Quantization of Discontinuous Symbols
And Quantum Ergodic Theorems

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Contents

1. Introduction

Let M be a compact Riemannian manifold, Λ a positive, self-adjoint, first-order elliptic pseudodifferential operator ($\Lambda \in OPS^1(M)$, for example, $\Lambda = \sqrt{-\Delta}$, where Δ is the Laplace-Beltrami operator on M), and $\{\varphi_k\}$ an orthonormal basis of $L^2(M)$ consisting of eigenfunctions of Λ, with eigenvalues $\lambda_k \nearrow +\infty$. If $X \subset T^*M$ denotes the set on which the principal symbol σ_Λ of Λ is equal to 1, then X carries a natural Liouville measure dS, which we normalize to have total mass 1, and the Hamiltonian flow $\{G_t : t \in \mathbb{R}\}$ generated by σ_Λ acts on X,

$$G_t : X \longrightarrow X, \tag{1.1}$$

preserving the Liouville measure dS. In [CV] (following work of [Shn] and [Ze]) the following classical quantum ergodic theorem was established.

Proposition 1.1. *Assume the flow (1.1) is ergodic. Then there is a subset $\mathcal{N} \subset \mathbb{N}$, of density 0, such that the following holds. Given $A \in OPS^0(M)$, with principal symbol $a \in C^\infty(X)$, we have*

$$\lim_{k\to\infty, k\notin\mathcal{N}} (A\varphi_k, \varphi_k)_{L^2} = \int_X a\, dS. \tag{1.2}$$

The result (1.2) is a microlocalization of the local equidistribution phenomenon, which is the special case of (1.2) when A is multiplication by a smooth function:

$$\lim_{k\to\infty, k\notin\mathcal{N}} \int_M b(x)|\varphi_k(x)|^2\, dV = \int_X b\circ\pi\, dS, \tag{1.3}$$

for $b \in C^\infty(M)$, $\pi : X \to M$ the natural projection, assuming the flow (1.1) is ergodic. In case $\Lambda = \sqrt{-\Delta}$, this takes the form

$$\lim_{k\to\infty, k\notin\mathcal{N}} \int_M b(x)|\varphi_k(x)|^2 \, dV = \frac{1}{V(M)} \int_M b(x)\, dV. \tag{1.4}$$

Proposition 1.1 has been extended to compact Riemannian manifolds with piecewise smooth boundary in [ZZ], in the setting where $\Lambda = \sqrt{-\Delta}$, with the Dirichlet boundary condition.

Proposition 1.1 has also been extended to cases where the flow (1.1) is not ergodic, in [ST] and [T2], and in [Riv] and [Gal]. Results of [T2] treat cases where the ergodicity hypothesis on $\{G_t\}$ is replaced by the hypothesis that

$$Pa \in C(X), \tag{1.5}$$

where P is the orthogonal projection of $L^2(X)$ onto the subspace of G_t-invariant elements. See Proposition 4.1 of this paper for a detailed statement. The paper [T2] gives examples where P maps $C(X)$ to $C(X)$ but it does not map $C^\infty(X)$ to $C^\infty(X)$, or even to the space of Hölder continuous functions. Thus one is motivated to quantize non smooth functions in $C(X)$. In this paper, we take this further, quantizing elements of $L^\infty(X)$, and making use of such a quantization to produce further quantum ergodic theorems.

To give another motivation for dealing with discontinuous symbols, we mention the following simple extension of (1.3).

Proposition 1.2. *In the setting of Proposition 1.1, the limiting result (1.3) holds whenever*

$$b \in \mathcal{R}(M). \tag{1.6}$$

Here $\mathcal{R}(M)$ denotes the space of bounded functions on M that are *Riemann integrable*. To prove Proposition 1.2, it suffices to treat the case where b is real valued. Then, given $\varepsilon > 0$, we can pick $b_1, b_2 \in C^\infty(M)$ such that

$$b_1 \le b \le b_2, \quad \text{and} \quad \int_M (b_2 - b_1)\, dV < \varepsilon, \tag{1.7}$$

hence

$$\int_X (b_2 - b_1)\circ\pi \, dS < C\varepsilon. \tag{1.8}$$

We know from Proposition 1.1 that (1.3) holds with b replaced by b_j. Hence the lim inf and the lim sup of the left side of (1.3) are squeezed between

$$\int_X b_1 \circ \pi \, dS \quad \text{and} \quad \int_X b_2 \circ \pi \, dS, \tag{1.9}$$

and taking $\varepsilon \searrow 0$ yields (1.3) for $b \in \mathcal{R}(M)$.

