Riemann Localization of Fourier Series Beyond L^1: A Distributional Approach

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Abstract. We establish a Riemann localization result for a class of distributions larger than $L^1(\mathbb{T}^1)$, which we denote $\mathcal{V}(\mathbb{T}^1)$, and compare this with a localization result going back to Riemann, as presented in [Z]. We explore related results on $\mathcal{V}(\mathbb{T}^1)$, in particular taking advantage of distribution theory to provide short proofs of generalizations of a number of results on trigonometric series presented in [Z].

Contents

1. Introduction

Basic texts in Fourier analysis typically state the Riemann localization principle as follows. Suppose f is integrable on $\mathbb{T}^1 = \mathbb{R}/(2\pi\mathbb{Z})$, i.e., $f \in L^1(\mathbb{T}^1)$. Assume $f = g$ on an open set $\mathcal{O} \subset \mathbb{T}^1$, and that $g \in C(\mathbb{T}^1)$ has a Fourier series that converges uniformly on $\mathbb{T}^1$. Then the Fourier series of f converges to f uniformly on compact subsets of $\mathcal{O}$.

An example of a series to which the principle as just stated does not apply is

$$\sum_{n=2}^{\infty} \frac{1}{\log n} \sin n\theta. \tag{1.1}$$

This appears in [L] as an example of a trigonometric series that converges pointwise, for each $\theta \in \mathbb{T}^1$, but is not the Fourier series of an L^1 function. This convergence can be demonstrated using the Dirichlet test for convergence of an infinite series (cf. [WW], §2.31). As for the sum, it can be shown to be

$$-\frac{1}{\theta \log|\theta|} + O\Big(\frac{1}{|\theta|(\log|\theta|)^2}\Big) \tag{1.2}$$

near $\theta = 0$ (see §5). In the terminology of the time of [L], (1.1) was said not to be a Fourier series, a conclusion repeated in [WW] (§9.12) and in [Z]. In these works, series like (1.1) fell under the more general rubric of "trigonometric series."

Of course, post L. Schwartz, we say (1.1) is the Fourier series of a *distribution*, call it u_L, and we can say quite a bit about this distribution. Having in hand the Schwartz theory of Fourier analysis on the space $\mathcal{D}'(\mathbb{T}^1)$ of distributions on $\mathbb{T}^1$, we can readily compute the Fourier series of $(1-e^{i\theta})u_L$ and see that

$$(1-e^{i\theta})u_L \in \mathcal{A}(\mathbb{T}^1), \tag{1.3}$$

where

$$\mathcal{A}(\mathbb{T}^1) = \Bigl\{ f \in \mathcal{D}'(\mathbb{T}^1) : \sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty \Bigr\} \tag{1.4}$$

(so $\mathcal{A}(\mathbb{T}^1) \subset C(\mathbb{T}^1)$). See Proposition 3.1 for a general version of this phenomenon. Since convergence of (1.1) at $\theta = 0$ is trivial, one could retrieve the convergence of (1.1) everywhere, given an appropriate extension of the localization principle stated in the first paragraph.

In fact, such localization extends to a class of distributions larger than $L^1(\mathbb{T}^1)$, to which u_L belongs, namely

$$\mathcal{V}(\mathbb{T}^1) = \{ u \in \mathcal{D}'(\mathbb{T}^1) : \lim_{|n|\to\infty} \hat{u}(n) = 0 \}. \tag{1.5}$$

Under $u \mapsto \hat{u}$ this space is isomorphic (as a Banach space) to the sequence space $c_0(\mathbb{Z})$, while $\mathcal{A}(\mathbb{T}^1)$ is isomorphic to $\ell^1(\mathbb{Z})$, with norms

$$\|f\|_{\mathcal{A}} = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|, \quad \|u\|_{\mathcal{V}} = \sup_n |\hat{u}(n)|. \tag{1.6}$$

The Parseval identity leads to the duality

$$\mathcal{V}(\mathbb{T}^1)' = \mathcal{A}(\mathbb{T}^1). \tag{1.7}$$

Given $u \in \mathcal{V}(\mathbb{T}^1)$, the sequence of partial sums

$$S_N u(\theta) = \sum_{n=-N}^{N} \hat{u}(n) e^{in\theta} \tag{1.8}$$

converges, as $N \to \infty$, to u in $\mathcal{V}$-norm, and hence in the L^2-Sobolev space $H^s(\mathbb{T}^1)$, for each $s < -1/2$. We have the following localization result.

Theorem 1.1. *Let $u \in \mathcal{V}(\mathbb{T}^1)$ and let $\mathcal{O} \subset \mathbb{T}^1$ be open. Assume there exists $f \in C(\mathbb{T}^1)$ such that $S_N f \to f$ uniformly on $\mathbb{T}^1$ and $u = f$ on $\mathcal{O}$. Then, for each compact $K \subset \mathcal{O}$,*

$$S_N u(\theta) \longrightarrow u(\theta), \quad \textit{uniformly for } \theta \in K. \tag{1.9}$$

We will prove Theorem 1.1 in §2. Given that we work in the framework of distribution theory, the proof is fairly short and simple. The following result will be useful.

Proposition 1.2. *Given $f \in \mathcal{A}(\mathbb{T}^1)$ and $u \in \mathcal{V}(\mathbb{T}^1)$, we have $fu \in \mathcal{V}(\mathbb{T}^1)$.*

Proof. To start, given $f \in C^\infty(\mathbb{T}^1)$ and $u \in \mathcal{D}'(\mathbb{T}^1)$, we have $fu \in \mathcal{D}'(\mathbb{T}^1)$ and

$$\widehat{fu}(n) = \sum_{k=-\infty}^{\infty} \hat{f}(k)\hat{u}(n-k), \tag{1.10}$$

the sum being absolutely convergent since $\hat{f}$ is rapidly decreasing on $\mathbb{Z}$ and $\hat{u}$ is polynomially bounded. If in fact $u \in \mathcal{V}(\mathbb{T}^1)$, then (1.10) implies that (for $f \in C^\infty(\mathbb{T}^1)$)

$$|\widehat{fu}(n)| \le \|\hat{f}\|_{\ell^1}\|\hat{u}\|_{\ell^\infty} = \|f\|_{\mathcal{A}}\|u\|_{\mathcal{V}}. \tag{1.11}$$

Now clearly $fu \in \mathcal{V}(\mathbb{T}^1)$ for $f, u \in C^\infty(\mathbb{T}^1)$. Since $C^\infty(\mathbb{T}^1)$ is dense in $\mathcal{A}(\mathbb{T}^1)$ (in $\mathcal{A}$-norm) and in $\mathcal{V}(\mathbb{T}^1)$ (in $\mathcal{V}$-norm), the conclusion that the product $(f,u) \mapsto fu$ extends uniquely by continuity from

$$C^\infty(\mathbb{T}^1) \times C^\infty(\mathbb{T}^1) \to \mathcal{V}(\mathbb{T}^1) \ \text{ to } \ \mathcal{A}(\mathbb{T}^1) \times \mathcal{V}(\mathbb{T}^1) \to \mathcal{V}(\mathbb{T}^1)$$

follows.

Remark. Use of (1.10) also yields the well known result that

$$f, u \in \mathcal{A}(\mathbb{T}^1) \Longrightarrow fu \in \mathcal{A}(\mathbb{T}^1) \ \text{ and } \ \|fu\|_{\mathcal{A}} \le \|f\|_{\mathcal{A}}\|u\|_{\mathcal{A}},$$

so $\mathcal{A}(\mathbb{T}^1)$ is a Banach algebra, and the content of Proposition 1.2 is that $\mathcal{V}(\mathbb{T}^1)$ is a module over $\mathcal{A}(\mathbb{T}^1)$.

Now an object closely related to $\mathcal{V}(\mathbb{T}^1)$ has been studied for a long time, namely infinite series of the form

$$\sum_{n=-\infty}^{\infty} a_n e^{in\theta}, \quad \text{given} \quad \lim_{|n|\to\infty} |a_n| = 0. \tag{1.12}$$

This study goes back to Riemann himself, and a number of results can be found in [Z], particularly in Chapters 5 and 9, where the study is called the *Riemann theory of trigonometic series*. Objects of the form (1.12) are not treated as distributions in [Z], but various results given there can be seen as precursors to distribution theory.

In §3 we prove a number of results involving the space $\mathcal{V}(\mathbb{T}^1)$, and draw comparisons with results in Chapter 5 of [Z]. Proposition 3.1 generalizes (1.3), and Corollary 3.2 applies Theorem 1.1 to elements of $\mathcal{V}(\mathbb{T}^1)$ covered by Proposition 3.1. A special case is that the series (1.1) converges locally uniformly on $\mathbb{T}^1 \setminus 0$. Another is that so does the series

$$\sum_{n=2}^{\infty} \frac{1}{\log n} \cos n\theta, \tag{1.13}$$

though of course this series diverges at $\theta = 0$. Proposition 3.3 and Corollary 3.4 give conditions on an element $u \in \mathcal{V}(\mathbb{T}^1)$ that guarantee $u \in L^1(\mathbb{T}^1)$, with Fourier series converging locally uniformly on $\mathbb{T}^1 \setminus 0$. A special case is that (1.13) defines an element of $L^1(\mathbb{T}^1)$ (as opposed to (1.1)). Proposition 3.6 has as a special case that one can go from (1.2) to

$$u_L + \mathrm{PV}\,\frac{1}{\theta \log|\theta|} \in L^1\Big(\Big[-\frac{1}{2}, \frac{1}{2}\Big]\Big). \tag{1.14}$$

As we have mentioned, results of §3 are seen to generalize various results in Chapter 5 of [Z]. However, the proofs here, making use of distribution theory, are different, and shorter.

In §4 we discuss a localization result from Chapter 9 of [Z], which goes back to Riemann. The statement of the result, given in Theorem 4.1, is less direct than Theorem 1.1, but we show that it is equivalent to Theorem 1.1, given the theory of distributions. We also present a distributional approach to the proof of Riemann localization given in Chapter 9 of [Z], which is different from our proof in §2.

Section 5 is devoted to a closer study of u_L, defined by (1.1). We complement (1.3) with a proof that

$$u_L \in C^\infty(\mathbb{T}^1 \setminus 0). \tag{1.15}$$

We also prove (1.2). In addition we give a proof that $u_L \notin L^1(\mathbb{T}^1)$ that does not use (1.2).

In §6 we reverse course from specifying a Fourier series (as in (1.1)) and deriving consequences. Instead, we look at distributions on $\mathbb{T}^1$ of the form

$$u = \mathrm{PV}\,\frac{f(\theta)}{\theta}, \tag{1.16}$$

with

$$f \in C([-\pi, \pi]), \quad \text{even}. \tag{1.17}$$

We show that, in such cases,

$$f \in \mathcal{A}(\mathbb{T}^1),\ f(0) = 0 \Longrightarrow u \in \mathcal{V}(\mathbb{T}^1). \tag{1.18}$$

Methods of [T2] imply that if

$$f \in C^\infty(\mathbb{T}^1 \setminus 0) \ \text{ and } \ f(\theta) = \frac{1}{\log|\theta|} \ \text{ for } \ |\theta| \le \frac{1}{2}, \tag{1.19}$$

then

$$\hat{f}(n) \sim \frac{C}{|n|(\log|n|)^2}, \quad \text{as } |n| \to \infty, \tag{1.20}$$

so this result provides a class of elements of $\mathcal{V}(\mathbb{T}^1)$ that includes (1.1) as a special case.

In §7 we discuss the space $\mathcal{M}^b(\mathbb{T}^1)$ of finite Borel measures on $\mathbb{T}^1$ that belong to $\mathcal{V}(\mathbb{T}^1)$. In particular, we consider certain measures naturally associated with various Cantor sets in $\mathbb{T}^1$ and show that some of them belong to $\mathcal{V}(\mathbb{T}^1)$ and some do not, giving a sample of results explored more systematically in classical work of R. Salem.

In §8 we discuss the Riemann summation of the Fourier series of an element of $\mathcal{V}(\mathbb{T}^1)$. Results obained here will be valuable for work in §9.

Section 9 contains some results on sets of uniqueness, or U-sets, in $\mathbb{T}^1$. We say $\Sigma \subset \mathbb{T}^1$ is a U-set provided that, for each $u \in \mathcal{V}(\mathbb{T}^1)$,

$$\lim_{N\to\infty} S_N u(\theta) = 0, \quad \forall\, \theta \in \mathbb{T}^1 \setminus \Sigma \Longrightarrow u = 0. \tag{1.21}$$

Otherwise, we say Σ is an M-set. We establish that a closed set $K \subset \mathbb{T}^1$ is an M-set if and only if there is a nonzero

$$u \in \mathcal{V}(\mathbb{T}^1) \ \text{ such that } \ \operatorname{supp} u \subset K. \tag{1.22}$$

This is a distributional variant of Theorem 6.8 in Chapter 9 of [Z]. From this result it is easy to deduce that if $K \subset \mathbb{T}^1$ is a closed M-set, then K contains a perfect set that is an M-set. We discuss how some of the Cantor sets introduced in §7 are U-sets and others are M-sets.

In §10 we establish the following variant of a theorem of A. Rajchman. We consider

$$u \in C^{-2}(\mathbb{T}^1) = \{u = v'' + c : v \in C(\mathbb{T}^1),\, c \in \mathbb{C}\}, \tag{1.23}$$

and show that if

$$\lim_{r\nearrow 1} \mathcal{A}(r)u(\theta) = 0 \tag{1.24}$$

for all $\theta \in \mathbb{T}^1$, then $u = 0$. Here, for $r \in (0,1)$,

$$\mathcal{A}(r)u(\theta) = \sum_k \hat{u}(k) r^{|k|} e^{ik\theta}. \tag{1.25}$$

We also discuss localizations of this result. We introduce the notion of $(U, \mathcal{A}, \mathfrak{X})$ sets and $(M, \mathcal{A}, \mathfrak{X})$ sets in $\mathbb{T}^1$, respectively sets of uniqueness and of non-uniqueness for Abel summability, for elements of a linear subspace $\mathfrak{X}$ of $C^{-2}(\mathbb{T}^1)$. Notable examples include $\mathfrak{X} = C^{-1}(\mathbb{T}^1)$ and $\mathfrak{X} = \mathcal{W}(\mathbb{T}^1)$, where a distribution u belongs to $\mathcal{W}(\mathbb{T}^1)$ if and only if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{|k|\le N} |\hat{u}(k)|^2 = 0. \tag{1.26}$$

We also look at Besov spaces $\mathfrak{X} = B^{(\gamma-1)/2}_{2,\infty}(\mathbb{T}^1)$, which arise as spaces to which Cantor measures from §7 belong, and also the L^2-Sobolev space $\mathfrak{X} = H^{-1/2}(\mathbb{T}^1)$. We see that a compact $K \subset \mathbb{T}^1$ is a $(U, \mathcal{A}, H^{-1/2}(\mathbb{T}^1))$-set if and only if

$$\operatorname{Cap}_{1/2}(K) = 0. \tag{1.27}$$

Remark. More than enough background in distribution theory for our needs here can be found in each of the following references: [S], Chapter 7 of [H], Chapter 6 of [Y], or Chapter 3 of [T1].

2. Proof of Theorem 1.1

To prove Theorem 1.1, it suffices to work with the case $f = 0$, and show that $S_N u(\theta) \to 0$ uniformly on K. Given $u \in \mathcal{D}'(\mathbb{T}^1)$, we have

$$S_N u(\theta) = \frac{1}{2\pi} \langle u_\theta, D_N \rangle, \tag{2.1}$$

where $u \mapsto u_\theta$ is the extension to $\mathcal{D}'(\mathbb{T}^1)$ of $f \mapsto f_\theta$, given by $f_\theta(\varphi) = f(\theta - \varphi)$, and

$$\begin{aligned} D_N(\varphi) &= \sum_{n=-N}^{N} e^{in\varphi} \\ &= \frac{\sin(N+1/2)\varphi}{\sin \varphi/2} \\ &= \Bigl(\cot \frac{\varphi}{2}\Bigr) \sin N\varphi + \cos N\varphi. \end{aligned} \tag{2.2}$$

It follows that

$$S_N u(\theta) = \frac{1}{2\pi} \Bigl\langle u_\theta, \Bigl(\cot \frac{\varphi}{2}\Bigr) \sin N\varphi \Bigr\rangle + \frac{1}{2\pi} \Bigl\langle u_\theta, \cos N\varphi \Bigr\rangle. \tag{2.3}$$

To analyze the right side of (2.3), note that

$$\begin{aligned} u \in \mathcal{V}(\mathbb{T}^1) &\Longrightarrow \theta \mapsto u_\theta \text{ is continuous from } \mathbb{T}^1 \text{ to } \mathcal{V}(\mathbb{T}^1) \\ &\Longrightarrow \langle u_\theta, \cos N\varphi \rangle \to 0, \text{ uniformly in } \theta \in \mathbb{T}^1, \text{ as } N \to \infty, \end{aligned} \tag{2.4}$$

the second implication as a consequence of the readily established fact that

$$\begin{aligned} &\mathcal{K} \subset \mathcal{V}(\mathbb{T}^1) \text{ compact } \Longrightarrow \\ &\quad \langle v, e^{iN\varphi} \rangle \to 0 \text{ as } |N| \to \infty, \text{ uniformly for } v \in \mathcal{K}. \end{aligned} \tag{2.5}$$

Furthermore,

$$\cot \frac{\varphi}{2} \in C^\infty(\mathbb{T}^1 \setminus 0), \tag{2.6}$$

so, thanks to Proposition 1.2,

$$\begin{aligned} &u \in \mathcal{V}(\mathbb{T}^1), \quad u\big|_{\mathcal{O}} = 0, \quad \theta \in K \subset\subset \mathcal{O} \\ &\quad \Longrightarrow u_\theta \cot \frac{\varphi}{2} \in \mathcal{V}(\mathbb{T}^1), \quad \text{and} \\ &\quad \Bigl\{ u_\theta \cot \frac{\varphi}{2} : \theta \in K \Bigr\} \text{ is compact in } \mathcal{V}(\mathbb{T}^1). \end{aligned} \tag{2.7}$$

This implies that the first term on the right side of (2.3) tends to 0 as $N \to \infty$, uniformly for $\theta \in K$. Thus one has Theorem 1.1.

