Rayleigh Waves in Linear Elasticity as a Propagation of Singularities Phenomenon

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Abstract. We examine the propagation of surface waves known as Rayleigh waves from the perspective of microlocal analysis. This paper is a TeXed version of Taylor (1979).

1. Introduction

We consider here the propagation of elastic waves in a bounded medium. We assume our medium is isotropic, and that the displacement u satisfies the equation of linear elasticity

$$\frac{\partial^2 u}{\partial t^2} = (\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}\,u + \mu \Delta u, \tag{1.1}$$

where λ and μ are certain scalar quantities called the "Lamé constants." We assume λ and μ are positive. Then, as is well known, there are two sound speeds, $\sqrt{\mu}$ and $\sqrt{\lambda + 2\mu}$, associated respectively with shear waves (s-waves) and pressure waves (p-waves). Now, it has long been observed that a discontinuous pulse on the surface ∂K of the body K gives rise to a third singular wave, traveling along the boundary at a third, slower speed. This wave is called a Rayleigh wave, and it is of considerable importance in seismology. For example, if the impulse is caused by a sudden break near the surface of the earth, giving rise to an earthquake, the p-waves and s-waves disperse rapidly, having amplitudes that vary inversely as the square of the distance of the epicenter, while the Rayleigh waves only go down like one over the distance.

In the case ∂K is flat, the propagation of Rayleigh waves has been analyzed in some detail. See Landau and Lifschitz (1970), Love (1944), and Rayleigh (1885). The purpose of this paper is to give a rigorous treatment of the singularity that travels along ∂K, at Rayleigh sound speed, in the case when ∂K is curved. For simplicity in exposition, we carry out the calculations in two space dimensions, but a similar approach will work in three space dimensions. In such a case, we can evidently have a phenomenon which would not occur in the case of a flat boundary. Namely, a point source on ∂K can give rise to caustics, on which the Rayleigh wave would have fairly large amplitude.

We use the method of geometrical optics and the calculus of Fourier integral operators to analyze the singularities of a solution to (1.1), assumed to satisfy the free boundary condition on ∂K, namely the normal components of the stress tensor shall vanish:

$$\sum_i n_i \sigma_{ij} = 0 \quad \text{on} \quad \partial K, \tag{1.2}$$

where the stress tensor is

$$\sigma_{ij} = \lambda(\operatorname{div} u)\delta_{jk} + \mu\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big),$$

and n is the normal to ∂K. The method of geometrical optics reduces the problem to studying a certain pseudodifferential operator P on $\mathbb{R} \times \partial K$, similar to cases studied by Majda and Osher (1975) and Taylor (1975) for analyses of reflection of singularities. Now, the results of these two papers dealt with the case when P was elliptic, or at least hypoelliptic, and, as we will see, the operator P in the current situation has characteristics that are simple, and near which $\det P$ is real valued. This, together with the fact that t is monotonic on each null bicharacteristic of P, will allow us to construct the appropriate parametrix and analyze the singularities. Aside from the physical interest of this problem, I think this additional case in the analysis of propagation of singularities in a domain with boundary is very interesting mathematically.

In Section 2, we give a brief account of the basic existence and uniqueness theory for solutions to the Cauchy problem for (1.1)–(1.2). This will serve to acquaint the reader who is familiar with the general theory of linear PDE with the peculiarities of the equations of linear elesticity, and also to give a concise description of the tools needed to justify our geometrical optics construction of Section 3. The solvability and analysis of the pseudodifferential equations obtained in Section 3 can be obtained as a special case of the work of Duistermaat and Hörmander (1972), but, in fact, a simpler construction of global parametrices will suffice (due to the monotonicity of t along null bicharacteristics). We will construct such parametrices in Section 4, both in order to make this exposition more self conained and to point out the possibility of constructing global parametrices without necessarily using all the global machinery of Duistermaat and Hörmander (1972) and Hörmander (1971). In Section 5, we put this phenomenon of Rayleigh waves in a general context, complementing our work in 1975.

Finally, let me remark that the phenomenon of Rayleigh waves is connected to the failure of the Kreiss-Sakamoto condition for (1.1)–(1.2) in the "elliptic region." In Taylor (1976) we treated the diffraction problem for first order systems satisfying the Kreiss condition (which generalizes to higher order systems with no difficulty) and since that analysis is microlocal and the system (1.1)–(1.2) near the "characteristic variety" in $T^*(\mathbb{R} \times \partial K)$ (over which the grazing rays pass) does satisfy the Kreiss-Sakamoto consition, it follows that we obtain a complete analysis of the singularities of solutions to (1.1)–(1.2), with no restriction on the wave front set of the initial data such as we introduce in Section 4 to avoid grazing rays, provided ∂K is convex with respect to the null bicharacteristics of (1.1). Thus, much of the scattering theory developed by Lax and Phillips (1967) for the acoustic equation, and also much of the analysis of Majda and Taylor (1977), goes through for the scattering of elastic waves off a convex obstacle.

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2. Basic existence and uniqueness, and smoothness of solution

In order to analyze the initial value problem, prescribing u and $\partial u/\partial t$ at $t = 0$, for solutions to (1.1)–(1.2), we consider the operator

$$Lu = (\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}\, u + \mu \Delta u.$$

The boundary condition (1.2) makes L a symmetric operator, i.e., $(Lu, v) = (u, Lv)$ for all smooth u and v on K, with bounded support, which satisfy the boundary condition (1.2).