We take up the quantization of discontinuous symbols in §2. We use the Friedrichs quantization, $\mathrm{op}_F : C^\infty(X) \to OPS^0_{1,0}(M)$, which enjoys the positivity property

$$a \ge 0 \Longrightarrow \mathrm{op}_F(a) \ge 0. \tag{1.10}$$

This has a unique continuous extension to $\mathrm{op}_F : C(X) \to \mathcal{L}(L^2(M))$, and from there to

$$\mathrm{op}_F : L^\infty(X) \longrightarrow \mathcal{L}(L^2(M)), \tag{1.11}$$

still obeying (1.10). In this setting,

$$\begin{aligned} &a_\nu \to a \text{ weak}^* \text{ in } L^\infty(X) \\ &\Longrightarrow \mathrm{op}_F(a_\nu) \to \mathrm{op}_F(a) \text{ in the weak operator topology.} \end{aligned} \tag{1.12}$$

In §3 we establish the Weyl law

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} (B\varphi_k, \varphi_k)_{L^2} = \int_X b\, dS, \tag{1.13}$$

given

$$b \in \mathcal{R}(X), \quad B = \mathrm{op}_F(b), \tag{1.14}$$

which is useful for the analysis in §4.

In §4 we establish quantum ergodic theorems. One result, Proposition 4.2, is that there is a subset $\mathcal{N} \subset \mathbb{N}$, of density 0, such that if

$$a \in C(X), \quad Pa \in \mathcal{R}(X), \tag{1.15}$$

then

$$\lim_{k\to\infty, k\notin\mathcal{N}} (A\varphi_k, \varphi_k)_{L^2} - (A_p\varphi_k, \varphi_k)_{L^2} = 0, \tag{1.16}$$

where $A = \mathrm{op}_F(a)$, $A_p = \mathrm{op}_F(Pa)$. Another, Proposition 4.3, is the following extension of Proposition 1.2: under the hypothesis that (1.1) is ergodic, (1.2) holds for $A = \mathrm{op}_F(a)$ whenever $a \in \mathcal{R}(X)$. A further extension, Proposition 4.5, assumes $\{G_t\}$ acts ergodically on an open set $U \subset X$. In such a case, if

$$a, b \in \mathcal{R}(X) \ \text{ are supported on a compact subset of } \ U, \tag{1.17}$$

then

$$\begin{aligned} &\int_X a\, dS = \int_X b\, dS \\ &\qquad \Longrightarrow \lim_{k\to\infty, k\notin\mathcal{N}} (A\varphi_k, \varphi_k)_{L^2} - (B\varphi_k, \varphi_k)_{L^2} = 0, \end{aligned} \tag{1.18}$$

for $A = \mathrm{op}_F(a)$, $B = \mathrm{op}_F(b)$.

2. Quantization of discontinuous symbols

Let M be a compact Riemannian manifold, $X \subset T^*M$ as in §1. A *quantization* of $C^\infty(X)$ is a continuous linear map

$$\text{op} : C^\infty(X) \longrightarrow OPS^0_{1,0}(M), \tag{2.1}$$

with the property that, given $a \in C^\infty(X)$, $A = \text{op}(a)$ has principal symbol a (mod $S^{-1}_{1,0}(M)$). We insist that $\text{op}(1) = I$, the identity map. The existence of quantizations follows via local coordinate charts and partitions of unity from the calculus of pseudodifferential operators on Euclidean space. There are many different quantizations. Each one gives rise to a family of elements

$$\mu_{u,v} \in \mathcal{D}'(X), \quad \forall\, u, v \in L^2(M), \tag{2.2}$$

defined by

$$\langle a, \mu_{u,v} \rangle = (\text{op}(a)u, v)_{L^2}. \tag{2.3}$$

Basic examples are "Kohn-Nirenberg" quantizations and "Weyl" quantizations:

$$\text{op}_{KN},\ \text{op}_W : C^\infty(X) \longrightarrow OPS^0(M) \subset OPS^0_{1,0}(M). \tag{2.4}$$

