Random Fields: Stationarity, Ergodicity, and Spectral Behavior

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1. Introduction and definitions

A random field Z on n-dimensional Euclidean space $\mathbb{R}^n$ or the lattice $\mathbb{Z}^n$ (also called a random function, or a stochastic process indexed by $\mathbb{R}^n$ or $\mathbb{Z}^n$) assigns to each $x \in \mathbb{R}^n$ (resp., $\mathbb{Z}^n$) a random variable $Z(x)$ on some probability space (Ω, μ) (where μ is a probability measure on the set Ω). For definiteness, let us say

$$Z : \mathbb{F}^n \longrightarrow L^2(\Omega, \mu), \tag{1.1}$$

where $L^2(\Omega, \mu)$ denotes the space of square-integrable functions (random variables with finite first and second moments) on Ω. Here and below, $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{Z}$. We assume the random variables $Z(x)$ are real valued.

We use Z to assign a probability measure ν on the set $\mathcal{O} = \mathbb{R}^{\mathbb{F}^n}$ of all functions from $\mathbb{F}^n$ to $\mathbb{R}$, as follows. First, Z gives rise to a map

$$F : \Omega \longrightarrow \mathcal{O}, \tag{1.2}$$

defined as follows. If $\xi \in \Omega$, $F(\xi) \in \mathcal{O}$ is a function on $\mathbb{F}^n$ whose value at $x \in \mathbb{F}^n$ is $Z(x)(\xi)$, i.e.,

$$F(\xi)(x) = Z(x)(\xi), \quad \xi \in \Omega,\ x \in \mathbb{F}^n. \tag{1.3}$$

(Recall that $Z(x)$ is a function on Ω, for each $x \in \mathbb{F}^n$.) Then ν is defined by

$$\nu(S) = \mu(F^{-1}(S)), \tag{1.4}$$

when $S \subset \mathcal{O}$ is a measurable set. The probability measure ν incorporates the joint probability distributions of the random variables $Z(x)$, as x runs over $\mathbb{F}^n$, as we indicate below. Another way to write (1.4) is as

$$\int_{\mathcal{O}} \varphi(\eta)\, d\nu(\eta) = \int_{\Omega} \varphi(F(\xi))\, d\mu(\xi). \tag{1.5}$$

Let us consider some special cases. Pick $x_1, x_2 \in \mathbb{F}^n$ and set

$$\varphi_1(\eta) = \eta(x_1), \quad \varphi_2(\eta) = \eta(x_1)\eta(x_2). \tag{1.6}$$

Then

$$\varphi_1(F(\xi)) = Z(x_1)(\xi), \quad \varphi_2(F(\xi)) = Z(x_1)(\xi)\, Z(x_2)(\xi), \tag{1.7}$$

so

$$\int_{\mathcal{O}} \varphi_1 \, d\nu = \int_{\Omega} Z(x_1)(\xi) \, d\mu(\xi) = \langle Z(x_1) \rangle, \tag{1.8}$$

and

$$\int_{\mathcal{O}} \varphi_2 \, d\nu = \int_{\Omega} Z(x_1)(\xi) \, Z(x_2)(\xi) \, d\mu(\xi) = \langle Z(x_1) Z(x_2) \rangle. \tag{1.9}$$

We see that (1.8) is the mean of the random variable $Z(x_1)$. The quantity (1.9) together with the means of $Z(x_1)$ and of $Z(x_2)$ are ingredients in the formula for the covariance of $Z(x_1)$ and $Z(x_2)$.

In further preparation for defining the concepts of stationarity and ergodicity, we bring in the action of $\mathbb{F}^n$ on $\mathcal{O}$,

$$\tau_y : \mathcal{O} \longrightarrow \mathcal{O}, \quad y \in \mathbb{F}^n, \tag{1.10}$$

defined as follows. If $y \in \mathbb{F}^n$ and $\eta \in \mathcal{O}$ (so η is a function, $\eta : \mathbb{F}^n \to \mathbb{R}$), $\tau_y \eta \in \mathcal{O}$ is given by

$$\tau_y \eta(x) = \eta(x + y), \quad x, y \in \mathbb{F}^n. \tag{1.11}$$

Definition 1.1. The random field Z is *stationary* provided τ_y preserves the probability measure ν, for each $y \in \mathbb{F}^n$. Equivalently, if $\varphi \in L^1(\mathcal{O}, \nu)$,

$$\int_{\mathcal{O}} \varphi(\tau_y \eta) \, d\nu(\eta) = \int_{\mathcal{O}} \varphi(\eta) \, d\nu(\eta), \quad \forall \, y \in \mathbb{F}^n. \tag{1.12}$$

