Quantum Ergodic Theorems for $e^{-it\Lambda}Ae^{it\Lambda}$

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Contents

1. Introduction

Let M be a compact Riemannian manifold, with Laplace-Beltrami operator Δ, and set $\Lambda = \sqrt{-\Delta}$. If A is a bounded linear map on $L^2(M)$ (we write $A \in \mathcal{L}(L^2(M))$), and $T \in (0,\infty)$, we set

$$A_T = \frac{1}{2T}\int_{-T}^{T} e^{-it\Lambda}Ae^{it\Lambda}\,dt. \tag{1.1}$$

Our goal is to study the behavior of A_T as $T \to \infty$ and relate this study to classical ergodic theory.

To get started, we recall the abstract mean ergodic theorem of von Neumann. Let U^t be a strongly continuous unitary group on a Hilbert space $\mathcal{H}$, and set

$$\mathcal{A}_T f = \frac{1}{2T}\int_{-T}^{T} U^t f\,dt, \quad f \in \mathcal{H}. \tag{1.2}$$

Then $U^t = e^{itB}$, where B is a self-adjoint operator on $\mathcal{H}$, and the spectral theorem yields

$$\begin{aligned}\mathcal{A}_T f &= \frac{1}{2T}\int_{-T}^{T} e^{itB} f\,dt \\ &= \frac{\sin TB}{TB}\,f,\end{aligned} \tag{1.3}$$

and hence, for each $f \in \mathcal{H}$,

$$\mathcal{A}_T f \longrightarrow P_0 f \ \text{ in } \mathcal{H}\text{-norm, as } \ T \to \infty, \tag{1.4}$$

where

$$P_0 = \text{ orthogonal projection of } \mathcal{H} \text{ onto } \operatorname{Ker} B, \quad \operatorname{Ker} B = \{f \in \mathcal{H} : U^t f = f,\ \forall\, t \in \mathbb{R}\}. \tag{1.5}$$

In other words, $\mathcal{A}_T$ converges to P_0 in the strong operator topology of $\mathcal{L}(\mathcal{H})$.

To relate this to (1.1), we look at

$$W^t : \mathcal{L}(L^2(M)) \longrightarrow \mathcal{L}(L^2(M)), \quad W^t(A) = e^{-it\Lambda} A e^{it\Lambda}. \tag{1.6}$$

Clearly $\{W^t\}$ is a group of isometries of $\mathcal{L}(L^2(M))$, and, for each $A \in \mathcal{L}(L^2(M))$, $W^t(A)$ is continuous in t with values in $\mathcal{L}(L^2(M))$, provided with the strong operator topology. On the other hand, clearly $W^t(A)$ is continuous from $t \in \mathbb{R}$ to $\mathcal{L}(L^2(M))$, with the *norm* topology, if A has finite rank, and hence if A is compact. We also have

$$W^t(A)\langle\Lambda\rangle^{-\kappa} = W^t(A\langle\Lambda\rangle^{-\kappa}), \tag{1.7}$$

where $\langle\Lambda\rangle = \sqrt{1-\Delta}$. Now $\mathcal{L}(L^2(M))$ is not a Hilbert space, so it is convenient to focus on the Hilbert space

$$\mathcal{H} = \mathrm{HS}(L^2(M)), \tag{1.8}$$

of Hilbert-Schmidt operators on $L^2(M)$, a Hilbert space with inner product

$$(A, B)_{\mathrm{HS}} = \operatorname{Tr} B^* A. \tag{1.9}$$

The restriction $U^t = W^t|_{\mathrm{HS}(L^2(M))}$ is a strongly continuous group of unitary operators on $\mathrm{HS}(L^2(M))$:

$$U^t(A) = e^{it \operatorname{ad} \Lambda}(A). \tag{1.10}$$

Consequently, for $A \in \mathrm{HS}(L^2(M))$,

$$A_T = \frac{\sin T \operatorname{ad} \Lambda}{T \operatorname{ad} \Lambda}\, A. \tag{1.11}$$

We have the following result.

Proposition 1.1. *If $A \in \mathrm{HS}(L^2(M))$, then, for A_T as in (1.1),*

$$A_T \longrightarrow \Pi_0(A) \ \text{ in HS-norm}, \tag{1.12}$$

where

$$\Pi_0 = \text{ orthogonal projection of } \mathrm{HS}(L^2(M)) \text{ onto } \mathcal{K}_0 = \{A \in \mathrm{HS}(L^2(M)) : e^{-it\Lambda} A e^{it\Lambda} = A,\ \forall\, t \in \mathbb{R}\}. \tag{1.13}$$

Remark. Under the natural isomorphism $\mathrm{HS}(L^2(M)) \approx L^2(M\times M)$, we have

$$\operatorname{ad}\Lambda = \Lambda_x - \Lambda_y. \tag{1.14}$$

Using (1.7), we see that, whenever $\kappa > n/2$, with $n = \dim M$ (so $\langle\Lambda\rangle^{-\kappa}$ is Hilbert-Schmidt),

$$\begin{aligned} A \in \mathcal{L}(L^2(M)) \Longrightarrow A_T\langle\Lambda\rangle^{-\kappa} &= \big(A\langle\Lambda\rangle^{-\kappa}\big)_T \\ &\to \Pi_0(A\langle\Lambda\rangle^{-\kappa}), \end{aligned} \tag{1.15}$$

in HS-norm, and a fortiori in operator norm in $\mathcal{L}(L^2(M))$. Consequently, for all $A \in \mathcal{L}(L^2(M))$,