Another family, of particular interest to us here, is the family of "Friedrichs quantizations,"

$$\text{op}_F : C^\infty(X) \longrightarrow OPS^0_{1,0}(M), \tag{2.5}$$

which has the property

$$a \ge 0 \Longrightarrow \text{op}_F(a) \ge 0. \tag{2.6}$$

This is constructed on the Euclidean space level from op_{KN} via Friedrichs symmetrization. See [T1], Chapter 7. It has the property that

$$a \in C^\infty(X) \Longrightarrow \text{op}_F(a) - \text{op}_{KN}(a) \in OPS^{-1}_{1,0}(M), \tag{2.7}$$

which plays an important role in the Friedrichs approach to the proof of the sharp Garding inequality. From (2.6) it follows that

$$\|\,\text{op}_F(a)\|_{\mathcal{L}(L^2)} \le \sup_X |a|, \tag{2.8}$$

and hence (2.5) has a unique continuous linear extension to

$$\mathrm{op}_F : C(X) \longrightarrow \mathcal{L}(L^2(M)). \tag{2.9}$$

The image of $C(X)$ in (2.9) is contained in the C^*-algebra of operators on $L^2(M)$ generated by $OPS^0(M)$, which we denote $\Psi(M)$. If we compose the map

$$\mathrm{op}_F : C(X) \longrightarrow \Psi(M) \tag{2.10}$$

with taking the quotient by $\mathcal{K}(L^2(M))$, the space of compact operators on $L^2(M)$, we get an isomorphism of C^*-algebras:

$$C(X) \xrightarrow{\approx} \Psi(M)/\mathcal{K}(L^2(M)). \tag{2.11}$$

We see from (2.6) that, for $u \in L^2(M)$, the distribution $\mu_u = \mu_{u,u} \in \mathcal{D}'(X)$ given by (2.3), with $u = v$, has the property that

$$a \geq 0 \Longrightarrow \langle a, \mu_u \rangle \geq 0. \tag{2.12}$$

Also $\langle 1, \mu_u \rangle = (u,u)_{L^2} = \|u\|_{L^2}^2$. Consequently, for each $u \in L^2(M)$,

$$\mu_u \text{ is a positive measure on } X, \text{ of mass } \|u\|_{L^2}^2. \tag{2.13}$$

Going further, we have from Cauchy's inequality that, for $u, v \in L^2(M)$,

$$\begin{aligned} |\langle a, \mu_{u,v} \rangle| &= |(\mathrm{op}_F(a)u, v)_{L^2}| \\ &\leq \big(\sup_X |a|\big) \|u\|_{L^2} \|v\|_{L^2}, \end{aligned} \tag{2.14}$$

so

$$\begin{aligned} \mu_{u,v} &\text{ is a complex measure on } X, \\ &\text{of total mass } \leq \|u\|_{L^2} \|v\|_{L^2}. \end{aligned} \tag{2.15}$$

Thus we can extend op_F from (2.9) to

$$\mathrm{op}_F : \mathcal{B}(X) \longrightarrow \mathcal{L}(L^2(M)), \tag{2.16}$$

where

$$\mathcal{B}(X) = \text{space of bounded Borel functions } a : X \to \mathbb{C}, \tag{2.17}$$

by

$$(\mathrm{op}_F(a)u, v) = \int_X a \, d\mu_{u,v}. \tag{2.18}$$

We continue to have (2.6) and (2.8), now for $a \in \mathcal{B}(X)$. We investigate the action of op_F on sequences a_ν.

Proposition 2.1. *Let* $a_\nu \in \mathcal{B}(X),\ \nu \in \mathbb{N}$. *Assume* $|a_\nu(z)| \le M < \infty$ *for all* ν, z, *and that*

$$a_\nu(z) \longrightarrow a(z), \quad \forall\, z \in X, \quad \text{as} \quad \nu \to \infty. \tag{2.19}$$

Then $\operatorname{op}_F(a_\nu) \to \operatorname{op}_F(a)$ *in the weak operator topology, i.e., for all* $u, v \in L^2(M)$,

$$(\operatorname{op}_F(a_\nu)u, v)_{L^2} \longrightarrow (\operatorname{op}_F(a)u, v)_{L^2}, \quad \text{as} \quad \nu \to \infty. \tag{2.20}$$

Proof. The desired conclusion (2.20) is equivalent to

$$\int_X a_\nu \, d\mu_{u,v} \longrightarrow \int_X a \, d\mu_{u,v}, \quad \text{as} \quad \nu \to \infty, \tag{2.21}$$

which follows from the Lebesgue dominated convergence theorem.