An alternative label for such a field Z is *homogeneous.*

If φ_1 and φ_2 are defined as in (1.6), then

$$\varphi_1(\tau_y \eta) = \eta(x_1 + y), \quad \text{and} \quad \varphi_2(\tau_y \eta) = \eta(x_1 + y)\eta(x_2 + y), \tag{1.13}$$

so, parallel to (1.8)–(1.9), we have

$$\begin{aligned} \int_{\mathcal{O}} \varphi_1(\tau_y \eta) \, d\nu(\eta) &= \int_{\Omega} Z(x_1 + y)(\xi) \, d\mu(\xi) \\ &= \langle Z(x_1 + y) \rangle, \end{aligned} \tag{1.14}$$

and

$$\begin{aligned} \int_{\mathcal{O}} \varphi_2(\tau_y \eta) \, d\nu(\eta) &= \int_{\Omega} Z(x_1 + y)(\xi) \, Z(x_2 + y)(\xi) \, d\mu(\xi) \\ &= \langle Z(x_1 + y) Z(x_2 + y) \rangle, \end{aligned} \tag{1.15}$$

so stationarity implies

$$\langle Z(x_1)\rangle = \langle Z(x_1+y)\rangle, \quad \langle Z(x_1)Z(x_2)\rangle = \langle Z(x_1+y)Z(x_2+y)\rangle, \tag{1.16}$$

for each $x_1, x_2, y \in \mathbb{F}^n$.

Definition 1.2. The action $\{\tau_y : y \in \mathbb{F}^n\}$ on $(\mathcal{O}, \nu)$ is *ergodic* provided it preserves the measure ν and the following holds. If $\varphi \in L^1(\mathcal{O}, \nu)$ and

$$\varphi \circ \tau_y = \varphi \quad \text{in} \quad L^1(\mathcal{O}, \nu), \quad \forall\, y \in \mathbb{F}^n, \tag{1.17}$$

then φ must be constant (ν-a.e.).

Definition 1.3. Assume Z is a stationary random field. Then Z is ergodic if and only if the action $\{\tau_y : y \in \mathbb{F}^n\}$ on $(\mathcal{O}, \nu)$ is ergodic.

It is useful to introduce the following auxiliary random field, namely

$$\mathcal{Z} : \mathbb{F}^n \longrightarrow L^2(\mathcal{O}, \nu), \tag{1.18}$$

given by

$$\mathcal{Z}(x)(\eta) = \eta(x), \quad x \in \mathbb{F}^n, \quad \eta \in \mathcal{O}. \tag{1.19}$$

By (1.3),

$$\mathcal{Z}(x)(F(\xi)) = Z(x)(\xi), \quad x \in \mathbb{F}^n, \ \xi \in \Omega. \tag{1.20}$$

The process $\mathcal{Z}$ has the same joint distributions as Z. In fact, given $x_1, \dots, x_k \in \mathbb{F}^n$ and suitable $\psi : \mathbb{R}^k \to \mathbb{R}$, we have

$$\begin{aligned} &\int_{\mathcal{O}} \psi(\mathcal{Z}(x_1), \dots, \mathcal{Z}(x_k))\, d\nu \\ &\quad = \int_{\Omega} \psi\big(\mathcal{Z}(x_1)(F(\xi)), \dots, \mathcal{Z}(x_k)(F(\xi))\big)\, d\mu(\xi) \\ &\quad = \int_{\Omega} \psi(Z(x_1), \dots, Z(x_k))\, d\mu, \end{aligned} \tag{1.21}$$

the first identity by (1.5) and the second by (1.20). It follows that the construction described in the first paragraph yields again the same space $(\mathcal{O}, \nu)$. In particular, if Z is stationary and ergodic, so is $\mathcal{Z}$.