3. Contact with Chapter 5 of [Z]

We aim to extend a number of results from Chapter 5 of [Z], which tie in with localization. Theorems from [Z] cited in this section will all be from this Chapter 5. We start with a result implying some regularity on $\mathbb{T}^1 \setminus 0$.

Proposition 3.1. *Take $u \in \mathcal{D}'(\mathbb{T}^1)$ with Fourier series*

$$u = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}. \tag{3.1}$$

Assume (a_n) has the bounded variation property

$$\sum_{n=-\infty}^{\infty} |a_n - a_{n-1}| < \infty. \tag{3.2}$$

Then $u \in C(\mathbb{T}^1 \setminus 0)$. More precisely,

$$(1 - e^{i\theta})u \in \mathcal{A}(\mathbb{T}^1). \tag{3.3}$$

Proof. Indeed,

$$(1 - e^{i\theta})u = \sum_n (a_n - a_{n-1})e^{in\theta}. \tag{3.4}$$

Remark 3.1. In light of (3.4), we actually see that (3.2) and (3.3) are equivalent.

Remark 3.2. If (3.2) holds and also $u \in \mathcal{V}(\mathbb{T}^1)$, then not only is the right side of (3.4) continuous, but it vanishes at $\theta = 0$. Hence

$$\lim_{\theta \to 0} \theta u(\theta) = 0. \tag{3.4A}$$

Together with Theorem 1.1, Proposition 3.1 implies the following.

Corollary 3.2. *If $u \in \mathcal{V}(\mathbb{T}^1)$ satisfies (3.1)–(3.2), then*

$$S_N u \longrightarrow u \quad \text{uniformly on compact sets} \quad K \subset \mathbb{T}^1 \setminus 0. \tag{3.5}$$

Second proof. A computation gives

$$(1 - e^{i\theta})S_N u = S_N(1 - e^{i\theta})u - \hat{u}(N)e^{i(N+1)\theta} + \hat{u}(-N-1)e^{-iN\theta}, \tag{3.5A}$$

which implies the following stronger version of (3.5),

$$(1 - e^{i\theta})S_N u \longrightarrow (1 - e^{i\theta})u \text{ uniformly on } \mathbb{T}^1, \tag{3.5B}$$

under the hypotheses of Corollary 3.2.

Remark. Proposition 3.1 and Corollary 3.2 apply to

$$\sum_{n=\ell}^{\infty} a_n \cos n\theta, \quad a_n \searrow 0, \tag{3.6}$$

and to

$$\sum_{n=\ell}^{\infty} a_n \sin n\theta, \quad a_n \searrow 0, \tag{3.7}$$

given $\ell \in \mathbb{Z}^+$. This contains Theorem 1.8 and Theorem 1.15 of [Z].

The next result extends Theorem 1.5 of [Z].

Proposition 3.3. *Let $u \in \mathcal{V}(\mathbb{T}^1)$ satisfy (3.1)–(3.2). Suppose in addition that the sequence (a_n) is positive definite, i.e., for each sequence (ξ_n), nonvanishing for only finitely many n,*

$$\sum_{m,n} a_{m-n} \xi_m \overline{\xi}_n \geq 0. \tag{3.8}$$

Then

$$u \text{ is a non-negative element of } L^1(\mathbb{T}^1) \cap C(\mathbb{T}^1 \setminus 0), \tag{3.9}$$

and

$$S_N u \longrightarrow u \text{ uniformly on each compact } K \subset \mathbb{T}^1 \setminus 0. \tag{3.10}$$

Proof. Bochner's theorem (which has a short proof via distribution theory) implies u is a finite positive measure on $\mathbb{T}^1$. Proposition 3.1 implies $u \in C(\mathbb{T}^1 \setminus 0)$. Hence $f = u|_{\mathbb{T}^1 \setminus 0}$ satisfies

$$f \geq 0, \quad \int_0^{2\pi} f(\theta)\, d\theta < \infty. \tag{3.11}$$

Then

$$u - f \text{ is a measure supported on } \{0\}, \tag{3.12}$$

and the hypothesis $u \in \mathcal{V}(\mathbb{T}^1)$ implies this measure is 0, so $u = f \in L^1(\mathbb{T}^1)$. Finally, (3.10) follows from Corollary 3.2.

The actual setting of Theorem 1.5 of [Z] is captured by the following.

Corollary 3.4. *Consider $u \in \mathcal{V}(\mathbb{T}^1)$ given by*

$$u = \frac{b_0}{2} + \sum_{k\ge 1} b_k \cos k\theta, \tag{3.13}$$

with

$$b_k \searrow 0, \quad (b_k) \ \ \text{convex.} \tag{3.14}$$

Then (3.9)–(3.10) hold.

Proof. We then have u given by (3.1), with $a_n = b_{|n|}/2$, and (3.14) implies (3.8).

Remark 3.3. Corollary 3.4 applies to

$$u = \frac{b_0}{2} + b_1 \cos\theta + \sum_{n\ge 2} \frac{\cos n\theta}{\log n}, \tag{3.15}$$

for a suitable choice of $b_0 > b_1 > 1/\log 2$. Contrast this with u_L, given by (1.1), which does not belong to $L^1(\mathbb{T}^1)$. On the other hand, as seen in (3.6)–(3.7), Proposition 3.1 and Corollary 3.2 apply to both (3.15) and (1.1).

The proof of Theorem 1.5 in [Z] did not use Bochner's theorem, but proceeded as follows. Summing $S_N u$ by parts twice yields

$$S_N u(\theta) = \sum_{k=0}^{N-2} (k+1)(\Delta^2 b)_k K_k(\theta) + N(\Delta b)_{N-1} K_{N-1}(\theta) + b_N D_N(\theta), \tag{3.16}$$

when u has the form (3.13), where

$$(\Delta b)_k = b_k - b_{k+1}, \quad (\Delta^2 b)_k = (\Delta b)_k - (\Delta b)_{k+1}, \tag{3.17}$$

$D_N(\theta)$ is the Dirichlet kernel, as in (2.2), and $K_k(\theta)$ is the Fejer kernel:

$$\begin{aligned} K_k(\theta) &= \frac{1}{k+1} \sum_{n=0}^{k} D_n(\theta) \\ &= \frac{2}{k+1} \Bigl(\frac{\sin(k+1)\theta/2}{2\sin\theta/2}\Bigr)^2. \end{aligned} \tag{3.18}$$

As [Z] notes, elementary bounds on $D_N(\theta)$ and $K_{N-1}(\theta)$ show that, as long as $b_N \to 0$ and $(\Delta b)_{N-1} \to 0$, the last two terms in (3.16) tend to 0 pointwise on $\mathbb{T}^1 \setminus 0$, as $N \to \infty$. We find it useful to make some complementary estimates.

First, clearly

$$\|b_N D_N\|_{\mathcal{V}(\mathbb{T}^1)} = |b_N|, \tag{3.19}$$

so if we set

$$S_N^{\#} u(\theta) = \sum_{k=0}^{N-2} (k+1)(\Delta^2 b)_k K_k(\theta) + N(\Delta b)_{N-1} K_{N-1}(\theta), \tag{3.20}$$

we see that, whenever $u \in \mathcal{D}'(\mathbb{T}^1)$ is given by (3.13),

$$u \in \mathcal{V}(\mathbb{T}^1) \Longrightarrow S_N^{\#} u \to u \text{ in } H^s(\mathbb{T}^1),\ \forall\, s < -\frac{1}{2}. \tag{3.21}$$

Clearly the hypotheses of (3.14) imply

$$S_N^{\#} u(\theta) \geq 0 \text{ on } \mathbb{T}^1, \quad \forall\, N. \tag{3.22}$$

It is also clear that

$$\frac{1}{2\pi} \int\limits_{\mathbb{T}^1} S_N^{\#} u(\theta)\, d\theta = \frac{1}{2\pi} \int\limits_{\mathbb{T}^1} S_N u(\theta)\, d\theta - b_N = \hat{u}(0) - b_N, \tag{3.23}$$

so $\{S_N^{\#} u\}$ is bounded in L^1-norm, and hence one has that, under the hypotheses of Corollary 3.4, u must be a finite positive measure on $\mathbb{T}^1$.

Note that integrating (3.20) term by term gives

$$\frac{1}{2\pi} \int\limits_{\mathbb{T}^1} S_N^{\#} u(\theta)\, d\theta = \sum_{k=0}^{N-2} (k+1)(\Delta^2 b)_k + N(\Delta b)_{N-1}. \tag{3.24}$$

If (3.14) holds, this is a sum of positive terms, uniformly $\leq \hat{u}(0)$, so in particular

$$N(\Delta b)_{N-1} \leq A < \infty, \quad \forall\, N. \tag{3.25}$$

Now we can set

$$S_N^b u(\theta) = \sum_{k=0}^{N-2} (k+1)(\Delta^2 b)_k K_k(\theta), \tag{3.26}$$

and deduce that if $u \in \mathcal{D}'(\mathbb{T}^1)$ is given by (3.13), and if hypothesis (3.14) holds, then $S_N^b u$ is a monotone increasing sequence of positive functions with an L^1 upper bound, and hence

$$S_N^b u \longrightarrow u^b \in L^1(\mathbb{T}^1), \text{ in } L^1\text{-norm}. \tag{3.27}$$

Meanwhile, the bound (3.25) implies

$$N(\Delta b)_{N-1} K_{N-1} \text{ is bounded in } L^1\text{-norm}, \tag{3.28}$$

and (via (3.18)) converges to 0 uniformly on compact subsets of $\mathbb{T}^1 \setminus 0$. Comparison of (3.21) and (3.27) shows that, if (3.14) holds,

$$u = u^b + \mu_0, \tag{3.29}$$

where μ_0 is a measure on $\mathbb{T}^1$ supported on $\{0\}$, hence equal to 0 given $u \in \mathcal{V}(\mathbb{T}^1)$. We have:

Proposition 3.5. *In the setting of Corollary 3.4, we have in addition to (3.9)–(3.10) that*

$$S_N^b u \longrightarrow u \quad \text{in } L^1\text{-norm.} \tag{3.30}$$

Note that since $\mu_0 = 0$, we also have

$$N(\Delta b)_{N-1} \longrightarrow 0, \quad \text{as} \quad N \to \infty, \tag{3.31}$$

given (3.14). Thus, in the setting of Corollary 3.4,

$$\limsup_{N\to\infty} \|u - S_N u\|_{L^1} = \limsup_{N\to\infty} |b_N| \cdot \|D_N\|_{L^1}. \tag{3.32}$$

As is well known,

$$\|D_N\|_{L^1} \sim \frac{4}{\pi} \log N, \tag{3.33}$$

as $N \to \infty$. Hence, with u as in Corollary 3.4,

$$\lim_{N\to\infty} \|u - S_N u\|_{L^1} = 0 \Longleftrightarrow |b_N| = o(|\log N|^{-1}).$$

This gives Theorem 1.12 of [Z].

Remark. Having gone on about the alternative proof of Corollary 3.4 described above, we emphasize that Proposition 3.3 is a substantially more general result.

We next derive a result complementary to Proposition 3.1, involving the following notion of a PV distribution. Let $f \in C(\mathbb{T}^1 \setminus 0)$ have the following properties:

$$f(\theta) = -f(-\theta), \quad |f(\theta)| \le \frac{C}{|\theta|}, \tag{3.34}$$

where we identify $\mathbb{T}^1$ with $[-\pi, \pi]$ (and $\pi \sim -\pi$). More generally, f could be measurable on $\mathbb{T}^1$ and satisfy

$$f(\theta) = -f(-\theta), \quad \int_{-\pi}^{\pi} |\theta f(\theta)| \, d\theta < \infty. \tag{3.35}$$

We define $\mathrm{PV}\, f \in \mathcal{D}'(\mathbb{T}^1)$ by

$$\langle \mathrm{PV}\, f, g \rangle = \int_{\mathbb{T}^1} f(\theta)(g(\theta) - g(0)) \, d\theta, \tag{3.36}$$

for $g \in C^\infty(\mathbb{T}^1)$. The integral on the right side of (3.36) is absolutely integrable for each $g \in C^\infty(\mathbb{T}^1)$, in fact for all Lipschitz continuous g on $\mathbb{T}^1$. It follows that

$$\begin{aligned} \langle \mathrm{PV}\, f, g \rangle &= \lim_{\varepsilon \searrow 0} \int_{\mathbb{T}^1 \setminus [-\varepsilon, \varepsilon]} f(\theta)(g(\theta) - g(0)) \, d\theta \\ &= \lim_{\varepsilon \searrow 0} \int_{\mathbb{T}^1 \setminus [-\varepsilon, \varepsilon]} f(\theta) g(\theta) \, d\theta. \end{aligned} \tag{3.37}$$

Proposition 3.6. *Assume $u \in \mathcal{D}'(\mathbb{T}^1)$ satisfies the hypotheses of Proposition 3.1 and is odd with respect to $\theta \mapsto -\theta$. Set $f = u|_{\mathbb{T}^1\setminus 0} \in C(\mathbb{T}^1 \setminus 0)$. Then f satisfies (3.34) and*

$$u = \mathrm{PV}\, f. \tag{3.38}$$

Proof. That $f \in C(\mathbb{T}^1 \setminus 0)$ satisfies $|f(\theta)| \le C/|\theta|$ follows from (3.3). It is obvious that f is odd if u is odd. Now

$$u - \mathrm{PV}\, f \in \mathcal{D}'(\mathbb{T}^1) \ \text{ is supported on } \ \{0\}. \tag{3.39}$$

More precisely, again by (3.3), we have

$$(1 - e^{i\theta})(u - \mathrm{PV}\, f) = 0, \tag{3.40}$$

so, as in (3.4), for $v = u - \mathrm{PV}\, f$,

$$v = \sum b_n e^{in\theta} \Longrightarrow \sum (b_n - b_{n-1}) e^{in\theta} = 0, \tag{3.41}$$

which implies $b_n = b_0$ for all n, hence $u - \mathrm{PV}\, f = c\delta$ for some $c \in \mathbb{C}$. But δ is even under $\theta \mapsto -\theta$, and u and $\mathrm{PV}\, f$ are odd, so $c = 0$, and we have (3.38).

Proposition 3.6 applies to distributions of the form (3.7), and we have a refinement of Theorem 1.15 of [Z].

4. Riemann's version of Theorem 1.1

Here we will compare Theorem 1.1 with Riemann's own localization result, as described in Chapter 9 of [Z], except that the discussion here will be presented in the language of distribution theory.

Take $u \in \mathcal{V}(\mathbb{T}^1)$, or more generally in

$$\mathcal{B}(\mathbb{T}^1) = \{u \in \mathcal{D}'(\mathbb{T}^1) : \sup_n |\hat{u}(n)| < \infty\}. \tag{4.1}$$

Note that under $u \mapsto \hat{u}$ this space is isomorphic (as a Banach space) to the sequence space $\ell^\infty(\mathbb{T}^1)$, and it is the dual of $\mathcal{A}(\mathbb{T}^1)$:

$$\mathcal{A}(\mathbb{T}^1)' = \mathcal{B}(\mathbb{T}^1). \tag{4.2}$$

With Riemann (and a little help from L. Schwartz) we define

$$G : \mathcal{B}(\mathbb{T}^1) \longrightarrow \mathcal{A}(\mathbb{T}^1) \tag{4.3}$$

by

$$Gu = \sum_{n \neq 0} n^{-2} \hat{u}(n) e^{in\theta}. \tag{4.4}$$

Note that

$$\frac{d^2}{d\theta^2} Gu = -u + \hat{u}(0). \tag{4.5}$$

The following is essentially Theorem 4.3 in Chapter 9 of [Z].

Theorem 4.1. *Let $u \in \mathcal{V}(\mathbb{T}^1)$. Let (a, b) be an interval in $\mathbb{T}^1$, and assume*

$$Gu \ \text{ is linear on } \ (a, b). \tag{4.6}$$

Then $S_N u \to u$ uniformly on each compact subset K of (a, b).