$$\begin{aligned} &A_T \longrightarrow \Pi(A) = \Pi_0(A\langle\Lambda\rangle^{-\kappa})\langle\Lambda\rangle^{\kappa} \\ &\text{in operator norm in } \mathcal{L}(H^\kappa(M), L^2(M)), \end{aligned} \tag{1.16}$$

for each $\kappa > n/2$, where $H^\kappa(M) = \mathcal{D}(\langle\Lambda\rangle^\kappa)$ is an L^2-Sobolev space. Here the identity

$$\Pi(A) = \Pi_0(A\langle\Lambda\rangle^{-\kappa})\langle\Lambda\rangle^{\kappa}, \quad \kappa > \frac{n}{2}, \tag{1.17}$$

defines

$$\Pi : \mathcal{L}(L^2(M)) \longrightarrow \mathcal{L}(H^\kappa(M), L^2(M)). \tag{1.18}$$

The action of Π is independent of $\kappa > n/2$. In addition, the uniform operator norm bounds $\|A_T\|_{\mathcal{L}(L^2)} \le \|A\|_{\mathcal{L}(L^2)}$, plus denseness of $H^\kappa(M)$ in $L^2(M)$, yield

$$\Pi : \mathcal{L}(L^2(M)) \longrightarrow \mathcal{L}(L^2(M)), \quad \|\Pi(A)\|_{\mathcal{L}(L^2(M))} \le \|A\|_{\mathcal{L}(L^2(M))}. \tag{1.19}$$

Going further, using this denseness and uniform bounds on A_T, we have:

Proposition 1.2. *For $A \in \mathcal{L}(L^2(M))$, A_T as in (1.2),*

$$A_T \longrightarrow \Pi(A) \ \ \textit{in the strong operator topology of} \ \ \mathcal{L}(L^2(M)). \tag{1.20}$$

There is another formula for $\Pi(A)$, which will prove useful. To state it let

$$\begin{aligned} &P_\lambda = \text{ orthogonal projection of } L^2(M) \text{ onto } \operatorname{Eigen}(\Lambda,\lambda), \\ &\Sigma_N = \{\lambda \in \operatorname{Spec}\Lambda : \lambda \le N\}, \end{aligned} \tag{1.21}$$

and set

$$S_N(A) = \sum_{\lambda\in\Sigma_N} P_\lambda A P_\lambda. \tag{1.22}$$

We see that $\|S_N(A)\|_{\mathcal{L}(L^2)} \le \|A\|_{\mathcal{L}(L^2)}$, that $S_N(A\langle\Lambda\rangle^{-\kappa}) = S_N(A)\langle\Lambda\rangle^{-\kappa}$, and that

$$S_N(A) \longrightarrow \Pi(A) \text{ in HS-norm, if } A \in \mathrm{HS}(L^2(M)). \tag{1.23}$$

It follows that

$$S_N(A) \longrightarrow \Pi(A), \text{ in the strong operator topology }, \quad \forall\, A \in \mathcal{L}(L^2(M)). \tag{1.24}$$

In other words,

$$\Pi(A) = \sum_{\lambda\in\operatorname{Spec}\Lambda} P_\lambda A P_\lambda. \tag{1.25}$$

Compare (2.24) of [Z1], and also the material in [Su].

Our main goal here will be to prove the following (cf. Proposition 3.5).

Theorem A. *If* $a, Pa \in C(S^*M)$, *and* $A = \mathrm{op}_F(a)$, *then*

$$\lim_{N\to\infty} \frac{1}{d_N} \|[\Pi(A) - \Pi(\mathrm{op}_F(Pa))]Q_N\|^2_{\mathrm{HS}} = 0. \tag{1.26}$$

Here, S^*M is the cosphere bundle of M, $P : L^2(S^*M) \to L^2(S^*M)$ is the orthogonal projection onto the space of functions on S^*M invariant under the geodesic flow,

$$Q_N = \sum_{\lambda\le N} P_\lambda, \quad d_N = \operatorname{Tr} Q_N, \tag{1.27}$$

and $\mathrm{op}_F : C(S^*M) \to \mathcal{L}(L^2(M))$ is a quantization operator, discussed in §2. A particular case of (1.26) is

$$a \in C(S^*M),\ Pa = \overline{a} \Rightarrow \lim_{N\to\infty} \frac{1}{d_N} \|[\Pi(A) - \overline{a}I]Q_N\|^2_{\mathrm{HS}} = 0, \tag{1.28}$$

where $\overline{a}$ is the mean value of a over S^*M. The hypothesis $Pa = \overline{a}$ for all $a \in C(S^*M)$ holds provided the geodesic flow on S^*M is ergodic. In the ergodic case, (1.28) is due to [Su] (see also [Z1]). Our extension beyond the case of an ergodic geodesic flow is done in the spirit of [ST] and [T1]. In this connection, we mention the following result (cf. Proposition 3.6).