Lemma 1.2. *L is elliptic and the boundary condition (1.2) coercive.*

Proof. The ellipticity of L is obvious. To check the coerciveness of (1.2), we can assume K is a half space, and check the Lopatinsky condition. Thus, if K is defined by $x_1 \geq 0$ and if $u(x_1)$ is a bounded solution to

$$L\Big(\frac{1}{i}\frac{\partial}{\partial x_1}, \alpha_2, \alpha_3\Big) u(x_1) = 0,$$

(obtained by replacing $(1/i)(\partial/\partial x_j)$ by α_j for $j = 2, 3$ in the formula for L) such that the boundary condition

$$\sum_i n_i \sigma_{ij}\Big(\frac{1}{i}\frac{\partial}{\partial x_1}, \alpha_2, \alpha_3\Big) u(x_1) = 0, \text{ at } \ x_1 = 0,$$

is satisfied, we need to show that $u \equiv 0$, provided $(\alpha_2, \alpha_3) \in \mathbb{R}^2 \setminus 0$. We may suppose

$$u = E e^{i\alpha_1 x_1} + \widetilde{E} e^{i\tilde{\alpha}_1 x_1},$$

with $E, \widetilde{E} \in \mathbb{C}^3$. First of all,

$$L\Big(\frac{1}{i}\frac{\partial}{\partial x_1}, \alpha_2, \alpha_3\Big) u(x_1) = 0$$

implies

$$\begin{aligned}(2.1)\qquad &\big[(\lambda+\mu) M E + \mu(\alpha_1^2 + \alpha_2^2 + \alpha_3^2) E\big] e^{i\alpha_1 x_1} \\ &+ \big[(\lambda+\mu)\widetilde{M}\widetilde{E} + \mu(\tilde{\alpha}_1^2 + \alpha_2^2 + \alpha_3^2) E\big] e^{i\tilde{\alpha}_1 x_1} = 0,\end{aligned}$$

where

$$M = \big(m_{ij}\big) = \big(\alpha_i \alpha_j\big),$$

and $\widetilde{M}$ is given by the same expression, with α_1 replaced by $\tilde{\alpha}_1$. Thus $-\mu(\lambda + \mu)^{-1}(\alpha_1^2 + \alpha_2^2 + \alpha_3^2)$ is required to be an eigenvalue of M (or $E = 0$), and $-\mu(\lambda + \mu)^{-1}(\tilde{\alpha}_1^2 + \alpha_2^2 + \alpha_3^2)$ must be an eigenvalue of $\widetilde{M}$ (or $\widetilde{E} = 0$). It is easy to see that the eigenvalues of M are $0, 0$, and $\alpha_1^2 + \alpha_2^2 + \alpha_3^2$, so (2.1) yields $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 0$, and similarly $\tilde{\alpha}_1^2 + \alpha_2^2 + \alpha_3^2 = 0$, so

$$\alpha_1 = i\sqrt{\alpha_2^2 + \alpha_3^2}, \quad \tilde{\alpha}_1 = -i\sqrt{\alpha_2^2 + \alpha_3^2}.$$

The boundedness hypothesis implies $\widetilde{E} = 0$, so we are left with

$$u = Ee^{-\eta x_1}, \quad \eta = \sqrt{\alpha_2^2 + \alpha_3^2}.$$

Now, the boundary condition $\sigma_{11} = \sigma_{12} = \sigma_{13} = 0$ at $x_1 = 0$ implies

$$\begin{aligned} -(\lambda + 2\mu)\eta E_1 + i\lambda\alpha_2 E_2 + i\lambda\alpha_3 E_3 &= 0, \\ -\eta E_2 + i\alpha_2 E_1 &= 0, \\ -\eta E_3 + i\alpha_3 E_1 &= 0, \end{aligned} \tag{2.2}$$

but, if $E \neq 0$, we see from (2.2) that η must satisfy the system

$$\det \begin{pmatrix} -(\lambda + 2\mu)\eta & i\lambda\alpha_2 & i\lambda\alpha_3 \\ i\alpha_2 & -\eta & 0 \\ i\alpha_3 & 0 & -\eta \end{pmatrix} = 0,$$

which reduces to

$$(\lambda + 2\mu)\eta^2 + \lambda(\alpha_2^2 + \alpha_3^2) = 0.$$

But, since $\lambda, \mu > 0$, this is not possible, so the Lopatinsky condition is verified.

If we assume K is a bounded domain, with smooth boundary, it follows from standard elliptic theory that L_s, defined on those $u \in C^\infty(K)$ satisfying the boundary condition (1.2), has a unique positive self adjoint extension, which we will denote L, and then the unique solvability of the cauchy problem, given $u(0) \in \mathcal{D}(L),\ u_t(0) \in L^2(K)$ for (1.1)–(1.2) is an exercise in spectral theory.

Having existence, we now want to show that a function u that solves such a mixed problem, with a smooth error, must differ from the exact solution by a smooth function. This is a standard consequence of the coerciveness of L, and we sketch the argument briefly.