We show that Theorem 1.1 implies Theorem 4.1. In fact, by (4.5), we know that (4.6) implies $u = \hat{u}(0)$ on (a, b), so indeed the conclusion of Theorem 4.1 follows from (1.9).

Conversely, Theorem 4.1 implies Theorem 1.1. To get this, it suffices to treat the case $\hat{u}(0) = 0$. Also, as noted in §2, it suffices to prove Theorem 1.1 when $u|_{\mathcal{O}} = 0$. If (a, b) is some connected component of $\mathcal{O}$, then, by (4.5), Gu must be linear on (a, b), and we are done.

Remark. Clearly we could replace the hypothesis (4.6) by

$$\begin{aligned} &Gu = h \text{ on } (a, b), \quad h \in C^2(\mathbb{T}^1), \text{ and} \\ &\qquad h'' \text{ has a uniformly convergent Fourier series on } \mathbb{T}^1. \end{aligned} \tag{4.7}$$

We discuss the proof of Riemann localization given in Chapter 9 of [Z], translated into the distributional setting, and applied directly to Theorem 1.1, rather than going through Theorem 4.1. We start with a variant of Theorem 4.9 in Chapter 9 of [Z], due to (Zygmund's teacher) A. Rajchman.

Proposition 4.2. *If* $u \in \mathcal{V}(\mathbb{T}^1)$ *and* $f' \in \mathcal{A}(\mathbb{T}^1)$*, then*

$$\lim_{N\to\infty} \|S_N(fu) - fS_N u\|_{L^\infty} = 0. \tag{4.8}$$

Proof. A computation parallel to (3.5A) gives

$$\begin{aligned} S_N(e^{ik\theta}u) - e^{ik\theta}S_N u = &\sum_{\ell=-N-k}^{-N-1} \hat{u}(\ell)e^{i(\ell+k)\theta} \\ &- \sum_{\ell=N-k+1}^{N} \hat{u}(\ell)e^{i(\ell+k)\theta}. \end{aligned} \tag{4.9}$$

Thus, if

$$f(\theta) = \sum_{k=-\infty}^{\infty} b_k e^{ik\theta}, \tag{4.10}$$

then

$$\begin{aligned} S_N(fu) - fS_N u = &\sum_k b_k \sum_{\ell=-N-k}^{-N-1} \hat{u}(\ell)e^{i(\ell+k)\theta} \\ &- \sum_k b_k \sum_{\ell=N-k+1}^{N} \hat{u}(\ell)e^{i(\ell+k)\theta}. \end{aligned} \tag{4.11}$$

It follows that

$$\begin{aligned} \sup_{\mathbb{T}^1} |S_N(fu) - fS_N u| \le I_N = &\sum_k |kb_k| \Bigl(\max_{\ell\in\{-N-k,\dots,-N-1\}} |\hat{u}(\ell)| \Bigr) \\ &+ \sum_k |kb_k| \Bigl(\max_{\ell\in\{N-k+1,\dots,N\}} |\hat{u}(\ell)| \Bigr). \end{aligned} \tag{4.12}$$

The discrete version of the dominated convergence theorem shows that, if $\sum |kb_k| < \infty$, then

$$u \in \mathcal{V}(\mathbb{T}^1) \Longrightarrow \lim_{N\to\infty} I_N = 0, \tag{4.13}$$

and we have (4.8).

Let us note that (4.11)–(4.12) hold for $f' \in \mathcal{A}(\mathbb{T}^1)$ and $u \in \mathcal{B}(\mathbb{T}^1)$, and this yields the following.

Proposition 4.3. *If* $u \in \mathcal{B}(\mathbb{T}^1)$ *and* $f' \in \mathcal{A}(\mathbb{T}^1)$*, then, for each* N*,*

$$\|S_N(fu) - fS_N u\|_{L^\infty} \le 2\|f'\|_{\mathcal{A}} \|u\|_{\mathcal{B}}. \tag{4.14}$$

We now derive Theorem 1.1 from Proposition 4.2. As before, it suffices to treat the case where $u = 0$ on an open set $\mathcal{O}$ and deduce (1.8) for compact $K \subset \mathcal{O}$. So take $g \in C_0^\infty(\mathcal{O})$ such that $g = 1$ on K. By Proposition 4.2, $S_N(gu) - gS_N u \to 0$ uniformly on $\mathbb{T}^1$ as $N \to \infty$. But $gu = 0$, so this yields $gS_N u \to 0$ uniformly on $\mathbb{T}^1$, and hence $S_N u \to 0$ uniformly on K, as desired.

The proof of Theorem 4.1 occupies pages 330–334 of [Z], much of which is devoted to the development of material needed in the absence of an existing theory of distributions.

5. More on u_L

To explore u_L in more detail, we find it convenient to work with Fourier integrals instead of Fourier series, so take

$$a \in C^\infty(\mathbb{R}), \quad a(\xi) = -a(-\xi), \quad a(\xi) = \frac{1}{\log \xi} \ \text{ for } \ \xi \geq 2, \qquad a(\xi) = 0 \ \text{ for } \ |\xi| \leq 1, \tag{5.1}$$

and consider

$$U_L(x) = \int_0^\infty a(\xi) \sin x\xi \, d\xi, \tag{5.2}$$

i.e.,

$$U_L(x) = \frac{1}{2i} \int_{-\infty}^\infty a(\xi) e^{ix\xi} \, d\xi. \tag{5.3}$$

This integral is not absolutely convergent, but it exists as an *oscillatory integral.* In more detail, the function $a(\xi)$ in (5.1) is a tempered distribution, i.e., $a \in \mathcal{S}'(\mathbb{R})$, and the Fourier transform maps $\mathcal{S}'(\mathbb{R})$ to itself, so (5.3) defines $U_L \in \mathcal{S}'(\mathbb{R})$. Further structure follows from the fact that

$$|a^{(k)}(\xi)| \leq C_k (1 + |\xi|)^{-k}, \tag{5.4}$$

and

$$x^k U_L(x) = \frac{i^k}{2i} \int_{-\infty}^\infty a^{(k)}(\xi) e^{ix\xi} \, d\xi, \tag{5.5}$$

and more generally

$$\Bigl(\frac{d}{dx}\Bigr)^\ell x^k U_L(x) = \frac{i^{k+\ell}}{2i} \int_{-\infty}^\infty \xi^\ell a^{(k)}(\xi) e^{ix\xi} \, d\xi, \tag{5.6}$$

so

$$k \geq \ell + 2 \Longrightarrow \Bigl| \Bigl(\frac{d}{dx}\Bigr)^\ell x^k U_L(x) \Bigr| \leq C_{k\ell} < \infty. \tag{5.7}$$

It follows that U_L is C^∞ on $\mathbb{R} \setminus 0$ and rapidly decreasing, with all its derivatives, as $|x| \to \infty$.

With these estimates in hand, we can use the Poisson summation formula to write

$$u_L(\theta) = \sum_{k=-\infty}^\infty U_L(\theta + 2\pi k), \tag{5.8}$$

and see that (1.15) holds and that the singularity of u_L at $\theta = 0$ coincides with that of U_L. In particular, (1.2) is equivalent to

$$U_L(x) = -\frac{1}{x\log|x|} + O\Bigl(\frac{1}{|x|(\log|x|)^2}\Bigr), \tag{5.9}$$

as $x \to 0$.

We now provide a proof of (5.9). The argument we use is parallel to that used for the proof of Theorem 2.17 in Chapter 5 of [Z], except that we deal with Fourier integrals, rather than Fourier series, which allows for some simplifications of the details. We first note that (5.9) is equivalent to the result

$$S_0(x) = -\frac{1}{x\log|x|} + O\Bigl(\frac{1}{|x|(\log|x|)^2}\Bigr), \tag{5.10}$$

as $x \to 0$, where

$$S_0(x) = \int_2^\infty \frac{1}{\log\xi}\sin x\xi\,d\xi. \tag{5.11}$$

Since $S_0(x)$ is odd, it suffices to treat it for $x > 0$. Note that

$$\begin{aligned} xS_0(x) &= -\int_2^\infty \frac{1}{\log\xi}\frac{d}{d\xi}\cos x\xi\,d\xi \\ &= -\int_2^\infty \frac{1}{\xi(\log\xi)^2}\cos x\xi\,d\xi + \frac{\cos 2x}{\log 2}, \end{aligned} \tag{5.12}$$

the latter identity by integration by parts. Now, for $u > 1$,

$$\int_u^\infty \frac{d\xi}{\xi(\log\xi)^2} = -\int_u^\infty \frac{d}{d\xi}\frac{1}{\log\xi}\,d\xi = \frac{1}{\log u}, \tag{5.13}$$

so (5.12) yields

$$xS_0(x) = \int_2^\infty \frac{1}{\xi(\log\xi)^2}(1-\cos x\xi)\,d\xi - \frac{1-\cos 2x}{\log 2}. \tag{5.14}$$

Assuming $0 < x < 1/2$, we break this integral into an integral over $[2, 1/x]$ and an integral over $[1/x, \infty)$, and we separate out the terms in the integrand of the latter integral, obtaining

$$\begin{aligned} xS_0(x) &+ \frac{1-\cos 2x}{\log 2} \\ &= \int_2^{1/x} \frac{1}{\xi(\log\xi)^2}(1-\cos x\xi)\,d\xi \\ &\quad + \int_{1/x}^\infty \frac{1}{\xi(\log\xi)^2}\,d\xi \\ &\quad - \int_{1/x}^\infty \frac{1}{\xi(\log\xi)^2}\cos x\xi\,d\xi \\ &= r_1(x) + v(x) - r_2(x). \end{aligned} \tag{5.15}$$

By (5.13), for $0 < x < 1$,

$$v(x) = \frac{1}{\log 1/x}. \tag{5.16}$$

Next, since

$$|1 - \cos x\xi| \le x^2\xi^2 \quad \text{for} \quad |x\xi| \le 1, \tag{5.17}$$

we have

$$\begin{aligned} |r_1(x)| &\le x^2 \int_2^{1/x} \frac{\xi}{(\log \xi)^2}\, d\xi \\ &\le Cx^2 \cdot \frac{1}{x} \cdot \frac{1/x}{(\log 1/x)^2} \\ &= \frac{C}{(\log 1/x)^2}, \end{aligned} \tag{5.18}$$

the second inequality because the integrand is monotonically increasing for large ξ. It remains to treat

$$\begin{aligned} r_2(x) &= \frac{1}{x} \int_{1/x}^{\infty} \frac{1}{\xi(\log \xi)^2} \frac{d}{d\xi} \sin x\xi\, d\xi \\ &= -\frac{1}{x} \int_{1/x}^{\infty} \frac{d}{d\xi} \frac{1}{\xi(\log \xi)^2} \sin x\xi\, d\xi + \frac{\sin 1}{(\log 1/x)^2}, \end{aligned} \tag{5.19}$$

the latter identity by integration by parts. A computation gives

$$\Bigl|\frac{d}{d\xi} \frac{1}{\xi(\log \xi)^2}\Bigr| \le \frac{C}{\xi^2(\log \xi)^2}, \tag{5.20}$$

which readily yields

$$|r_2(x)| \le \frac{C}{(\log 1/x)^2}. \tag{5.21}$$

This proves (5.10), so we have (1.2).

We now provide an alternative proof that u_L does not belong to $L^1(\mathbb{T}^1)$, which avoids the asymptotic evaluation established above. To begin, we note that u_L is the distributional derivative of

$$f_L(\theta) = -\sum_{n=2}^{\infty} \frac{1}{n \log n} \cos n\theta. \tag{5.22}$$

If u_L were in $L^1(\mathbb{T}^1)$, then f_L would be absolutely continuous. More generally, if u_L were a finite measure on $\mathbb{T}^1$, then f_L would have bounded variation. Now the analysis behind the Gibbs phenomenon shows that

$$f \in BV(\mathbb{T}^1) \Longrightarrow \sup_{N,\theta} |S_N f(\theta)| < \infty. \tag{5.23}$$

In more detail, say $f' = \mu$ is a finite measure on $\mathbb{T}^1$, and define $\psi \in \mathrm{BV}(\mathbb{T}^1)$ by

$$\psi(\theta) = \frac{1}{2\pi}(\pi - \theta), \quad 0 < \theta < 2\pi. \tag{5.23A}$$

Then

$$f(\theta) = \mu * \psi(\theta) + \hat{f}(0), \tag{5.23B}$$

so

$$S_N f(\theta) = \hat{f}(0) + \mu * S_N \psi(\theta). \tag{5.23C}$$

The analysis of the Gibbs phenomenon yields $A < \infty$ such that

$$\sup_\theta |S_N \psi(\theta)| \le A, \quad \forall\, N, \tag{5.23D}$$

and this establishes (5.23). In contrast to (5.23), we have

$$S_N f_L(0) = -\sum_{n=2}^{N} \frac{1}{n \log n} \to -\infty \quad \text{as} \quad N \to \infty. \tag{5.24}$$

Hence $u_L \notin L^1(\mathbb{T}^1)$.

To be sure, (1.2) gives a more precise picture of how u_L fails to be integrable. A related result is

$$f_L(\theta) = -\log\log \frac{1}{|\theta|} + O(1), \tag{5.25}$$

as $\theta \to 0$, which follows from (1.2) by integrating.

We refer to [T2] for a derivation of much more precise asymptotics on u_L, f_L and related distributions, including results in higher dimensions.

6. PV distributions and other distributions in $\mathcal{V}(\mathbb{T}^1)$

We will construct elements of $\mathcal{V}(\mathbb{T}^1)$ that are products of the form fv, with

$$v = \sum_{n=1}^{\infty} \sin n\theta \in \mathcal{B}(\mathbb{T}^1), \tag{6.1}$$

where $\mathcal{B}(\mathbb{T}^1)$ is as in (4.1). Note that

$$v = \frac{1}{2i} \sum_{n=1}^{\infty} e^{in\theta} - \frac{1}{2i} \sum_{n=1}^{\infty} e^{-in\theta}. \tag{6.2}$$

Multiplying by $e^{i\theta}$ and by $e^{-i\theta}$ and subtracting, we obtain

$$(\sin\theta) v = \frac{1}{2}(1 + \cos\theta). \tag{6.3}$$

We claim that

$$v = \frac{1}{2}\,\mathrm{PV}\,\frac{1+\cos\theta}{\sin\theta} = \frac{1}{2}\,\mathrm{PV}\cot\frac{\theta}{2}, \tag{6.4}$$

the PV distribution defined as in (3.34)–(3.37). In fact, denoting the right side of (6.4) by v_1, we see that also v_1 satisfies (6.3), so $(\sin\theta)(v - v_1) = 0$, hence the Fourier coefficients of $w = v - v_1$ satisfy $\hat{w}(n) = \hat{w}(n+2)$. This implies that w is even, but $v - v_1$ is odd, so $w = 0$. We see from (6.4) that $v \in C^\infty(\mathbb{T}^1 \setminus 0)$ and

$$v - \mathrm{PV}\,\frac{1}{\theta} \in C^\infty([-\pi, \pi]). \tag{6.5}$$

We will establish the following.

Proposition 6.1. *Given v as in (6.1),*

$$f \in \mathcal{A}(\mathbb{T}^1),\ f(0) = 0 \Longrightarrow fv \in \mathcal{V}(\mathbb{T}^1). \tag{6.6}$$

Before giving the proof, we make some comments about $\mathcal{B}(\mathbb{T}^1)$, which is a Banach space, with norm

$$\|u\|_{\mathcal{B}} = \sup_n |\hat{u}(n)|. \tag{6.7}$$

Under $u \mapsto \hat{u}$ this space is isomorphic to $\ell^\infty(\mathbb{Z})$, and, parallel to (1.7), we have the duality (noted already in (4.2))

$$\mathcal{A}(\mathbb{T}^1)' = \mathcal{B}(\mathbb{T}^1). \tag{6.8}$$

Also, parallel to Proposition 1.2, we have

$$\begin{aligned} &f \in \mathcal{A}(\mathbb{T}^1),\ u \in \mathcal{B}(\mathbb{T}^1) \Longrightarrow fu \in \mathcal{B}(\mathbb{T}^1),\ \text{and} \\ &\qquad\qquad \widehat{fu}(n) = \sum_{k=-\infty}^{\infty} \hat{f}(k)\hat{u}(n-k). \end{aligned} \tag{6.9}$$

In this case, we can use (6.8) to characterize $fu \in \mathcal{B}(\mathbb{T}^1)$ by

$$\langle fu, g\rangle = \langle u, fg\rangle, \quad \forall\, g \in \mathcal{A}(\mathbb{T}^1), \tag{6.16}$$

using the standard fact that $\mathcal{A}(\mathbb{T}^1)$ is a Banach algebra under the pointwise product.