Corollary B. *Let $U \subset S^*M$ be open and assume the geodesic flow $\mathcal{G}_t : U \to U$ and that the action on U is ergodic. Let $a, b \in C(S^*M)$ be supported in a compact subset of U, $A = \mathrm{op}_F(a)$, $B = \mathrm{op}_F(b)$. Then*

$$\int\limits_{S^*M} a\,dS = \int\limits_{S^*M} b\,dS \Longrightarrow \lim_{N\to\infty} \frac{1}{d_N} \|[\Pi(A) - \Pi(B)]Q_N\|_{\mathrm{HS}}^2 = 0. \tag{1.29}$$

As indicated above, these results are established in §3 of this paper. In §2 we set up tools needed to accomplish this, including a discussion of the quantization $a \mapsto \mathrm{op}_F(a)$ (part of pseudodifferential operator calculus on M), and of the Weyl law and Egorov's theorem. A key ingredient in the proof of Theorem A is that, given $a, Pa \in C(S^*M)$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} |(A\varphi_k, \varphi_k) - (\mathrm{op}_F(Pa)\varphi_k, \varphi_k)|^2 = 0, \tag{1.30}$$

where $\{\varphi_k\}$ is an orthonormal basis of $L^2(M)$ satisfying $\Lambda\varphi_k = \lambda_k\varphi_k$, $\lambda_k \nearrow \infty$. Cf. (3.18). If the geodesic flow on S^*M is ergodic, then $Pa = \overline{a}$, and (1.30) is a standard version of quantum ergodicity (cf. [CdV]).

In §§4–5 we discuss eigenfunction concentration effects in some cases where the geodesic flow is not ergodic, due to the existence of a nontrivial Killing field on M. Section 4 treats spherical harmonics on the standard sphere S^n, and §5 treats much more general cases.

Remark. While the conjugate $e^{-it\Lambda}Ae^{it\Lambda}$ is natural to work with due to its connection to Egorov's theorem, it is also quite natural to consider

$$e^{it\Delta}Ae^{-it\Delta}, \tag{1.31}$$

in view of its quantum mechanical significance, and to replace (1.1) by

$$A_T^\# = \frac{1}{2T} \int_{-T}^{T} e^{it\Delta}Ae^{-it\Delta}\,dt. \tag{1.32}$$

Arguments parallel to those leading to Proposition 1.2 also yield

$$A_T^\# \longrightarrow \Pi(A), \tag{1.33}$$

in the strong operator topology, for each $A \in \mathcal{L}(L^2(M))$. The limit here is the same as in (1.20), as one verifies that it satisfies (1.25).

2. Quantization of $X = S^*M$

With M as in §1, let $X = S^*M$. A quantization of X is a continuous linear map

$$\mathrm{op} : C^\infty(X) \longrightarrow OPS^0_{1,0}(M), \tag{2.1}$$

with the property that for each $a \in C^\infty(X)$, the principal symbol of $\mathrm{op}(A)$ is a. We also require $\mathrm{op}(1) = I$. Examples include the Kohn-Nirenberg quantization op_{KN} and the Weyl quantization op_W. We will focus on another, the Friedrichs quantization,

$$\mathrm{op}_F : C^\infty(X) \longrightarrow OPS^0_{1,0}(M), \tag{2.2}$$

which has the special property that, for each $a \in C^\infty(X)$,

$$a \geq 0 \Longrightarrow \mathrm{op}_F(a) \geq 0. \tag{2.3}$$

It also satisfies

$$\mathrm{op}_F(a) - \mathrm{op}_{KN}(a) \in OPS^{-1}_{1,0}(M), \tag{2.4}$$

for all $a \in C^\infty(X)$. Thanks to (2.2)–(2.3), there is a unique continuous linear extension

$$\mathrm{op}_F : C(X) \longrightarrow \mathcal{L}(L^2(M)), \tag{2.5}$$

and it also satisfies (2.3). Furthermore, as shown in [T2], there is a unique extension to

$$\mathrm{op}_F : L^\infty(X) \longrightarrow \mathcal{L}(L^2(M)), \tag{2.6}$$

having the property that

$$\begin{aligned} &a_\nu \in L^\infty(X),\ a_\nu \to a \text{ weak}^* \text{ in } L^\infty(X) \\ &\Longrightarrow \mathrm{op}_F(a_\nu) \to \mathrm{op}_F(a) \text{ in the weak operator topology of } \mathcal{L}(L^2(M)). \end{aligned} \tag{2.7}$$

The positivity condition (2.3) continues to hold. Furthermore, for $a \in L^\infty(X)$,

$$\| \mathrm{op}_F(a) \|_{\mathcal{L}(L^2(M))} \leq \|a\|_{L^\infty(X)}. \tag{2.8}$$

We mention that a special case of (2.6) is

$$\mathrm{op}_F : \mathcal{R}(X) \longrightarrow \mathcal{L}(L^2(M)), \tag{2.9}$$

where $\mathcal{R}(X)$ denotes the space of bounded functions on X that are Riemann integrable.

We next describe some results that are useful for the analysis of A_T, defined as in (1.1), when $A = \mathrm{op}_F(a)$. First is the Weyl law. To state it, let $\{\varphi_k\}$ be an orthonormal basis of $L^2(M)$ consisting of eigenfunctions of Δ:

$$\Delta \varphi_k = -\lambda_k^2 \varphi_k, \quad 0 \leq \lambda_k \nearrow +\infty. \tag{2.10}$$

Proposition 2.1. *We have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} (B\varphi_k, \varphi_k)_{L^2} = \int_X b\, dS, \tag{2.11}$$

for $B = \mathrm{op}_F(b)$, $b \in C^\infty(X)$, *where* dS *is the Liouville measure on* X, *normalized so that* $\int_X dS = 1$. *More generally, (2.11) holds for all* $b \in \mathcal{R}(X)$.

Proposition 2.1 is classical for $b \in C^\infty(X)$. It is extended to $b \in \mathcal{R}(X)$ in §3 of [T2].

