have mentioned, sometimes $\mathcal{D}$ has no boundary conditions of the form (4.3.2)–(4.3.4) satisfying the regularity condition (4.3.5)–(4.3.6). In §4.4 we shall give important examples (well-known for smooth boundaries) of regular boundary conditions for $\mathcal{D} = d + d^*$, acting on differential forms. Here, we record a simple example (also well known) of a first order elliptic operator with no such regular boundary condition. Namely, we take a bounded $\Omega \subset \mathbb{R}^2$ (possibly with smooth boundary) and set

$$\mathcal{D} = \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}, \tag{4.3.17}$$

acting on complex valued u, so $E_x = \mathbb{C}$. In this case, $\sigma_{\mathcal{D}}(x,\xi)u = i(\xi_1 + i\xi_2)u$, or, if we identify $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ with $\xi_1 + i\xi_2 \in \mathbb{C}$, $\sigma_{\mathcal{D}}(x,\xi)u = i\xi u$, hence

$$M(x,\xi) = \nu^{-1}\xi. \tag{4.3.18}$$

Now ξ runs over the orthogonal complement of ν, i.e., over real multiples of $i\nu$. We have

$$M(x, i\nu) = i, \quad M(x, -i\nu) = -i, \tag{4.3.19}$$

so

$$P_+\big(M(x, i\nu)\big) = I, \quad P_+\big(M(x, -i\nu)\big) = 0. \tag{4.3.20}$$

Since the ranges have different dimensions, there is no way to achieve (4.3.5) both for $\xi = i\nu$ and for $\xi = -i\nu$.

Returning to the setting of Proposition 4.10 and Theorem 4.11, we see from (4.3.9) that the operator B in (4.3.8) has a right Fredholm inverse that is an element of $\mathrm{OP}(L^\infty \cap \mathrm{vmo})S^0_{\mathrm{cl}}$, and that this operator is independent of $p \in (1,\infty)$. Since B in (4.3.8) is Fredholm, this right Fredholm inverse is also a left Fredholm inverse, for each $p \in (1,\infty)$. Call it

$$H : L^p(\partial\Omega, F) \longrightarrow P_{\mathcal{D}} L^p(\partial\Omega, E). \tag{4.3.21}$$

Using this observation, we can prove the following.

Proposition 4.12. *Under the hypotheses of Proposition 4.10, the index of B in* (4.3.8)*, hence the index of $B \circ \tau$ in* (4.3.11)*, is independent of p.*

Proof. Setting $V_p = P_{\mathcal{D}} L^p(\partial\Omega, E)$ and $W_p = L^p(\partial\Omega, F)$, our setup is

$$B : V_p \to W_p, \quad H : W_p \to V_p, \quad \text{Fredholm inverses}, \tag{4.3.22}$$

for $p \in (1,\infty)$. Setting

$$\mathrm{Ker}_p\, B := \big\{ f \in V_p : \ Bf = 0 \big\}, \quad \mathrm{Coker}_p\, B := \big\{ \varphi \in W_p' : \ B^*\varphi = 0 \big\}, \tag{4.3.23}$$

we have

$$\begin{aligned} 1 < p < q < \infty &\Longrightarrow \mathrm{Ker}_q\, B \subset \mathrm{Ker}_p\, B,\ \mathrm{Coker}_p\, B \subset \mathrm{Coker}_q\, B \\ &\Longrightarrow \mathrm{index}_q\, B \le \mathrm{index}_p\, B. \end{aligned} \tag{4.3.24}$$

The same argument gives

$$1 < p < q < \infty \Longrightarrow \mathrm{index}_q\, H \le \mathrm{index}_p\, H, \tag{4.3.25}$$

and since $\mathrm{index}_p\, B = -\mathrm{index}_p\, H$, we have

$$1 < p, q < \infty \Longrightarrow \mathrm{index}_p\, B = \mathrm{index}_q\, B, \tag{4.3.26}$$

as desired. $\square$

The results (4.3.24)–(4.3.26) also imply that

$$1 < p, q < \infty \Longrightarrow \operatorname{Ker}_p B = \operatorname{Ker}_q B. \tag{4.3.27}$$

Let us set

$$\mathcal{H}^p_B(\Omega) := \{u \in \mathcal{H}^p(\Omega, \mathcal{D}) : \, Bu = 0 \ \text{ on } \ \partial\Omega\}. \tag{4.3.28}$$

Then the isomorphism (4.3.15) gives

$$\tau : \mathcal{H}^p_B(\Omega) \xrightarrow{\approx} P_{\mathcal{D}} L^p(\partial\Omega, E) \cap \operatorname{Ker} B = \operatorname{Ker}_p B. \tag{4.3.29}$$

Thus (4.3.27) yields the following.

Corollary 4.13. *Under the hypotheses of Proposition 4.10, the space $\mathcal{H}^p_B(\Omega)$ defined in* (4.3.28) *is independent of $p \in (1, \infty)$.*

4.4 Absolute and relative boundary conditions for the Hodge-Dirac operator

Let Ω be a $\mathrm{Lip} \cap \mathrm{vmo}_1$ domain in a smooth Riemannian manifold M. Let d denote the exterior derivative on M, denote by $\delta = d^*$ its adjoint, then define the Hodge-Dirac operator

$$\mathcal{D} := d + \delta, \tag{4.4.1}$$

acting on sections of

$$E := \Lambda^*_{\mathbb{C}} M. \tag{4.4.2}$$

We take $F := \Lambda^*_{\mathbb{C}} \partial^* \Omega$ and

$$Bu := j^* u, \tag{4.4.3}$$

the pull-back associated to $j : \partial^*\Omega \hookrightarrow M$. We claim that $(\mathcal{D}, B)$, given by (4.4.1) and (4.4.3), satisfy the regularity conditions (4.3.5)–(4.3.6), i.e.,

$$\sigma_B(x, \xi) : P_+(M(x, \xi)) E_x \longrightarrow F_x, \ \text{ isomorphically,} \tag{4.4.4}$$

for almost every $(x, \xi) \in T^*\partial^*\Omega \setminus 0$, with a uniform lower bound, of the form

$$v \in F_x, \ P_+(M(x, \xi)) v = v \Longrightarrow \|\sigma_B(x, \xi) v\| \geq C\|v\|. \tag{4.4.5}$$

Recall that $P_+(M(x, \xi))$ is the projection of E_x onto the span of the generalized eigenvectors of $M(x, \xi)$ associated with eigenvalues with positive imaginary part, annihilating those associated with eigenvalues with negative imaginary part, where

$$M(x, \xi) = \sigma_{\mathcal{D}}(x, \nu)^{-1} \sigma_{\mathcal{D}}(x, \xi). \tag{4.4.6}$$

Checking (4.4.4)–(4.4.5) is a purely algebraic problem, and to do this algebra, it suffices to take the case

$$M := \mathbb{R}^{n+1}, \qquad \Omega := \{x \in \mathbb{R}^{n+1} : \, x_{n+1} < 0\}. \tag{4.4.7}$$

Let $\wedge$ and $\vee$ denote, respectively, the exterior and interior product of forms. The following calculation shows that we have symbols independent of x:

$$\sigma_{\mathcal{D}}(\xi) u = i\xi \wedge u - i\xi \vee u, \quad \sigma_B(\xi) u = j^* u = \nu \vee (\nu \wedge u). \tag{4.4.8}$$

In addition, $\sigma_{\mathcal{D}}(\xi)^2 = |\xi|^2 I$ and, more generally, the anti-commutator identity holds:

$$\sigma_{\mathcal{D}}(\xi)\sigma_{\mathcal{D}}(\eta) + \sigma_{\mathcal{D}}(\eta)\sigma_{\mathcal{D}}(\xi) = 2\langle \xi, \eta\rangle I. \tag{4.4.9}$$

Consequently $\sigma_{\mathcal{D}}(\nu)^{-1} = \sigma_{\mathcal{D}}(\nu)$ and, for $\xi \in T^*\partial\Omega \setminus 0$,

$$M(\xi) = \sigma_{\mathcal{D}}(\nu)\sigma_{\mathcal{D}}(\xi) = -\sigma_{\mathcal{D}}(\xi)\sigma_{\mathcal{D}}(\nu), \tag{4.4.10}$$

hence

$$M(\xi)^2 = -|\xi|^2 I, \tag{4.4.11}$$

so

$$\operatorname{Spec} M(\xi) = \{i|\xi|, -i|\xi|\}. \tag{4.4.12}$$

Note that if ξ, η belong to $T^*\partial\Omega = \mathbb{R}^n$ and have the same length, then $M(\xi)$ and $M(\eta)$ are conjugate, if $n \geq 2$, since then one can pass from ξ to η by an element of $SO(n)$. On the other hand, $M(-\xi) = -M(\xi)$. It follows that

$$\dim P_+(M(\xi)) = \tfrac{1}{2}\dim E_x = \dim F_x, \tag{4.4.13}$$

for all $\xi \neq 0$. For $n = 1$, this can be checked by a simple direct calculation.

Having this, all we need to show to establish (4.4.4)–(4.4.5) is that

$$v \in \Lambda^*_{\mathbb{C}}\mathbb{R}^{n+1}, \quad \xi \in \mathbb{R}^n, \quad |\xi| = 1, \quad M(\xi)v = iv, \quad j^*v = 0 \tag{4.4.14}$$

implies

$$v = 0. \tag{4.4.15}$$

Indeed, (4.4.14) implies

$$\sigma_{\mathcal{D}}(\xi)v = i\sigma_{\mathcal{D}}(\nu)v = -\nu \wedge v + \nu \vee v, \tag{4.4.16}$$

hence, since $j^*v = 0$ forces $\nu \wedge v = 0$, we obtain

$$\sigma_{\mathcal{D}}(\xi)v = \nu \vee v. \tag{4.4.17}$$

Now the right-hand side of (4.4.17) belongs to $\Lambda^*_{\mathbb{C}}\mathbb{R}^n$. But if $\nu \wedge v = 0$ and $\xi \in \mathbb{R}^n \setminus 0$, the left-hand side of (4.4.17) cannot belong to $\Lambda^*_{\mathbb{C}}\mathbb{R}^n$, unless it is zero. This implies $\sigma_{\mathcal{D}}(\xi)v = 0$, and hence (4.4.15) follows.

A similar argument applies if we replace B in (4.4.3) by

$$Bu = \nu \vee u\Big|^{\text{n.t.}}_{\partial\Omega}. \tag{4.4.18}$$

Then we need to show that

$$v \in \Lambda^*_{\mathbb{C}}\mathbb{R}^{n+1}, \quad \xi \in \mathbb{R}^n, \quad |\xi| = 1, \quad M(\xi)v = iv, \quad \nu \vee v = 0 \tag{4.4.19}$$

implies (4.4.15). Indeed, (4.4.19) implies

$$\sigma_{\mathcal{D}}(\xi)v = -\nu \wedge v. \tag{4.4.20}$$

If $\nu \vee v = 0$ and $\xi \in \mathbb{R}^n \setminus 0$, one cannot factor out a ν on the left side of (4.4.20) unless this term vanishes, so again we get (4.4.15).

The boundary condition (4.4.3) is called the relative boundary condition for $d + \delta$, and (4.4.18) is called the absolute boundary condition for $d + \delta$. The arguments above establish the following.