We are ready to prove Proposition 6.1. Given v as in (6.1) and $f \in \mathcal{A}(\mathbb{T}^1)$, (6.9) implies $fv \in \mathcal{B}(\mathbb{T}^1)$ and

$$\widehat{fv}(n) = \sum_{k=-\infty}^{\infty} \hat{f}(k)\hat{v}(n-k). \tag{6.11}$$

If also $f(0) = 0$, we have

$$\hat{f} \in \ell^1(\mathbb{Z}), \quad \sum_{k=-\infty}^{\infty} \hat{f}(k) = 0. \tag{6.12}$$

Now $\hat{v}$ is constant on $\{1, 2, 3, \dots\}$ and on $\{-1, -2, -3, \dots\}$, so (6.12) readily yields

$$\widehat{fv}(n) \longrightarrow 0, \quad \text{as} \quad |n| \to \infty, \tag{6.17}$$

and we have (6.6).

If in the setting of Proposition 6.1 one also has that f is even, it is readily verified that fv in (6.6) coincides with

$$\mathrm{PV}\, fv, \tag{6.18}$$

as defined in (3.34)–(3.37). As noted in (1.19)–(1.20), one has such behavior for

$$f \in C^\infty(\mathbb{T}^1 \setminus 0), \quad f(\theta) = \frac{1}{\log|\theta|} \text{ for } |\theta| \le \frac{1}{2}. \tag{6.19}$$

In such a case,

$$fv \in \mathcal{V}(\mathbb{T}^1), \quad \text{but} \quad fv \notin L^1(\mathbb{T}^1). \tag{6.20}$$

One can also find *odd* $f \in \mathcal{A}(\mathbb{T}^1)$ such that (6.20) holds. We describe how to obtain such f. First, Theorem 1.9 (Chapter 5) of [Z] provides examples of

$$w_0 = \sum_{n\ge 0} a_n \cos n\theta, \quad a_n \searrow 0, \quad w_0 \notin L^1(\mathbb{T}^1), \tag{6.21}$$

while of course $w_0 \in \mathcal{V}(\mathbb{T}^1)$. By Proposition 3.1,

$$f_0 = (1 - e^{i\theta})w_0 \in \mathcal{A}(\mathbb{T}^1), \tag{6.22}$$

so w_0 is continuous on $\mathbb{T}^1 \setminus 0$. To produce an odd f, we multiply (6.22) by

$$2i\frac{1+\cos\theta}{1+e^{i\theta}}, \tag{6.23}$$

which belongs to $C^\infty(\mathbb{T}^1)$ since the zero at $\theta = \pi$ in the numerator cancels the zero in the denominator. Thus

$$f = 2i\frac{1+\cos\theta}{1+e^{i\theta}} f_0 \in \mathcal{A}(\mathbb{T}^1), \tag{6.24}$$

and we have

$$\begin{aligned} f &= 2i(1+\cos\theta)\frac{1-e^{i\theta}}{1+e^{i\theta}} w_0 \\ &= 2(1+\cos\theta)\Bigl(\tan\frac{\theta}{2}\Bigr) w_0, \end{aligned} \tag{6.25}$$

which implies that f is odd. We can write

$$f = 2\Bigl(\tan\frac{\theta}{2}\Bigr) w, \quad w = (1+\cos\theta) w_0. \tag{6.26}$$

Comparison with (6.4) gives

$$fv = w, \tag{6.27}$$

and by (6.21) plus the observation that the nonintegrable singularity of w_0 occurs at $\theta = 0$, we see that $w \notin L^1(\mathbb{T}^1)$ (though $w \in \mathcal{V}(\mathbb{T}^1)$).

We make additional contact with §3.

Proposition 6.2. *In the setting of Proposition 6.1, $u = fv$ satisfies the hypotheses of Corollary 3.2. Hence, as $N \to \infty$,*

$$(1-e^{i\theta}) S_N(fv) \longrightarrow (1-e^{i\theta}) fv, \quad \textit{uniformly on} \ \ \mathbb{T}^1. \tag{6.28}$$

Proof. By (6.4) we have $(1-e^{i\theta})v \in C^\infty(\mathbb{T}^1)$, hence

$$(1-e^{i\theta}) fv \in \mathcal{A}(\mathbb{T}^1). \tag{6.29}$$

By Remark 3.1, this implies that fv satisfies the hypotheses of Proposition 3.1. Also, by (6.6), $fv \in \mathcal{V}(\mathbb{T}^1)$, so fv satisfies the hypotheses of Corollary 3.2. The convergence (6.28) then follows by the second proof of Corollary 3.2.

7. Measures in $\mathcal{V}(\mathbb{T}^1)$

Let $\mathcal{M}(\mathbb{T}^1)$ denote the space of finite (complex) Borel measures on $\mathbb{T}^1$, and set

$$\mathcal{M}^b(\mathbb{T}^1) = \mathcal{V}(\mathbb{T}^1) \cap \mathcal{M}(\mathbb{T}^1). \tag{7.1}$$

In view of the estimate

$$|\hat{\mu}(k)| \le \|\mu\|_{\mathrm{TV}}, \tag{7.2}$$

we have the following simple but useful result.

Proposition 7.1. *The space $\mathcal{M}^b(\mathbb{T}^1)$ is a linear subspace of $\mathcal{M}(\mathbb{T}^1)$ that is closed in the* TV*-norm topology.*

Recall that, for $u \in \mathcal{V}(\mathbb{T}^1)$, $f \in \mathcal{A}(\mathbb{T}^1)$, we have $fu \in \mathcal{V}(\mathbb{T}^1)$. Since $\mathcal{A}(\mathbb{T}^1)$ is dense in $C(\mathbb{T}^1)$, it follows readily from Proposition 7.1 that

$$f \in C(\mathbb{T}^1),\ \mu \in \mathcal{M}^b(\mathbb{T}^1) \Longrightarrow f\mu \in \mathcal{M}^b(\mathbb{T}^1). \tag{7.3}$$

We will take this much further. Let $|\mu|$ denote the total variation measure associated to μ. By the Radon-Nikodym theorem,

$$\mu = \varphi|\mu|, \quad |\varphi| = 1,\ |\mu|\text{-a.e. on } \ \mathbb{T}^1. \tag{7.4}$$

The following result is a version of Theorem 10.2 in Chapter 12 of [Z].

Proposition 7.2. *If $\mu \in \mathcal{M}^b(\mathbb{T}^1)$, then $|\mu| \in \mathcal{M}^b(\mathbb{T}^1)$.*

Proof. Since $|\varphi^{-1}| = 1$, $|\mu|$-a.e., we can take $f_\nu \in C(\mathbb{T}^1)$ such that

$$f_\nu \to \varphi^{-1} \ \text{ in } \ L^1(\mathbb{T}^1, |\mu|), \ \text{and} \ \sup_\theta |f_\nu(\theta)| \le 1. \tag{7.5}$$

Passing to a subsequence, which we continue to denote (f_ν), we have

$$f_\nu \to \varphi^{-1}, \ \ |\mu|\text{-a.e. on } \mathbb{T}^1, \text{ so } \ f_\nu\varphi \to 1,\ |\mu|\text{-a.e.}, \tag{7.6}$$

and boundedly, so the Lebesgue dominated convergence theorem then yields

$$f_\nu\varphi \longrightarrow 1 \ \text{ in } L^1(\mathbb{T}, |\mu|)\text{-norm}, \tag{7.7}$$

hence

$$f_\nu\mu = f_\nu\varphi|\mu| \longrightarrow |\mu| \text{ in TV-norm}. \tag{7.8}$$

Since each $f_\nu\mu$ belongs to $\mathcal{M}^b(\mathbb{T}^1)$, the conclusion follows from Propsition 7.1.

The next result is equivalent to Theorem 10.9 in Chapter 12 of [Z].

Corollary 7.3. *In the setting of Proposition 7.2,*

$$g \in L^1(\mathbb{T}^1, |\mu|) \Longrightarrow g|\mu| \in \mathcal{M}^b(\mathbb{T}^1). \tag{7.9}$$

Proof. Given such g, there exist $g_\nu \in C(\mathbb{T}^1)$ such that $g_\nu \to g$ in $L^1(\mathbb{T}^1, |\mu|)$-norm. Hence $g_\nu|\mu| \to g|\mu|$ in TV-norm. Now, by Proposition 7.2, (7.3) applies to $|\mu|$, so each $g_\nu|\mu| \in \mathcal{M}^b(\mathbb{T}^1)$, and the conclusion follows from Proposition 7.1.

We now construct some Cantor-like measures on $\mathbb{T}^1$ (more precisely, on $[0, 2\pi]$), some of which belong to $\mathcal{V}(\mathbb{T}^1)$ and some of which do not. Let

$$X = \prod_{\ell \geq 1} \{0, 1\} \tag{7.10}$$

be endowed with the product probability measure λ, whose factor measures on each copy of $\{0, 1\}$ assign measure $1/2$ to each point in such a factor. Let

$$\varphi : X \longrightarrow [0, 2\pi] \tag{7.11}$$

be a continuous map. Then the push-forward $\mu_\varphi = \varphi_*(\lambda)$ is a probability measure on $[0, 2\pi]$ defined by

$$\int_0^{2\pi} f\, d\mu_\varphi = \int_X f(\varphi(x))\, d\lambda(x), \quad f \in C([0, 2\pi]). \tag{7.12}$$

Particular examples include

$$\varphi((a_1, a_2, a_3, \dots)) = \sum_{\ell \geq 1} a_\ell b_\ell, \quad b_\ell > 0, \quad \sum_\ell b_\ell = A \leq 2\pi. \tag{7.13}$$

Note that such φ is one-to-one provided

$$\sum_{\ell \geq m+1} b_\ell < b_m, \quad \forall\, m \in \mathbb{N}. \tag{7.14}$$

In such a case, φ is a homeomorphism of X onto its image K_φ, which is then a compact, totally disconnected subset of $[0, 2\pi]$, with no isolated points. In such a case, μ_φ has no atoms.

A special case of (7.13) is $\varphi_b : X \to [0, 2\pi]$, given by

$$\varphi_b((a_1, a_2, a_3, \dots)) = \sum_{\ell \geq 1} a_\ell b^\ell, \quad 0 < b < \frac{1}{2}, \tag{7.15}$$

which then satisfies (7.14), yielding a Cantor set K_b and a non-atomic probability measure μ_b supported on K_b. (The limiting case $\varphi_{1/2}$ maps X onto $[0, 1]$.) For each $b \in (0, 1/2)$, K_b can be seen to have Lebesgue measure 0. The set $K_{1/3}$ is, up to a dilation, the standard Cantor middle third set.

In (7.16)–(7.19), we reproduce some calculations from Chapter 5, §3, of [Z]. First, for φ of the form (7.13), we have

$$\begin{aligned} 2\pi\hat{\mu}_\varphi(\xi) &= \int e^{-it\xi}\,d\mu_\varphi(t) \\ &= \int_X e^{-i\varphi(x)\xi}\,d\lambda(x) \\ &= \prod_{\ell=1}^{\infty}\Bigl(\frac{1}{2}\sum_{a_\ell\in\{0,1\}} e^{-ia_\ell b_\ell \xi}\Bigr). \end{aligned} \tag{7.16}$$

Using

$$\frac{1}{2}\Bigl(1+e^{-ib_\ell\xi}\Bigr) = e^{-ib_\ell\xi/2}\cos\frac{b_\ell\xi}{2}, \tag{7.17}$$

we obtain

$$2\pi\hat{\mu}_\varphi(2\xi) = e^{-iA\xi}\prod_{\ell=1}^{\infty}\cos(b_\ell\xi). \tag{7.18}$$

In case $b_\ell = b^\ell$, $b\in(0,1/2)$, we get

$$2\pi\hat{\mu}_b(2\xi) = e^{-iA\xi}\Phi_b(\xi), \quad \Phi_b(\xi) = \prod_{\ell=1}^{\infty}\cos(b^\ell\xi). \tag{7.19}$$

Note that the infinite product for $\Phi_b(\xi)$ is absolutely convergent, defining Φ_b as a bounded, continuous function on $[0,\infty)$, satisfying $\Phi_b(0) = 1$. We have the following.

Proposition 7.4. *For the measure μ_b defined above, $b\in(0,1/2)$,*

$$\mu_b\in\mathcal{V}(\mathbb{T}^1) \Longleftrightarrow \lim_{|\xi|\to\infty}\prod_{\ell=1}^{\infty}\cos(b^\ell\xi) = 0. \tag{7.20}$$

Proof. The proof of $\Leftarrow$ is immediate from (7.19). For $\Rightarrow$, we bring in the following.

Lemma 7.5. *Let $u\in\mathcal{D}'(\mathbb{T}^1)$ and assume* $\operatorname{supp} u\subset[0,\gamma)$, $\gamma<2\pi$. *Then u naturally gives rise to $v\in\mathcal{D}'(\mathbb{R})$ such that* $\operatorname{supp} v\subset[0,\gamma)$, *and*

$$u\in\mathcal{V}(\mathbb{T}^1) \Longleftrightarrow \lim_{|\xi|\to\infty}|\hat{v}(\xi)| = 0. \tag{7.21}$$

Again the only part that needs an argument is $\Rightarrow$, and we leave this as an exercise for the reader, with the hint that Proposition 1.2 should be useful.

To study further when $\Phi_b(\xi)\to\infty$ as $|\xi|\to\infty$, we set

$$\beta = \frac{1}{b}\in(2,\infty). \tag{7.22}$$

Then, for all $\xi \in \mathbb{R}$,

$$\begin{aligned}\Phi_{1/\beta}(\beta^m \xi) &= \prod_{\ell=1}^{\infty} \cos(\beta^{m-\ell}\xi) \\ &= \Phi_{1/\beta}(\xi) \prod_{k=0}^{m-1} \cos(\beta^k \xi).\end{aligned} \tag{7.23}$$

It follows from (7.19) that

$$\Phi_{1/\beta}(\xi) \neq 0, \quad \text{for } \ 0 \leq \xi < \frac{\pi}{2}\beta, \tag{7.24}$$

hence $\Phi_{1/\beta}(\xi) \neq 0$ for $1 \leq \xi \leq \beta$. We deduce the following from Proposition 7.4

Proposition 7.6. *For $\beta \in (2, \infty)$,*

$$\mu_{1/\beta} \in \mathcal{V}(\mathbb{T}^1) \Longleftrightarrow \lim_{m \to \infty} \prod_{k=0}^{m-1} \cos(\beta^k \xi) = 0, \quad \forall\, \xi \in [1, \beta]. \tag{7.25}$$

Proof. It follows directly from the analysis above that the left side of (7.25) holds if and only if the right side converges *uniformly* to 0, for $\xi \in [1, \beta]$. However, Dini's theorem, applied to the absolute value of the product, which is monotone in m, yields that pointwise convergence for $\xi \in [1, \beta]$ implies uniform convergence.

Note that

$$\begin{aligned}\beta \in \mathbb{N} &\Rightarrow \cos(\beta^k \pi) = (-1)^{\beta}, \quad \forall\, k \in \mathbb{N} \\ &\Rightarrow \Phi_{1/\beta}(\beta^m \pi) = \pm \Phi_{1/\beta}(\pi), \quad \forall\, m \in \mathbb{N}.\end{aligned} \tag{7.26}$$

By (7.24), $\Phi_{1/\beta}(\pi) \neq 0$ if $\beta > 2$, so we deduce that

$$\frac{1}{b} \in \{3, 4, 5, \dots\} \Longrightarrow \mu_b \notin \mathcal{V}(\mathbb{T}^1). \tag{7.27}$$

For example, the standard measure $\mu_{1/3}$ associated to the Cantor middle third set does not belong to $\mathcal{V}(\mathbb{T}^1)$. See §9 for results that imply that, for b as in (7.27), K_b supports no nonzero elements of $\mathcal{V}(\mathbb{T}^1)$.

On the other hand, for most $\beta > 2$, one does have $\mu_{1/\beta} \in \mathcal{V}(\mathbb{T}^1)$. In fact, suppose there exists $\xi \in [1, \beta]$ such that

$$\prod_{k=0}^{m-1} \left|\cos(\beta^k \xi)\right| \geq A > 0, \quad \forall\, m \in \mathbb{N}. \tag{7.28}$$

Then, for all $k \in \mathbb{N}$, we must have

$$\beta^k \xi = \nu_k \pi + \delta_k, \quad \nu_k \in \mathbb{N}, \ \delta_k \to 0 \ \text{ as } \ k \to \infty. \tag{7.29}$$

Comparison with

$$\beta^{k+1} \xi = \nu_{k+1} \pi + \delta_{k+1} \tag{7.30}$$

gives

$$\beta \nu_k = \nu_{k+1} + \varepsilon_k, \quad \varepsilon_k = \frac{1}{\pi}(\delta_{k+1} - \beta \delta_k) \to 0 \ \text{ as } \ k \to \infty. \tag{7.31}$$

(In fact, if (7.28) holds, then $\sum_k \delta_k^2 < \infty$, hence $\sum_k \varepsilon_k^2 < \infty$.) The result (7.31) has the following simple consequence.