We can rewrite (2.11) as follows. Let

$$Q_N = \text{ orthogonal projection of } L^2(M) \text{ onto } \mathrm{Span}\{\varphi_k : 1 \le k \le N\}. \tag{2.12}$$

Then (2.11) says

$$\lim_{N\to\infty} \frac{1}{N} \,\mathrm{Tr}\, BQ_N = \int_X b\, dS. \tag{2.13}$$

The next result is a Weyl/Szegö type result.

Proposition 2.2. *We have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \|B\varphi_k\|_{L^2}^2 = \int_X |b|^2\, dS, \tag{2.14}$$

for $B = \mathrm{op}_F(b)$, $b \in C^\infty(X)$. *More generally, (2.14) holds for* $b \in C(X)$.

Proof. The left side of (2.14) is equal to

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N} \|BQ_N\|_{\mathrm{HS}}^2 &= \lim_{N\to\infty} \frac{1}{N} \,\mathrm{Tr}\, Q_N B^* B Q_N \\ &= \lim_{N\to\infty} \frac{1}{N} \,\mathrm{Tr}\, B^* B Q_N. \end{aligned} \tag{2.15}$$

If $b \in C^\infty(X)$, then

$$B^* B = \mathrm{op}_F(|b|^2), \quad \mathrm{mod}\ OPS^{-1}_{1,0}(M), \tag{2.16}$$

and the result follows from Proposition 2.1, with b replaced by $|b|^2 \in C^\infty(X)$. The extension of (2.14) to $b \in C(X)$ follows from the denseness of $C^\infty(X)$ in $C(X)$ and the estimate (2.8).

Remark. Unlike Proposition 2.1, I have not extended Proposition 2.2 to work for $b \in \mathcal{R}(X)$.

Another important tool is Egorov's theorm, which implies

$$e^{-it\Lambda}\,\mathrm{op}_F(a)e^{it\Lambda} - \mathrm{op}_F(a\circ\mathcal{G}_t) \text{ compact on } L^2(M), \quad \forall\, t\in\mathbb{R}, \tag{2.17}$$

for all $a \in C(X)$, where

$$\mathcal{G}_t \text{ is the geodesic flow on } X = S^*M, \tag{2.18}$$

generated by the Hamiltonian vector field associated to the principal symbol of Λ, a smooth flow on X that preserves the Liouville measure. For $a \in C^\infty(X)$, this difference belongs to $OPS^{-1}_{1,0}(M)$ for all t. Extension of (2.17) to $a \in C(X)$ follows readily from the denseness of $C^\infty(X)$ in $C(X)$ and (2.8). As a corollary, we have the following.

Proposition 2.3. *Let $a \in C(X)$, and, for $T \in (0,\infty)$, set*

$$a_T = \frac{1}{2T}\int_{-T}^{T} a\circ\mathcal{G}_t\,dt, \tag{2.19}$$

and

$$A_T = \frac{1}{2T}\int_{-T}^{T} e^{-it\Lambda}Ae^{it\Lambda}\,dt, \quad A = \mathrm{op}_F(a). \tag{2.20}$$

Then, for each such T,

$$A_T - \mathrm{op}_F(a_T) \;\; \textit{is compact on } L^2(M). \tag{2.21}$$

Corollary 2.4. *In the setting of Proposition 2.3,*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\|[A_T - \mathrm{op}_F(a_T)]\varphi_k\|^2_{L^2} = 0, \quad \forall\, T<\infty. \tag{2.22}$$

Proof. Given $\{\varphi_k\}$ is an orthonormal set in $L^2(M)$, then $\varphi_k \to 0$ weakly, as $k\to\infty$, so $\|K\varphi_k\|_{L^2} \to 0$ as $k \to \infty$ for each compact operator on $L^2(M)$. Hence, for each $T < \infty$,

$$\|[A_T - \mathrm{op}_F(a_T)]\varphi_k\|_{L^2} \longrightarrow 0 \;\text{ as }\; k\to\infty, \tag{2.23}$$

and (2.22) follows.

3. Ergodic theorems

Before discussing quantum ergodic theorems, we recall some classical ergodic theorems, as applied to the group $\mathcal{G}_t : X \to X$ of measure preserving homeomorphisms of X. Von Neumann's mean ergodic theorem yields for a_T in (2.19)

$$a_T \longrightarrow Pa, \quad \text{in } L^2(X)\text{-norm}, \tag{3.1}$$

as $T \to \infty$, for all $a \in L^2(X)$, where

$$P = \text{ orthogonal projection of } L^2(X) \text{ onto } \{a \in L^2(X) : a \circ \mathcal{G}_t = a, \ \forall\, t \in \mathbb{R}\}. \tag{3.2}$$

Birkhoff's ergodic theorem then yields

$$a_T \longrightarrow Pa, \quad \text{a.e. on } X, \tag{3.3}$$

for all $a \in L^1(X)$. We also have

$$P : L^p(X) \longrightarrow L^p(X), \quad \forall\, p \in [1, \infty], \tag{3.4}$$

and

$$a_T \longrightarrow Pa \ \text{ in } L^p\text{-norm, for } a \in L^p(X), \quad 1 \le p < \infty, \tag{3.5}$$

while

$$\begin{aligned} a \in L^\infty(X) &\Longrightarrow a_T \to Pa \text{ pointwise a.e. and boundedly} \\ &\Longrightarrow a_T \to Pa \text{ weak}^* \text{ in } L^\infty(X), \end{aligned} \tag{3.6}$$

as $T \to \infty$. In light of (2.7), we deduce from (3.6) that

$$\mathrm{op}_F(a_T) \longrightarrow \mathrm{op}_F(Pa) \ \text{ in the weak operator topology of } \mathcal{L}(L^2(M)), \tag{3.7}$$

as $T \to \infty$, given $a \in L^\infty(X)$.