It suffices to show that, if $u = 0$ for $t < 0$, and

$$\frac{\partial^2 u}{\partial t^2} - Lu = f, \quad \text{on } \mathbb{R} \times K, \tag{2.3}$$

$$\sum n_i \sigma_{ij} = g, \quad \text{on } \mathbb{R} \times \partial K, \tag{2.4}$$

where $f \in C^\infty(\mathbb{R}\times\overline{K})$, $g \in C^\infty(\mathbb{R}\times\partial K)$, then $u \in C^\infty(\mathbb{R}\times\overline{K})$. First, since $\mathbb{R}\times\partial K$ is noncharacteristic for $\partial^2/\partial t^2 - L$, we may use the formal Cauchy-Kowalevsky process and Borel's theorem to solve (2.3) to infinite order at $\mathbb{R}\times\partial K$, with (2.4) satisfied (and you could specify both u and $\partial u/\partial\nu$ on $\mathbb{R}\times\partial K$). Thus, it suffices to show that solutions to (2.3)–(2.4) with $u = 0$ for $t < 0$ are smooth, assuming that $g = 0$ and that f vanishes to infinite order on $\mathbb{R}\times\partial K$. In such a case, write down u using Duhamel's formula:

$$u(t) = \int_0^t \frac{\sin((t-s)\sqrt{-L})}{\sqrt{-L}} f(s)\, ds. \tag{2.5}$$

Here, L is the self adjoint operator on $L^2(K)$ with domain $\mathcal{D}(L)$ specified by the boundary condition (1.2). The operator in the integrand is defined by the spectral theorem and is a bounded family of operators on $L^2(K)$, and also on each Hilbert space $\mathcal{D}(L^k)$, $k = 1, 2, 3, \ldots$. Now, since $f \in C^\infty(\mathbb{R}\times\overline{K})$ and vanishes on $\mathbb{R}\times\partial K$ to infinite order, it follows that

$$f(s) \in \bigcap_{k\ge 1} \mathcal{D}(L^k),$$

as do all its derivatives. Hence, by (2.5), we see that $u(t)$ is a smooth function of t taking values in $\mathcal{D}(L^k)$ for each k. Since the boundary condition (1.2) is coercive, it follows that $\mathcal{D}(L^k) \subset H^{2k}(K)$. Thus u is C^∞ on $\mathbb{R}\times\overline{K}$, as desired.

3. Construction of parametrices

We construct an approximation to (1.1) satisfying the inhomogeneous boundary condition

$$\sum n_i\sigma_{ij} = f_j \quad \text{on} \quad \mathbb{R}\times\partial K, \tag{3.1}$$

where $f_j \in \mathcal{E}'(\mathbb{R}\times\partial K)$ vanish for $t < 0$. We assume $K \subset \mathbb{R}^2$ for convenience in calculation. We look for the unique outgoing solution, i.e., we require that $u = 0$ for $t < 0$. We assume that $\mathrm{WF}(f_j)$ is contained in a small conic neighborhood of a point $(x_0, t_0, \xi_0, \tau_0) \in T^*(\mathbb{R}\times\partial K)\setminus 0$. We look for an approximate solution of the form

$$u = \int a(t,x_1,x_2,\zeta)e^{i\varphi(t,x_1,x_2,\zeta)}\widehat{F}(\zeta)\,d\zeta + \int b(t,x_1,x_2,\zeta)e^{i\psi(t,x_1,x_2,\zeta)}\widehat{G}(\zeta)\,d\zeta, \tag{3.2}$$

where $\zeta \in \mathbb{R}^2$, F and G are scalar valued distributions to be determined by the boundary condition (3.1), a and b are vector valued amplitudes, and φ and ψ are certain phase functions, satisfying the eikonal equations of geometrical optics,

$$\varphi_t^2 = (\lambda + 2\mu)\nabla_x\varphi\cdot\nabla_x\varphi, \tag{3.3}$$

$$\psi_t^2 = \mu\nabla_x\psi\cdot\nabla_x\psi. \tag{3.4}$$

Away from the characteristic variety in $T^*(\mathbb{R}\times\partial K)$, the boundary $\mathbb{R}\times\partial K$ is noncharacteristic for each of these eikonal equations. We would like to specify that both φ and ψ equal some given function $\gamma(t,x,\xi)$ on $\mathbb{R}\times\partial K$. There are three cases:

(i) Over $((t,x),\nabla_{\tan}\gamma(t,x,\xi))\in T^*(\mathbb{R}\times\partial K)$ pass four rays. This is the hyperbolic region. The eikonal equations (3.3) and (3.4) can both be solved exactly here.

(ii) Over $((t,x),\nabla_{\tan}\gamma(t,x,\xi))\in T^*(\mathbb{R}\times\partial K)$ pass two rays. This is the "mixed" region. Here, (3.4) can be solved exactly, but (3.3) cannot. However, one can solve (3.3) to infinite order on $\mathbb{R}\times\partial K$, which will suffice for the construction of the parametrix. We demand that $\operatorname{Im}\varphi\geq 0$.

(iii) Over $((t,x),\nabla_{\tan}\gamma(t,x,\xi))\in T^*(\mathbb{R}\times\partial K)$ pass no rays. This is the "elliptic region." Here, (3.3) and (3.4) can both be solved to infinite order on $\mathbb{R}\times\partial K$, with $\operatorname{Im}\varphi\geq 0$ and $\operatorname{Im}\psi\geq 0$ everywhere.