Proposition 7.7. *If $\beta > 2$ is a rational numner but is not an integer, then $\mu_{1/\beta} \in \mathcal{V}(\mathbb{T}^1)$.*

Proof. Suppose $\beta = a/b$, with $a, b \in \mathbb{N},\ a > b > 1$, and a and b have no common factors. Then, given $\nu_k, \nu_{k+1} \in \mathbb{N}$,

$$\begin{aligned} \beta\nu_k = \nu_{k+1} + \varepsilon_k, \quad |\varepsilon_k| < \frac{1}{b} \\ \Rightarrow \varepsilon_k = 0 \Rightarrow \frac{a}{b}\nu_k = \nu_{k+1} \Rightarrow b|\nu_k. \end{aligned} \tag{7.32}$$

If p is a prime factor of b and the prime factorization of ν_k contains exactly α factors of p, then that of ν_{k+1} must contain fewer than α factors of p. By the same reasoning, if also $|\varepsilon_{k+1}| < 1/b$, we also have $b|\nu_{k+1}$, and hence ν_{k+2} contains still fewer factors of p. Iterating this through $b|\nu_{k+j}$ for sufficiently large j yields a contradiction.

Proposition 7.7 is a special case of a more thorough analysis of the right side of (7.20), due to [Sal]. The result is that, if $0 < b < 1/2$,

$$\lim_{|\xi|\to\infty} \prod_{\ell=1}^{\infty} \cos(b^\ell \xi) = 0 \Longleftrightarrow \beta = \frac{1}{b} \ \text{ is not a Pisot number,} \tag{7.33}$$

where a Pisot number is an algebraic integer $\beta > 1$ all of whose conjugates (of which there are none if β is an integer) have absolute value < 1. See Chapter 12, §11 of [Z] for a treatment.

8. Riemann summation of trigonometric series

Given $u \in \mathcal{V}(\mathbb{T}^1)$, $\varepsilon > 0$, we set

$$\mathcal{R}(\varepsilon)u(\theta) = \sum_{k=-\infty}^{\infty} \hat{u}(k)\Bigl(\frac{\sin k\varepsilon}{k\varepsilon}\Bigr)^2 e^{ik\theta}, \tag{8.1}$$

where $(\sin k\varepsilon)/(k\varepsilon)$ is set equal to 1 at $k = 0$. Note that

$$\mathcal{R}(\varepsilon) : \mathcal{V}(\mathbb{T}^1) \longrightarrow \mathcal{A}(\mathbb{T}^1), \quad \forall\, \varepsilon > 0, \tag{8.2}$$

and

$$u \in \mathcal{V}(\mathbb{T}^1) \Longrightarrow \mathcal{R}(\varepsilon)u \to u \text{ in } \mathcal{V}(\mathbb{T}^1)\text{-norm, as } \varepsilon \searrow 0. \tag{8.3}$$

Now (8.1) is a special case of the following construction. Take

$$\begin{gathered} \varphi : \mathbb{R} \longrightarrow \mathbb{R}, \ \ \text{continuous at } 0, \quad \varphi(0) = 1, \quad \varphi(-s) = \varphi(s), \\ |\varphi(s)| \le C(1 + |s|)^{-\gamma}, \quad \gamma > 1, \end{gathered} \tag{8.4}$$

and set

$$\mathcal{S}_{\varphi_\varepsilon} u(\theta) = \sum_{k=-\infty}^{\infty} \hat{u}(k)\varphi_\varepsilon(k) e^{ik\theta}, \quad \varphi_\varepsilon(k) = \varphi(\varepsilon k). \tag{8.5}$$

Then we have analogues of (8.2)–(8.3). Note that

$$\mathcal{R}(\varepsilon) = \mathcal{S}_{\varphi_\varepsilon} \ \text{ with } \ \varphi(s) = \Bigl(\frac{\sin s}{s}\Bigr)^2. \tag{8.6}$$

A crucial ingredient in Riemann's investigation of $\mathcal{R}(\varepsilon)$ is that, for each $u \in \mathcal{V}(\mathbb{T}^1)$, $\theta_0 \in \mathbb{T}^1$,

$$\lim_{N\to\infty} S_N u(\theta_0) = L_0 \Longrightarrow \lim_{\varepsilon\searrow 0} \mathcal{R}(\varepsilon)u(\theta_0) = L_0. \tag{8.7}$$

This is a special case of the following result.

Proposition 8.1. *Let φ satisfy (8.4) and assume also that*

$$\varphi' \in L^1(\mathbb{R}), \quad \textit{or more generally,} \ \ \varphi \in \mathrm{BV}(\mathbb{R}). \tag{8.8}$$

Then, for each $u \in \mathcal{V}(\mathbb{T}^1)$ and $\theta_0 \in \mathbb{T}^1$,

$$\lim_{N\to\infty} S_N u(\theta_0) = L_0 \Longrightarrow \lim_{\varepsilon\searrow 0} \mathcal{S}_{\varphi_\varepsilon} u(\theta_0) = L_0. \tag{8.9}$$

In fact, this is just a result about infinite series, so let us take $a = (a_k) \in c_0(\mathbb{Z})$ (i.e., $|a_k| \to 0$ as $|k| \to \infty$) and define

$$S_N a = \sum_{|k| \le N} a_k, \quad \mathcal{S}_{\varphi_\varepsilon} a = \sum_k a_k \varphi(\varepsilon k). \tag{8.10}$$

The claim is that

$$\lim_{N\to\infty} S_N a = L_0 \Longrightarrow \lim_{\varepsilon \searrow 0} \mathcal{S}_{\varphi_\varepsilon} a = L_0, \tag{8.11}$$

as long as (8.4) and (8.8) hold. To get this, set

$$S(0) = 0, \quad S(x) = S_N a \ \text{ for } \ N < x \le N+1. \tag{8.12}$$

Then, with the first integral denoting a Stieltjes integral, we have

$$\begin{aligned} \mathcal{S}_{\varphi_\varepsilon} a &= \int_0^\infty \varphi_\varepsilon(x)\, dS(x) \\ &= -\int_0^\infty S(x)\varphi_\varepsilon'(x)\, dx \\ &= L_0 + \int_0^\infty (L_0 - S(x))\varphi_\varepsilon'(x)\, dx \\ &= L_0 + \int_0^\infty \Bigl(L_0 - S\Bigl(\frac{x}{\varepsilon}\Bigr)\Bigr)\varphi'(x)\, dx. \end{aligned} \tag{8.13}$$

In case $\varphi \in \mathrm{BV}(\mathbb{R})$, replace $\varphi'(x)\, dx$ by $d\varphi(x)$. Now the hypothesis $S_N a \to L_0$ yields

$$L_0 - S\Bigl(\frac{x}{\varepsilon}\Bigr) \longrightarrow 0, \quad \text{boundedly}, \quad \forall\, x \in (0,\infty), \tag{8.14}$$

as $\varepsilon \searrow 0$, so the convergence of (8.13) to L_0 follows from (8.8) and the dominated convergence theorem.

Other examples besides $\mathcal{R}(\varepsilon)$ to which Proposition 8.1 applies include

$$\begin{aligned} \varphi(s) = {} & (1-|s|)_+, \\ & (1-s^2)_+^\alpha, \quad \alpha > 0, \\ & e^{-|s|}, \end{aligned} \tag{8.15}$$

and many others. The following calculations spotlight a singular property of $\mathcal{R}(\varepsilon)$. Given $u \in \mathcal{V}(\mathbb{T}^1)$, set

$$F(\theta) = Gu(\theta) = \sum_{k\ne 0} \frac{\hat{u}(k)}{k^2} e^{ik\theta}, \tag{8.16}$$

with $G : \mathcal{V}(\mathbb{T}^1) \to \mathcal{A}(\mathbb{T}^1)$ as in (4.4). Then

$$F(\theta \pm \varepsilon) = \sum_{k\ne 0} \frac{\hat{u}(k)}{k^2} e^{\pm ik\varepsilon} e^{ik\theta}, \tag{8.17}$$

so, with

$$\Delta_\varepsilon^2 F(\theta) = \varepsilon^{-2}\Big[F(\theta+\varepsilon) - 2F(\theta) + F(\theta-\varepsilon)\Big], \tag{8.18}$$

we have

$$\begin{aligned} \Delta_{2\varepsilon}^2 F(\theta) &= \sum_{k\neq 0} \hat{u}(k) \frac{2\cos 2k\varepsilon - 2}{(2k\varepsilon)^2} e^{ik\theta} \\ &= -\sum_{k\neq 0} \hat{u}(k) \Big(\frac{\sin k\varepsilon}{k\varepsilon}\Big)^2 e^{ik\theta}, \end{aligned} \tag{8.19}$$

since $1-\cos 2\beta = 2\sin^2\beta$. Consequently,

$$\mathcal{R}(\varepsilon)(u - \hat{u}(0)) = -\Delta_{2\varepsilon}^2 F(\theta). \tag{8.20}$$

The results (8.7) and (8.20) play key roles in the proof of the Riemann uniqueness theorem, discussed in §9.

For reasons related to getting rid of the term $\hat{u}(0)$ in (8.20), we bring in the following "zero-mode elimination trick." Namely, given $u \in \mathcal{V}(\mathbb{T}^1)$, we set

$$v = \sum_{k=-\infty}^{\infty} \hat{u}(k) e^{(2k+1)i\theta}, \tag{8.21}$$

which also belongs to $\mathcal{V}(\mathbb{T}^1)$, and satisfies $\hat{v}(0)=0$. Note that, given $\theta_0 \in \mathbb{T}^1$,

$$\lim_{N\to\infty} S_N u(2\theta_0) = L_0 \Longleftrightarrow \lim_{N\to\infty} S_N v(\theta_0) = e^{i\theta_0} L_0. \tag{8.22}$$

We return to the setting of $\mathcal{S}_{\varphi_\varepsilon}$, with the goal of proving the following extension of Proposition 4.2, which will be useful for certain localization results in §9.

Proposition 8.2. *Let φ satisfy (8.4) and (8.8), and let $f \in C(\mathbb{T}^1)$ satisfy $f' \in \mathcal{A}(\mathbb{T}^1)$. Then*

$$u \in \mathcal{V}(\mathbb{T}^1) \Longrightarrow \lim_{\varepsilon \searrow 0} \|\mathcal{S}_{\varphi_\varepsilon}(fu) - f\mathcal{S}_{\varphi_\varepsilon} u\|_{\mathcal{A}} = 0. \tag{8.23}$$

Proof. A calculation gives

$$\begin{aligned} \mathcal{S}_{\varphi_\varepsilon}(e^{i\ell\theta} u) &= \sum_k \hat{u}(k)\varphi_\varepsilon(k+\ell) e^{i(k+\ell)\theta}, \\ e^{i\ell\theta}\mathcal{S}_{\varphi_\varepsilon} u &= \sum_k \hat{u}(k)\varphi_\varepsilon(k) e^{i(k+\ell)\theta}. \end{aligned} \tag{8.24}$$

It follows that

$$[\mathcal{S}_{\varphi_\varepsilon}, f]u = \sum_{k,\ell} \hat{u}(k)\big[\varphi_\varepsilon(k+\ell) - \varphi_\varepsilon(k)\big]\hat{f}(\ell) e^{i(k+\ell)\theta}. \tag{8.25}$$

Therefore

$$\begin{aligned} \|[\mathcal{S}_{\varphi_\varepsilon}, f]u\|_{\mathcal{A}} &\le \sum_{k,\ell} |\hat{u}(k)| \cdot |\varphi_\varepsilon(k+\ell) - \varphi_\varepsilon(k)| \cdot |\hat{f}(\ell)| \\ &\le \|u\|_{\mathcal{B}} \sum_{k,\ell} |\varphi_\varepsilon(k+\ell) - \varphi_\varepsilon(k)| \cdot |\hat{f}(\ell)|. \end{aligned} \tag{8.26}$$

Now

$$\begin{aligned} \sum_k |\varphi_\varepsilon(k+\ell) - \varphi_\varepsilon(k)| &\le \sum_k \int_k^{k+\ell} |\varphi_\varepsilon'(t)|\, dt \\ &\le |\ell| \int_{-\infty}^{\infty} |\varphi_\varepsilon'(t)|\, dt \\ &= |\ell| \int_{-\infty}^{\infty} |\varphi'(t)|\, dt, \end{aligned} \tag{8.27}$$

the latter integral interpreted as $\|\varphi'\|_{\mathrm{TV}}$ if one uses the latter hypothesis in (8.8). Since $\sum_\ell |\ell \hat{f}(\ell)| = \|f'\|_{\mathcal{A}}$, we have the following extension of (4.14):

$$\|[\mathcal{S}_{\varphi_\varepsilon}, f]u\|_{\mathcal{A}} \le \|\varphi'\|_{\mathrm{TV}} \|f'\|_{\mathcal{A}} \|u\|_{\mathcal{B}}. \tag{8.28}$$

This is a uniform estimate, independent of $\varepsilon > 0$.

In view of the uniform bound in (8.28), since the space of finite linear combinations of $\{e^{ik\theta} : k \in \mathbb{Z}\}$ is dense in $\mathcal{V}(\mathbb{T}^1)$, it suffices to verify (8.23) for $u = e^{ik\theta}$. In such a case, (8.25) gives

$$\|[\mathcal{S}_{\varphi_\varepsilon}, f]e^{ik\theta}\|_{\mathcal{A}} \le \sum_\ell |\varphi_\varepsilon(k+\ell) - \varphi_\varepsilon(k)| \cdot |\hat{f}(\ell)|. \tag{8.29}$$

Now it follows from (8.4) that, for each $k, \ell \in \mathbb{Z}$,

$$|\varphi_\varepsilon(k+\ell) - \varphi_\varepsilon(k)| \le 2C, \tag{8.30}$$

and, as $\varepsilon \searrow 0$,

$$|\varphi_\varepsilon(k+\ell) - \varphi_\varepsilon(k)| \le |1 - \varphi(\varepsilon k + \varepsilon \ell)| + |1 - \varphi(\varepsilon k)| \to 0, \tag{8.31}$$

so, by the discrete version of the dominated convergence theorem,

$$f \in \mathcal{A}(\mathbb{T}^1) \Longrightarrow \lim_{\varepsilon \searrow 0} \sum_\ell |\varphi_\varepsilon(k+\ell) - \varphi_\varepsilon(k)| \cdot |\hat{f}(\ell)| = 0. \tag{8.32}$$

This finishes the proof of Proposition 8.2.

Given that $\mathcal{R}(\varepsilon)$ is going to play an important role in the next section, we want to supplement Proposition 8.1 with a study of when

$$\lim_{\varepsilon \searrow 0} \mathcal{R}(\varepsilon)u(\theta_0) = L_0 \Longrightarrow \lim_{\varepsilon \searrow 0} \mathcal{S}_{\varphi_\varepsilon} u(\theta_0) = L_0, \tag{8.33}$$

given $u \in \mathcal{V}(\mathbb{T}^1)$, $\theta_0 \in \mathbb{T}^1$, or equivalently, when

$$\lim_{\varepsilon \searrow 0} \mathcal{R}(\varepsilon)a = L_0 \Longrightarrow \lim_{\varepsilon \searrow 0} \mathcal{S}_{\varphi_\varepsilon} a = L_0, \tag{8.33A}$$

given $a \in c_0(\mathbb{Z})$, with $\mathcal{S}_{\varphi_\varepsilon} a = \sum_k a_k \varphi(\varepsilon k)$ and $\mathcal{R}(\varepsilon)a$ similarly defined, with $\varphi(\varepsilon k)$ replaced by $\rho_\varepsilon(k) = (\sin \varepsilon k/\varepsilon k)^2$. We claim a sufficient condition is that

$$\begin{aligned} \varphi(s) &= \int_0^\infty \rho_t(s)\alpha(t)\,dt \\ &= \frac{1}{2s^2}\int_0^\infty \frac{1-\cos 2st}{t^2}\alpha(t)\,dt, \end{aligned} \tag{8.34}$$

with

$$\alpha \in L^1(\mathbb{R}^+), \qquad \int_0^\infty \alpha(t)\,dt = 1. \tag{8.35}$$

In such a case,

$$\mathcal{S}_{\varphi_\varepsilon} a = \int_0^\infty \mathcal{R}(\varepsilon t)a\,\alpha(t)\,dt. \tag{8.36}$$

Now $\mathcal{R}(t) : c_0(\mathbb{Z}) \to \mathbb{C}$ satisfies $\|\mathcal{R}(t)\| \le Ct^{-2}$, and hence

$$\lim_{t \searrow 0} \mathcal{R}(t)a = L_0 \Longrightarrow \sup_t |\mathcal{R}(t)a| \le C < \infty. \tag{8.37}$$

Thus if $a \in c_0(\mathbb{Z})$ satisfies the hypotheses of (8.33A),

$$\mathcal{R}(\varepsilon t)a \longrightarrow L_0, \quad \text{boundedly} \quad \forall\, t \in (0,\infty), \quad \text{as} \quad \varepsilon \searrow 0, \tag{8.38}$$

so the dominated convergence theorem gives

$$\lim_{\varepsilon \searrow 0} \mathcal{S}_{\varphi_\varepsilon} a = \int_0^\infty L_0 \alpha(t)\,dt = L_0, \tag{8.39}$$

provided (8.35) holds. Thus our task is to see for which φ (satisfying (8.4) and (8.8)) we have (8.34)–(8.35). We will try for $\alpha(t)$ in the form

$$\alpha(t) = t^2 \beta'(t), \quad \beta, \beta' \in L^1(\mathbb{R}^+), \quad \beta(0) = \beta(\infty) = 0. \tag{8.40}$$

Then (8.34) becomes

$$\begin{aligned} \varphi(s) &= \frac{1}{2s^2}\int_0^\infty (1-\cos 2st)\beta'(t)\,dt \\ &= \frac{1}{s}\int_0^\infty (\sin 2st)\beta(t)\,dt, \end{aligned} \tag{8.41}$$

or

$$s\varphi(s) = \frac{1}{2}\int_0^\infty (\sin st)\beta\Big(\frac{t}{2}\Big)\,dt. \tag{8.42}$$

The Fourier inversion formula gives

$$\beta\Big(\frac{t}{2}\Big) = -\frac{4}{\pi}\int_0^\infty s\varphi(s)(\sin st)\,ds. \tag{8.43}$$

We thus have the following.