The relevance of (3.1) in particular arises from applying Proposition 2.2 to

$$b = a_T - Pa. \tag{3.8}$$

We have

Proposition 3.1. *If $a \in C(X)$, then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \| \operatorname{op}_F(a_T - Pa)\varphi_k\|^2_{L^2(M)} = \int_X |a_T - Pa|^2 \, dS, \tag{3.9}$$

provided that also

$$Pa \in C(X). \tag{3.10}$$

We can use this, in combination with Corollary 2.4, to establish the following.

Proposition 3.2. *If $a, Pa \in C(X)$, $A = \operatorname{op}_F(a)$, and A_T is as in (2.20), then*

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \|[A_T - \operatorname{op}_F(Pa)]\varphi_k\|^2_{L^2(M)} \le 2\int_X |a_T - Pa|^2 \, dS, \tag{3.11}$$

for each $T < \infty$.

Proof. Write

$$A_T - \operatorname{op}_F(Pa) = [A_T - \operatorname{op}_F(a_T)] + [\operatorname{op}_F(a_T) - \operatorname{op}_F(Pa)], \tag{3.12}$$

and use $(\alpha+\beta)^2 \le 2\alpha^2 + 2\beta^2$ for $\alpha, \beta \ge 0$, to dominate the left side of (3.11) by

$$\begin{aligned} &\limsup_{N\to\infty} \frac{2}{N} \sum_{k=1}^{N} \|[A_T - \operatorname{op}_F(a_T)]\varphi_k\|^2_{L^2} \\ &+ \limsup_{N\to\infty} \frac{2}{N} \sum_{k=1}^{N} \| \operatorname{op}_F(a_T - Pa)\varphi_k\|^2_{L^2}. \end{aligned} \tag{3.13}$$

Apply (2.22) to the first limsup in (3.13). Then apply Proposition 2.2 with $b = a_T - Pa$ to get

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \| \operatorname{op}_F(a_T - Pa)\varphi_k\|^2_{L^2} = \int_X |a_T - Pa|^2 \, dS. \tag{3.14}$$

Then we have (3.11).

As we have seen,

$$A_T \longrightarrow \Pi(A) \text{ in the strong operator topology,} \tag{3.15}$$

as $T \to \infty$, for each $A \in \mathcal{L}(L^2(M))$, and in particular for $A = \operatorname{op}_F(a),\ a \in L^\infty(X)$. This leads to the following.

Problem 1. Given $a, Pa \in C(X)$, compare

$$\Pi(\mathrm{op}_F(a)) \quad \text{and} \quad \mathrm{op}_F(Pa). \tag{3.16}$$

The result (3.11) looks relevant to this task, but I have not seen how to use it to solve the problem. See [Su] and [Z1] for related results, particularly when $\{\mathcal{G}_t\}$ acts ergodically on X.

We pursue consequences of Proposition 3.2. Since $\|B\varphi_k\|_{L^2}^2 \geq |(B\varphi_k, \varphi_k)|^2$ and $(A_T\varphi_k, \varphi_k) = (A\varphi_k, \varphi_k)$, for φ_k as in (2.10), we deduce from (3.11) that

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} |([A - \mathrm{op}_F(Pa)]\varphi_k, \varphi_k)|^2 \leq 2 \int\limits_X |a_T - Pa|^2 \, dS, \tag{3.17}$$

for each $T < \infty$. Taking $T \to \infty$, we have

Corollary 3.3. *In the setting of Proposition 3.2,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} |(A\varphi_k, \varphi_k) - (\mathrm{op}_F(Pa)\varphi_k, \varphi_k)|^2 = 0. \tag{3.18}$$

This is essentially the "standard" quantum ergodic theorem, in the formulation given in [T1].

To proceed with consequences of (3.11), let us rewrite it as

$$\limsup_{N\to\infty} \frac{1}{d_N} \sum_{\lambda\in\Sigma_N} \|[A_T - \mathrm{op}_F(Pa)]P_\lambda\|_{\mathrm{HS}}^2 \leq 2 \int\limits_X |a_T - Pa|^2 \, dS, \tag{3.19}$$

where, as in §1, P_λ is the orthogonal projection of $L^2(M)$ onto $\mathrm{Eigen}(\Lambda, \lambda)$, and

$$d_N = \sum_{\lambda\leq N} \dim \mathrm{Eigen}(\Lambda, \lambda) = \mathrm{Tr} \sum_{\lambda\leq N} P_\lambda. \tag{3.20}$$

Now

$$P_\lambda e^{-it\Lambda} A e^{it\Lambda} P_\lambda = P_\lambda A P_\lambda, \quad \forall\, t \in \mathbb{R}, \tag{3.21}$$

hence

$$P_\lambda A_T P_\lambda = P_\lambda A P_\lambda. \tag{3.22}$$

Hence, for all $T < \infty$,

$$\begin{aligned} \|P_\lambda A P_\lambda\|_{\mathrm{HS}}^2 &= \|P_\lambda A_T P_\lambda\|_{\mathrm{HS}}^2 \\ &\le \|A_T P_\lambda\|_{\mathrm{HS}}^2. \end{aligned} \tag{3.23}$$

Consequently, by (3.19),

$$\begin{aligned} Pa = 0 \Longrightarrow \limsup_{N\to\infty} \frac{1}{d_N} \sum_{\lambda\in\Sigma_N} &\|P_\lambda A P_\lambda\|_{\mathrm{HS}}^2 \\ &\le \inf_{T<\infty} \int\limits_X |a_T - Pa|^2 \, dS \\ &= 0. \end{aligned} \tag{3.24}$$

We hence have the following.