Proposition 4.14. *The absolute boundary condition* (4.4.18) *and the relative boundary condition* (4.4.3) *are each regular boundary conditions for the elliptic operator $d+\delta$. Consequently, specializing* (4.3.28)*, the spaces*

$$\begin{aligned}\mathcal{H}_A(\Omega) &:= \Big\{u \in \mathcal{H}^p(\Omega, d+\delta) :\, \nu \vee u\Big|_{\partial\Omega}^{\rm n.t.} = 0\Big\},\\ \mathcal{H}_R(\Omega) &:= \Big\{u \in \mathcal{H}^p(\Omega, d+\delta) :\, \nu \wedge u\Big|_{\partial\Omega}^{\rm n.t.} = 0\Big\},\end{aligned} \tag{4.4.21}$$

where $p \in (1,\infty)$ and, as in (4.3.14),

$$\mathcal{H}^p(\Omega, d+\delta) := \big\{u \in \mathscr{C}^1(\Omega, \Lambda^*_{\mathbb{C}}) :\, (d+\delta)u = 0 \text{ in } \Omega,\ \mathcal{N}u \in L^p(\partial\Omega)\big\}, \tag{4.4.22}$$

are finite dimensional. Furthermore, by Corollary 4.13, the spaces in (4.4.21) *are independent of $p \in (1,\infty)$.*

Here, $\Lambda^*_{\mathbb{C}} := \oplus_{\ell=0}^n \Lambda^\ell_{\mathbb{C}}$, where $n := \dim\Omega$. We also set

$$\Lambda^o_{\mathbb{C}} := \bigoplus_{\ell \text{ odd}} \Lambda^\ell_{\mathbb{C}}, \quad \Lambda^e_{\mathbb{C}} := \bigoplus_{\ell \text{ even}} \Lambda^\ell_{\mathbb{C}}, \tag{4.4.23}$$

$$\mathcal{H}^p_\sigma(\Omega, d+\delta) := \mathcal{H}^p(\Omega, d+\delta) \cap \mathscr{C}^0(\Omega, \Lambda^\sigma_{\mathbb{C}}), \quad \sigma = o \text{ or } e, \tag{4.4.24}$$

$$\mathcal{H}^\sigma_b(\Omega) := \mathcal{H}_b(\Omega) \cap \mathscr{C}^0(\Omega, \Lambda^\sigma_{\mathbb{C}}), \quad b = A \text{ or } R, \quad \sigma = o \text{ or } e. \tag{4.4.25}$$

Note that

$$\begin{aligned} d+\delta &: \mathscr{C}^1(\Omega, \Lambda^o_{\mathbb{C}}) \longrightarrow \mathscr{C}^0(\Omega, \Lambda^e_{\mathbb{C}}),\\ d+\delta &: \mathscr{C}^1(\Omega, \Lambda^e_{\mathbb{C}}) \longrightarrow \mathscr{C}^0(\Omega, \Lambda^o_{\mathbb{C}}),\end{aligned} \tag{4.4.26}$$

so

$$\mathcal{H}^p(\Omega, d+\delta) = \mathcal{H}^p_e(\Omega, d+\delta) \oplus \mathcal{H}^p_o(\Omega, d+\delta), \tag{4.4.27}$$

$$\mathcal{H}_b(\Omega) = \mathcal{H}^e_b(\Omega) \oplus \mathcal{H}^o_b(\Omega), \quad b = A \text{ or } R. \tag{4.4.28}$$

In this vein, we wish to note that if we also consider

$$\widetilde{\mathcal{H}}_A(\Omega) := \Big\{u \in \mathcal{H}^p(\Omega, d\oplus\delta) :\, \nu \vee u\Big|_{\partial\Omega}^{\rm n.t.} = 0\Big\}, \tag{4.4.29}$$

$$\widetilde{\mathcal{H}}_R(\Omega) := \Big\{u \in \mathcal{H}^p(\Omega, d\oplus\delta) : \nu \wedge u\Big|_{\partial\Omega}^{\rm n.t.} = 0\Big\}, \tag{4.4.30}$$

where

$$\mathcal{H}^p(\Omega, d\oplus\delta) := \big\{u \in \mathscr{C}^1(\Omega, \Lambda^*_{\mathbb{C}}) : du = \delta u = 0 \text{ on } \Omega,\ \mathcal{N}u \in L^p(\partial\Omega)\big\}, \tag{4.4.31}$$

then from [25, Theorem 6.1] it follows that

$$\widetilde{\mathcal{H}}_A(\Omega) = \mathcal{H}_A(\Omega) \quad \text{and} \quad \widetilde{\mathcal{H}}_R(\Omega) = \mathcal{H}_R(\Omega). \tag{4.4.32}$$

In more detail, (4.4.32) was demonstrated for p close to 2 in [25], in the setting of a general Lipschitz domain. However, the independence of $\mathcal{H}_A(\Omega)$ and $\mathcal{H}_R(\Omega)$ from p, plus the obvious inclusions $\widetilde{\mathcal{H}}_A(\Omega) \subset \mathcal{H}_A(\Omega)$ and $\widetilde{\mathcal{H}}_R(\Omega) \subset \mathcal{H}_R(\Omega)$, imply that $\widetilde{\mathcal{H}}_A(\Omega)$ and $\widetilde{\mathcal{H}}_R(\Omega)$ are also independent of p.

A Auxiliary results

In this appendix, consisting of several subsections, we collect a number of auxiliary results that are useful in the body of the paper.

A.1 Spectral theory for the Dirichlet Laplacian

Specifically, fix an arbitrary bounded open set $\mathcal{O} \subseteq \mathbb{R}^n$ and, for any given $p \in (1, \infty)$ and $k \in \mathbb{Z}$, denote by $W^{k,p}(\mathcal{O})$ the standard L^p-based Sobolev space of smoothness order k. Also, let $\mathring{W}^{k,p}(\mathcal{O})$ be the closure of $\mathscr{C}_0^\infty(\mathcal{O})$ in $W^{k,p}(\mathcal{O})$.

Let Δ_D be the realization of the Laplacian with (homogeneous) Dirichlet boundary condition as an unbounded linear operator in the context of the Hilbert space $L^2(\mathcal{O})$, with domain

$$\mathrm{Dom}\,(\Delta_D) := \{u \in \mathring{W}^{1,2}(\mathcal{O}) : \Delta u \in L^2(\mathcal{O})\}. \tag{A.1.1}$$

Then $-\Delta_D$ is a nonnegative self-adjoint operator mapping $\mathrm{Dom}\,(\Delta_D)$ isomorphically onto $L^2(\mathcal{O})$, and its inverse

$$G_D := (-\Delta_D)^{-1} : L^2(\mathcal{O}) \longrightarrow L^2(\mathcal{O}) \tag{A.1.2}$$

is self-adjoint, nonnegative, and compact. In particular, $-\Delta_D$ has a pure point spectrum

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \lambda_{j+1} \leq \cdots \tag{A.1.3}$$

listed according to their (finite) multiplicities. See, e.g., [8, p. 82].

Let us temporarily write $\lambda_j(\mathcal{O})$ in place of λ_j in order to emphasize the dependence on the underlying domain $\mathcal{O}$. The classical Rayleigh-Ritz min-max principle asserts (cf., e.g., [8, Theorem 10, p. 102]) that for each $j \in \mathbb{N}$,

$$\lambda_j(\mathcal{O}) = \min_{\substack{V_j \subseteq \mathring{W}^{1,2}(\mathcal{O}) \\ \dim V_j = j}} \ \max_{u \in V_j \setminus \{0\}} \frac{\int_{\mathcal{O}} |\nabla u|^2}{\int_{\mathcal{O}} |u|^2}. \tag{A.1.4}$$

Assume now that $\widetilde{\mathcal{O}}$ is a bounded open subset of $\mathbb{R}^n$ such that $\mathcal{O} \subseteq \widetilde{\mathcal{O}}$. Given that extension by zero is a well-defined norm-preserving mapping from $\mathring{W}^{1,2}(\mathcal{O})$ into $\mathring{W}^{1,2}(\widetilde{\mathcal{O}})$, it readily follows from (A.1.4) that the following domain monotonicity property holds:

$$\lambda_j(\mathcal{O}) \geq \lambda_j(\widetilde{\mathcal{O}}), \qquad \forall\, j \in \mathbb{N}. \tag{A.1.5}$$

In this vein, let us also mention that each $\lambda_j(\mathcal{O})$ is invariant with respect to translations and rotations of $\mathcal{O}$, and one has the scaling property

$$\lambda_j(c\mathcal{O}) = c^{-2}\lambda_j(\mathcal{O}), \qquad \forall\, c \in (0, \infty),\ \ \forall\, j \in \mathbb{N}. \tag{A.1.6}$$

Finally, pick a complete set of normalized eigenfunctions $\{\vartheta_j\}_{j \in \mathbb{N}} \subset L^2(\mathcal{O})$ for $-\Delta_D$. Thus,

$$\vartheta_j \in \mathring{W}^{1,2}(\mathcal{O}), \quad \|\vartheta_j\|_{L^2(\mathcal{O})} = 1, \quad \text{and} \quad -\Delta\vartheta_j = \lambda_j\vartheta_j, \quad \text{for each} \ \ j \in \mathbb{N}. \tag{A.1.7}$$

Lemma A.1. *Let $\mathcal{O}$ be a bounded open subset of $\mathbb{R}^n$.*

Then there exist $c_1, c_2 \in (0, \infty)$ depending only on n and $\mathcal{O}$ such that

$$c_1 j^{2/n} \leq \lambda_j \leq c_2 j^{2/n} \ \ \textit{for each} \ \ j \in \mathbb{N}. \tag{A.1.8}$$

Also, there exists $C_{\mathcal{O},n} \in (0,\infty)$ *with the property that*

$$\|\vartheta_j\|_{L^\infty(\mathcal{O})} \leq C_{\mathcal{O},n}\, j^{1/2+2/n} \quad \textit{for each} \quad j \in \mathbb{N}. \tag{A.1.9}$$

Moreover, for each $j \in \mathbb{N}$ *one has*

$$\vartheta_j \in \mathscr{C}^\infty_{loc}(\mathcal{O}) \tag{A.1.10}$$

and for every compact subset K *of* $\mathcal{O}$ *and every multi-index* $\alpha \in \mathbb{N}_0^n$ *there exists a constant* $C_{\mathcal{O},K,\alpha} \in (0,\infty)$ *with the property that*

$$\|\partial^\alpha \vartheta_j\|_{L^\infty(K)} \leq C_{\mathcal{O},K,\alpha}\, j^{1/2+2/n}. \tag{A.1.11}$$

Proof. When $\mathcal{O}$ is the cube $(0,1)^n$ in $\mathbb{R}^n$, the pure point spectrum of the Dirichlet Laplacian is given by

$$\{\lambda_j\big((0,1)^n\big)\}_{j\in\mathbb{N}} = \big\{4\pi^2(k_1^2 + \cdots + k_n^2) : \, k_i \in \mathbb{N}, \ 1 \leq i \leq n\big\}, \tag{A.1.12}$$

an identification that takes into account multiplicities. From this one can deduce Weyl's asymptotic formula

$$\lambda_j\big((0,1)^n\big) \approx \frac{4\pi^2 j^{2/n}}{\pi^{n/2}\Gamma(n/2+1)}, \tag{A.1.13}$$

valid for large values of $j \in \mathbb{N}$, and the estimates in (A.1.8) follow in this scenario from (A.1.13). The general situation when $\mathcal{O}$ is an arbitrary bounded open set in $\mathbb{R}^n$ may be then handled based on the special case just treated and the comments in (A.1.5)-(A.1.6).

The operator G_D in (A.1.2) is an integral operator whose kernel is minus the Green function for $\mathcal{O}$, i.e.,

$$G_D u(x) = -\int_{\mathcal{O}} G(x,y)u(y)\,dy, \qquad x \in \mathcal{O}, \tag{A.1.14}$$

for each $u \in L^2(\mathcal{O})$. Since (cf. [11]) we have

$$|G(x,y)| \leq \frac{C_n}{|x-y|^{n-2}}, \qquad x,y \in \mathcal{O}, \tag{A.1.15}$$

(assuming $n > 2$; the case $n = 2$, when a logarithm is involved, is treated analogously), it follows that G_D behaves like a fractional integral operator of order 2, hence (cf. [32])

$$\begin{gathered} G_D : L^p(\mathcal{O}) \longrightarrow L^q(\mathcal{O}) \ \text{ linearly and boundedly} \\ \text{if either } q < \infty \text{ and } 1/q \geq 1/p - 2/n, \text{ or } q = \infty \text{ and } p > n/2. \end{gathered} \tag{A.1.16}$$

Iterating, it follows that

$$(G_D)^k : L^2(\mathcal{O}) \longrightarrow L^\infty(\mathcal{O}) \ \text{ boundedly if } \ k > n/4. \tag{A.1.17}$$

On the other hand, for each fixed $j \in \mathbb{N}$, from (A.1.7) we have $\vartheta_j = \lambda_j G_D \vartheta_j$ which, inductively, implies $\vartheta_j = \lambda_j^k (G_D)^k \vartheta_j$ for each $k \in \mathbb{N}$. Consequently, if $k := \big[n/4\big] + 1$ then $k \in \mathbb{N}$ satisfies $k \in \big(n/4, n/4+1\big]$, hence we may estimate

$$\begin{aligned} \|\vartheta_j\|_{L^\infty(\mathcal{O})} &= \|\lambda_j^k (G_D)^k \vartheta_j\|_{L^\infty(\mathcal{O})} \\ &\leq \|(G_D)^k\|_{\mathcal{L}(L^2(\mathcal{O}),L^\infty(\mathcal{O}))} \lambda_j^k \|\vartheta_j\|_{L^2(\mathcal{O})} \\ &\leq C_{\mathcal{O},n}\, j^{2k/n} \leq C_{\mathcal{O},n}\, j^{1/2+2/n}, \end{aligned} \tag{A.1.18}$$

by (A.1.17), (A.1.7), and (A.1.8). This proves (A.1.9).