Proposition 8.3. *Let φ satisfy (8.4) and (8.8). Assume in addition that $s\varphi(s) \in L^1(\mathbb{R}^+)$ and that β, given by (8.43), satisfies*

$$\beta, \quad \beta', \quad t^2\beta' \in L^1(\mathbb{R}^+). \tag{8.44}$$

Then, for $u \in \mathcal{V}(\mathbb{T}^1),\ \theta_0 \in \mathbb{T}^1$,

$$\lim_{\varepsilon \searrow 0} \mathcal{R}(\varepsilon)u(\theta_0) = L_0 \Longrightarrow \lim_{\varepsilon \searrow 0} \mathcal{S}_{\varphi_\varepsilon} u(\theta_0) = L_0. \tag{8.45}$$

Remark. If, in the setting of Proposition 8.3, we add the hypotheses

$$s^2\varphi, \quad \Big(\frac{d}{ds}\Big)^2(s^2\varphi) \in L^1(\mathbb{R}), \tag{8.46}$$

then we have

$$\begin{aligned} \beta'(t) &= -\frac{4}{\pi}\int_{-\infty}^{\infty} s^2\varphi(s)\ \cos 2st\, ds, \\ t^2\beta'(t) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \Big(\frac{d}{ds}\Big)^2 (s^2\varphi(s))\ \cos 2st\, ds. \end{aligned} \tag{8.47}$$

If one furthermore strengthens (8.46) to

$$s\varphi, \quad s^2\varphi, \quad \Big(\frac{d}{ds}\Big)^2(s^2\varphi) \in \mathcal{A}(\mathbb{R}) = \{g \in L^1(\mathbb{R}) : \hat{g} \in L^1(\mathbb{R})\}, \tag{8.48}$$

then (8.44) holds.

Corollary 8.4. *The implication (8.45) holds in the following cases:*

$$\begin{aligned} &\varphi(s) = e^{-|s|}, \\ &\varphi(s) = (1-s^2)_+^{\alpha}, \quad \alpha > 2. \end{aligned} \tag{8.49}$$

9. Localization and sets of uniqueness

We begin by recalling Riemann's global uniqueness theorem.

Theorem 9.1. *Given $u \in \mathcal{V}(\mathbb{T}^1)$, if*

$$S_N u(\theta) \to 0 \quad \text{as} \quad N \to \infty \quad \text{for each} \quad \theta \in \mathbb{T}^1, \tag{9.1}$$

then each $\hat{u}(k) = 0$, so $u = 0$.

Proof. Using the zero-mode elimination trick from §8, it suffices to prove the result for $u \in \mathcal{V}(\mathbb{T}^1)$ satisfying $\hat{u}(0) = 0$. By Proposition 8.1, the hypothesis (9.1) implies

$$\mathcal{R}(\varepsilon)u(\theta) \longrightarrow 0 \ \text{ as } \ \varepsilon \searrow 0, \quad \forall\, \theta \in \mathbb{T}^1, \tag{9.2}$$

with $\mathcal{R}(\varepsilon)u$ defined as in (8.1). As also seen in §8, for each $\theta \in \mathbb{T}^1$,

$$\mathcal{R}(\varepsilon)u(\theta) = -\Delta^2_{2\varepsilon} F(\theta), \tag{9.3}$$

where $F = Gu \in \mathcal{A}(\mathbb{T}^1)$, and

$$\Delta^2_\varepsilon F(\theta) = \varepsilon^{-2}\Big[F(\theta+\varepsilon) - 2F(\theta) + F(\theta-\varepsilon)\Big]. \tag{9.4}$$

Given (9.2), we have

$$\lim_{\varepsilon \searrow 0} \Delta^2_\varepsilon F(\theta) = \mathcal{D}^2 F(\theta) = 0, \quad \forall\, \theta \in \mathbb{T}^1. \tag{9.5}$$

The next lemma will imply that F is constant on $\mathbb{T}^1$, and since by construction $\widehat{F}(0) = 0$, the constant must be 0. But clearly

$$F = 0 \text{ in } \mathcal{A}(\mathbb{T}^1) \Longrightarrow u = 0 \text{ in } \mathcal{V}(\mathbb{T}^1), \tag{9.6}$$

so we have Theorem 9.1, modulo the following classical elementary result.

Lemma 9.2. *Given an interval $[a, b] \subset \mathbb{R}$ and $F \in C([a, b])$, if*

$$\lim_{\varepsilon \searrow 0} \Delta^2_\varepsilon F(\theta) = 0 \quad \forall\, \theta \in (a, b), \tag{9.7}$$

then F is linear on $[a, b]$.

Proof. It suffices to treat the case where F is real. By a simple change of variable, we can reduce to the case $[a, b] = [-1, 1]$. Also by adding a linear term, which does not affect (9.7), we can arrange that

$$F(-1) = F(1) = 0. \tag{9.8}$$

Then we want to show that (9.7) implies $F \equiv 0$ on $[-1, 1]$. It suffices to show that

$$F_\delta(\theta) = F(\theta) + \delta(1 - \theta^2) \Longrightarrow \tag{9.9}$$

$$F_\delta(\theta) \ge 0, \quad \forall\, \delta > 0, \ \theta \in [-1, 1], \ \text{ and} \tag{9.10}$$

$$F_\delta(\theta) \le 0, \quad \forall\, \delta < 0, \ \theta \in [-1, 1]. \tag{9.11}$$

Note that $\lim_{\varepsilon \searrow 0} \Delta^2_\varepsilon F_\delta(\theta) = -2\delta$, and $F_\delta(-1) = F_\delta(1) = 0$. If (9.10) fails, then there must be $\theta_0 \in (-1, 1)$ such that $F_\delta(\theta_0) < 0$ and $F_\delta(\theta_0)$ is minimal. But it is clear from the defining formula that, at such θ_0,

$$\Delta^2_\varepsilon F_\delta(\theta_0) \ge 0, \quad \forall \text{ small } \varepsilon > 0, \tag{9.12}$$

contradicting the identity $\mathcal{D}^2 F_\delta(\theta_0) = -2\delta < 0$. A similar argument gives (9.11), and we have the lemma.

Now that we have Theorem 9.1, the result localizes as follows.

Proposition 9.3. *Let $\mathcal{O} \subset \mathbb{T}^1$ be open, with complement K. Let $u \in \mathcal{V}(\mathbb{T}^1)$ and assume*

$$S_N u(\theta) \to 0 \quad \text{for each} \quad \theta \in \mathcal{O}. \tag{9.13}$$

Then

$$\operatorname{supp} u \subset K, \tag{9.14}$$

and consequently

$$S_N u \to 0 \quad \text{uniformly on compact subsets of} \quad \mathcal{O}. \tag{9.15}$$

Proof. Pick f such that $f' \in \mathcal{A}(\mathbb{T}^1)$ and $\operatorname{supp} f \subset \mathcal{O}$, for example $f \in C_0^2(\mathcal{O})$. Then (9.13) implies $f S_N u \to 0$ pointwise on $\mathbb{T}^1$, so Proposition 4.2 implies

$$S_N(fu)(\theta) \to 0 \quad \text{for each} \quad \theta \in \mathbb{T}^1. \tag{9.16}$$

Now $fu \in \mathcal{V}(\mathbb{T}^1)$, so Theorem 9.1 implies $fu = 0$. This gives (9.14), and then (9.15) follows from Theorem 1.1.

Generally, a set $\Sigma \subset \mathbb{T}^1$ is called an M-set if there is a nonzero u such that

$$u \in \mathcal{V}(\mathbb{T}^1), \quad S_N u(\theta) \to 0, \ \forall\, \theta \in \mathbb{T}^1 \setminus \Sigma, \tag{9.17}$$

and Σ is called a U-set otherwise, i.e., when (9.17) implies $u = 0$. The following result is a distributional version of Theorem 6.8 in Chapter 9 of [Z].

Proposition 9.4. *Let $K \subset \mathbb{T}^1$ be closed. Then K is an M-set if and only if*

$$\text{there exists a nonzero} \quad u \in \mathcal{V}(\mathbb{T}^1) \quad \text{such that} \quad \operatorname{supp} u \subset K. \tag{9.18}$$

Proof. First, if there is a u as in (9.18), Theorem 1.1 implies that (9.17) holds, with $\Sigma = K$.

For the converse, if there is a nonzero $u \in \mathcal{V}(\mathbb{T}^1)$ such that (9.17) holds, with $\Sigma = K$, then Proposition 9.3 yields $\operatorname{supp} u \subset K$.

Now, given $u \in \mathcal{D}'(\mathbb{T}^1)$, $\operatorname{supp} u$ is uniquely characterized as the minimal closed subset of $\mathbb{T}^1$ off which u vanishes. In light of this, we have the following.

Proposition 9.5. *If $u \in \mathcal{V}(\mathbb{T}^1)$, then $\operatorname{supp} u$ contains no isolated points.*

Proof. If p is such an isolated point, we can write $\operatorname{supp} u = K_0 \cup \{p\}$, a disjoint union, and, using a smooth cutoff, write $u = u_0 + u_1$, with $u_j \in \mathcal{V}(\mathbb{T}^1)$, $\operatorname{supp} u_0 \subset K_0$, and $\operatorname{supp} u_1 \subset \{p\}$. But if $u_1 \in \mathcal{D}'(\mathbb{T}^1)$ is supported at p, it must be a finite linear combination of derivatives of δ_p, so it cannot belong to $\mathcal{V}(\mathbb{T}^1)$ unless it vanishes. This requires $u = u_0$, leading to the contradiction that $\operatorname{supp} u \subset \operatorname{supp} u \setminus \{p\}$.

A compact subset of $\mathbb{T}^1$ with no isolated points is equal to its limit set, that is, it is a *perfect set.* Thus Propositions 9.4 and 9.5 yield the following.

Proposition 9.6. *Let $K \subset \mathbb{T}^1$ be closed. If K is an M-set, then there exists $K_0 \subset K$ such that*

$$K_0 \text{ is a perfect set and an M-set.} \tag{9.19}$$

We return to the global uniqueness issue, and bring in the following concept.

Definition. Let $\varphi : \mathbb{R} \to \mathbb{R}$ satisfy (8.4) and (8.8). We say φ is a *Rajchman multiplier* provided

$$u \in \mathcal{V}(\mathbb{T}^1), \ \lim_{\varepsilon \searrow 0} \mathcal{S}_{\varphi_\varepsilon} u(\theta) \equiv 0 \Longrightarrow u = 0. \tag{9.20}$$

The content of Theorem 9.1 is that $\chi_{[0,1]}(s)$ is a Rajchman multiplier. In fact, the proof goes through the result that $u = 0$ whenever $u \in \mathcal{V}(\mathbb{T}^1)$ and (9.2) holds, so in fact

$$\rho(s) = \Bigl(\frac{\sin s}{s}\Bigr)^2 \tag{9.21}$$

is a Rajchman multiplier. It was shown by M. Riesz that if

$$\varphi(s) = (1 - |s|)_+, \tag{9.22}$$

then (9.20) holds. These works preceeded that of [R], who demonstrated the much stronger result that if

$$\varphi(s) = e^{-|s|}, \tag{9.23}$$

then (9.20) holds. We will take this up in §10. Our current goal is to extend Proposition 9.3, as follows.

Proposition 9.7. *Let $\mathcal{O} \subset \mathbb{T}^1$ be open, with complement K, let φ be a Rajchman multiplier, and let $u \in \mathcal{V}(\mathbb{T}^1)$. Then*

$$\mathcal{S}_{\varphi_\varepsilon} u(\theta) \to 0 \ \text{ for each } \ \theta \in \mathcal{O} \Longrightarrow \operatorname{supp} u \subset K. \tag{9.24}$$

Proof. As in the proof of Proposition 9.3, pick f such that $f' \in \mathcal{A}(\mathbb{T}^1)$ and $\operatorname{supp} f \subset \mathcal{O}$. Then the hypothesis in (9.24) implies $f \mathcal{S}_{\varphi_\varepsilon} u \to 0$ pointwise on $\mathbb{T}^1$, so Proposition 8.2 implies

$$\mathcal{S}_{\varphi_\varepsilon}(fu)(\theta) \longrightarrow 0 \ \text{ for each } \ \theta \in \mathbb{T}^1, \tag{9.25}$$

and hence $fu = 0$. This yields the conclusion of (9.24).

In parallel to (9.17), it is natural to make the following:

Definition. Let $\varphi : \mathbb{R} \to \mathbb{R}$ satisfy (8.4) and (8.8). We say a set $\Sigma \subset \mathbb{T}^1$ is an (M, φ)-set if there is a nonzero u such that

$$u \in \mathcal{V}(\mathbb{T}^1), \quad \mathcal{S}_{\varphi_\varepsilon} u(\theta) \to 0, \quad \forall\, \theta \in \mathbb{T}^1 \setminus \Sigma, \tag{9.26}$$

and we say Σ is a (U, φ)-set otherwise, i.e., when (9.26) implies $u = 0$.

The following result extends Proposition 9.4.

Proposition 9.8. *Let φ be a Rajchman multiplier, and let $K \subset \mathbb{T}^1$ be closed. Then K is an (M, φ)-set if and only if*

$$\text{(9.27)} \qquad \textit{there is a nonzero } u \in \mathcal{V}(\mathbb{T}^1) \textit{ such that } \operatorname{supp} u \subset K.$$

Proof. First, if there is a u as in (9.27), Theorem 1.1 inplies that (9.17) holds, with $\Sigma = K$, and then Proposition 8.1 implies (9.26) holds.

For the converse, if there is a nonzero $u \in \mathcal{V}(\mathbb{T}^1)$ such that (9.26) holds, with $\Sigma = K$, then Proposition 9.7 yields $\operatorname{supp} u \subset K$.

Corollary 9.9. *If φ is a Rajchman multiplier and $K \subset \mathbb{T}^1$ is closed, then K is an (M, φ)-set if and only if K is an M-set.*

Returning to the setting of Proposition 9.4, we have the following, which is a variant of Theorem 6.1 in Chapter 9 of [Z].

Proposition 9.10. *Let $K \subset \mathbb{T}^1$ be compact. Assume that there exist $f_k \in \mathcal{A}(\mathbb{T}^1)$ satisfying*

$$\text{(9.28)} \qquad \{f_k\} \ \textit{bounded in} \ \mathcal{A}(\mathbb{T}^1),$$

$$\text{(9.29)} \qquad f_k \longrightarrow 1 \ \textit{ in } \ \mathcal{D}'(\mathbb{T}^1), \quad \textit{and}$$

$$\text{(9.30)} \qquad \operatorname{supp} f_k \subset \mathcal{O} = \mathbb{T}^1 \setminus K.$$

Then K is a U-set.

Proof. Take $u \in \mathcal{V}(\mathbb{T}^1)$, $\varphi \in C^\infty(\mathbb{T}^1)$. Note that (9.28)–(9.29) imply $f_k \to 1$ weak* in $\mathcal{A}(\mathbb{T}^1) = \mathcal{V}(\mathbb{T}^1)'$. Now, as $k \to \infty$,

$$\text{(9.31)} \qquad \langle f_k u, \varphi \rangle = \langle \varphi u, f_k \rangle \to \langle \varphi u, 1 \rangle = \langle u, \varphi \rangle,$$

so

$$\text{(9.32)} \qquad u \in \mathcal{V}(\mathbb{T}^1) \Longrightarrow f_k u \to u \ \text{ in } \ \mathcal{D}'(\mathbb{T}^1).$$

It then follows from (9.30) that

$$\text{(9.33)} \qquad u \in \mathcal{V}(\mathbb{T}^1), \ \operatorname{supp} u \subset K \Longrightarrow u = \lim_{k \to \infty} f_k u = 0,$$

and the conclusion that K is a U-set follows from Proposition 9.4.

The following variant is actually closer to the Theorem 6.1 of [Z] cited above.