Proposition 3.4. *In the setting of Proposition 3.2, if also $Pa = 0$, then*

$$\lim_{N\to\infty} \frac{1}{d_N} \sum_{\lambda\in\Sigma_N} \|P_\lambda A P_\lambda\|_{\mathrm{HS}}^2 = 0. \tag{3.25}$$

If we apply this result with a replaced by $a - Pa$, we have the following.

Proposition 3.5. *If $a, Pa \in C(X)$, $A = \mathrm{op}_F(a)$, then*

$$\lim_{N\to\infty} \frac{1}{d_N} \sum_{\lambda\in\Sigma_N} \|P_\lambda [A - \mathrm{op}_F(Pa)] P_\lambda\|_{\mathrm{HS}}^2 = 0. \tag{3.26}$$

That is,

$$\lim_{N\to\infty} \frac{1}{d_N} \|[\Pi(A) - \Pi(\mathrm{op}_F(Pa))] Q_N\|_{\mathrm{HS}}^2 = 0, \tag{3.27}$$

where

$$Q_N = \sum_{\lambda\le N} P_\lambda. \tag{3.28}$$

This brings us back to Problem 1, which we can restate as

Problem 2. Assume $b \in C(X)$ and $Pb = b$. Then estimate

$$B - \Pi(B), \quad B = \mathrm{op}_F(b). \tag{3.29}$$

Note that

$$Pa = \overline{a} \Rightarrow \mathrm{op}_F(Pa) = \overline{a} I \Rightarrow \Pi(\mathrm{op}_F(Pa)) = \overline{a} I. \tag{3.30}$$

This always holds when $\{\mathcal{G}_t\}$ is ergodic. In such a case, the conclusion of Proposition 3.5 is contained in Theorem 2 of [Z1].

The following result (potentially) addresses cases where $\{\mathcal{G}_t\}$ is not ergodic. It refines Proposition 4.4 of [T2].

Proposition 3.6. *Let $U \subset X$ be open and assume $\mathcal{G}_t : U \to U$, and that the action on U is ergodic. Let $a, b \in C(X)$ be supported on a compact subset of U, $A = \mathrm{op}_F(a)$, $B = \mathrm{op}_F(b)$. Then*

$$\int_X a\,dS = \int_X b\,dS \Longrightarrow \lim_{N\to\infty} \frac{1}{d_N} \|[\Pi(A) - \Pi(B)]Q_N\|_{\mathrm{HS}}^2 = 0. \tag{3.31}$$

Proof. Under these hypotheses, $P(a-b) = 0$, so (3.27) applies to $A - B$.

Remark. We have not established a corresponding refinement of Proposition 4.5 of [T2], which allows $a, b \in \mathcal{R}(X)$.

4. Concentration of eigenfunctions on S^n

Here we work on S^n, the unit sphere in $\mathbb{R}^{n+1}$, with its standard metric. Then the geodesic flow $\{\mathcal{G}_t\}$ is periodic of period 2π. It is convenient to take

$$\Lambda = \sqrt{-\Delta + \Bigl(\frac{n-1}{2}\Bigr)^2} - \frac{n-1}{2}, \tag{4.1}$$

so $e^{it\Lambda}$ is also periodic of period 2π (cf. (4.8) below). Then, given $A \in \mathcal{L}(L^2(S^n))$,

$$\Pi(A) = \frac{1}{2\pi}\int_0^{2\pi} e^{-it\Lambda} A e^{it\Lambda}\, dt. \tag{4.2}$$

In case $a \in C^\infty(S^*S^n)$, we have

$$Pa(x,\xi) = \frac{1}{2\pi}\int_0^{2\pi} a(\mathcal{G}_t(x,\xi))\, dt, \tag{4.3}$$

and it is a straightforward consequence of Egorov's theorem that, if $A = \mathrm{op}_F(a)$,

$$\Pi(A) - \mathrm{op}_F(Pa) \in OPS^{-1}(S^n). \tag{4.4}$$

We now specialize to the case where A is a multiplication operator,

$$Au(x) = a(x)u(x), \quad a \in C^\infty(S^n), \tag{4.5}$$

and, to keep things simple, assume that

$$n = 2, \text{ and } a(x) \text{ is invariant under } R(t), \tag{4.6}$$

where $R(t)$ is the group of rotations about the x_3-axis. Then A commutes with the associated unitary group $R(t)$ on $L^2(S^2)$, which we write as

$$R(t) = e^{itX}, \tag{4.7}$$

where $iX = Y$ is the real vector field on S^2 generating the rotation. This group is also periodic, of period 2π. We note that

$$\operatorname{Spec}\Lambda = \{k \in \mathbb{Z} : k \geq 0\}, \tag{4.8}$$

and if V_k denotes the k-eigenspace of Λ, then

$$\dim V_k = 2k + 1, \tag{4.9}$$

and

$$\operatorname{Spec} X\big|_{V_k} = \{\ell \in \mathbb{Z} : -k \le \ell \le k\}. \tag{4.10}$$

Let us note that Λ and X commute, and that the pair $\{\Lambda, X\}$ has simple spectrum. Also, under the hypothesis (4.5)–(4.6), $\Pi(A)$ commutes with X as well as with Λ. Hence $\Pi(A)$ is a function of (Λ, X),

$$\Pi(A) = F(\Lambda, X). \tag{4.11}$$

Also, given $a \in C^\infty(S^*S^2)$, we have

$$\Pi(A) \in OPS^0(S^2), \tag{4.12}$$

with principal symbol given by (4.3).