Finally, (A.1.10)-(A.1.11) follow from (A.1.7), (A.1.9), and elliptic regularity. □

A.2 Truncating singular integrals

If $U \subseteq \mathbb{R}^n$, call $\Phi : U \to \mathbb{R}^m$ bi-Lipschitz if there exist $0 < M_1 \leq M_2 < \infty$ such that

$$M_1|x-y| \leq |\Phi(x) - \Phi(y)| \leq M_2|x-y|, \qquad \forall\, x, y \in U. \tag{A.2.1}$$

When U is an open set, it is known (cf. [29]) that necessarily $m \geq n$, Φ is an open mapping, the Jacobian matrix $D\Phi = (\partial_k \Phi_j)_{1\leq j\leq m,\, 1\leq k\leq n}$ exists a.e. in U and

$$\operatorname{rank} D\Phi(x) = n \text{ for a.e. } x \in U. \tag{A.2.2}$$

Lemma A.2. *Let $A : \mathbb{R}^n \to \mathbb{R}^m$ and $B : \mathbb{R}^n \to \mathbb{R}^{m'}$ be functions satisfying*

$$|A(x) - A(y)| \leq M|x-y| \quad \textit{and} \tag{A.2.3}$$

$$M^{-1}|x-y| \leq |B(x) - B(y)| \leq M|x-y|, \quad \forall\, x, y \in \mathbb{R}^n, \tag{A.2.4}$$

for some positive constant M. Also let $F : \mathbb{R}^m \to \mathbb{R}$ be an odd function of class $\mathscr{C}^1$. Finally, fix a point $x \in \mathbb{R}^n$ where both $DA(x), DB(x)$ exist, $\operatorname{rank} DB(x) = n$*, and for each $\varepsilon > 0$ consider*

$$\begin{aligned} U(\varepsilon) &:= \{y \in \mathbb{R}^n : 1 > |x-y| > \varepsilon\}, \\ V(\varepsilon) &:= \{y \in \mathbb{R}^n : |DB(x)(x-y)| > \varepsilon,\ |x-y| < 1\}, \\ W(\varepsilon) &:= \{y \in \mathbb{R}^n : |B(x) - B(y)| > \varepsilon,\ |x-y| < 1\}. \end{aligned} \tag{A.2.5}$$

Then whenever any of the following three limits exists (in $\mathbb{R}$)

$$\lim_{\varepsilon \downarrow 0} \int_{U(\varepsilon)} \frac{1}{|x-y|^n} F\Big(\frac{A(x) - A(y)}{|x-y|}\Big)\, dy, \tag{A.2.6}$$

$$\lim_{\varepsilon \downarrow 0} \int_{V(\varepsilon)} \frac{1}{|x-y|^n} F\Big(\frac{A(x) - A(y)}{|x-y|}\Big)\, dy, \tag{A.2.7}$$

$$\lim_{\varepsilon \downarrow 0} \int_{W(\varepsilon)} \frac{1}{|x-y|^n} F\Big(\frac{A(x) - A(y)}{|x-y|}\Big)\, dy, \tag{A.2.8}$$

it follows that all exist and are equal.

Proof. Without loss of generality we can take $x = 0$ and assume that $A(0) = 0$, $B(0) = 0$. Note that as a consequence of this normalization and (A.2.3) we have

$$\frac{|A(y)|}{|y|} \leq M, \qquad \forall\, y \in \mathbb{R}^n \setminus \{0\}. \tag{A.2.9}$$

The fact that $DA(0), DB(0)$ exist implies that we can find a function $\eta : (0, \infty) \to [0, \infty)$ with the property that $\eta(t) \downarrow 0$ as $t \downarrow 0$ and

$$|B(y) - DB(0)y| + |A(y) - DA(0)y| \leq |y|\, \eta(|y|), \qquad \forall\, y \in \mathbb{R}^n. \tag{A.2.10}$$

In particular,

$$\begin{aligned} |A(y) + A(-y)| &= \big|(A(y) - DA(0)y) + (A(-y) - DA(0)(-y))\big| \\ &\leq |A(y) - DA(0)y| + |A(-y) - DA(0)(-y)| \\ &\leq 2|y|\, \eta(|y|), \qquad \forall\, y \in \mathbb{R}^n. \end{aligned} \tag{A.2.11}$$

Recall that the matrix $DB(0)$ is assumed to have rank n. Hence, $\|DB(0)\| > 0$ and if for each $\varepsilon > 0$ we now define

$$\Delta(\varepsilon) := \big\{y \in \mathbb{R}^n : \varepsilon \geq |y| \geq \varepsilon/\|DB(0)\|\big\} \tag{A.2.12}$$

then

$$V(\varepsilon) \setminus U(\varepsilon) \subseteq \Delta(\varepsilon), \qquad \forall\, \varepsilon > 0. \tag{A.2.13}$$

Observing that $U(\varepsilon)$ and $V(\varepsilon)$ are symmetric with respect to the origin, employing the properties of F and η, and keeping in mind (A.2.10), (A.2.13), (A.2.11), and (A.2.9), we may use the mean value theorem in order to estimate the absolute value of the difference of the limits in (A.2.6) and (A.2.7) by

$$\begin{aligned}
&\lim_{\varepsilon \downarrow 0} \left| \int_{V(\varepsilon)\setminus U(\varepsilon)} \frac{1}{|y|^n} F\Big(\frac{A(y)}{|y|}\Big)\, dy \right| && \text{(A.2.14)}\\
&\quad = \lim_{\varepsilon \downarrow 0} \frac{1}{2} \left| \int_{V(\varepsilon)\setminus U(\varepsilon)} \frac{1}{|y|^n} \Big[F\Big(\frac{A(y)}{|y|}\Big) + F\Big(\frac{A(-y)}{|y|}\Big)\Big]\, dy \right| \\
&\quad = \lim_{\varepsilon \downarrow 0} \frac{1}{2} \left| \int_{V(\varepsilon)\setminus U(\varepsilon)} \frac{1}{|y|^n} \Big[F\Big(\frac{A(y)}{|y|}\Big) - F\Big(-\frac{A(-y)}{|y|}\Big)\Big]\, dy \right| \\
&\quad \leq \Big[\sup_{|\xi| \leq M} |\nabla F(\xi)|\Big] \lim_{\varepsilon \downarrow 0} \int_{\Delta(\varepsilon)} \eta(|y|)|y|^{-n}\, dy \\
&\quad \leq C \lim_{\varepsilon \downarrow 0} \eta(\varepsilon) = 0. && \text{(A.2.15)}
\end{aligned}$$

This proves that the limits in (A.2.6) and (A.2.7) exist simultaneously and are equal.

In order to prove the simultaneous existence and coincidence of the limits in (A.2.7) and (A.2.8), observe that for each $y \in V(\varepsilon)\backslash W(\varepsilon)$ we have $M^{-1}|y| \leq |B(y)| \leq \varepsilon$, hence $|y| \leq \varepsilon M$. That is,

$$y \in V(\varepsilon)\backslash W(\varepsilon) \Longrightarrow |y| \leq \varepsilon M. \tag{A.2.16}$$

In turn, this forces

$$|(DB)(0)y| \leq |(DB)(0)y - B(y)| + |B(y)| \leq \varepsilon M\, \eta(\varepsilon M) + \varepsilon \tag{A.2.17}$$

and, further,

$$y \in V(\varepsilon)\backslash W(\varepsilon) \Longrightarrow \varepsilon < |(DB)(0)y| \leq \varepsilon M\, \eta(\varepsilon M) + \varepsilon. \tag{A.2.18}$$

From (A.2.16) and (A.2.18) we may therefore conclude that

$$V(\varepsilon)\backslash W(\varepsilon) \subseteq Z\big[\varepsilon; M\eta(\varepsilon M)\big] \tag{A.2.19}$$

where, in general, we define

$$Z[\varepsilon; a] := \{y \in \mathbb{R}^n : \varepsilon < |DB(0)y| \leq \varepsilon a + \varepsilon\}, \qquad \forall\, \varepsilon > 0,\ \forall\, a > 0. \tag{A.2.20}$$

Let $\mathcal{H}^k_N$ be the k-dimensional Hausdorff measure in $\mathbb{R}^N$. To estimate the n-dimensional Lebesgue measure of $Z[\varepsilon; a]$, note first that for each $a > 0$ fixed,

$$Z[\varepsilon; a] = \varepsilon\, Z[1; a], \qquad \forall\, \varepsilon > 0. \tag{A.2.21}$$

On the other hand, if we set $H_n := \{DB(0)y : y \in \mathbb{R}^n\} \subseteq \mathbb{R}^{m'}$ then, since $DB(0)$ is a rank n matrix, it follows that H_n is an n-dimensional plane in $\mathbb{R}^{m'}$ and $DB(0) : \mathbb{R}^n \to H_n$ is a linear isomorphism. As such, we obtain

$$\begin{aligned}\mathcal{H}^n_n\big(Z[1;a]\big) &= \mathcal{H}^n_n\big(\{y \in \mathbb{R}^n : 1 < |DB(0)y| \leq a+1\}\big)\\ &\leq C\,\mathcal{H}^n_{m'}\big(\{z \in H_n : 1 < |z| \leq a+1\}\big).\end{aligned} \tag{A.2.22}$$

A moment's reflection shows that

$$\lim_{a\to 0^+} \mathcal{H}^n_{m'}\big(\{z \in H_n : 1 < |z| \leq a+1\}\big) = 0. \tag{A.2.23}$$

From this, (A.2.21), (A.2.19), and the fact that $\eta(\varepsilon M) \to 0$ as $\varepsilon \to 0^+$, we may finally conclude that

$$\lim_{\varepsilon\to 0^+} \frac{\mathcal{H}^n_n\big(V(\varepsilon)\backslash W(\varepsilon)\big)}{\varepsilon^n} = 0. \tag{A.2.24}$$

Since the expression $\frac{1}{|y|^n}F\Big(\frac{A(y)}{|y|}\Big)$ restricted to $V(\varepsilon)\backslash W(\varepsilon)$ is pointwise of the order ε^{-n} in a uniform fashion, we deduce from (A.2.24) that

$$\lim_{\varepsilon\to 0^+} \int_{V(\varepsilon)\backslash W(\varepsilon)} \frac{1}{|y|^n}F\Big(\frac{A(y)}{|y|}\Big)\,dy = 0, \tag{A.2.25}$$

as desired.

Finally, an argument analogous to (A.2.18) gives that

$$\varepsilon - \varepsilon M\eta(\varepsilon M) < |(DB)(0)y| \leq \varepsilon, \qquad \forall\, y \in W(\varepsilon)\backslash V(\varepsilon). \tag{A.2.26}$$

Thus, for reasons similar to those discussed above, we also have

$$\lim_{\varepsilon\to 0^+} \int_{W(\varepsilon)\backslash V(\varepsilon)} \frac{1}{|y|^n}F\Big(\frac{A(y)}{|y|}\Big)\,dy = 0, \tag{A.2.27}$$

which completes the proof of the lemma. □

The main result in this subsection, pertaining to the manner in which singular integrals are truncated, reads as follows.

Proposition A.3. *Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz function and assume that $F : \mathbb{R}^m \to \mathbb{R}$ is an odd function of class $\mathscr{C}^N$ for some sufficiently large integer $N = N(m)$. Also, suppose $B : \mathbb{R}^n \to \mathbb{R}^{m'}$ is a bi-Lipschitz function, and pick $p \in (1,\infty)$. Then for each fixed $f \in L^p(\mathbb{R}^n)$ the limit*

$$\lim_{\varepsilon\downarrow 0} \int_{\{y\in\mathbb{R}^n :\, |B(x)-B(y)|>\varepsilon\}} \frac{1}{|x-y|^n}F\Big(\frac{A(x)-A(y)}{|x-y|}\Big)f(y)\,dy, \tag{A.2.28}$$

exists at a.e. point $x \in \mathbb{R}^n$. Moreover, this limit is independent of the choice of the function B, in the sense that for each given $f \in L^p(\mathbb{R}^n)$ the limit (A.2.28) *is equal to*

$$\lim_{\varepsilon\downarrow 0} \int_{\{y\in\mathbb{R}^n :\, |x-y|>\varepsilon\}} \frac{1}{|x-y|^n}F\Big(\frac{A(x)-A(y)}{|x-y|}\Big)f(y)\,dy \tag{A.2.29}$$

for a.e. $x \in \mathbb{R}^n$.

As a preamble, we deal with a simple technical result. In the sequel, we agree to let $\mathcal{M}$ stand for the usual Hardy-Littlewood maximal operator.