Proposition 9.11. *In the setting of Proposition 9.10, replace the hypothesis (9.30) by*

$$\text{(9.34)} \qquad f_k' \in \mathcal{A}(\mathbb{T}^1), \quad f_k = 0 \ \textit{ on } \ K.$$

Then the conclusion that K is a U-set still holds.

Proof. We still have (9.32). This time, if $u \in \mathcal{V}(\mathbb{T}^1)$ and $\operatorname{supp} u \subset K$, Theorem 1.1 and Proposition 4.2 imply that, for each k,

$$\text{(9.35)} \qquad S_N(f_k u)(\theta) \longrightarrow 0 \ \text{ as } \ N \to \infty, \quad \forall\, \theta \in \mathbb{T}^1,$$

and then Theorem 9.1 implies $f_k u = 0$. Thus we again have (9.33).

Here is a way to construct a sequence (f_k) to which Propositions 9.10–9.11 apply. Start with

$$f \in C^2(\mathbb{T}^1), \quad \int_{\mathbb{T}^1} f(\theta)\, d\theta = 1, \quad f = 0 \ \text{ on } \ \mathcal{K}, \tag{9.36}$$

where $\mathcal{K} \subset \mathbb{T}^1$ is closed. Regard f as defined on $\mathbb{R}$, periodic of period 2π. Pick a sequence $n_k \nearrow \infty$ of positive integers, and set

$$f_k(\theta) = f(n_k\theta) = \sum_\ell \hat{f}(\ell) e^{i\ell n_k \theta}. \tag{9.37}$$

Then (f_k) satisfies (9.28)–(9.29), and also $f_k' \in C^1(\mathbb{T}^1) \subset \mathcal{A}(\mathbb{T}^1)$. From Proposition 9.11 we immediately have the following.

Proposition 9.12. *Let (f_k) be as in (9.36)–(9.37). If $K \subset \mathbb{T}^1$ is a compact set such that $f_k = 0$ on K for each k, then K is a U-set.*

In this regard, note that

$$f_k\big|_K = 0 \Longleftarrow n_k\theta \in \mathcal{K}^\#, \quad \forall\, \theta \in K, \tag{9.38}$$

where $\mathcal{K}$ is as in (9.36) and

$$\mathcal{K}^\# = P^{-1}(\mathcal{K}) \tag{9.39}$$

is the inverse image of $\mathcal{K}$ under the projection $P : \mathbb{R} \to \mathbb{R}/(2\pi\mathbb{Z}) = \mathbb{T}^1$.

The following is essentially a restatement of Proposition 9.12.

Corollary 9.13. *Let $\mathcal{K}^\#$ be a closed, proper subset of $\mathbb{R}$, invariant under the translation $x \mapsto x + 2\pi$. If $n_k \in \mathbb{N}$, $n_k \nearrow \infty$, and*

$$K^\# = \bigcap_k n_k^{-1} \mathcal{K}^\#, \tag{9.40}$$

(which is also invariant under such a translation), then $P(K^\#) = K \subset \mathbb{T}^1$ is a U-set.

Proof. It suffices to note that, if $\mathcal{K} \subset \mathbb{T}^1$ is closed and $\mathbb{T}^1 \setminus \mathcal{K} \neq \emptyset$, then one can find f as in (9.36).

We can apply Corollary 9.13 to certain families of fractal sets. Let $K \subset \mathbb{T}^1$ be compact, and let $n > 1$ be an integer. We say K is of class $\mathcal{F}(n, 2\pi)$ provided

$$K \neq \mathbb{T}^1, \ \text{ and } \ nK^\# \subset K^\#, \tag{9.41}$$

where $K^\# = P^{-1}(K)$.

Proposition 9.14. *If $K \subset \mathbb{T}^1$ is compact and of class $\mathcal{F}(n, 2\pi)$ for an integer $n \geq 2$, then K is a U-set.*

Proof. Apply Corollary 9.13 with $n_k = n^k$, noting that (9.41) implies $n^k K^\# \subset K^\#$, for each $k \in \mathbb{N}$.

Examples of sets satisfying the hypotheses of Proposition 9.14 include the following Cantor sets $\mathcal{C}_n$, given an integer $n \geq 3$. To construct $\mathcal{C}_n$, remove from $[0, 2\pi]$ the central interval $(2\pi/n, 2\pi(1 - 1/n))$, of length $2\pi(1 - 2/n)$ (which is positive if $n > 2$), leaving two closed intervals, each of length $2\pi/n$. From each of these, remove the central open interval of length $(2\pi/n)(1-2/n)$, leaving a total of 4 closed intervals, each of length $2\pi/n^2$. Continue, obtaining a shrinking family of compact subsets of $[0, 2\pi]$, whose intersection defines $\mathcal{C}_n$. The set $\mathcal{C}_3$ is the standard Cantor middle third set, up to a dilation. We see that $n\mathcal{C}_n^\# \subset \mathcal{C}_n^\#$, where $\mathcal{C}_n^\# = P^{-1}(\mathcal{C}_n)$, and deduce that

$$\text{for each integer } n \geq 3,\ \mathcal{C}_n \text{ is a U-set.} \tag{9.42}$$

Note that each $\mathcal{C}_n$ is a dilate of $K_{1/n}$, constructed in §7, via (7.15). The calculation there showing that the natural Cantor measure on $K_{1/n}$ does not belong to $\mathcal{V}(\mathbb{T}^1)$ is consistent with (9.42).

The sets $\mathcal{C}_n$ are natural dilates of the sets $K_{1/n}$. The class of compact U-sets is invariant under translations and dilations, as is the class of compact M-sets. The proof, with some help from Lemma 7.5, is left to the reader.

Proposition 7.7 implies that $K_{1/\beta}$ is an M-set whenever $\beta > 2$ is a rational number that is not an integer. These are examples of perfect sets of Lebesgue measure 0 that are M-sets. D. Mensov first produced examples of such M-sets (hence the name "M-sets"). Extending (7.33), which implies that the natural Cantor measure μ_b on K_b (for $0 < b < 1/2$) belongs to $\mathcal{V}(\mathbb{T}^1)$ if and only if

$$\frac{1}{b} \text{ is not a Pisot number,} \tag{9.43}$$

[Sal] showed that, for $0 < b < 1/2$, K_b is an M-set if and only if (9.43) holds. This can be found in Chapter 12, §11 of [Z]. See also [KS], for further results.

There has been much work on subsets of $\mathbb{T}^1$ that are U-sets and are not closed. W. H. Young showed that every countable subset of $\mathbb{T}^1$ is a U-set. If $u \in \mathcal{V}(\mathbb{T}^1)$, the set of points where its Fourier series does not converge to 0 is a Borel set, and if it is not countable it must contain a perfect set. A recent survey on the issue of U-sets and M-sets, emphasizing modern descriptive set theory, is given in [K].

10. Rajchman's global uniqueness theorem and localizations

Here we establish a version of A. Rajchman's improvement of Riemann's global uniqueness theorem. It concerns Abel summability,

$$\mathcal{A}(r)u(\theta) = \sum_k \hat{u}(k) r^{|k|} e^{ik\theta} = \mathcal{S}_{\phi_y} u(\theta), \tag{10.1}$$

with

$$\varphi(s) = e^{-|s|}, \quad r = e^{-y} \in (0,1). \tag{10.2}$$

The result works for u in the following space, which is quite a bit larger than $\mathcal{V}(\mathbb{T}^1)$,

$$C^{-2}(\mathbb{T}^1) = \{u \in \mathcal{D}'(\mathbb{T}^1) : Gu \in C(\mathbb{T}^1)\}, \tag{10.3}$$

where, extenidng (4.3), we define

$$G : \mathcal{D}'(\mathbb{T}^1) \longrightarrow \mathcal{D}'(\mathbb{T}^1), \quad Gu = \sum_{k \neq 0} k^{-2} \hat{u}(k) e^{ik\theta}. \tag{10.4}$$

Equivalently, $C^{-2}(\mathbb{T}^1) = \{u = v'' + c : v \in C(\mathbb{T}^1),\, c \in \mathbb{C}\}$. Here is the result.

Theorem 10.1. *Let $u \in C^{-2}(\mathbb{T}^1)$ and assume*

$$\lim_{r \nearrow 1} \mathcal{A}(r)u(\theta) = 0, \quad \forall\, \theta \in \mathbb{T}^1. \tag{10.5}$$

Then $u = 0$.

To see how close to sharp Theorem 10.1 is, note that the hypothesis (10.5) holds for $u = \delta_0' \in \mathcal{D}'(\mathbb{T}^1)$, but $G\delta_0'$ has a jump discontinuity, so the hypothesis $u \in C^{-2}(\mathbb{T}^1)$ barely fails. In the classical version presented in Chapter 9 of [Z], the hypothesis on u is

$$\hat{u}(k) = o(k), \tag{10.6}$$

which is somewhat similar to our hypothesis $u \in C^{-2}(\mathbb{T}^1)$, though neither hypothesis implies the other. Note that (10.6) also barely fails for $u = \delta_0'$.

Our proof of Theorem 10.1 is adapted from the treatment given in Chapter 9 of [Z], with some simplifications resulting from our use of y rather than r as the governing variable. We start with the following variant of Lemma 7.6 in Chapter 9 of [Z].

Lemma 10.2. *Let $u \in C^{-2}(\mathbb{T}^1)$ be real valued, and assume $\hat{u}(0) = 0$. Consider*

$$F(\theta) = -Gu(\theta) + C, \tag{10.7}$$

where C is a constant. Set

$$v(y,\theta) = \mathcal{A}(e^{-y})u(\theta), \quad y > 0. \tag{10.8}$$

Let

$$
\underline{\mathcal{D}}^2 F(\theta) = \liminf_{\varepsilon \to 0} \Delta_\varepsilon^2 F(\theta), \qquad (10.9)
$$
$$
\overline{\mathcal{D}}^2 F(\theta) = \limsup_{\varepsilon \to 0} \Delta_\varepsilon^2 F(\theta).
$$

Then, for each $\theta \in \mathbb{T}^1$,

$$
\underline{\mathcal{D}}^2 F(\theta) \le v^*(\theta) = \limsup_{y \searrow 0} v(y, \theta), \quad \textit{and} \qquad (10.10)
$$
$$
\overline{\mathcal{D}}^2 F(\theta) \ge v_*(\theta) = \liminf_{y \searrow 0} v(y, \theta).
$$

Proof. It suffices to prove the first part of (10.10). Also, it suffices to treat $\theta = 0$. Since $\Delta_\varepsilon^2 F(0)$ and $v(y,0)$ are unchanged if $F(\theta)$ is replaced by $F(-\theta)$, we can furthermore just treat the case $F(\theta) = F(-\theta)$. We can also assume $F(0) = 0$.

If our desired conclusion in (10.10) fails, there must exist $m \in \mathbb{R}$ such that $\underline{\mathcal{D}}^2 F(0) > m$ but $v^*(0) < m$. Replacing $F(\theta)$ by $F(\theta) + m\cos\theta$ and u by $u + m\cos\theta$, we can assume $m = 0$, so

$$
\underline{\mathcal{D}}^2 F(0) > 0, \quad \text{but} \quad v^*(0) < 0. \qquad (10.11)
$$

Our task is to show that (10.11) is impossible. Let us set

$$
G(y,\theta) = \mathcal{A}(e^{-y}) F(\theta), \quad g(y) = G(y, 0). \qquad (10.12)
$$

Note that

$$
\frac{\partial^2 G}{\partial \theta^2} + \frac{\partial^2 G}{\partial y^2} = 0, \qquad (10.13)
$$

and, since $\partial_\theta^2 F = u$,

$$
\frac{\partial^2}{\partial \theta^2} G(y,\theta) = \mathcal{A}(e^{-y}) u(\theta) = v(y,\theta), \quad \text{hence} \qquad (10.14)
$$
$$
v(y,\theta) = -\frac{\partial^2}{\partial y^2} G(y,\theta).
$$

Thus, if (10.11) holds, then there exist $\delta > 0$ and $\alpha > 0$ such that

$$
\Delta_\varepsilon^2 F(0) \ge \delta > 0, \quad \forall\, |\varepsilon| \le \alpha, \quad \text{and} \qquad (10.15)
$$
$$
g''(y) \ge \delta > 0, \quad \forall\, y \in (0, \alpha).
$$

We want to show this is impossible.

Note that $F(0) = 0 \Rightarrow g(0) = 0$. It follows from the mean value theorem that, for each $y > 0$,

$$
\frac{g(y)}{y} = g'(\rho), \quad \text{for some} \quad \rho \in (0, y), \qquad (10.16)
$$

and, similarly, $g(\rho)/\rho = g'(\sigma)$ for some $\sigma \in (0,\rho)$, so

$$\frac{g(y)}{y} - \frac{g(\rho)}{\rho} = g'(\rho) - g'(\sigma). \tag{10.17}$$

Now, for $y \in (0,\alpha)$, $g''(y) \geq \delta > 0$ implies $g'(y) \nearrow$, hence $g'(\rho) - g'(\sigma) > 0$. To get the desired contradiction of (10.15), we will show that

$$\frac{d}{dy}\frac{g(y)}{y} \leq -\varepsilon < 0, \quad \text{for } y > 0 \text{ small}, \tag{10.18}$$

which will contradict the positivity of (10.17), for $0 < \rho < y$ sufficiently small.

If we regard F as a function on $\mathbb{R}$ that is periodic of period 2π, we can write

$$G(y,\theta) = \int_{-\infty}^{\infty} F(\varphi)P(y,\theta-\varphi)\,d\varphi, \tag{10.19}$$

with

$$P(y,\theta) = \frac{1}{\pi}\frac{y}{y^2+\theta^2}, \tag{10.20}$$

hence

$$\frac{g(y)}{y} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{F(\theta)}{y^2+\theta^2}\,d\theta, \tag{10.21}$$

so

$$\frac{d}{dy}\frac{g(y)}{y} = -\frac{2}{\pi}\int_{-\infty}^{\infty} F(\theta)\frac{y}{(y^2+\theta^2)^2}\,d\theta. \tag{10.22}$$

We see that, for each $\eta > 0$,

$$\lim_{y\to 0} \int\limits_{|y|>\eta} |F(\theta)|\frac{y}{(y^2+\theta^2)^2}\,d\theta = 0. \tag{10.23}$$

On the other hand, with $\eta = \alpha$ as in (10.15),

$$2F(\theta) \geq \delta\theta^2, \quad \text{for } |\theta| \leq \eta, \tag{10.24}$$

so

$$\begin{aligned} \int_{-\eta}^{\eta} F(\theta)\frac{y}{(y^2+\theta^2)^2}\,d\theta &\geq \delta\int_{-\eta}^{\eta}\frac{y\theta^2}{(y^2+\theta^2)^2}\,d\theta \\ &= \delta\int_{-\eta/y}^{\eta/y} x^2(1+x^2)^{-2}\,dx, \end{aligned} \tag{10.25}$$

which is positive and bounded away from 0 as $y \searrow 0$. This establishes (10.18) and hence completes the proof of Lemma 10.2.

The next ingredient in the proof of Theorem 10.1 is the following extension of Lemma 9.2.

Lemma 10.3. *Given an interval* $[a,b] \subset \mathbb{R}$ *and real valued* $F \in C([a,b])$, *if*

$$\underline{\mathcal{D}}^2 F(\theta) \le 0 \le \overline{\mathcal{D}}^2 F(\theta), \quad \forall\, \theta \in (a,b), \tag{10.26}$$

then F *is linear on* $[a,b]$.

Proof. As before, we can assume $[a,b] = [-1,1]$ and $F(-1) = F(1) = 0$. Then we want to show that (10.26) implies $F \equiv 0$ on $[-1,1]$. Again, it suffices to show that if $F_\delta(\theta) = F(\theta) + \delta(1-\theta^2)$, then (9.10) and (9.11) hold. Now (10.26) implies that, for all $\theta \in (-1,1)$,

$$\underline{\mathcal{D}}^2 F_\delta(\theta) \le -2\delta \le \overline{\mathcal{D}}^2 F_\delta(\theta), \quad \text{and} \quad F_\delta(-1) = F_\delta(1) = 0. \tag{10.27}$$

If (9.10) fails, there exist $\delta > 0$ and $\theta_0 \in (-1,1)$ such that $F_\delta(\theta_0) < 0$ and $F_\delta(\theta_0)$ is minimal. But, for such θ_0, $\Delta_\varepsilon^2 F_\delta(\theta_0) \ge 0$ for all sufficiently small $\varepsilon > 0$, contradicting $\underline{\mathcal{D}}^2 F_\delta(\theta_0) \le -2\delta < 0$. A similar argument gives (9.11), and we have the lemma.

Proof of Theorem 10.1. It suffices to work with real u. If $a = \hat{u}(0)$, (10.5) implies $\mathcal{A}(r)(u(\theta) - a) \to -a$ as $r \nearrow 1$, for each $\theta \in \mathbb{T}^1$. Then Lemma 10.2, applied to $u - a$, gives

$$\underline{\mathcal{D}}^2 F(\theta) \le -a \le \overline{\mathcal{D}}^2 F(\theta), \quad \forall\, \theta \in \mathbb{T}^1. \tag{10.28}$$

Then Lemma 10.3, applied to $F(\theta) + (a/2)\theta^2$, yields

$$Gu(\theta) = \frac{a}{2}\theta^2 + b\theta + c, \quad \text{for} \quad -2\pi \le \theta \le 2\pi, \tag{10.29}$$

for certain constants b and c. That Gu is continuous and periodic of period 2π yields $a = b = 0$. In particular, $\hat{u}(0) = 0$. Now the formula (10.4) implies Gu has mean value 0, so $c = 0$. Hence $Gu = 0$, so $u = 0$.