Given these facts, we can use results of Chapter 12 of [T0] to analyze F in (4.11). These results yield

$$F \in S^0(\mathbb{R}^2) \Longrightarrow F(\Lambda, X) = B \in OPS^0(S^2), \tag{4.13}$$

with principal symbol

$$b(x, \xi) = F(|\xi|, \langle Y, \xi\rangle). \tag{4.14}$$

Recall that $Y = iX$ is a real vector field. Note that it suffices to specify F on $\{(\lambda_1, \lambda_2) : \lambda_1 \ge 0,\ |\lambda_2| \le \lambda_1\}$, in light of (4.8)–(4.10), and also taking into account that $|Y| \le 1$ on S^2. We want the principal part of (4.14) to match up with (4.3) on S^*S^2.

Thus, we want to define $F_0(\lambda_1, \lambda_2)$, homogeneous of degree 0 in (λ_1, λ_2), so that

$$F_0(1, \langle Y, \xi\rangle) = Pa(x, \xi), \quad \text{for} \ \ (x, \xi) \in S^*S^2. \tag{4.15}$$

Now $F_0(1, \lambda_2)$ is a function of $\lambda_2 \in [-1, 1]$, while Pa is a function on S^*S^2, which has dimension 3. However, Pa is invariant under the flows $\mathcal{G}_t$ and $R(t)$, and in fact it is uniquely specified by its behavior on $S^*_{x_0}S^2$, where x_0 is an arbitrarily chosen point on the equator of S^2. At x_0, Y is a unit vector parallel to the equator, and (4.15) becomes

$$F_0(1, \lambda_2) = Pa\big(x_0, (\lambda_2, \sqrt{1 - \lambda_2^2})\big). \tag{4.16}$$

At first glance, this looks non-smooth at $\lambda_2 = \pm 1$, but in fact we have

$$Pa(x_0, (\xi_1, \xi_2)) = Pa(x_0, (\xi_1, -\xi_2)). \tag{4.17}$$

Such an identity is clear if $a(x)$ is even under $x_3 \mapsto -x_3$. On the other hand, if $a(x)$ is odd under this transformation its invariance under $R(t)$ guarantees that (4.3) vanishes, so we have (4.17) for general $R(t)$-invariant $a \in C^\infty(S^2)$. From (4.17) we have that (4.16) defines a smooth function of $\lambda_2 \in [-1,1]$. Then

$$\begin{aligned} &F_0(\Lambda, X) \in OPS^0(S^2), \quad \text{and} \\ &\Pi(A) - F_0(\Lambda, X) \in OPS^{-1}(S^2). \end{aligned} \tag{4.18}$$

Note that

$$F_0(\Lambda, X) = g(\Lambda^{-1}X), \tag{4.19}$$

where $g(\lambda) = F_0(1, \lambda)$, i.e.,

$$g(\lambda) = Pa(x_0, (\lambda, \sqrt{1-\lambda^2})). \tag{4.20}$$

Results just described have implications for concentration of spherical harmonics. In fact, we can take an orthonormal basis

$$\{\varphi_{k\ell} : k, \ell \in \mathbb{Z},\ k \ge 0,\ |\ell| \le k\} \tag{4.21}$$

of $L^2(S^2)$, satisfying

$$\Lambda \varphi_{k\ell} = k\varphi_{k\ell}, \quad X\varphi_{k\ell} = \ell \varphi_{k\ell}. \tag{4.21A}$$

Then

$$\begin{aligned} \int_{S^2} a(x)|\varphi_{k\ell}(x)|^2\, dS(x) &= (A\varphi_{k\ell}, \varphi_{k\ell})_{L^2} \\ &= (\Pi(A)\varphi_{k\ell}, \varphi_{k\ell})_{L^2} \\ &= (F_0(\Lambda, X)\varphi_{k\ell}, \varphi_{k\ell})_{L^2} + R_{k\ell}, \end{aligned} \tag{4.22}$$

where

$$R_{k\ell} \longrightarrow 0, \quad \text{as} \ \ k \to \infty. \tag{4.25}$$

Hence

$$\int_{S^2} a(x)|\varphi_{k\ell}(x)|^2\, dS(x) = g\Big(\frac{\ell}{k}\Big) + R_{k\ell}, \tag{4.24}$$

with $g(\lambda)$ given by (4.20).

Let us pick $\beta \in (0,1)$ and take $a \in C^\infty(S^2)$, invariant under $R(t)$, and satisfying

$$a(x) = 0, \quad \text{for } |x_3| \le \beta. \tag{4.25}$$

It follows from (4.20) and (4.3) that

$$g(\lambda) = 0, \quad \text{for } \sqrt{1-\lambda^2} \le \beta, \tag{4.26}$$

i.e., for $|\lambda| \ge \sqrt{1-\beta^2}$. Hence

$$\begin{aligned} \int_{S^2} a(x)|\varphi_{k\ell}(x)|^2 \, dS(x) = R_{k\ell} \to 0, \quad \text{as } k \to 0, \\ \text{for } |\ell|/k \ge \sqrt{1-\beta^2}. \end{aligned} \tag{4.27}$$

Conclusion. The orthonormal eigenfunctions $\varphi_{k\ell}$ concentrate on the strip $|x_3| \le \beta$ as $k \to \infty$, for $|\ell|/k \ge \sqrt{1-\beta^2}$.