Lemma A.4. *Assume that*

$$C_1|x-y| \leq \rho(x,y) \leq C_2|x-y|, \qquad \forall\, x,y \in \mathbb{R}^n, \tag{A.2.30}$$

and

$$|k(x,y)| \leq \frac{C_0}{|x-y|^n}, \qquad \forall\, x,y \in \mathbb{R}^n, \tag{A.2.31}$$

for some finite positive constants C_0, C_1, C_2. Then

$$\begin{aligned}\Delta(x) &:= \left| \int\limits_{\substack{|x-y|>\varepsilon \\ y\in\mathbb{R}^n}} k(x,y)f(y)dy - \int\limits_{\substack{\rho(x-y)>\varepsilon \\ y\in\mathbb{R}^n}} k(x,y)f(y)dy \right| \\ &\leq C_0(C_1^{-n} + C_2^n)\mathcal{M}f(x), \end{aligned} \tag{A.2.32}$$

for all $x \in \mathbb{R}^n$.

Proof. A direct size estimate gives

$$\begin{aligned}\Delta(x) &\leq \int\limits_{\substack{|x-y|>\varepsilon,\, \rho(x,y)<\varepsilon \\ y\in\mathbb{R}^n}} \frac{C_0}{|x-y|^n}|f(y)|\,dy + \int\limits_{\substack{|x-y|<\varepsilon,\, \rho(x,y)>\varepsilon \\ y\in\mathbb{R}^n}} \frac{C_0}{|x-y|^n}|f(y)|\,dy \\ &:= I + II, \end{aligned} \tag{A.2.33}$$

where the last equality defines I, II. We have:

$$I \leq \frac{C_0}{\varepsilon^n}\int_{C_1|x-y|<\varepsilon} |f(y)|\,dy \leq \frac{C_0}{C_1^n}\mathcal{M}f(x), \tag{A.2.34}$$

and

$$II \leq \frac{C_0 C_2^n}{\varepsilon^n}\int_{|x-y|<\varepsilon} |f(y)|\,dy \leq C_0 C_2^n \mathcal{M}f(x). \tag{A.2.35}$$

The desired conclusion follows. $\square$

Below, we shall also make use of the following standard result.

Lemma A.5. *Let $\{T_\varepsilon\}_{\varepsilon>0}$ be a family of operators with the following properties:*

(1) There exists a dense subset $\mathcal{V}$ of $L^p(\mathbb{R}^n)$ such that for any $f \in \mathcal{V}$ the limit $\lim\limits_{\varepsilon\to 0^+} T_\varepsilon f(x)$ exists for almost every $x \in \mathbb{R}^n$.

(2) The maximal operator $T_* f(x) := \sup\big\{|T_\varepsilon f(x)| : \varepsilon > 0\big\}$ is bounded on $L^p(\mathbb{R}^n)$.

Then, the limit $\lim\limits_{\varepsilon\to 0^+} T_\varepsilon f(x)$ exists for any $f \in L^p(\mathbb{R}^n)$ at almost any $x \in \mathbb{R}^n$, and the operator

$$Tf(x) := \lim_{\varepsilon\to 0^+} T_\varepsilon f(x) \tag{A.2.36}$$

is bounded on $L^p(\mathbb{R}^n)$.

Proof. The boundedness of the operator T is an immediate consequence of *(2)*, once we prove the existence of the limit in (A.2.36). In this regard, having fixed $f \in L^p(\mathbb{R}^n)$, we aim to show that

$$\big|\{x \in \mathbb{R}^n : \limsup_{\varepsilon\to 0^+} T_\varepsilon f(x) \neq \liminf_{\varepsilon\to 0^+} T_\varepsilon f(x)\}\big| = 0. \tag{A.2.37}$$

Fix $\theta > 0$ and consider

$$S := \Big\{x \in \mathbb{R}^n : \big|\limsup_{\varepsilon\to 0^+} T_\varepsilon f(x) - \liminf_{\varepsilon\to 0^+} T_\varepsilon f(x)\big| > \theta\Big\}. \tag{A.2.38}$$

Also, fix $\delta > 0$ and select $h \in \mathcal{V}$ such that $\|f - h\|_{L^p(\mathbb{R}^n)} < \delta$. Then

$$S \subset S_1 \cup S_2 \tag{A.2.39}$$

where

$$\begin{aligned} S_1 &:= \Big\{x \in \mathbb{R}^n : \big|\limsup_{\varepsilon\to 0^+} T_\varepsilon f(x) - \lim_{\varepsilon\to 0^+} T_\varepsilon h(x)\big| > \theta/2\Big\}, \\ S_2 &:= \Big\{x \in \mathbb{R}^n : \big|\liminf_{\varepsilon\to 0^+} T_\varepsilon f(x) - \lim_{\varepsilon\to 0^+} T_\varepsilon h(x)\big| > \theta/2\Big\}. \end{aligned} \tag{A.2.40}$$

Then the measure of the set S_1 can be estimated by

$$\begin{aligned} |S_1| &\le \big|\{x \in \mathbb{R}^n : T_*(f-h)(x) > \theta/2\}\big| \le \left(\frac{2}{\theta}\right)^p \int_{\mathbb{R}^n} |T_*(f-h)(x)|^p \, dx \\ &\le C\left(\frac{2}{\theta}\right)^p \|f-h\|^p_{L^p(\mathbb{R}^n)} \le C\left(\frac{2}{\theta}\right)^p \delta^p. \end{aligned} \tag{A.2.41}$$

Since $\delta > 0$ was arbitrary, this proves that $|S_1| = 0$. The same consideration works for the set S_2, hence also $|S| = 0$ by (A.2.39). This concludes the proof of Lemma A.5. $\square$

We are now ready to present the

Proof of Proposition A.3. For each bi-Lipschitz function B defined in $\mathbb{R}^n$, consider the truncated singular integral operator

$$T_{B,\varepsilon} f(x) := \int_{\{y\in\mathbb{R}^n : |B(x)-B(y)|>\varepsilon\}} \frac{1}{|x-y|^n} F\Big(\frac{A(x)-A(y)}{|x-y|}\Big) f(y)\, dy, \qquad x \in \mathbb{R}^n, \tag{A.2.42}$$

where $\varepsilon > 0$. The maximal operator associated with the family $\{T_{B,\varepsilon}\}_{\varepsilon>0}$ is defined as

$$T_{B,*} f(x) := \sup_{\varepsilon>0} \big|T_{B,\varepsilon} f(x)\big|, \qquad x \in \mathbb{R}^n. \tag{A.2.43}$$

In particular, corresponding to the case when $B = I$, the identity on $\mathbb{R}^n$, we have

$$T_{I,\varepsilon} f(x) = \int_{\{y\in\mathbb{R}^n : |x-y|>\varepsilon\}} \frac{1}{|x-y|^n} F\Big(\frac{A(x)-A(y)}{|x-y|}\Big) f(y)\, dy, \qquad x \in \mathbb{R}^n, \tag{A.2.44}$$

and

$$T_{I,*} f(x) = \sup_{\varepsilon>0} \big|T_{I,\varepsilon} f(x)\big|, \qquad x \in \mathbb{R}^n. \tag{A.2.45}$$

We proceed is a number of steps.

`Step 1:` *Given $p \in (1,\infty)$ there exists a constant $C \in (0,\infty)$ with the property that for each Lipschitz function $A : \mathbb{R} \to \mathbb{R}$ and for each $\varepsilon > 0$ the truncated Cauchy integral operator*

$$\mathcal{C}_{A,\varepsilon} f(x) := \int_{\{y\in\mathbb{R}:\, |x-y|>\varepsilon\}} \frac{f(y)}{x-y+i(A(x)-A(y))}\, dy, \qquad x \in \mathbb{R}, \tag{A.2.46}$$

satisfies

$$\|\mathcal{C}_{A,\varepsilon} f\|_{L^p(\mathbb{R})} \leq C\big(1+\|A'\|_{L^\infty(\mathbb{R})}\big)\|f\|_{L^p(\mathbb{R})}. \tag{A.2.47}$$

This is the Coifman, McIntosh, Meyer theorem (cf. [5]). An elegant proof is given by M. Melnikov and J. Verdera in [20].

`Step 2:` *Given $p \in (1,\infty)$ there exists a constant $C \in (0,\infty)$ with the property that if $\beta \in (1,\infty)$ and if $B : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function satisfying $\beta^{-1} < B'(x) < \beta$ for a.e. $x \in \mathbb{R}$, then for each $\varepsilon > 0$ and each $\eta \in [-1,1]$ the operator*

$$\widetilde{\mathcal{C}}_{B,\eta,\varepsilon} f(x) := \int_{\{y\in\mathbb{R}:\, |x-y|>\varepsilon\}} \frac{f(y)}{\eta(x-y)i+B(x)-B(y)}\, dy, \qquad x \in \mathbb{R}, \tag{A.2.48}$$

satisfies

$$\big\|\widetilde{\mathcal{C}}_{B,\eta,\varepsilon} f\big\|_{L^p(\mathbb{R})} \leq C\beta^4\|f\|_{L^p(\mathbb{R})}. \tag{A.2.49}$$

To prove (A.2.49), changing variables $s := B(x)$ and $t := B(y)$ allows us to write

$$\big(\widetilde{\mathcal{C}}_{B,\eta,\varepsilon} f\big)\big(B^{-1}(s)\big) = \int_{|B^{-1}(s)-B^{-1}(t)|>\varepsilon} \frac{f(B^{-1}(t))[B'(B^{-1}(t))]^{-1}}{s-t+i\eta(B^{-1}(s)-B^{-1}(t))}\, dt. \tag{A.2.50}$$

Based on this and Lemma A.4 we then obtain the pointwise estimate

$$\big|\big(\widetilde{\mathcal{C}}_{B,\eta,\varepsilon} f\big)\big(B^{-1}(s)\big)\big| \leq \big|\mathcal{C}_{\eta B^{-1},\varepsilon}\big((f/B')\circ B^{-1}\big)(s)\big| + C\beta^3\mathcal{M}f\big(B^{-1}(s)\big), \tag{A.2.51}$$

for all $s \in \mathbb{R}$. Then (A.2.49) follows from (A.2.51) with the help of (A.2.47).

`Step 3:` *Suppose $F(z)$ is an analytic function in the open strip $\{z \in \mathbb{C} : |\mathrm{Im}\, z| < 2\}$. Let $A : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function with $\|A'\|_{L^\infty(\mathbb{R})} \leq M$. Then for each $p \in (1,\infty)$ there exists a constant $C = C_p \in (0,\infty)$ such that, for each $\varepsilon > 0$, the operator*

$$K_{A,F,\varepsilon} f(x) := \int_{|x-y|>\varepsilon} \frac{1}{x-y} F\Big(\frac{A(x)-A(y)}{x-y}\Big) f(y)\, dy, \qquad x \in \mathbb{R}, \tag{A.2.52}$$

satisfies

$$\|K_{A,F,\varepsilon} f\|_{L^p(\mathbb{R})} \leq C(1+M^4)\sup\big\{|F(z)| :\, z \in \mathbb{C},\, |\mathrm{Im}\, z| < 2\big\}\|f\|_{L^p(\mathbb{R})}. \tag{A.2.53}$$

To justify (A.2.53), let $\gamma^1_\pm := \{\zeta = u \pm i :\, |u| \leq 2M\}$, $\gamma^2_\pm := \{\zeta = \pm 2M + iv :\, |v| \leq 1\}$, and set $\gamma := \gamma^1_+ \cup \gamma^2_+ \cup \gamma^1_- \cup \gamma^2_-$. Since F is analytic for $z \in \mathbb{C}$ with $|\mathrm{Im}\, z| < 2$, Cauchy's reproducing formula yields

$$F(s) = \frac{1}{2\pi i}\int_\gamma \frac{F(\zeta)}{\zeta - s}\, d\zeta = \frac{1}{2\pi i}\int_{\gamma^1_+\cup\gamma^1_-} \frac{F(\zeta)}{\zeta - s}\, d\zeta + \frac{1}{2\pi i}\int_{\gamma^2_+\cup\gamma^2_-} \frac{F(\zeta)}{\zeta - s}\, d\zeta. \tag{A.2.54}$$

Accordingly,

$$
\begin{aligned}
K_{A,F,\varepsilon}f(x) &= \frac{1}{2\pi\, i}\int_{\gamma^1_+\cup\gamma^1_-} F(\zeta)\int_{|x-y|>\varepsilon}\frac{1}{x-y}\frac{f(y)}{\zeta-\frac{A(x)-A(y)}{x-y}}\,dy\,d\zeta\\
&\quad+\frac{1}{2\pi\, i}\int_{\gamma^2_+\cup\gamma^2_-} F(\zeta)\int_{|x-y|>\varepsilon}\frac{1}{x-y}\frac{f(y)}{\zeta-\frac{A(x)-A(y)}{x-y}}\,dy\,d\zeta\\
&= I_+ + I_- + II_+ + II_-,
\end{aligned}
\tag{A.2.55}
$$

where

$$
I_\pm := \mp\frac{1}{2\pi}\int_{\gamma^1_\pm} F(\zeta)\int_{|x-y|>\varepsilon}\frac{f(y)}{x-y+i[A^\pm_\zeta(x)-A^\pm_\zeta(y)]}\,dy\,d\zeta
\tag{A.2.56}
$$

with $A^\pm_\zeta(x) := \mp[A(x)-(\operatorname{Re}\zeta)x]$, and

$$
II_\pm := \frac{1}{2\pi\, i}\int_{\gamma^2_\pm} F(\zeta)\int_{|x-y|>\varepsilon}\frac{f(y)}{(\operatorname{Im}\zeta)(x-y)i+[B^\pm(x)-B^\pm(y)]}\,dy\,d\zeta
\tag{A.2.57}
$$

with $B^\pm(x) := -[A(x)\mp 2Mx]$. At this point, the proof of (A.2.53) is concluded by invoking the results from Steps 1-2.