We have the following localization of Theorem 10.1.

Proposition 10.4. *Let* $\mathcal{O} \subset \mathbb{T}^1$ *be open, with complement* K. *Assume* $u \in C^{-2}(\mathbb{T}^1)$ *and*

$$\lim_{r \nearrow 1} \mathcal{A}(r)u(\theta) = 0, \quad \forall\, \theta \in \mathcal{O}. \tag{10.30}$$

Then

$$\operatorname{supp} u \subset K, \tag{10.31}$$

and (10.30) holds locally uniformly on $\mathcal{O}$.

Proof. It suffices to treat the case where u is real. Let $I \subset \mathcal{O}$ be an interval. As in the proof of Theorem 10.1, there exists $a \in \mathbb{R}$ such that $F = -Gu$ satisfies $\underline{\mathcal{D}}^2 F(\theta) \le -a \le \overline{\mathcal{D}}^2 F(\theta)$, for all $\theta \in I$, and hence Gu has the form (10.29) for $\theta \in I$. In particular, Gu is *smooth* on I, hence on $\mathcal{O}$, and consequently u itself is smooth on $\mathcal{O}$. The structure of the Poisson kernel implies that, as $r \nearrow 1$, $\mathcal{A}(r)u$

converges locally uniformly to u on $\mathcal{O}$. By (10.30), this limit must be 0, and (10.31) follows.

Expanding notions of M-sets and U-sets from §9, we make the following:

Definition. Let $\mathfrak{X}$ be a linear subspace of $C^{-2}(\mathbb{T}^1)$. We say a set $\Sigma \subset \mathbb{T}^1$ is an $(M, \mathcal{A}, \mathfrak{X})$-set if there is a nonzero u such that

$$u \in \mathfrak{X}, \quad \mathcal{A}(r)u(\theta) \to 0, \quad \forall\, \theta \in \mathbb{T}^1 \setminus \Sigma, \tag{10.32}$$

and we say Σ is a $(U, \mathcal{A}, \mathfrak{X})$-set otherwise, i.e., when (10.32) implies $u = 0$.

As the example $u = \delta_0$ shows, there are no non-empty $(U, \mathcal{A}, C^{-2}(\mathbb{T}^1))$-sets. We do however have the following counterpart of Proposition 9.8.

Proposition 10.5. *Let $K \subset \mathbb{T}^1$ be closed, and let $\mathfrak{X}$ be a linear subspace of $C^{-2}(\mathbb{T}^1)$. Then K is an $(M, \mathcal{A}, \mathfrak{X})$-set if and only if*

$$\text{there is a nonzero } u \in \mathfrak{X} \text{ such that } \operatorname{supp} u \subset K. \tag{10.33}$$

This leads to the following counterpart of Proposition 9.6.

Corollary 10.6. *Let $\mathfrak{X}$ be a linear subspace of $C^{-2}(\mathbb{T}^1)$ with the following property:*

$$\text{if } u \in \mathfrak{X}, \text{ then } \operatorname{supp} u \text{ contains no isolated points.} \tag{10.34}$$

Let $K \subset \mathbb{T}^1$ be closed. If K is an $(M, \mathcal{A}, \mathfrak{X})$-set, then there exists $K_0 \subset K$ such that

$$K_0 \text{ is a perfect set and an } (M, \mathcal{A}, \mathfrak{X})\text{-set.} \tag{10.35}$$

An example of a linear subspace of $C^{-2}(\mathbb{T}^1)$ satisfying (10.34) is $C^{-1}(\mathbb{T}^1)$, where, for a positive integer k, we set

$$C^{-k}(\mathbb{T}^1) = \{v^{(k)} + c : v \in C(\mathbb{T}^1),\, c \in \mathbb{C}\}. \tag{10.36}$$

We mention that an inductive argument establishes that, for each $k \in \mathbb{N}$,

$$f \in C^k(\mathbb{T}^1),\ u \in C^{-k}(\mathbb{T}^1) \Longrightarrow fu \in C^{-k}(\mathbb{T}^1). \tag{10.37}$$

If $u \in C^{-1}(\mathbb{T}^1)$ and if $p \in \mathbb{T}^1$ were an isolated point in $\operatorname{supp} u$, then we could pick a cut-off $f \in C^1(\mathbb{T}^1)$ such that $\operatorname{supp} fu = \{p\}$. Then fu would have to be a finite linear combination of δ_p and its derivatives, and such a distribution cannot belong to $C^{-1}(\mathbb{T}^1)$.

Note that if μ is a finite (complex) measure on $\mathbb{T}^1$, then $\mu \in C^{-1}(\mathbb{T}^1)$ if and only if μ has no atoms. Hence $C^{-1}(\mathbb{T}^1) \cap \mathcal{M}(\mathbb{T}^1)$ is somewhat larger than $\mathcal{V}(\mathbb{T}^1) \cap \mathcal{M}(\mathbb{T}^1)$, considered in §7. In connection with this, we mention the following result.

Lemma 10.7. *If $K_0 \subset \mathbb{T}^1$ is a perfect set, then there is a nonzero measure μ, with no atoms, supported in K_0.*

This has the following application.

Corollary 10.8. *Let $\mathfrak{X}$ be a linear subspace of $C^{-2}(\mathbb{T}^1)$ satisfying (10.34) and such that*

$$\mathfrak{X} \text{ contains all finite measures with no atoms.} \tag{10.38}$$

Let $K \subset \mathbb{T}^1$ be closed. Then K is a $(U, \mathcal{A}, \mathfrak{X})$-set if and only if K is countable.

In particular, this result holds for

$$\mathfrak{X} = C^{-1}(\mathbb{T}^1). \tag{10.39}$$

Proof. Every uncountable compact subset of $\mathbb{T}^1$ contains a perfect set.

Here is another perspective on the atoms of a measure μ on $\mathbb{T}^1$, due to N. Wiener. We have, for $N \geq 1$,

$$\begin{aligned} \frac{1}{N} \sum_{|k| \leq N} |\hat{\mu}(k)|^2 &= \frac{1}{4\pi^2 N} \sum_{|k| \leq N} \int e^{-i\theta k}\, d\mu(\theta) \int e^{i\varphi k}\, \overline{d\mu(\varphi)} \\ &= \frac{1}{4\pi^2 N} \iint D_N(\varphi - \theta)\, d\mu(\theta)\, \overline{d\mu(\varphi)}. \end{aligned} \tag{10.40}$$

As in (2.2),

$$\frac{1}{N} D_N(\varphi) = \frac{1}{N} \sum_{|k| \leq N} e^{ik\varphi} = \frac{1}{N} \frac{\sin(N + 1/2)\varphi}{\sin \varphi/2}. \tag{10.41}$$

This is uniformly bounded in absolute value and tends to 0 as $N \to \infty$, for $\varphi \neq 0$, while tending to 2 as $N \to \infty$ for $\varphi = 0$. Hence the dominated convergence theorem gives

$$\begin{aligned} \lim_{N \to \infty} \frac{1}{N} \sum_{|k| \leq N} |\hat{\mu}(k)|^2 &= \frac{1}{2\pi^2} \iint_{\varphi = \theta} d\mu(\theta)\, \overline{d\mu(\varphi)} \\ &= \frac{1}{2\pi^2} \sum_{\theta} |\mu(\{\theta\})|^2, \end{aligned} \tag{10.42}$$

i.e., the limit of the left side of (10.42) is equal to the sum of the squares of the point masses of μ (times $1/2\pi^2$). Consequently a measure μ on $\mathbb{T}^1$ belongs to $C^{-1}(\mathbb{T}^1)$ if and only if

$$\lim_{N \to \infty} \frac{1}{N} \sum_{|k| \leq N} |\hat{\mu}(k)|^2 = 0. \tag{10.43}$$

This tempts us to define the following space of distributions, which is larger than $\mathcal{V}(\mathbb{T}^1)$:

Definition. Given $u \in \mathcal{D}'(\mathbb{T}^1)$, we say $u \in \mathcal{W}(\mathbb{T}^1)$ provided

$$\lim_{N \to \infty} \frac{1}{N} \sum_{|k| \leq N} |\hat{u}(k)|^2 = 0. \tag{10.44}$$

The space $\mathcal{W}(\mathbb{T}^1)$ is a Banach space, with norm defined by

$$\|u\|_{\mathcal{W}}^2 = \sup_N p_N(u)^2, \quad p_N(u)^2 = \frac{1}{N}\sum_{|k|\le N} |\hat{u}(k)|^2. \tag{10.45}$$

Note that, for $N \ge 1$,

$$p_N(e^{i\ell\theta}u)^2 \le \frac{N+|\ell|}{N} p_{N+|\ell|}(u)^2, \tag{10.46}$$

so

$$\|e^{i\ell\theta}u\|_{\mathcal{W}}^2 \le (|\ell|+1)\|u\|_{\mathcal{W}}^2, \quad \text{and} \;\; |\hat{u}(\ell)| \le \|e^{i\ell\theta}u\|_{\mathcal{W}}. \tag{10.47}$$

It follows that

$$\mathcal{W}(\mathbb{T}^1) \subset C^{-2}(\mathbb{T}^1), \tag{10.48}$$

and

$$u \in \mathcal{W}(\mathbb{T}^1), \;\; \sum_\ell (|\ell|+1)^{1/2}|\hat{f}(\ell)| < \infty \Longrightarrow fu \in \mathcal{W}(\mathbb{T}^1). \tag{10.49}$$

From here, the same argument that established (10.34) for $\mathfrak{X} = C^{-1}(\mathbb{T}^1)$ gives the following.

Proposition 10.9. *If $u \in \mathcal{W}(\mathbb{T}^1)$, then* supp u *contains no isolated points. Consequently (by Corollary 10.8) a closed $K \subset \mathbb{T}^1$ is a $(U, \mathcal{A}, \mathcal{W}(\mathbb{T}^1))$-set if and only if K is countable.*

We see that the $(U, \mathcal{A}, \mathfrak{X})$-sets assciated to $\mathfrak{X} = C^{-1}(\mathbb{T}^1)$ and $\mathfrak{X} = \mathcal{W}(\mathbb{T}^1)$ have a simple structure, as opposed to the rich variety that occurs when $\mathfrak{X} = \mathcal{V}(\mathbb{T}^1)$. One might seek more general spaces

$$\mathfrak{X} \subset \mathcal{W}(\mathbb{T}^1) \tag{10.50}$$

for which the $(U, \mathcal{A}, \mathfrak{X})$-sets have a richer structure than in Corollary 10.8 and Proposition 10.9. A place to start would be to identify some spaces $\mathfrak{X}$ that satisfy (10.50) but not (10.38).

To get started, we recall the measures $\mu_{1/\ell}$, supported on the Cantor sets $K_{1/\ell}$, for $\ell \in \mathbb{N}, \; \ell \ge 3$, considered in (7.16)–(7.27), and shown not to belong to $\mathcal{V}(\mathbb{T}^1)$. Further analysis of (7.20)–(7.27) shows that

$$\frac{1}{N}\sum_{N \le |k| \le 2N} |\hat{\mu}_{1/\ell}(k)|^2 \approx CN^{-\gamma(\ell)}, \quad \gamma(\ell) = \frac{\log 2}{\log \ell}, \tag{10.51}$$

in the sense that one has two-sided bounds. This leads us to bring in the Besov spaces

$$B^s_{2,\infty}(\mathbb{T}^1) = \Big\{u \in \mathcal{D}'(\mathbb{T}^1) : \sum_{N \le |k| \le 2N} |\hat{u}(k)|^2 \le CN^{-2s}\Big\} \tag{10.52}$$

or, for short,

$$\mathcal{W}_\gamma(\mathbb{T}^1) = B^{(\gamma-1)/2}_{2,\infty}(\mathbb{T}^1), \tag{10.53}$$

so that

$$u \in \mathcal{W}_\gamma(\mathbb{T}^1) \Longleftrightarrow \frac{1}{N} \sum_{N\le|k|\le 2N} |\hat{u}(k)|^2 \le CN^{-\gamma}, \tag{10.54}$$

given $u \in \mathcal{D}'(\mathbb{T}^1)$. The space $\mathcal{W}_0(\mathbb{T}^1)$ contains $\mathcal{W}(\mathbb{T}^1)$, but $\mathcal{W}_\gamma(\mathbb{T}^1) \subset \mathcal{W}(\mathbb{T}^1)$ for $\gamma > 0$. Note that, for $\gamma > 0$,

$$u \in \mathcal{W}_\gamma(\mathbb{T}^1) \Longleftrightarrow \Lambda^{\gamma/2} u \in \mathcal{W}_0(\mathbb{T}^1) = B^{-1/2}_{2,\infty}(\mathbb{T}^1), \tag{10.55}$$

where

$$\Lambda^{\gamma/2} : \mathcal{D}'(\mathbb{T}^1) \to \mathcal{D}'(\mathbb{T}^1), \quad \Lambda^{\gamma/2} u = \sum_k (1+|k|)^{\gamma/2} \hat{u}(k) e^{ik\theta}. \tag{10.56}$$

By (10.51), for $\ell \in \mathbb{N},\ \ell \ge 3$,

$$\mu_{1/\ell} \in \mathcal{W}_\gamma(\mathbb{T}^1) \Longleftrightarrow \gamma < \frac{\log 2}{\log \ell}. \tag{10.57}$$

Thus, for $\gamma \in (0,1)$, there are some measures $\mu_{1/\ell}$ that belong to $\mathcal{W}_\gamma(\mathbb{T}^1)$ and some that do not. By (10.57),

$$\gamma < \frac{\log 2}{\log \ell} \Rightarrow K_{1/\ell} \text{ is an } (M, \mathcal{A}, \mathcal{W}_\gamma(\mathbb{T}^1))\text{-set}. \tag{10.58}$$

On the other hand, we are tempted to make the following:

Conjecture 10.10. *Given an integer $\ell \ge 3$ and $\gamma \in (0,1)$, the compact set $K_{1/\ell} \subset \mathbb{T}^1$ is a $(U, \mathcal{A}, \mathcal{W}_\gamma(\mathbb{T}^1))$-set provided*

$$\gamma > \frac{\log 2}{\log \ell}. \tag{10.59}$$

In connection with (10.51)–(10.59), let us note that

$$\gamma > 0 \Longrightarrow B^{(\gamma-1)/2}_{2,\infty}(\mathbb{T}^1) \subset H^{-1/2}(\mathbb{T}^1), \tag{10.60}$$

where $H^\sigma(\mathbb{T}^1)$ denotes the L^1-Sobolev space, defined by

$$u \in H^\sigma(\mathbb{T}^1) \Longleftrightarrow \sum_k (1+|k|)^{2\sigma} |\hat{u}(k)|^2 < \infty. \tag{10.61}$$

Hence, given $\gamma > 0$, each compact $(M, \mathcal{A}, \mathcal{W}_\gamma(\mathbb{T}^1))$-set is an $(M, \mathcal{A}, H^{-1/2}(\mathbb{T}^1))$-set, and equivalently each compact $(U, \mathcal{A}, H^{-1/2}(\mathbb{T}^1))$-set is a $(U, \mathcal{A}, \mathcal{W}_\gamma(\mathbb{T}^1))$-set. By Proposition 10.5, a compact $K \subset \mathbb{T}^1$ is a $(U, \mathcal{A}, H^{-1/2}(\mathbb{T}^1))$-set if and only if

$$K \text{ supports no nontrivial elements of } H^{-1/2}(\mathbb{T}^1). \tag{10.62}$$

There is a family of capacities, Cap_σ, associated to the operators $\Lambda^{-\sigma}$, for $\sigma \in (0,1/2]$, and one has, for compact $K \subset \mathbb{T}^1$,

$$(10.62) \iff \mathrm{Cap}_{1/2}(K) = 0. \tag{10.63}$$

See [D], p. 311, particularly the Lemma and subsequent comments, complemented by the Theorem on p. 305.

Using the readily established fact that $\mathrm{Cap}_{1/2}(\{p\}) = 0$ for each point $p \in \mathbb{T}^1$, and standard general properties of capacities, which imply that $\mathrm{Cap}_{1/2}(K_j) \to 0$ when a sequence of compact sets K_j shrinks to p, and the countable subadditivity of the set function $\mathrm{Cap}_{1/2}$, one can readily construct "ultra-thin" Cantor sets K^b with the property that

$$\mathrm{Cap}_{1/2}(K^b) = 0. \tag{10.64}$$

Note that these sets are uncountable. In view of (10.63), such sets satisfy (10.62), and hence are $(U, \mathcal{A}, H^{-1/2}(\mathbb{T}^1))$-sets.

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