5. More general concentration results

Let M be a compact, connected Riemannian manifold, and assume M has a nonzero Killing field Y, generating a 1-parameter family of isometries of M. We will also make the hypothesis that

$$A_0 = \min_{x \in M} |Y(x)| < \max_{x \in M} |Y(x)| = A_1. \tag{5.1}$$

The operator $X = iY$ is self adjoint on $L^2(M)$ and commutes with $\Lambda = \sqrt{-\Delta}$. Thus there is an orthonormal basis $\{\varphi_k\}$ of $L^2(M)$ consisting of joint eigenfunctions,

$$\Lambda \varphi_k = \lambda_k \varphi_k, \quad X \varphi_k = \mu_k \varphi_k, \tag{5.2}$$

with $\lambda_k \nearrow +\infty$, as in (2.10). Note that

$$\begin{aligned} \mu_k^2 = \|X\varphi_k\|_{L^2}^2 \le A_1^2 \|\nabla \varphi_k\|_{L^2}^2 &= A_1^2(-\Delta\varphi_k, \varphi_k) \\ = A_1^2 \|\Lambda \varphi_k\|_{L^2}^2 = A_1^2 \lambda_k^2, & \end{aligned} \tag{5.3}$$

i.e.,

$$|\mu_k| \le A_1 \lambda_k. \tag{5.4}$$

We can define a function $F(\Lambda, X)$ by

$$F(\Lambda, X)\varphi_k = F(\lambda_k, \mu_k)\varphi_k. \tag{5.5}$$

Then, as shown in Chapter 12 of [T0],

$$\begin{aligned} F \in S^0(\mathbb{R}^2) \Longrightarrow F(\Lambda, X) \in OPS^0(M), \quad \text{and} \\ \sigma_{F(\Lambda,X)}(x,\xi) = F(|\xi|, \langle Y, \xi\rangle). \end{aligned} \tag{5.6}$$

From here on, we assume $F \in C^\infty(\mathbb{R}^2 \setminus 0)$ is homogeneous of degree 0, and note that only its behavior on the wedge $\{(\lambda, \mu) : |\mu| \le A_1 \lambda\}$ is significant for the behavior of $F(\Lambda, X)$. We set

$$\varphi(\mu) = F(1, \mu), \quad \text{so} \quad F(\Lambda, X) = \varphi(\Lambda^{-1} X). \tag{5.7}$$

Note that only the behavior of φ on $\mu \in [-A_1, A_1]$ is significant. The Weyl law (2.11) (or (2.14)) yields

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \|F(\Lambda, X)\varphi_k\|_{L^2}^2 = \int_{S^*M} |\varphi(\langle Y, \xi\rangle)|^2 \, dS, \tag{5.8}$$

where dS is the Liouville measure on S^*M, normalized so that $\int_{S^*M} dS = 1$. This gives information on the joint spectrum of the pair (Λ, X), in connection with the classical result

$$\lambda_k \sim (Ck)^{1/n}, \quad \text{as} \quad k \to \infty, \tag{5.9}$$

where $n = \dim M$ and $C = \Gamma(n/2+1)(4\pi)^{n/2}/\mathrm{Vol}\, M$. Another application of the Weyl formula is that, for $a \in C^\infty(M)$,

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \int_M a(x)|F(\Lambda, X)\varphi_k|^2\, dV \\ = \int_{S^*M} a(x)|\varphi(\langle Y, \xi\rangle)|^2\, dS. \end{aligned} \tag{5.10}$$

We are ready to obtain some general concentration results, parallel to those of §4, but valid in much greater generality. The key to this result is the observation that, if $A_0 < B < A_1$,

(5.11)

$$\begin{aligned} &\varphi(\mu) = 0 \quad \text{for} \quad |\mu| \le B \\ &\Longrightarrow \varphi(\langle Y, \xi\rangle) = 0, \quad \forall (x,\xi) \in S^*M \text{ such that } x \in M_B = \{x \in M : |Y(x)| \le B\}. \end{aligned}$$

Hence we have the following conclusion.

Proposition 5.1. *With $a \in C^\infty(M)$, set $Au(x) = a(x)u(x)$. Then*

$$\begin{aligned} &\varphi(\mu) = 0 \quad \text{for} \quad |\mu| \le B, \ \operatorname{supp} a \subset M_B \\ &\qquad \Longrightarrow F(\Lambda, X)^* A F(\Lambda, X) \in OPS^{-1}(M). \end{aligned} \tag{5.12}$$

Hence, when these hypotheses hold,

$$\begin{aligned} &\lim_{k\to\infty} \int_M a(x)|F(\Lambda, X)\varphi_k|^2\, dV \\ &\quad = \lim_{k\to\infty} (F(\Lambda, X)^* A F(\Lambda, X)\varphi_k, \varphi_k)_{L^2} = 0. \end{aligned} \tag{5.13}$$

Equivalently,

$$\lim_{k\to\infty} |\varphi(\lambda_k^{-1}\mu_k)|^2 \int_M a(x)|\varphi_k(x)|^2\, dV(x) = 0. \tag{5.14}$$

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