`Step 4:` *Suppose $F\in\mathscr{C}^N(\mathbb{R})$, $N\geq 6$ and assume that $A:\mathbb{R}\to\mathbb{R}$ is a Lipschitz function with $\|A'\|_{L^\infty(\mathbb{R})}\leq M$. Then for each $p\in(1,\infty)$ there exists a constant $C=C_p\in(0,\infty)$ such that the operator* (A.2.52) *satisfies, for each $\varepsilon>0$,*

$$
\|K_{A,F,\varepsilon}f\|_{L^p(\mathbb{R})}\leq C(1+M^4)\sup\big\{|F^{(k)}(x)|:\,|x|\leq M+1,\,0\leq k\leq 6\big\}\|f\|_{L^p(\mathbb{R})}.
\tag{A.2.58}
$$

In dealing with (A.2.58) there is no loss of generality in assuming that F is supported in the interval $[-M-1,M+1]$. With 'hat' denoting the Fourier transform we have

$$
K_{A,F,\varepsilon}f(x) = \int_{\mathbb{R}}\widehat{F}(\xi)\Big(\int_{\{y\in\mathbb{R}:\,|x-y|>\varepsilon\}}\frac{1}{x-y}e^{i\xi\frac{A(x)-A(y)}{x-y}}f(y)\,dy\Big)d\xi.
\tag{A.2.59}
$$

Note that the inner integral above is precisely the truncated Cauchy operator (A.2.46) corresponding to the choice $F(z):=\exp(iz)$ and with A replaced by ξA. Consequently, (A.2.58) follows from (A.2.59) with the help of (A.2.53).

`Step 5:` *Suppose $F\in\mathscr{C}^N(\mathbb{R}^m)$, $N\geq m+5$, F is odd, and assume that $A:\mathbb{R}^n\to\mathbb{R}^m$ is a Lipschitz function with $\|DA\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^m)}\leq M$. Then for each $p\in(1,\infty)$ there exists a constant $C=C_p\in(0,\infty)$ such that for each $\varepsilon>0$ the operator*

$$
K_{A,F,\varepsilon}f(x) := \int_{|x-y|>\varepsilon}\frac{1}{|x-y|^n}F\Big(\frac{A(x)-A(y)}{|x-y|}\Big)f(y)\,dy,\qquad x\in\mathbb{R}^n,
\tag{A.2.60}
$$

satisfies

$$
\big\|K_{A,F,\varepsilon}f\big\|_{L^p(\mathbb{R}^n)}\leq C(1+M^4)\sup\big\{|\partial^\alpha F(x)|:\,|x|\leq M+1,\,|\alpha|\leq m+5\big\}\|f\|_{L^p(\mathbb{R}^n)}.
\tag{A.2.61}
$$

In the case $n=1$, since F is odd we may write

$$
\frac{1}{|x-y|}F\Big(\frac{A(x)-A(y)}{|x-y|}\Big) = \frac{1}{x-y}F\Big(\frac{A(x)-A(y)}{x-y}\Big)
\tag{A.2.62}
$$

so (A.2.61) follows from an argument similar to the one used in the treatment of Step 4, based on writing

$$K_{A,F,\varepsilon} f(x) = \int_{\mathbb{R}^m} \widehat{F}(\xi) \Big(\int_{\{y \in \mathbb{R} :\, |x-y| > \varepsilon\}} \frac{1}{x-y} e^{i \left\langle \xi, \frac{A(x)-A(y)}{x-y} \right\rangle} f(y)\, dy \Big) d\xi \tag{A.2.63}$$

and invoking the result established in Step 3. For $n > 1$ we can reduce the problem to the one-dimensional case by the classical method of rotation.

`Step 6:` *Retain the same assumptions as in Step 5. Then there is a constant C such that*

$$\big| \big\{ x \in \mathbb{R}^n : |K_{A,F,\varepsilon} f(x)| > \lambda \big\} \big| \leq \frac{C}{\lambda} \|f\|_{L^1(\mathbb{R}^n)} \tag{A.2.64}$$

for every function $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and every positive number λ. In particular, $K_{A,F,\varepsilon}$ extends to a bounded operator from $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$ (where $L^{1,\infty}(\mathbb{R}^n)$ stands for the weak-L^1 space in $\mathbb{R}^n$).

This follows from Step 5 (with $p = 2$) and the classical Calderón-Zygmund lemma.

`Step 7:` *Retain the same assumptions as in Step 5. There exists a finite constant $C > 0$ depending only on the dimension with the property that for each fixed $\varepsilon_0 > 0$ the following Cotlar-type estimate holds*

$$K^{(\varepsilon)}_{A,F,*} f(x) \leq C \mathcal{M} f(x) + 2 \mathcal{M} \big(K_{A,F,\varepsilon_0} f \big)(x), \qquad \forall\, \varepsilon > \varepsilon_0, \tag{A.2.65}$$

for each $f \in \mathrm{Lip}_{comp}(\mathbb{R}^n)$ and each $x \in \mathbb{R}^n$, where

$$K^{(\varepsilon)}_{A,F,*} f(x) := \sup_{\varepsilon' > \varepsilon} \big| K_{A,F,\varepsilon'} f(x) \big|. \tag{A.2.66}$$

Without loss of generality, it suffices to prove (A.2.65) for $x = 0$, in which case we focus on showing that

$$\big| K_{A,F,\varepsilon} f(0) \big| \leq C \mathcal{M} f(0) + 2 \mathcal{M} (K_{A,F,\varepsilon_0}) f(0), \qquad \forall\, \varepsilon > \varepsilon_0. \tag{A.2.67}$$

Then (A.2.67) implies (A.2.65) by suitably taking the supremum.

The first step is to observe that for all $x \in \mathbb{R}^n$ and for all $\varepsilon > 0$,

$$\big| K_{A,F,\varepsilon} f(x') - K_{A,F,\varepsilon} f(x) \big| \leq C \mathcal{M} f(0), \qquad \text{provided } |x - x'| \leq \varepsilon/2. \tag{A.2.68}$$

To see that this is the case, abbreviate $k(x,y) := \frac{1}{|x-y|^n} F \left(\frac{A(x)-A(y)}{|x-y|} \right)$, then write

$$\begin{aligned} \Big| K_{A,F,\varepsilon} f(x') - K_{A,F,\varepsilon} f(x) \Big| &\leq \Big| \int_{|x-y| \geq \varepsilon} \big(k(x',y) - k(x,y) \big) f(y)\, dy \Big| \\ &\quad + \Big| \int_{|x'-y| \geq \varepsilon} k(x',y) f(y)\, dy - \int_{|x-y| \geq \varepsilon} k(x',y) f(y)\, dy \Big| \\ &=: I + II. \end{aligned} \tag{A.2.69}$$

The term II can be bounded by a multiple of $\mathcal{M} f(0)$ using the argument similar to that in Lemma A.4. The estimate for I follows from the Mean Vale Theorem, the nature of the kernel $k(x,y)$, and the standard inequality

$$\varepsilon \int_{|y| \geq \varepsilon} |y|^{-n-1} |f(y)|\, dy \leq C \mathcal{M} f(0), \qquad \forall \varepsilon > 0. \tag{A.2.70}$$

Turning to the proof of (A.2.67) in earnest, fix $\varepsilon > \varepsilon_0 > 0$ then introduce $f_1 := f\chi_{B(0,\varepsilon)}$ and set $f_2 := f - f_1$. In particular, this entails

$$K_{A,F,\varepsilon} f(0) = K_{A,F,\varepsilon_0} f_2(0). \tag{A.2.71}$$

Then for each $x \in B(0, \varepsilon/2)$, by (A.2.68) we have

$$\big|K_{A,F,\varepsilon_0} f_2(x) - K_{A,F,\varepsilon_0} f_2(0)\big| \le C\mathcal{M}f(0), \tag{A.2.72}$$

therefore,

$$|K_{A,F,\varepsilon_0} f_2(0)| \le |K_{A,F,\varepsilon_0} f(x)| + |K_{A,F,\varepsilon_0} f_1(x)| + C\mathcal{M}f(0) \quad \text{for a.e. } \ x \in B(0,\varepsilon/2). \tag{A.2.73}$$

We finish the proof by analyzing the weak-L^1 norms of the above functions. To this end, define

$$N(f) := \sup_{\lambda>0} \big[\lambda\, \mu(\{x \in B : \, |f(x)| > \lambda\})\big], \tag{A.2.74}$$

where $B := B(0, \varepsilon/2)$, and μ stands for the n-dimensional Lebesgue measure restricted to the ball B of constant density $|B|^{-1}$. Observe that $f(x) = \alpha$ on B implies $N(f) = \alpha$ for any constant α, and that $N(f_1 + f_2 + f_3) \le 2N(f_1) + 4N(f_2) + 4N(f_3)$ for every functions f_1, f_2 and f_3. Then the estimate

$$|K_{A,F,\varepsilon} f(0)| = |K_{A,F,\varepsilon_0} f_2(0)| \le 2N\big(K_{A,F,\varepsilon_0} f\big) + 4N\big(K_{A,F,\varepsilon_0} f_1\big) + 4C\mathcal{M}f(0) \tag{A.2.75}$$

follows from (A.2.71), these observations, and (A.2.73). At this stage, there remains to note that the right-hand side above can be further bounded using Chebyshev's inequality, which yields $N(K_{A,F,\varepsilon_0} f) \le C\mathcal{M}(K_{A,F,\varepsilon_0} f)(0)$, and the weak-$L^1$ boundedness result from Step 6, which eventually gives $N(K_{A,F,\varepsilon_0} f_1) \le C\mathcal{M}f(0)$. From these, (A.2.67) follows.

`Step 7:` *Retain the same assumptions as in Step 5, and consider the maximal operator*

$$K_{A,F,*} f(x) := \sup_{\varepsilon>0} \big|K_{A,F,\varepsilon} f(x)\big|, \qquad x \in \mathbb{R}^n. \tag{A.2.76}$$

Then for each $p \in (1,\infty)$ there exists a constant $C = C(F, A, m, n, p) \in (0,\infty)$ with the property that

$$\big\|K_{A,F,*} f\big\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}, \qquad \forall\, f \in L^p(\mathbb{R}^n). \tag{A.2.77}$$

To see this, fix an arbitrary $f \in L^p(\mathbb{R}^n)$ and first observe from (A.2.66) that for each $x \in \mathbb{R}^n$ we have

$$K^{(\varepsilon)}_{A,F,*} f(x) \nearrow K_{A,F,*} f(x) \ \text{ as } \ \varepsilon \searrow 0. \tag{A.2.78}$$

Based on this, Lebesgue's Monotone Convergence Theorem, (A.2.65), (A.2.61), and the boundedness of the Hardy-Littlewood maximal function, we obtain

$$\begin{aligned} \big\|K_{A,F,*} f\big\|_{L^p(\mathbb{R}^n)} &= \lim_{\varepsilon\to 0^+} \big\|K^{(\varepsilon)}_{A,F,*} f\big\|_{L^p(\mathbb{R}^n)} \\ &\le C \lim_{\varepsilon\to 0^+} \Big(\|\mathcal{M}f\|_{L^p(\mathbb{R}^n)} + \big\|\mathcal{M}\big(K_{A,F,\varepsilon/2} f\big)\big\|_{L^p(\mathbb{R}^n)}\Big) \le C\|f\|_{L^p(\mathbb{R}^n)}, \end{aligned} \tag{A.2.79}$$

completing the proof of (A.2.77).

In terms of the maximal operator $T_{I,*}$ from (A.2.45), estimate (A.2.77) yields

$$\big\|T_{I,*}f\big\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}, \qquad \forall\, f \in L^p(\mathbb{R}^n). \tag{A.2.80}$$

In order to show the existence of the pointwise limit in (A.2.29), the strategy is to return to the various particular operators discussed in Steps 1-5 and show that, in each case, such a pointwise convergence holds for such operators, acting on functions in L^p, almost everywhere in $\mathbb{R}^n$. In all cases, we shall make use of the abstract scheme described in Lemma A.5.