In each case, there are also the associated transport equations for the amplitudes a and b, which are treated similarly.

It follows that (3.2) satisfies (1.1) up to a smooth error. We can prescribe $a(t,x,\zeta)$ and $b(t,x,\zeta)$ for $x\in\partial K$ as long as they are respectively in the $(\lambda+2\mu)|\xi|^2$ and $\mu|\xi|^2$ eigenspaces of the symbol $-L(t,x,\tau,\xi)$ of $-(\lambda+2\mu)\operatorname{grad}\operatorname{div}-\mu\Delta$ evaluated respectively at $(\tau,\xi)=(\varphi_t,\varphi_x)$ and $(\tau,\xi)=(\psi_t,\psi_x)$. Naturally, we require that these vectors be nonzero on ∂K, so they span $\mathbb{R}^2$ there. Satisfying the boundary condition (3.1) leads to a pseudodifferential equation on $\mathbb{R}\times\partial K$ for (F,G) in terms of (f_1,f_2), which we derive as follows. The left side of (3.1) is

$$\lambda(\operatorname{div}u)n_j+\mu\sum_i n_i\Big(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}\Big).$$

Thus, the vector $\sum_i n_i\sigma_{ij}|_{\mathbb{R}\times\partial K}$ is given by

$$T\begin{pmatrix}F\\G\end{pmatrix}=\int\big[A(t,x,\zeta)\widehat{F}(\zeta)+B(t,x,\zeta)\widehat{G}(\zeta)\big]e^{i\gamma(t,x,\zeta)}\,d\zeta, \tag{3.5}$$

with A and B vector valued symbols of order 1. Their principal parts, homogeneous of degree 1 in ζ, are given by

$$A_1(t,x,\zeta)=i\lambda(\nabla_x\varphi\cdot a_0)n+i\mu\big((\nabla_x\varphi\cdot n)a_0+(a_0\cdot n)\nabla_x\varphi\big) \tag{3.6}$$

$$B_1(t,x,\zeta)=i\lambda(\nabla_x\psi\cdot b_0)n+i\mu\big((\nabla_x\psi\cdot n)b_0+(b_0\cdot n)\nabla_x\psi\big). \tag{3.7}$$

We now consider solving the system

$$T\begin{pmatrix}F\\G\end{pmatrix}=\begin{pmatrix}f_1\\f_2\end{pmatrix}\mod C^\infty, \tag{3.8}$$

given $f_j\in\mathcal{E}'(\mathbb{R}\times\partial K)$, supported in $t>0$, where we demand that F and G vanish for $t<0$. If coordinates $z=(t,x)$ are chosen on $\mathbb{R}\times\partial K$ such that $\gamma(t,x,\zeta)=z\cdot\zeta=x\cdot\xi+\tau t$, then (3.8) is a pseudodifferential equation for (F,G) in erms of (f_1,f_2). We will show that T has the following behavior.

Lemma 3.1. *In the hyperbolic and mixed regions, T is elliptic. In the elliptic region, the real valued symbol* $\det \sigma_T$ *has a simple zero on a hypersurface in* $T^*(\mathbb{R} \times \partial K) \setminus 0$. *On this surface*

$$\frac{\partial}{\partial \tau} \det \sigma_T \neq 0.$$

As we will see in Section 4, this leads to the following.

Lemma 3.2. *The system (3.8) has a unique solution, mod C^∞, which vanishes for $t < 0$, given f_j supported in $\{t > 0\}$. Then* $\mathrm{WF}(F)$ *and* $\mathrm{WF}(G)$ *are contained in the set Σ: the union of $S = \mathrm{WF}(f_1) \cup \mathrm{WF}(f_2)$ and the set of null bicharacteristics for* $\det \sigma_T$ *passing over S, traveling in the positive t direction.*

Plugging this result into (3.2), we immediately obtain our main result on propagation of singularities for solutions to (1.1) and (3.1).

Theorem 3.3. *Let u be the unique solution to (1.1) and (3.1), vanishing for $t < 0$, given $f_j \in \mathcal{E}'(\mathbb{R} \times \partial K)$ vanishing for $t < 0$. Assume* $\mathrm{WF}(f_j)$ *avoids the characteristic variety. Then, in $\mathbb{R} \times K$,* $\mathrm{WF}(u)$ *is contained in the set of null bicharacteristics of L passing over* $\mathrm{WF}(f_1) \cup \mathrm{WF}(f_2)$*, going in the positive t direction, as long as these bicharacteristics do not pass over $\mathbb{R} \times \partial K$ again. The solution u is smooth up to ∂K except at the singular supports of f_1, f_2 and at the image in $\mathbb{R} \times \partial K$ of the set Σ described in Lemma 3.2. If we consider $u|_{\mathbb{R} \times \partial K}$ and $\partial_\nu u|_{\mathbb{R} \times \partial K}$, the wave front sets of these distributions are contained in Σ.*

It is the set Σ that forms the wave front set of the Rayleigh wave produced by u. If the null bicharacteristics mentioned in Theorem 3.3 do pass over $\mathbb{R} \times \partial K$ again, propagation and reflection of singularities results continue to hold, as described in Taylor (1975), as long as they do not pass over the characteristic variety. Note that such rays cannot pass over the elliptic region, so no further Rayleigh waves are produced.