We can improve the conclusion of Proposition 2.1, given the following result on regularity of the measures $\mu_{u,v}$. This result follows by applying integretion by parts arguments to oscillatory integrals that yield the Euclidean space versions of $\operatorname{op}_F(a)$.

Proposition 2.2. *If* $u, v \in C^\infty(M)$, *then* $\mu_{u,v}$ *is a smooth multiple of Liouville measure on* X, *that is, for all* $a \in C^\infty(X)$, *hence for all* $a \in \mathcal{B}(X)$,

$$\int_X a \, d\mu_{u,v} = \int_X a\Phi(u, v) \, dS, \quad \text{given } u, v \in C^\infty(M), \tag{2.22}$$

where

$$\Phi : C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(X) \tag{2.23}$$

is a continuous bilinear map.

From here, we deduce that

$$a \in \mathcal{B}(X),\ a = 0 \ S\text{-a.e.} \implies (\operatorname{op}_F(a)u, v)_{L^2} = 0, \tag{2.24}$$

first for all $u, v \in C^\infty(M)$, and then for all $u, v \in L^2(M)$, by denseness of $C^\infty(M)$ in $L^2(M)$ and the estimate (2.8). Thus we can pass from (2.16) to

$$\operatorname{op}_F : L^\infty(X) \longrightarrow \mathcal{L}(L^2(M)), \tag{2.25}$$

satisfying

$$a \in L^\infty(X),\ a \ge 0 \ S\text{-a.e. on } X \implies \operatorname{op}_F(a) \ge 0, \tag{2.26}$$

and

$$a \in L^\infty(X) \implies \|\operatorname{op}_F(a)\|_{\mathcal{L}(L^2)} \le \|a\|_{L^\infty}. \tag{2.27}$$

We can then extend Proposition 2.1 as follows.

Proposition 2.3. *Let* $a_\nu \in L^\infty(X),\ \nu \in \mathbb{N}$. *Assume* $\|a_\nu\|_{L^\infty} \le M < \infty$ *for all* ν, *and that*

$$a_\nu \longrightarrow a,\ S\text{-a.e. on } X,\ \text{as } \nu \to \infty. \tag{2.28}$$

Then $\mathrm{op}_F(a_\nu) \to \mathrm{op}_F(a)$ *in the weak operator topology, i.e., for all* $u, v \in L^2(M)$, *(2.20) holds.*

Proof. If $u, v \in C^\infty(M)$, then (2.20) follows from

$$\int_X a_\nu\, \Phi(u,v)\, dS \longrightarrow \int_X a\, \Phi(u,v)\, dS, \quad \text{as} \ \ \nu \to \infty, \tag{2.29}$$

which in turn follows from the Lebesgue dominated convergence theorem. Then (2.20) follows for general $u, v \in L^2(M)$ via denseness of $C^\infty(M)$ in $L^2(M)$ and the operator bounds $\|\mathrm{op}_F(a_\nu)\|_{\mathcal{L}(L^2)} \le M$.

A similar argument yields the following.

Proposition 2.4. *Let* $a_\nu \in L^\infty(X)$ *for* $\nu \in \mathbb{N}$, *and* $a \in L^\infty(X)$. *If* $a_\nu \to a$ *weak** *in* $L^\infty(X)$, *then* $\mathrm{op}_F(a_\nu) \to \mathrm{op}_F(a)$ *in the weak operator topology.*

Proof. Again, (2.29) holds for each $u, v \in C^\infty(M)$. From here, we follow the rest of the proof of Proposition 2.3.

3. Weyl law

Recall that M is a compact Riemannian manifold, $\Lambda \in OPS^1(M)$ is a first-order, positive, self-adjoint elliptic operator, and $\{\varphi_k : k \in \mathbb{N}\}$ is an orthonormal basis of $L^2(M)$ consisting of eigenfunctions of Λ, with eigenvalues $\lambda_j \nearrow +\infty$. Also $X \subset T^*M$ is the level set on which the principal symbol of Λ is equal to 1.

This section is devoted to the proof of the following.