`Step 8:` *Pointwise convergence for the Cauchy operator from* (A.2.46). Let $\mathcal{V} := (1{+}iA')\mathrm{Lip}_{comp}(\mathbb{R})$, which is a dense subclass of $L^p(\mathbb{R})$, $1 < p < \infty$, since A is real-valued and Lipschitz. We claim that

$$\text{for any } h \in \mathcal{V} \text{ the limit } \lim_{\varepsilon\to 0^+} \mathcal{C}_{A,\varepsilon}h(x) \text{ exists for a.e. } x \in \mathbb{R}. \tag{A.2.81}$$

Indeed, if $h = (1 + iA')f$ with $f \in \mathrm{Lip}_{comp}(\mathbb{R})$, then we can write

$$\begin{aligned}
\mathcal{C}_{A,\varepsilon}h(x) &= \int_{1>|x-y|>\varepsilon} \frac{1+iA'(y)}{x-y+i(A(x)-A(y))}(f(y)-f(x))\,dy \\
&\quad - f(x)\int_{1>|x-y|>\varepsilon} \frac{-(1+iA'(y))}{x-y+i(A(x)-A(y))}\,dy \\
&\quad + \int_{|x-y|>1} \frac{1+iA'(y)}{x-y+i(A(x)-A(y))} f(y)\,dy \\
&=: I + II + III.
\end{aligned} \tag{A.2.82}$$

Using the fact that f is a compactly supported Lipschitz function, it is immediate that $\lim\limits_{\varepsilon\to 0^+} I$ and $\lim\limits_{\varepsilon\to 0^+} III$ exist at every $x \in \mathbb{R}$. Furthermore, the Fundamental Theorem of Calculus gives

$$II = -f(x)\ln\left(\frac{-1+i\frac{A(x)-A(x+\varepsilon)}{\varepsilon}}{1+i\frac{A(x)-A(x-\varepsilon)}{\varepsilon}}\right) \tag{A.2.83}$$

and the limit as $\varepsilon \to 0^+$ of the right-hand side exists for almost every $x \in \mathbb{R}$ since, by Rademacher's theorem, the Lipschitz function A is a.e. differentiable. This concludes the proof of (A.2.81).

Finally, a combination of (A.2.81), Lemma A.5, and (a suitable version of) the maximal inequality (A.2.80) gives that for $f \in L^p(\mathbb{R})$ the limit $\lim\limits_{\varepsilon\to 0^+} \mathcal{C}_{A,\varepsilon}f(x)$ exists for almost every $x \in \mathbb{R}$.

`Step 9:` *Pointwise convergence for the Cauchy operator* (A.2.48). Using Step 8, (A.2.50), and Lemma A.2 it follows that, for each function $f \in L^p(\mathbb{R})$, the limit $\lim\limits_{\varepsilon\to 0^+} \widetilde{\mathcal{C}}_{B,\eta,\varepsilon}f(x)$ exists for almost every $x \in \mathbb{R}$.

`Step 10:` *Pointwise convergence for the operator* (A.2.52). Specifically, we claim that if $f \in L^p(\mathbb{R})$, the limit $\lim\limits_{\varepsilon\to 0} K_{A,F,\varepsilon}f(x)$ exists for almost every $x \in \mathbb{R}$.

In order to prove this claim, fix $f \in L^p(\mathbb{R})$ and recall $I_\pm$, $II_\pm$ as defined in (A.2.55). The goal is to first show that $\lim\limits_{\varepsilon\to 0} I_+$ exists for almost every $x \in \mathbb{R}$. To this end, for x, $\zeta \in \mathbb{R}$ set

$$F_\varepsilon^{\zeta,x} := F(\zeta)\int_{|x-y|>\varepsilon} \frac{f(y)}{x-y+i[A_\zeta^\pm(x)-A_\zeta^\pm(y)]}\,dy. \tag{A.2.84}$$

Then employing Step 9 it follows that for each $\zeta \in \gamma_+^1$ the limit

$$\lim_{\varepsilon\to 0^+} F_\varepsilon^{\zeta,x} \tag{A.2.85}$$

exists for almost every $x \in \mathbb{R}$. Next we want to prove that $\sup_{\varepsilon>0} |F_\varepsilon^{\zeta,x}| \in L^1_\zeta(\gamma_+^1)$ for almost every $x \in \mathbb{R}$. To see the latter we write

$$\int_{\mathbb{R}} \left| \int_{\gamma_+^1} \sup_{\varepsilon>0} |F_\varepsilon^{\zeta,x}|\, d\zeta \right|^2 dx \leq \int_{\gamma_+^1} \int_{\mathbb{R}} \big(\sup_{\varepsilon>0} |F_\varepsilon^{\zeta,x}| \big)^2\, dx\, d\zeta \leq C\|f\|_{L^2(\mathbb{R})}. \tag{A.2.86}$$

The first inequality in (A.2.86) is standard, while for the second one we have used (a suitable version of) the maximal inequality (A.2.80). The above analysis provides all the ingredients necessary for invoking Lebesgue's Dominated Convergence Theorem which, in turn, allows us to conclude that

$$\lim_{\varepsilon\to 0^+} I_+ = \lim_{\varepsilon\to 0^+} \left\{ -\frac{1}{2\pi} \int_{\gamma_+^1} F_\varepsilon^{\zeta,x}\, d\zeta \right\} \quad \text{exists at almost every point } x \in \mathbb{R}. \tag{A.2.87}$$

Similarly, one shows that $\lim\limits_{\varepsilon\to 0^+} I_-$, $\lim\limits_{\varepsilon\to 0^+} II_\pm$ exist for almost every $x \in \mathbb{R}$, and thus the earlier claim is proved.

`Step 11:` *Pointwise convergence for the operator* (A.2.58). The fact that for $f \in L^p(\mathbb{R})$, the limit $\lim\limits_{\varepsilon\to 0^+} K_{A,F,\varepsilon} f(x)$ exists for almost every $x \in \mathbb{R}$ follows by a reasoning similar to the one in Step 10. This time the identity (A.2.59) replaces the expressions in (A.2.55) and the decay properties of the Fourier transform $\widehat{F}(\xi)$ in are used when applying Lebesgue's Dominated Convergence Theorem.

`Step 12:` *For each given $f \in L^p(\mathbb{R}^n)$, the limit* (A.2.29) *exists for a.e. $x \in \mathbb{R}^n$.* Indeed, the case $n = 1$ has been treated in Step 11. Finally, in the case $n > 1$, the existence of the limit in question for $f \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ follows via the rotation method from the one-dimensional result (and Lebesgue's Dominated Convergence Theorem). Granted this, we may invoke Lemma A.5 and the maximal inequality (A.2.80) in order to finish, keeping in mind that $\mathscr{C}_0^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.

In summary, at this point we know that

$$\text{for each } f \in L^p(\mathbb{R}^n), \text{ the limit } \lim_{\varepsilon\to 0^+} T_{I,\varepsilon} f(x) \text{ exists for a.e. } x \in \mathbb{R}^n. \tag{A.2.88}$$

In turn, this readily yields that

$$\lim_{\varepsilon\downarrow 0} \int_{\{y\in\mathbb{R}^n:\, 1>|x-y|>\varepsilon\}} \frac{1}{|x-y|^n} F\Big(\frac{A(x)-A(y)}{|x-y|}\Big)\, dy \quad \text{exists for a.e. } x \in \mathbb{R}^n. \tag{A.2.89}$$

With this in hand and relying on Lemma A.2 we deduce that for each bi-Lipschitz function B

$$\lim_{\varepsilon\downarrow 0} \int_{\{y\in\mathbb{R}^n:\, |B(x)-B(y)|>\varepsilon,\ |x-y|<1\}} \frac{1}{|x-y|^n} F\Big(\frac{A(x)-A(y)}{|x-y|}\Big)\, dy \quad \text{exists for almost every point } x \in \mathbb{R}^n \text{ and is equal to (A.2.89)}. \tag{A.2.90}$$

Having proved this, it follows that

$$\text{for each function } f \in \mathscr{C}_0^\infty(\mathbb{R}^n), \text{ the limit } \lim_{\varepsilon\to 0^+} T_{B,\varepsilon} f(x) \text{ exists for a.e. } x \in \mathbb{R}^n \text{ and is equal to } \lim_{\varepsilon\to 0^+} T_{I,\varepsilon} f(x). \tag{A.2.91}$$

Let us also note that, thanks to (A.2.80) and Lemma A.4,

$$\big\|T_{B,*}f\big\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}, \qquad \forall\, f \in L^p(\mathbb{R}^n). \tag{A.2.92}$$

From (A.2.91), (A.2.92), and Lemma A.5 we may finally conclude that for each fixed $f \in L^p(\mathbb{R}^n)$ the limit (A.2.28) exists at a.e. point $x \in \mathbb{R}^n$ and is equal to (A.2.29). This finishes the proof of Proposition A.3. □

A.3 Background on $\mathrm{OP}(L^\infty \cap \mathrm{vmo})S^0_{\mathrm{cl}}$

If X is a Banach space of functions on $\mathbb{R}^n$, we say a function p of $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ belongs to the symbol class $XS^m_{1,0}$,

$$p \in XS^m_{1,0}, \tag{A.3.1}$$

provided $p(\cdot,\xi) \in X$ for each $\xi \in \mathbb{R}^n$, and

$$\|\partial^\alpha_\xi p(\cdot,\xi)\|_X \leq C_\alpha \langle\xi\rangle^{m-|\alpha|}, \quad \forall\, \alpha \in \mathbb{N}^n_0, \tag{A.3.2}$$

where $\langle\xi\rangle := (1+|\xi|^2)^{1/2}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. If, in addition,

$$p(x,\xi) \sim \sum_{j\geq 0} p_j(x,\xi), \quad p_j(x,r\xi) = r^{m-j}p_j(x,\xi) \text{ for } r, |\xi| \geq 1, \tag{A.3.3}$$

in the sense that for every $k \in \mathbb{N}$ the difference $p - \sum_{j=0}^{k-1} p_j$ belongs to $XS^{m-k}_{1,0}$, we say

$$p \in XS^m_{\mathrm{cl}}. \tag{A.3.4}$$

The associated operator $p(x,D)$ is given by

$$p(x,D)u = (2\pi)^{-n/2} \int p(x,\xi)\hat{u}(\xi)e^{ix\cdot\xi}\, d\xi. \tag{A.3.5}$$

If (A.3.1) holds, we say $p(x,D) \in \mathrm{OP}XS^m_{1,0}$, and if (A.3.4) holds, we say $p(x,D) \in \mathrm{OP}XS^m_{\mathrm{cl}}$.

Here we single out the spaces

$$L^\infty(\mathbb{R}^n), \quad \mathrm{bmo}(\mathbb{R}^n), \quad \mathrm{vmo}(\mathbb{R}^n), \quad L^\infty(\mathbb{R}^n) \cap \mathrm{vmo}(\mathbb{R}^n), \tag{A.3.6}$$

to play the role of X. Here bmo is the localized variant of BMO, and vmo that of VMO. We summarize some results about the associated pseudodifferential operators. Details can be found in [36, Chapter 1, §11], which builds on work in [4] and in [35, §6]. A key ingredient in the proofs of these results is the classical commutator estimate of [6],

$$\big\|[M_g, B]u\big\|_{L^p} \leq C_p\|g\|_{\mathrm{bmo}}\|u\|_{L^p}, \tag{A.3.7}$$

given $B \in \mathrm{OP}S^0_{1,0}$. Here $M_g u := gu$ is the operator of multiplication by g.

The following extension appears in [36, Proposition 11.1].

Proposition A.6. *If $p(x,D) \in \mathrm{OP(bmo)}S^0_{\mathrm{cl}}$ and $B = b(x,D) \in \mathrm{OP}S^0_{1,\delta}$, $\delta < 1$, with B scalar, then*

$$[p(x,D), B] : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n), \quad 1 < p < \infty. \tag{A.3.8}$$

If $p \in \mathrm{vmo}S^0_{\mathrm{cl}}$ and $b \in S^0_{1,\delta}$ have compact x-support, this commutator is compact.

This result in turn helps prove the following, which may be found in [36, Proposition 11.3].

Proposition A.7. *Assume that*

$$p \in L^\infty S^0_{\mathrm{cl}}, \quad q \in (L^\infty \cap \mathrm{vmo}) S^0_{\mathrm{cl}}, \tag{A.3.9}$$

with compact x-support. Then

$$p(x,D)q(x,D) = a(x,D) + K, \quad a(x,\xi) = p(x,\xi)q(x,\xi), \tag{A.3.10}$$

with K compact on $L^p(\mathbb{R}^n)$, for $1 < p < \infty$.

The following result has a proof parallel to that of Proposition A.7.