We turn now to the proof of Lemma 3.1. In order to compute $\det \sigma_T = \det(A_1\ B_1)$, where $(A_1\ B_1)$ is the 2×2 matrix whose columns are A_1 and B_1, at a point (t_0, x_0, τ, ξ), we may as well assume Euclidean coordinates are chosen so that $x_0 = 0$ and the plane $\{x_1 = 0\}$ is tangent to ∂K at x_0, with $n(x_0)$ pointing in the positive x_1 direction. At x_0, the eikonal equations solved by φ and ψ can be rewritten, with $z = (x_2, t)$, as

$$\varphi_{x_1} = \lambda_1(x_1, z, \nabla_z \varphi), \quad \psi_{x_1} = \mu_1(x_1, z, \nabla_z \psi),$$

where, if we write $E(t, x, \tau, \xi)$ for the principal symbol of $\partial_t^2 - L$, λ_1 and λ_2 are the roots of $\det E(t, x, \tau, \lambda_1, \xi_2) = 0$. Then, the vectors a_0 and b_0 are picked so that

$$E(t, x, \tau, \lambda_1, \xi_2) a_0 = 0, \quad E(t, x, \tau, \mu_1, \xi_2) b_0 = 0.$$

A straightforward computation yields the following formulas:

$$\lambda_1^2 = \frac{1}{\mu} \tau_2 - \xi_2^2, \quad \mu_1^2 = \frac{1}{\lambda + 2\mu} \tau^2 - \xi_2^2, \tag{3.9}$$

$$a_0 = \begin{pmatrix} \xi_2 \\ -\lambda_1 \end{pmatrix}, \quad b_0 = \begin{pmatrix} \mu_1 \\ \xi_1 \end{pmatrix}, \tag{3.10}$$

and combining these formulas with (3.6) and (3.7) yields

$$(A_1\ B_1) = C\begin{pmatrix} \lambda_1\xi_2\Big(1-\frac{\lambda}{\lambda+2\mu}\Big) & \mu_1^2+\frac{\lambda}{\lambda+2\mu}\xi_2^2 \\ \xi_2^2-\lambda_1^2 & 2\mu_1\xi_2 \end{pmatrix},$$

where C is a nonvanishing scalar. Thus $\det\sigma_T$ is a nonvanishing multiple of

$$2\mu\lambda_1\xi_2^2+\mu\Big(2\xi_2^2-\frac{1}{\mu}\tau^2\Big)^2 = p(\xi_2,\tau). \tag{3.11}$$

Note that (ξ_2,τ) is the fibre variable of $T^*(\mathbb{R}\times\partial K)$, which is divided into three regions (excluding the characteristic variety).

I. $|\tau| > (\lambda+2\mu)^{1/2}|\xi_2|$ (hyperbolic region). Here λ_1^2 and μ_1^2 are positive, so the roots λ_1 and μ_1 are real. Representing outgoing waves, they must have the same sign.

II. $\mu^{1/2}|\xi_2| < |\tau| < (\lambda+2\mu)^{1/2}|\xi_2|$ (mixed region). Here $\lambda_1^2>0$, but $\mu_1^2<0$, so λ_1 is real but μ_1 is purely imaginary.

III. $|\tau| < \mu^{1/2}|\xi_2|$ (elliptic region). Here both λ_1^2 and μ_1^2 are negative, so all roots are purely imaginary.

The assertion of Lemma 3.1 is that $p(\xi_2,\tau)$ is nonvanishing in regions I and II, and in region III has a simple zero, at which $(\partial/\partial\tau)p\neq 0$.

The behavior of $p(\xi_2,\tau)$ in regions I and II is easy to investigate. In region I, we have $p(\xi_2,\tau)>0$. In region II, the term $4\mu\lambda_1\mu_1\xi_2^2$ is imaginary and the term $\mu(2\xi_2^2-\tau^2/\mu)^2$ is real, and again $p(\xi_2,\tau)\neq 0$.

In region III, $p(\xi_2,\tau)$ is also real valued. To simplify the analysis of $p(\xi_2,\tau)$ there, let $\xi_2=1$ and $\mathfrak{t}=\tau^2$. Since $p(\xi_2,\tau)=0$ in region III is equivalent to $16\lambda_1^2\mu_1^2\xi_2^4=(2\xi_2^2-\tau^2/\mu)^4$, this condition becomes

$$16\Big(\frac{1}{\mu}\mathfrak{t}-1\Big)\Big(\frac{1}{\lambda+2\mu}\mathfrak{t}-1\Big) = \Big(\frac{1}{\mu}\mathfrak{t}-2\Big)^4.$$

Multiplying this out and replacing $\mathfrak{t}$ by $s=\mathfrak{t}/\mu$ yields

$$s^4-8s^3+\Big(24-16\frac{\mu}{\lambda+2\mu}\Big)s^2-16\Big(1-\frac{\mu}{\lambda+2\mu}\Big)s=0,$$

which, upon division by s, reduces to the cubic equation

$$q(s)=s^3-8s^2+\Big(24-16\frac{\mu}{\lambda+2\mu}\Big)s-16\Big(1-\frac{\mu}{\lambda+2\mu}\Big)=0.$$