Proposition 3.1. *Given*

$$b \in \mathcal{R}(X), \quad B = \mathrm{op}_F(b), \tag{3.1}$$

we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} (B\varphi_k, \varphi_k)_{L^2} = \int_X b\, dS. \tag{3.2}$$

Here $\mathcal{R}(X)$ is the space of Riemann integrable functions on X. The result (3.2) is classical for $b \in C^\infty(X)$. In such a case, one can obtain an asymptotic expansion of

$$\mathrm{Tr}\, Be^{-t\Lambda}, \quad \text{as } \ t \searrow 0, \tag{3.3}$$

and deduce (3.2) via a Tauberian theorem.

The transition from $b \in C^\infty(X)$ to $b \in \mathcal{R}(X)$ is fairly straightforward, given the material of §2. It suffices to treat the case where $b \in \mathcal{R}(X)$ is real valued. Then, given $\varepsilon > 0$, we can pick $b_1, b_2 \in C^\infty(X)$ such that

$$b_1 \le b \le b_2, \quad \int_X (b_2 - b_1)\, dS < \varepsilon. \tag{3.4}$$

We have

$$\mathrm{op}_F(b_1) \le B \le \mathrm{op}_F(b_2), \tag{3.5}$$

and the classical result for $b_j \in C^\infty(X)$ gives

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} (\mathrm{op}_F(b_j)\varphi_k, \varphi_k)_{L^2} = \int_X b_j\, dS, \quad j = 1, 2. \tag{3.6}$$

Hence

$$\begin{aligned}\int_X b_1\, dS &\le \liminf_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N}(B\varphi_k,\varphi_k)_{L^2} \\ &\le \limsup_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N}(B\varphi_k,\varphi_k)_{L^2} \\ &\le \int_X b_2\, dS,\end{aligned} \tag{3.7}$$

and (3.2) follows, upon taking $\varepsilon \searrow 0$.

Remark. It would be interesting to know if one can extend this result, replacing the hypothesis $b \in \mathcal{R}(X)$ by $b \in L^\infty(X)$.

4. Quantum ergodic theorems

Our goal is to present various extensions of the following result, established in [ST] and [T2].

Proposition 4.1. *Given M and Λ, and φ_k as described in §1, there is a subset $\mathcal{N} \subset \mathbb{N}$, of density 0, such that the following holds. Let $a \in C(X)$, $A = \mathrm{op}_F(a)$. Assume*

$$Pa = \overline{a} = \int_X a\, dS. \tag{4.1}$$

Then

$$\lim_{k\to\infty, k\notin\mathcal{N}} (A\varphi_k, \varphi_k)_{L^2} = \overline{a}. \tag{4.2}$$

Going further, if we replace (4.1) by

$$Pa \in C(X), \tag{4.3}$$

then

$$\lim_{k\to\infty, k\notin\mathcal{N}} (A\varphi_k, \varphi_k)_{L^2} - (\mathrm{op}_F(Pa)\varphi_k, \varphi_k)_{L^2} = 0. \tag{4.4}$$

That (4.1) $\Rightarrow$ (4.2), for $a \in C^\infty(X)$, was noted in [ST], and the extension to $a \in C(X)$ was made in [T2]. The result (4.3) $\Rightarrow$ (4.4) follows by applying the first part of the proposition to $B = \mathrm{op}_F(b)$, with $b = a - Pa$, for which we have $Pb = \overline{b} = 0$. Our first goal here is to establish the following more general result.

Proposition 4.2. *There is a set $\mathcal{N} \subset \mathbb{N}$, of density 0, such that the following holds. Assume*

$$a \in C(X), \quad Pa \in \mathcal{R}(X), \tag{4.5}$$

where $\mathcal{R}(X)$ is the space of Riemann integrable functions on X. Then (4.4) continues to hold.