Proposition A.8. *Assume $q \in (L^\infty \cap \mathrm{vmo}) S^0_{\mathrm{cl}}$, with compact x-support, and set*

$$q^*(x,\xi) = q(x,\xi)^*. \tag{A.3.11}$$

Then

$$q(x,D)^* = q^*(x,D) + K, \tag{A.3.12}$$

with K compact on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

To proceed, we have the following useful result, which appears in [36, Proposition 11.4].

Proposition A.9. *The space $L^\infty \cap \mathrm{vmo}$ is a closed subalgebra of $L^\infty(\mathbb{R}^n)$.*

Putting Propositions A.7 and A.9 together yields the following.

Corollary A.10. *Assume that*

$$p, q \in (L^\infty \cap \mathrm{vmo}) S^0_{\mathrm{cl}}, \tag{A.3.13}$$

with compact x-support. Then

$$p(x,D)q(x,D) = a(x,D) + K, \tag{A.3.14}$$

with K compact on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, and

$$a = pq \in (L^\infty \cap \mathrm{vmo}) S^0_{\mathrm{cl}}. \tag{A.3.15}$$

Generally, if $\mathcal{A}$ is a C^*-algebra and $\mathcal{B}$ a closed $*$-subalgebra of $\mathcal{A}$ containing the identity element, and if $f \in \mathcal{B}$, then f is invertible in $\mathcal{B}$ if and only if it is invertible in $\mathcal{A}$. To see this, consider $h = f^* f$ and expand $H(z) = (h+1-z)^{-1}$ in a power series about $z = 0$. The radius of convergence is > 1, if f is invertible in $\mathcal{A}$. Clearly, $H(z) \in \mathcal{B}$ for $|z| < 1$, if $f \in \mathcal{B}$, so $H(1) \in \mathcal{B}$.

Consequently, we have

$$a \in L^\infty \cap \mathrm{vmo}, \quad a^{-1} \in L^\infty \Longrightarrow a^{-1} \in L^\infty \cap \mathrm{vmo}. \tag{A.3.16}$$

This holds for matrix valued $a(x)$. Similarly, if

$$p \in (L^\infty \cap \mathrm{vmo}) S^0_{\mathrm{cl}} \ \text{ is elliptic,} \tag{A.3.17}$$

so there exist $C_j < \infty$ such that

$$|p(x,\xi)^{-1}| \le C_1 \ \text{ for } \ |\xi| \ge C_2, \tag{A.3.18}$$

then

$$(1 - \varphi(\xi)) p(x,\xi)^{-1} \in (L^\infty \cap \mathrm{vmo}) S^0_{\mathrm{cl}}, \tag{A.3.19}$$

where $\varphi \in C_0^\infty(\mathbb{R}^n)$ is equal to 1 for $|\xi| \le C_2$. This allows the construction of Fredholm inverses of elliptic operators with coefficients in $L^\infty \cap \mathrm{vmo}$.

A.4 Analysis on spaces of homogeneous type

We begin by discussing a few results of a general nature, valid in the context of spaces of homogeneous type. Recall that (Σ, ρ) is a quasi-metric space if Σ is a set (of cardinality at least two) and the mapping $\rho : \Sigma \times \Sigma \to [0, \infty)$ is a quasi-distance, that is, there exists $C \in [1, \infty)$ such that for every $x, y, z \in \Sigma$ ρ satisfies:

$$\rho(x, y) = 0 \Leftrightarrow x = y, \quad \rho(y, x) = \rho(x, y), \quad \rho(x, y) \leq C\big(\rho(x, z) + \rho(z, y)\big). \tag{A.4.1}$$

A space of homogeneous type in the sense of Coifman and Weiss (cf. [7]) is a triplet (Σ, ρ, μ) such that (Σ, ρ) is a quasi-metric space and μ is a Borel measure on Σ (equipped with the topology canonically induced by ρ) that is doubling. That is, there exists $C \in (0, \infty)$ such that

$$0 < \mu\big(B_\rho(x, 2r)\big) \leq C\mu\big(B_\rho(x, r)\big), \qquad \forall\, x \in \Sigma,\ \forall\, r > 0, \tag{A.4.2}$$

where $B_\rho(x, r)$ is the ρ-ball of center x and radius r given by $\{y \in \Sigma :\ \rho(x, y) < r\}$.

Then the John-Nirenberg space of functions of bounded mean oscillations, $\mathrm{BMO}(\Sigma, \mu)$, consists of functions $f \in L^1_{\mathrm{loc}}(\Sigma, \mu)$ for which $\|f\|_{\mathrm{BMO}(\Sigma,\mu)} < +\infty$. As usual, we have set

$$\|f\|_{\mathrm{BMO}(\Sigma,\mu)} := \begin{cases} \sup_{R>0} M_1(f; R) \quad \text{if } \mu(\Sigma) = +\infty, \\ \Big|\int_\Sigma f\, d\mu\Big| + \sup_{R>0} M_1(f; R) \quad \text{if } \mu(\Sigma) < +\infty, \end{cases} \tag{A.4.3}$$

where, for $p \in [1, \infty)$, we have set

$$\begin{aligned} &M_p(f; R) := \sup_{x\in\Sigma}\ \sup_{r\in(0,R]} \Big(⨍_{B_\rho(x,r)} \Big|f - ⨍_{B_\rho(x,r)} f\, d\mu\Big|^p d\mu\Big)^{1/p}, \\ &\text{and} \qquad ⨍_{B_\rho(x,r)} f\, d\mu := \tfrac{1}{\mu(B_\rho(x,r))} \int_{B_\rho(x,r)} f\, d\mu. \end{aligned} \tag{A.4.4}$$

Following [30], if $\mathrm{UC}(\Sigma, \mu)$ stands for the space of uniformly continuous functions on X, we introduce $\mathrm{VMO}(\Sigma, \mu)$, the space of functions of vanishing mean oscillations on Σ, as

$$\mathrm{VMO}(\Sigma, \mu) := \text{ the closure of } \mathrm{UC}(\Sigma, \mu) \cap \mathrm{BMO}(\Sigma, \mu) \text{ in } \mathrm{BMO}(\Sigma, \mu). \tag{A.4.5}$$

We have the following useful equivalent characterization of VMO on compact spaces of homogeneous type. To state it, we denote by $\mathscr{C}^\alpha(\Sigma, \rho)$ the space of real-valued Hölder functions of order $\alpha > 0$ on the quasi-metric space (Σ, ρ). That is, $\mathscr{C}^\alpha(\Sigma, \rho)$ is the collection of all real-valued functions f on Σ with the property that

$$\|f\|_{\mathscr{C}^\alpha(\Sigma,\rho)} := \sup_{x\in\Sigma} |f(x)| + \sup_{x,y\in\Sigma,\, x\neq y} \frac{|f(x) - f(y)|}{\rho(x, y)^\alpha} < +\infty. \tag{A.4.6}$$

For further reference, let us also set

$$\mathscr{C}^\alpha_0(X, \rho) := \big\{f \in \mathscr{C}^\alpha(\Sigma, \rho) :\ \operatorname{supp} f \ \text{ bounded}\big\}. \tag{A.4.7}$$

The following two propositions contain results proved in [14], [21].

Proposition A.11. *Assume that (Σ, ρ, μ) is a compact space of homogeneous type. Then*

$$\mathrm{VMO}(\Sigma, \mu) \text{ is the closure of } \mathscr{C}^\alpha(\Sigma, \rho) \cap \mathrm{BMO}(\Sigma, \mu) \text{ in } \mathrm{BMO}(\Sigma, \mu), \tag{A.4.8}$$

for every $\alpha \in \mathbb{R}$ such that

$$0 < \alpha \leq \left[\log_2\Big(\sup_{\substack{x,y,z\in\Sigma \\ \text{not all equal}}} \frac{\rho(x,y)}{\max\{\rho(x,z), \rho(z,y)\}}\Big)\right]^{-1}. \tag{A.4.9}$$

Proposition A.12. *Let (Σ, ρ, μ) be a space of homogeneous type. Then for each $p \in [1, \infty)$,*

$$\begin{aligned}
&\mathrm{dist}_{\mathrm{BMO}}\,(f,\, \mathrm{VMO}\,(\Sigma, \mu)) \\
&\qquad \approx \limsup_{r\to 0^+}\left\{\sup_{x\in\Sigma} ⨍_{B_\rho(x,r)} ⨍_{B_\rho(x,r)} |f(y) - f(z)|^p \, d\mu(y)\, d\mu(z)\right\}^{1/p} \\
&\qquad \approx \limsup_{r\to 0^+}\left\{\sup_{x\in\Sigma} ⨍_{B_\rho(x,r)} \Big| f - ⨍_{B_\rho(x,r)} f\, d\mu\Big|^p \, d\mu\right\}^{1/p},
\end{aligned} \tag{A.4.10}$$

uniformly for $f \in \mathrm{BMO}(\Sigma, \mu)$ (i.e., the constants do not depend on f), where the distance is measured in the BMO *norm. In particular, for each $p \in [1, \infty)$,*

$$\mathrm{dist}_{\mathrm{BMO}}\,(f,\, \mathrm{VMO}\,(\Sigma, \mu)) \approx \lim_{R\to 0^+} M_p(f; R), \qquad \text{uniformly for } f \in \mathrm{BMO}(\Sigma, \mu), \tag{A.4.11}$$

where $M_p(f; R)$ is defined as in (A.4.4). *Moreover, for each function $f \in \mathrm{BMO}(\Sigma, \mu)$ and each $p \in [1, \infty)$,*

$$f \in \mathrm{VMO}(\Sigma, \mu) \iff \lim_{r\to 0^+}\left\{\sup_{x\in\Sigma} ⨍_{B_\rho(x,r)} \Big| f - ⨍_{B_\rho(x,r)} f\, d\mu\Big|^p \, d\mu\right\}^{1/p} = 0. \tag{A.4.12}$$

For future purposes, we find it convenient to restate (A.4.11) in a slightly different form. More specifically, in the context of Proposition A.12, given $f \in L^2_{\mathrm{loc}}(\Sigma, \mu)$, $x \in \Sigma$ and $R > 0$, we set

$$\|f\|_*(B_\rho(x, R)) := \sup_{B\subseteq B_\rho(x,R)} \Big(⨍_B |f - f_B|^2\, d\mu\Big)^{1/2}, \tag{A.4.13}$$

where the supremum is taken over all ρ-balls B included in $B_\rho(x, R)$, and $f_B := \mu(B)^{-1}\int_B f\, d\mu$. It is then clear from definitions that

$$\sup_{x\in\Sigma} \|f\|_*(B_\rho(x, R)) \approx M_2(f; R). \tag{A.4.14}$$

Consequently, (A.4.11) yields:

Corollary A.13. *With the above notation and conventions,*

$$\lim_{R\to 0^+}\Big[\sup_{x\in\Sigma} \|f\|_*(B_\rho(x, R))\Big] \approx \mathrm{dist}_{\mathrm{BMO}}\,(f,\, \mathrm{VMO}\,(\Sigma, \mu)), \tag{A.4.15}$$

uniformly for $f \in \mathrm{BMO}(\Sigma, \mu)$.

We continue by recoding the following useful counterpart of Proposition A.9 (formulated in the Euclidean context) to spaces of homogeneous type.

Proposition A.14. *Assume that* (Σ, ρ, μ) *is a space of homogeneous type. Then there exists a constant* $C \in (0, \infty)$ *such that*

$$\begin{aligned}
&\operatorname{dist}_{\mathrm{BMO}}\big(fg, \mathrm{VMO}\,(\Sigma, \mu)\big) \\
&\qquad \le C\|f\|_{L^\infty(\Sigma,\mu)} \operatorname{dist}_{\mathrm{BMO}}\big(g, \mathrm{VMO}\,(\Sigma, \mu)\big) \\
&\qquad\quad + C\|g\|_{L^\infty(\Sigma,\mu)} \operatorname{dist}_{\mathrm{BMO}}\big(f, \mathrm{VMO}\,(\Sigma, \mu)\big),
\end{aligned} \tag{A.4.16}$$

for any $f, g \in L^\infty(\Sigma, \mu)$*, where all distances are considered in the space* $\mathrm{BMO}\,(\Sigma, \mu)$*.*

Moreover,

$$\mathrm{VMO}\,(\Sigma, \mu) \cap L^\infty(\Sigma, \mu) \quad \textit{is a closed } C^* \textit{ subalgebra of } L^\infty(\Sigma, \mu), \tag{A.4.17}$$

and

$$\left.\begin{array}{r} f \in \mathrm{VMO}\,(\Sigma, \mu) \cap L^\infty(\Sigma, \mu) \\ \textit{and} \quad 1/f \in L^\infty(\Sigma, \mu) \end{array}\right\} \Longrightarrow 1/f \in \mathrm{VMO}\,(\Sigma, \mu) \cap L^\infty(\Sigma, \mu). \tag{A.4.18}$$

Proof. Note that (A.4.16) implies (A.4.17) and also (A.4.18), via the same type of argument used to establish (A.3.16). As such, it suffices to prove (A.4.16). To this end, if $f, g \in L^\infty(\Sigma, \mu)$ then for any $x \in \Sigma$ and $r > 0$ and $y, z \in B_\rho(x, r)$ we have

$$\begin{aligned}
|f(y)\,g(y) - f(z)\,g(z)| &\le |f(y)|\,|g(y) - g(z)| + |g(z)|\,|f(y) - f(z)| \\
&\le \|f\|_{L^\infty(X,\mu)}\,|g(y) - g(z)| + \|g\|_{L^\infty(X,\mu)}\,|f(y) - f(z)|
\end{aligned} \tag{A.4.19}$$

With this in hand, (A.4.16) follows with the help of the first equivalence in (A.4.10). □

Another useful result pertains to the manner in which one can control the distance to VMO under composition by a Lipschitz function.