This is the equation that occurs in the analysis of Raleigh waves in the half space (see Love, 1944), and the location of its zeros is easy. Our assertion boils down to showing that, for $0 < s < 1$, $q(s)$ has exactly one zero, where $q'(s) \neq 0$. (Outside this interval, $q(s)$ may have other zeros, but these do not correspond to zeros of $p(\xi_2, \tau)$, due to our having squared the equation $4\mu\lambda_1\mu_1\xi_2^2 = -\mu(2\xi_2^2 - \tau^2/\mu)$.) In fact, we readily see that

$$q(0) = -16\Big(1 - \frac{\mu}{\lambda + 2\mu}\Big) < 0, \quad q(1) = 1 > 0,$$

so $q(s)$ has at least one zero in this interval. Meanwhile one verifies that $q'(s) > 0$ for $0 \leq s \leq 1$. This completes the proof of Lemma 3.1.

4. Solution of the boundary equation

In this section, we give a construction for a parametrix for the pseudodifferential equation

$$Tu = f \quad (\mathrm{mod} \;\; C^\infty), \tag{4.1}$$

where we assume T is a $k \times k$ matrix of pseudodifferential operators, which we may suppose to be of order zero, on a manifold $\mathbb{R} \times X$ with coordinates (t, x), such that

$$p(t, x, \tau, \xi) = \det \sigma_T \;\; \text{is real}, \tag{4.2}$$

and

$$\frac{\partial}{\partial \tau} p \neq 0 \;\; \text{where} \;\; p = 0. \tag{4.3}$$

We assume that $f = 0$ for $t < 0$, and demand that

$$u \in C^\infty \;\; \text{for} \;\; t < 0. \tag{4.4}$$

We construct a *global* approximate solution, under the assumption that X is compact. This is a special case of a construction of Duistermaat and Hörmander (1972), which used somewhat heavier machinery.

Let ${}^{co}T$ be the cofactor matrix of T, so ${}^{co}TT = P + Q$, where $\sigma_P = p(t, x, \tau, \xi)$ and $Q \in OPS^{-1}$. Thus (4.1) implies ${}^{co}TTu = {}^{co}Tf = g$, or

$$(P + Q)u = g.$$

Letting $\Lambda \in OPS^1$ be some scalar, elliptic operator on $\mathbb{R} \times X$, with positive symbol, this is equivalent to

$$(q + B)u = \tilde{g} \quad \mathrm{mod} \;\; C^\infty, \tag{4.5}$$

where $\tilde{g} = \Lambda g$, $q = \Lambda P \in OPS^1$, $B = \Lambda Q \in OPS^0$. The hypothesis $f = 0$ for $t < 0$ implies $\tilde{g} \in C^\infty$ for $t < 0$.

To solve (4.5), let us proceed momentarily on a formal level. We have

$$\begin{aligned} u &= (iq + iB)^{-1}(i\tilde{g}) && \text{(formally)} \\ &= i \int_0^\infty e^{is(q+B)} \tilde{g}\, ds && \text{(formally)} \\ &= -i \int_{-\infty}^0 e^{is(q+B)} \tilde{g}\, ds && \text{(formally).} \end{aligned} \tag{4.6}$$

Now, (4.6) is not well defined. However, $e^{is(q+B)}\tilde{g} = w(s)$ solves the hyperbolic equation

$$\frac{\partial w}{\partial s} = i(q + B)w, \quad w(0) = \tilde{g}, \tag{4.7}$$

and we can construct a solution (mod C^∞) to this via the method of geometrical optics, for $|s|$ small, and then exploit the group properties to represent such an approximate solution for arbitrary s. Let us denote such an approximate solution to (4.7) by $e_{\mathrm{GO}}^{is(q+B)}\tilde{g}$. Then, as is well known, the wave front set of $e_{\mathrm{GO}}^{is(q+B)}\tilde{g}$ is obtained from $\mathrm{WF}(\tilde{g})$ by following the Hamiltonian flow generated by H_q for s units of "time." Since we are assuming (4.3), it follows that $\dot{t}$ is bounded away from zero on each orbit of this flow in $\gamma(q) := \{q = 0\} = \{p = 0\}$. Write

$$\gamma(q) = S_+ \cup S_-, \quad \dot{t} > 0 \text{ on } S_+, \quad \dot{t} < 0 \text{ on } S_-.$$

Write the identity operator I on $\mathcal{D}'(\mathbb{R} \times X)$ as a sum of three operators in OPS^0,

$$I = P_+ + P_- + P_0,$$

where σ_{P_+} is supported in a small conic neighborhood Γ^+ of S_+, with $\dot{t} > 0$ on Γ_+, $\sigma_{P_+} = 1$ on a smaller conic neighborhood of S_+; likewise σ_{P_-} is supported on a small conic neighborhood Γ_- of S_-, with $\dot{t} < 0$ on Γ_-, and $\sigma_{P_-} = 1$ on a smaller conic neighborhood of S_-. Note that $q + B$ is elliptic on the support of P_0, so if we write $\tilde{g} = P_+\tilde{g} + P_-\tilde{g} + P_0\tilde{g}$, it is easy to solve