Proof. The strategy will be to use results of §3 to extend the proof of Proposition 4.1 given in [T2] to handle the more general situation described above. We bring in the probability measures μ_k on X, given by $\mu_k = \mu_{\varphi_k} = \mu_{\varphi_k,\varphi_k}$. We will show that, given (4.5),

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \Big| \int_X (a - Pa)\, d\mu_k \Big| = 0. \tag{4.6}$$

One ingredient is the following consequence of Egorov's theorem:

$$\int_X (a_T - a)\, d\mu_k \longrightarrow 0, \quad \text{as} \quad k \to \infty, \quad \forall\, a \in C(X), \tag{4.7}$$

locally uniformly in $T \in [0, \infty)$, where

$$a_T = \frac{1}{T} \int_0^T a \circ G_t \, dt. \tag{4.8}$$

See (5.7) of [T2]. From (4.7), we have

$$\int_X (a_T - Pa)\, d\mu_k - \int_X (a - Pa)\, d\mu_k \longrightarrow 0, \quad \text{as} \quad k \to \infty, \tag{4.9}$$

locally uniformly in T. The next ingredient is the Weyl law:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^N \int_X b\, d\mu_k = \int_X b\, dS, \quad \forall\, b \in \mathcal{R}(X), \tag{4.10}$$

established in §3, as an extension of the classical version, for $b \in C^\infty(X)$. We apply this to

$$b = |a_T - Pa|, \tag{4.11}$$

which belongs to $\mathcal{R}(X)$ if (4.5) holds, and obtain

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^N \int_X |a_T - Pa|\, d\mu_k = \int_X |a_T - Pa|\, dS. \tag{4.12}$$

Now the mean ergodic theorem implies that, for each $\varepsilon > 0$, there exists $T_\varepsilon < \infty$ such that

$$\int_X |a_T - Pa|\, dS \le \varepsilon, \quad \forall\, T \ge T_\varepsilon. \tag{4.13}$$

Together, (4.9), (4.12), and (4.13) yield

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^N \Bigl| \int_X (a - Pa)\, d\mu_k \Bigr| \le \varepsilon, \quad \forall\, \varepsilon > 0, \tag{4.14}$$

which implies (4.6).

We deduce (4.4) from (4.6) as follows. Note that

$$E(X) = \{a \in C(X) : Pa \in \mathcal{R}(X)\} \tag{4.15}$$

is a closed linear subspace of $C(X)$. This follows from the fact that

$$\sup_z |Pa_\nu(z) - Pa(z)| \le \sup_z |a(z) - a_\nu(z)|, \tag{4.16}$$

and that uniform limits of elements of $\mathcal{R}(X)$ also belong to $\mathcal{R}(X)$. Since $C(X)$ is separable, so is $E(X)$, and we can take a countable dense subset $\{a_\nu : \nu \in \mathbb{N}\}$ of $E(X)$. It follows directly from (4.6), applied to a_ν, that there exist $\mathcal{N}_\nu \subset \mathbb{N}$, of density 0, such that

$$\lim_{k\to\infty, k\notin\mathcal{N}_\nu} \int_X (a_\nu - Pa_\nu)\, d\mu_k = 0. \tag{4.17}$$

Now we can take a set $\mathcal{N} \subset \mathbb{N}$, of density 0, such that $\mathcal{N} \setminus \mathcal{N}_\nu$ is finite, for each ν, so

$$\lim_{k\to\infty, k\notin\mathcal{N}} \int_X (a_\nu - Pa_\nu)\, d\mu_k = 0, \quad \forall \nu \in \mathbb{N}. \tag{4.18}$$

The denseness of $\{a_\nu : \nu \in \mathbb{N}\}$ in $E(X)$, together with (4.16), then implies

$$\lim_{k\to\infty, k\notin\mathcal{N}} \int_X (a - Pa)\, d\mu_k = 0, \quad \forall\, a \in E(X), \tag{4.19}$$

and this is equivalent to (4.4).

We now relax the hypothesis that the symbol a be continuous on X, first under an ergodicity hypothesis.

Proposition 4.3. *Assume $\{G_t : t \in \mathbb{R}^+\}$ acts ergodically on X. Then there is a set $\mathcal{N} \subset \mathbb{N}$, of density* 0*, such that*

$$\lim_{k\to\infty, k\notin\mathcal{N}} (\mathrm{op}_F(a)\varphi_k, \varphi_k)_{L^2} = \int_X a\, dS, \tag{4.20}$$

for all $a \in \mathcal{R}(X)$.