Proposition A.15. *Assume that* (Σ, ρ, μ) *is a space of homogeneous type. Let* $F : \mathbb{R}^m \to \mathbb{R}$ *be a Lipschitz function. Then there exists a constant* $C \in (0, \infty)$ *such that for every* $f : \Sigma \to \mathbb{R}^m$ *with components in* $\mathrm{BMO}\,(\Sigma, \mu)$ *there holds*

$$\operatorname{dist}_{\mathrm{BMO}}\big(F \circ f, \mathrm{VMO}\,(\Sigma, \mu)\big) \le C\|\nabla F\|_{L^\infty(\mathbb{R}^m)} \operatorname{dist}_{\mathrm{BMO}}\big(f, \mathrm{VMO}\,(\Sigma, \mu)\big). \tag{A.4.20}$$

where the distances are considered in the space $\mathrm{BMO}\,(\Sigma, \mu)$*. In particular,*

$$f \in \mathrm{VMO}\,(\Sigma, \mu) \Longrightarrow F \circ f \in \mathrm{VMO}\,(\Sigma, \mu). \tag{A.4.21}$$

Proof. Fix $x \in \Sigma$ and $r > 0$, arbitrary. Using the fact that F is Lipschitz we may then estimate for every $y, z \in B_\rho(x, r)$

$$|F(f(y)) - F(f(z))| \le \|\nabla F\|_{L^\infty(\mathbb{R}^m)}\,|f(y) - f(x)|. \tag{A.4.22}$$

Then the desired conclusion readily follows from this and the first equivalence in (A.4.10). □

A.5 On the class of $\mathrm{Lip} \cap \mathrm{vmo}_1$ domains

The starting point in this subsection is the following result.

Lemma A.16. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function, with graph*

$$\Sigma := \{(x, \varphi(x)) : x \in \mathbb{R}^n\} \subset \mathbb{R}^{n+1}. \tag{A.5.1}$$

Set $\mu := \mathcal{H}^n \lfloor \Sigma$, where $\mathcal{H}^n$ is the n-dimensional Hausdorff measure in $\mathbb{R}^{n+1}$. Then

$$f \in \mathrm{VMO}(\Sigma, \mu) \Longleftrightarrow f\big(\cdot, \varphi(\cdot)\big) \in \mathrm{VMO}(\mathbb{R}^n). \tag{A.5.2}$$

Proof. For each given point $X = (x, \varphi(x)) \in \Sigma$, with $x \in \mathbb{R}^n$, and each given radius $r > 0$ set $\Delta(X, r) := \{Y \in \Sigma : |Y - X| < r\}$. Fix then $X_0 = (x_0, \varphi(x_0)) \in \Sigma$, for some $x_0 \in \mathbb{R}^n$, and pick some $r > 0$. Consider $c := ⨍_{B(x_0,r)} f(x, \varphi(x))\, dx$. Then

$$\begin{aligned}
&⨍_{\Delta(X_0,r)} \Big| f - ⨍_{\Delta(X_0,r)} f\, d\mu \Big|\, d\mu \\
&\quad = ⨍_{\Delta(X_0,r)} \Big| (f - c) - ⨍_{\Delta(X_0,r)} (f - c)\, d\mu \Big|\, d\mu \leq 2 ⨍_{\Delta(X_0,r)} |f - c|\, d\mu \\
&\quad = 2 ⨍_{\{x \in \mathbb{R}^n :\, |x - x_0|^2 + (\varphi(x) - \varphi(x_0))^2 < r^2\}} \big| f(x, \varphi(x)) - c \big| \sqrt{1 + |\nabla \varphi(x)|^2}\, dx \\
&\quad \leq C ⨍_{\{x \in \mathbb{R}^n :\, |x - x_0| < r\}} \big| f(x, \varphi(x)) - c \big|\, dx.
\end{aligned} \tag{A.5.3}$$

Bearing in mind the significance of c, the left-pointing inequality in (A.5.2) follows from (A.4.12) (with $p = 1$). For the opposite implication, pick $c' := ⨍_{\Delta(X_0,r)} f\, d\mu$. Then for some sufficiently large $M > 0$, depending on the Lipschitz constant of φ, we have

$$\begin{aligned}
⨍_{B(x_0,r)} \Big| f(x, \varphi(x)) - ⨍_{B(x_0,r)} f(y, \varphi(y))\, dy \Big|\, dx &\leq 2 ⨍_{\{x \in \mathbb{R}^n :\, |x - x_0| < r\}} \big| f(x, \varphi(x)) - c' \big|\, dx \\
&\leq C ⨍_{\{x \in \mathbb{R}^n :\, |x - x_0|^2 + (\varphi(x) - \varphi(x_0))^2 < (Mr)^2\}} \big| f(x, \varphi(x)) - c' \big| \sqrt{1 + |\nabla \varphi(x)|^2}\, dx \\
&\leq C ⨍_{\Delta(X_0,r)} \Big| f - ⨍_{\Delta(X_0,r)} f\, d\mu \Big|\, d\mu.
\end{aligned} \tag{A.5.4}$$

Based on this and (A.4.12), the right-pointing inequality in (A.5.2) now follows. □

In turn, Lemma A.16 is an important ingredient in the proof of the following result.

Lemma A.17. *Assume that $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz function, and let Σ as in* (A.5.1) *denote its graph. Set $\mu := \mathcal{H}^n \lfloor \Sigma$, where $\mathcal{H}^n$ is the n-dimensional Hausdorff measure in $\mathbb{R}^{n+1}$, and let $\nu = (\nu_1, ..., \nu_{n+1})$ stand for the unit normal to Σ (defined ν-a.e.). Then*

$$\nu_j \in \mathrm{VMO}(\Sigma, \mu) \;\; \textit{for} \;\; 1 \leq j \leq n + 1 \Longleftrightarrow \partial_j \varphi \in \mathrm{VMO}(\mathbb{R}^n) \;\; \textit{for} \;\; 1 \leq j \leq n. \tag{A.5.5}$$

Proof. Recall that the components $\nu_j : \Sigma \to \mathbb{R}$ of the unit normal to the Lipschitz surface Σ satisfy

$$\nu_j\big(x, \varphi(x)\big) = \begin{cases} \dfrac{\partial_j \varphi(x)}{\sqrt{1 + |\nabla \varphi(x)|^2}} & \text{if } 1 \leq j \leq n, \\ \dfrac{-1}{\sqrt{1 + |\nabla \varphi(x)|^2}} & \text{if } j = n+1, \end{cases} \tag{A.5.6}$$

for a.e. $x \in \mathbb{R}^n$. As regards (A.5.5), assume first that

$$\partial_j \varphi \in \mathrm{VMO}(\mathbb{R}^n) \ \text{ for each } \ j \in \{1, ..., n\}, \tag{A.5.7}$$

and consider the functions $F_j : \mathbb{R}^n \to \mathbb{R}$, $1 \leq j \leq n+1$, given by

$$F_j(x) := \begin{cases} \dfrac{x_j}{\sqrt{1 + |x|^2}} & \text{if } 1 \leq j \leq n, \\ \dfrac{-1}{\sqrt{1 + |x|^2}} & \text{if } j = n+1, \end{cases} \tag{A.5.8}$$

for each $x = (x_1, ..., x_n) \in \mathbb{R}^n$. A straightforward computation gives that there exists a dimensional constant such that for every $x \in \mathbb{R}^n$

$$|\nabla F_j(x)| \leq \begin{cases} \dfrac{C_n}{\sqrt{1 + |x|^2}} & \text{if } 1 \leq j \leq n, \\ \dfrac{C_n}{1 + |x|^2} & \text{if } j = n+1. \end{cases} \tag{A.5.9}$$

In particular, each function $F_j : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz. Upon noting from (A.5.6) and (A.5.8) that $\nu_j\big(x, \varphi(x)\big) = F_j\big(\nabla \varphi(x)\big)$ for a.e. $x \in \mathbb{R}^n$, this implies, in concert with (A.5.7) and (A.4.21), that $\nu_j\big(\cdot, \varphi(\cdot)\big) \in \mathrm{VMO}(\mathbb{R}^n)$ for each $j \in \{1, ..., n+1\}$. Having established this, we may then conclude that $\nu_j \in \mathrm{VMO}(\Sigma, \mu)$ for $1 \leq j \leq n+1$ by invoking Lemma A.16. This proves the left-pointing implication in (A.5.5).

In the opposite direction, assume

$$\nu_j \in \mathrm{VMO}(\Sigma, \mu) \ \text{ for each } \ j \in \{1, ..., n+1\}. \tag{A.5.10}$$

Then Lemma A.16 gives

$$\nu_j\big(\cdot, \varphi(\cdot)\big) \in \mathrm{VMO}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \ \text{ for each } \ j \in \{1, ..., n+1\}. \tag{A.5.11}$$

Since from (A.5.6) and the fact that φ is Lipschitz we have

$$1/\nu_{n+1}\big(\cdot, \varphi(\cdot)\big) \in L^\infty(\mathbb{R}^n), \tag{A.5.12}$$

we deduce from (A.4.18), (A.5.11) with $j = n+1$, and (A.5.12) that

$$1/\nu_{n+1}\big(\cdot, \varphi(\cdot)\big) \in \mathrm{VMO}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n). \tag{A.5.13}$$

Given that $\mathrm{VMO}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is an algebra (cf. (A.4.17) in Proposition A.14), it follows from (A.5.11) and (A.5.13) that

$$\nu_j\big(\cdot, \varphi(\cdot)\big)/\nu_{n+1}\big(\cdot, \varphi(\cdot)\big) \in \mathrm{VMO}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \ \text{ for each } \ j \in \{1, ..., n\}. \tag{A.5.14}$$

In light of (A.5.6) this ultimately entails $\partial_j \varphi \in \mathrm{VMO}(\mathbb{R}^n)$ for $1 \leq j \leq n$, as wanted. $\square$

We are now in a position to define the class of $\mathrm{Lip} \cap \mathrm{vmo}_1$ domains.

Definition A.1. *Assume that $C \in (0, \infty)$ and let Ω be a nonempty, open subset of $\mathbb{R}^n$, with diameter $\leq C$. One calls Ω a bounded Lipschitz domain, with Lipschitz character controlled by C, if there exists $r \in (0, C)$ with the property that for every $x_0 \in \partial\Omega$ one can find a rigid transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ and a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ with $\|\nabla\varphi\|_{L^\infty(\mathbb{R}^{n-1})} \leq C$ such that*

$$T\big(\Omega \cap B(x_0, r)\big) = T\big(B(x_0, r)\big) \cap \big\{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n > \varphi(x')\big\}. \tag{A.5.15}$$

Whenever this is the case, call $\phi(x') := (x', \varphi(x'))$ a coordinate chart for $\partial\Omega$.

If, in addition, $\partial_j\varphi \in \mathrm{vmo}(\mathbb{R}^{n-1})$ for each $j \in \{1, ..., n-1\}$ then we shall say that Ω is a bounded $\mathrm{Lip} \cap \mathrm{vmo}_1$ *domain.*

Both the class of Lipschitz domains and the class of $\mathrm{Lip} \cap \mathrm{vmo}_1$ domains may be naturally defined in the manifold setting, by working in local coordinates, in a similar fashion as above (cf. also the discussion in [13]).

We conclude this subsection by proving the following characterization of the class of $\mathrm{Lip} \cap \mathrm{vmo}_1$ domains.

Proposition A.18. *Let Ω be a Lipschitz domain, with outward unit normal ν. Then*

$$\nu \in \mathrm{vmo}(\partial\Omega) \iff \Omega \ \textit{ is a } \mathrm{Lip} \cap \mathrm{vmo}_1 \textit{ domain.} \tag{A.5.16}$$

Proof. This is a consequence of Lemma A.17 and definitions. □

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