$$(q + B)u_0 = P_0\tilde{g}, \quad \text{mod } C^\infty.$$

With these facts in mind, we construct the actual parametrix for (4.5), replacing the formal calculation (4.6). Pick a $T_1 < \infty$, and we desire to solve (4.5) for $t < T_1$. Choose a $T_0 < \infty$ such that, for all $\zeta \in WF(\tilde{g})$, if $|s| > T_0$, the image of ζ under $C(s)$, the Hamiltonian flow on $T^*(\mathbb{R} \times X)$ generated by H_q, has t coordinate outside the interval $[0, T_1]$. We set $\psi \in C_0^\infty(\mathbb{R})$ equal to 1 for $|s| \le T_0$, and take

$$u = u_0 + i \int_0^\infty \psi(s) e_{\mathrm{GO}}^{is(q+B)} P_+\tilde{g}\, ds - i \int_{-\infty}^0 \psi(s) e_{\mathrm{GO}}^{is(q+B)} P_-\tilde{g}\, ds. \tag{4.8}$$

It is a simple matter to show that u verifies (4.5), mod C^∞, for $t < T_1$, and u is smooth for $t < 0$.

It remains to show that such u solves (4.1), mod C^∞. To see this, rewrite (4.5) as $\Lambda {}^{co}T(Tu - f) = 0$, or, since Λ is elliptic,

$$ {}^{co}T(Tu - f) = 0 \mod C^\infty. $$

Apply T to both sides of this, noting that $T {}^{co}T = P + \widetilde{Q}$, with P as before and $\widetilde{Q} \in OPS^{-1}$. Thus

$$ (P + \widetilde{Q})(Tu - f) = 0, \quad \mod C^\infty, $$

while $Tu - f$ is smooth for $t < 0$. Now, propagation of singularities results for solutions to $(P + \widetilde{Q})w = 0$ mod C^∞ yield that $Tu - f$ is smooth for all $t < T_1$, as desired. Thus we have our parametrix. Furthermore, the standard propagation of singularities results show that $\mathrm{WF}(u)$ is contained in the union of $\mathrm{WF}(g)$ with the set of orbits of H_p that pass over $\mathrm{WF}(g)$, so we have Lemma 3.2.

We emphasize that the construction described here uses only the *local* theory of Fourier integral operators.

5. Generalities

In this final section, we put the phenomenon, analyzed in §§2–4 for equations of linear elasticity, into a general framework, and also mention some additional phenomena that could occur for general systems. This section is complementary to our work on reflection of singularities (1975).

We consider a $k \times k$ matrix of first order pseudodifferential operators $G(y) = G(y, x, D_x)$, acting on functions on $\mathbb{R}^+ \times X$, and consider solutions to the boundary value problem for $u = u(y) = u(y, x)$,

$$ \frac{\partial u}{\partial y} = G(y)u, \tag{5.1} $$

$$ \beta u(0) = f, \tag{5.2} $$

where $\beta \in OPS^0(X)$. The hypothesis that all null bicharacteristics intersecting $\partial(\mathbb{R}^+ \times X)$ do so transversally and that G has simple characteristics implies that the principal symbol of $G(y)$ is similar to a matrix of the form

$$ \begin{pmatrix} i\lambda_1 & & & & \\ & \ddots & & & \\ & & i\lambda_j & & \\ & & & A & \\ & & & & B \end{pmatrix}, $$

where $\lambda_\nu(y,x,\xi)$ are real valued (scalar), the spectrum of $A(y,x,\xi)$ has negative real part, and the spectrum of $B(y,x,\xi)$ has positive real part. The complete decoupling procedure described in Taylor (1975) implies that any solution $u \in C([0,y_0),\mathcal{D}'(X))$ of (5.1) can be written in the form

$$u(y) = U(y)\begin{pmatrix} w_1 \\ \vdots \\ w_j \\ w_+ \\ w_- \end{pmatrix},$$

with elliptic $U(y) \in OPS^0(X)$, where w_ν solve the equation

$$\frac{\partial w_\nu}{\partial y} = i\mu_\nu(y,x,D_x)w_\nu,$$

and $w_\pm$ solve

$$\frac{\partial w_+}{\partial y} = a(y,x,D_x)w_+,$$
$$\frac{\partial w_-}{\partial y} = b(y,x,D_x)w_-.$$

and, furthermore, the principal symbol of μ_ν is λ_ν, the principal symbol of $a(y,x,D_x)$ is A, and the principal symbol of $b(y,x,D_x)$ is B. Actually, there might be global topological obstructions to the construction of these operators, but these can be avoided if one microlocalizes appropriately. The details are given in Taylor (1975). The boundary condition (5.2) becomes

$$\beta U(0)w(0) = f. \tag{5.3}$$

The reflection of singularities phenomenon we consider is described simply as follows. Suppose we know that u is smooth in a conic neighborhood of the rays $\gamma_1,\dots,\gamma_\ell$ $(0 \le \ell \le j)$ passing over $(x_0,\xi_0) \in T^*X \setminus 0$, where γ_ν is a null bicharacteristic strip associated to $\partial_y - i\lambda_\nu$. Note that this is equivalent to the microlocal smoothness (up to the boundary $y=0$) of $w_1,\dots,w_\ell$. More generally, suppose we know the nature of the singularities of u near $\gamma_1,\dots,\gamma_\ell$, i.e., suppose we know $w_1,\dots,w_\ell$, microlocally, mod C^∞. We want to construct a parametrix for u which, in particular, will tell us the nature of the singularities of $w_{\ell+1},\dots,w_j$, and also the nature of the boundary regularity of w_+. (Note that, since w_+ and w_- solve elliptic evolution equations that are forward and backward, respectively, they are automatically C^∞ inside $(0,y_0)\times X$, and w_- is smooth up to the boundary $y=0$.) This goal is achieved in Taylor (1976), granted the following hypothesis. (Here, let $w_\nu = P_\nu w,\ w_+ = P_+ w,\ w_- = P_- w$.)