Proof. For $a \in C(X)$, this is the classical quantum ergodic theorem (cf. [CV]). Note that it also follows from Proposition 4.1, since ergodicity implies $Pa = \overline{a}$. Now, given a real valued $a \in \mathcal{R}(X)$, and given $\varepsilon > 0$, we can pick

$$b_1, b_2 \in C(X) \text{ such that } b_1 \le a \le b_2 \ \text{ and } \int_X (b_2 - b_1)\, dS < \varepsilon. \tag{4.21}$$

We know that

$$\lim_{k\to\infty, k\notin\mathcal{N}} (\mathrm{op}_F(b_j)\varphi_k, \varphi_k)_{L^2} = \int_X b_j \, dS, \tag{4.22}$$

and that

$$\mathrm{op}_F(b_1) \le \mathrm{op}_F(a) \le \mathrm{op}_F(b_2), \tag{4.23}$$

so

$$\begin{aligned} \limsup_{k\to\infty, k\notin\mathcal{N}} (\mathrm{op}_F(a)\varphi_k, \varphi_k)_{L^2} &\le \int_X b_2 \, dS, \quad \text{and} \\ \liminf_{k\to\infty, k\notin\mathcal{N}} (\mathrm{op}_F(a)\varphi_k, \varphi_k)_{L^2} &\ge \int_X b_1 \, dS, \end{aligned} \tag{4.24}$$

and we have (4.20).

The following is a local equidistribution result, associated with ergodicity on an open subset of X. Compare results of [Riv] and [Gal].

Proposition 4.4. *Let $U \subset X$ be open and assume $G_t : U \to U$, and that the action on U is ergodic. Let $a, b \in C(X)$ be supported on a compact subset of U, $A = \mathrm{op}_F(a)$, $B = \mathrm{op}_F(b)$. Take $\mathcal{N}$ as above. Then*

$$\begin{aligned} &\int_X a \, dS = \int_X b \, dS \\ &\qquad \Longrightarrow \lim_{k\to\infty, k\notin\mathcal{N}} (A\varphi_k, \varphi_k)_{L^2} - (B\varphi_k, \varphi_k)_{L^2} = 0. \end{aligned} \tag{4.25}$$

Proof. The hypotheses yield

$$a - b \in C(X), \quad P(a-b) = 0, \tag{4.26}$$

so the conclusion (4.25) is a corollary of Proposition 4.1.

We next extend the scope of Proposition 4.4, along the lines of Proposition 4.3.

Proposition 4.5. *In the setting of Proposition 4.4, the implication (4.25) holds for all $a, b \in \mathcal{R}(X)$ that are supported on a compact subset of U.*

Proof. It suffices to show that, given

$$\text{real valued} \quad a \in \mathcal{R}(X), \quad \mathrm{supp}\, a \subset K \subset\subset U, \tag{4.27}$$

we have

$$\int\limits_X a\,dS = 0 \Longrightarrow \lim_{k\to\infty, k\notin\mathcal{N}} (A\varphi_k, \varphi_k)_{L^2} = 0. \tag{4.28}$$

To get this, let $\varepsilon > 0$ and take $b_1, b_2 \in C(X)$ such that

$$\operatorname{supp} b_j \subset\subset U, \quad b_1 \le a \le b_2, \quad \int\limits_X (b_2 - b_1)\,dS < \varepsilon. \tag{4.29}$$

Next, take $g_j \in C(X)$, $\operatorname{supp} g_j \subset\subset U$, such that

$$\sup |g_j| \le C\varepsilon, \quad \text{and} \quad \int\limits_X (b_j - g_j)\,dS = 0. \tag{4.30}$$

Then, by Proposition 4.4,

$$((\operatorname{op}_F(b_j) - \operatorname{op}_F(g_j))\varphi_k, \varphi_k)_{L^2} \longrightarrow 0, \quad \text{as} \ \ k \to \infty,\ k \notin \mathcal{N}. \tag{4.31}$$

Now

$$\operatorname{op}_F(b_1) \le A \le \operatorname{op}_F(b_2), \tag{4.32}$$

so

$$\operatorname{op}_F(b_1 - g_1) - C\varepsilon I \le A \le \operatorname{op}_F(b_2 - g_2) + C\varepsilon I, \tag{4.33}$$

hence

$$\begin{aligned} &\limsup_{k\to\infty, k\notin\mathcal{N}} (A\varphi_k, \varphi_k)_{L^2} \le C\varepsilon, \quad \text{and} \\ &\liminf_{k\to\infty, k\notin\mathcal{N}} (A\varphi_k, \varphi_k)_{L^2} \ge C\varepsilon. \end{aligned} \tag{4.34}$$

This gives (4.28).

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