(5.4) Given a knowledge of $w_1(0),\dots,w_\ell(0)$ and of $w_-(0)$ the system (5.3) is an elliptic system for $w_{\ell+1}(0),\dots,w_j(0),w_+(0)$.

More generally, as considered in Taylor (1975), the system was assumed to be hypoelliptic. However, in cases we have run across (e.g., Taylor, 1976) hypoelliptic equations seem to occur naturally for a number of grazing ray problems, but in the nongrazing case, one has to work to contrive such a problem (omitting such problems as the $\overline{\partial}$ Neumann problem, where reflection of singularities is not the issue). We are now in a position to generalize this result, as follows.

Theorem 5.1. *One can construct a solution to*

$$\frac{\partial u}{\partial y} = Gu, \qquad \operatorname{mod} C^\infty,$$
$$Bu(0) = f, \quad \operatorname{mod} C^\infty,$$

given $f \in \mathcal{E}'(X)$, with the property that u is smooth along the rays $\gamma_1, \dots, \gamma_\ell$, provided that, for specified $w_1(0), \dots, w_\ell(0), w_-(0) \in C^\infty(X)$, we can solve the system

$$\beta U(0) w(0) = f, \quad \operatorname{mod} C^\infty,$$

for $w_{\ell+1}(0), \dots, w_j(0), w_+(0)$.

If we can deduce that $\mathrm{WF}(w_\nu(0)) \subset \Gamma_\nu$, $\mathrm{WF}(w_+(0)) \in \Gamma_+$, where Γ_ν, $\nu = \ell+1, j$ and Γ_+ are closed conic subsets of T^*X obtained from $\mathrm{WF}(f)$ by some process, it follows that $\mathrm{WF}(u)$ is smooth except along those rays passing over Γ_ν and in $\gamma(\partial_y - i\lambda_\nu)$. Furthermore, w_+ is smooth up t the boundary $y = 0$ except at points $x \in X$ such that $(x, \xi) \in \Gamma_+$ for some ξ.

The proof of Theorem 5.1 is the same as the proof in the special case where (5.4) is satisfied. The context in which such a situation arises is when $\partial_y - G$ comes from reducing a hyperbolic equation to a first order system of pseudodifferential operators, the time variable being one of the x variables, say $t = x_1$. In such a case, typically the $\gamma_1, \dots \gamma_\ell$ are the null bicharacteristics on which t is decreasing (as they leave the boundary) and $\gamma_{\ell+1}, \dots, \gamma_j$ are the null bicharacteristics on which t is increasing. Granted appropriate energy estimates, the approximate solution constructed via Theorem 5.1 differs from the exact solution by a smooth error (recall the argument at the end of §2), so the description of singularities given in Theorem 5.1 will hold for the exact solution.

In such a case as the equations of linear elasticity in three space variables, we need to allow the w_ν to be vector valued, though the principal symbol of μ_ν needs to be scalar. This does not affect the discussion above at all.

In the case of the equations of linear elasticity considered in Sections 2 and 3, the boundary value problem (5.3) fails to be elliptic only in a region where all the eigenvalues of $G_1(y, x, \xi)$ have nonzero real part (the "elliptic region" mentioned in §3), so $j = 0$. Thus, in that example, the reflection of singularities phenomenon is the same as described in Taylor (1975), except for the Rayleigh waves, which travel along the boundary. No extra singularities propagate into the interior. Now, it is easy to concoct a boundary value problem for which this additional phenomenon will occur. For example, for vector valued u and v, consider the system

$$u_{tt} - (\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}\, u - \mu\Delta u = 0, \tag{5.5}$$

$$c^{-2} v_{tt} - \Delta v = 0, \tag{5.6}$$

with boundary conditions of the form

$$\sum_i n_i \sigma_{ij} = f \quad \text{on} \quad \partial K, \ \text{for} \quad u, \tag{5.7}$$

$$v = u \quad \text{on } \partial K. \tag{5.8}$$

We assume $f = 0$ for $t < 0$ and solve, requiring u and v to vanish for $t < 0$. Note that (5.5), (5.7) is precisely the boundary value problem considered in §§2–3. So solve it. The equation for v is coupled to that for u via the boundary condition (5.8). Having solved for u, we obtain v by solving the Dirichlet problem for the wave equation (5.6), (5.8). If the sound speed c in (5.6) is picked to be *less* than the propagation speed of the Rayleigh waves, it follows from the propagation of singularities results for the Dirichlet problem that v picks up singularities along rays going into K in the positive t direction passing over $\mathrm{WF}(u|_{\partial K})$, which includes the wave fronts of the Rayleigh waves. I do not know whether there is a physical process for which (5.5)–(5.8) is a model. It would be interesting to find boundary value problems for physical processes for which this additional propagation of singularities phenomenon does occur.

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