The following sequence of identities will prove to be valuable:

$$\mathcal{Z}(x)(\tau_y\eta) = (\tau_y\eta)(x) = \eta(x+y) = \mathcal{Z}(x+y)(\eta), \tag{1.22}$$

valid for $x, y \in \mathbb{F}^n,\ \eta \in \mathcal{O}$.

The rest of this note is structured as follows. In §2 we relate spatial averages and ensemble averages of quantities associated to a random field, particularly means and covariances, when the field is ergodic. In §3 we discuss stationary Gaussian fields, and in §4 we give a criterion, involving the behavior of the covariance, that such fields are ergodic. We note that stationary Gaussian fields with covariances given by (3.24), (3.25), (3.26), (3.27), or (when $n > 1$) by (3.33) are ergodic, while those with covariance given by (3.32) are not ergodic.

In §5 we consider stationary random fields on Lie groups, and in §6 we consider stationary random fields on homogeneous spaces. In §§5–6, we focus not on ergodicity but on spectra. In §7 we consider the inverse problem of constructing a random field on a compact homogeneous space, given spectral data.

In §8 we take a finite-dimensional vector space V and discuss V-valued random fields, defined first on a homogeneous space X, though we specialize to $X = \mathbb{R}^n$, with special attention to $V = \mathbb{R}^n$, i.e., to random vector fields. In §9 we discuss random divergence-free vector fields on $\mathbb{R}^n$.

In §10 we discuss generalized random fields on $\mathbb{R}^n$, which are distributions on $\mathbb{R}^n$ with values in $L^2(\Omega, \mu)$. We define stationary generalized random fields and develop some of their properties.

We have three appendices. Appendix A gives background on ergodic theorems, and Appendix B relates the criterion on the covariance function given in §4 to the behavior of its Fourier transform. Appendix C discusses the Fourier transform of a continuous, stationary field, first on $\mathbb{T}^n$ (obtaining a special case of results of §5) and then on $\mathbb{R}^n$, where we need to regard $\widehat{Z}$ as a vector-valued tempered distribution.

2. Implications of the ergodic theorem

The significance of the property of ergodicity, defined in §1, arises from the following result, known as the ergodic theorem. As before, $\mathbb{F}$ stands for $\mathbb{R}$ or $\mathbb{Z}$.

Theorem 2.1. *Let $\{\tau_y : y \in \mathbb{F}^n\}$ consist of measure preserving maps on the probability space $(\mathcal{O}, \nu)$, satisfying $\tau_{y_1+y_2} = \tau_{y_1}\tau_{y_2}$, for $y_1, y_2 \in \mathbb{F}^n$. Assume the action is ergodic. Take $\varphi \in L^1(\mathcal{O}, \nu)$.*

(A) If $\mathbb{F} = \mathbb{Z}$, then

$$\lim_{R\to\infty} \frac{1}{V(R)} \sum_{y\in\mathbb{Z}^n\cap B_R} \varphi(\tau_y \eta) = \int_{\mathcal{O}} \varphi \, d\nu, \tag{2.1}$$

for ν-almost every $\eta \in \mathcal{O}$.

(B) If $\mathbb{F} = \mathbb{R}$, and if the action of τ_y on $L^1(\mathcal{O}, \nu)$ is strongly continuous in y, then

$$\lim_{R\to\infty} \frac{1}{V(R)} \int_{B_R} \varphi(\tau_y \eta) \, dy = \int_{\mathcal{O}} \varphi \, d\nu, \tag{2.2}$$

for ν-almost every $\eta \in \mathcal{O}$.

Here, $B_R = \{y \in \mathbb{R}^n : |y| \le R\}$ is a ball and $V(R)$ is its volume (which is a good approximation to the number of points in $\mathbb{Z}^n \cap B_R$). The left sides of (2.1) and (2.2) are spatial averages, and the right sides are ensemble averages.

We apply Theorem 2.1 to results discussed in §1. First, take $\mathbb{F} = \mathbb{Z}$, so $Z : \mathbb{Z}^n \to L^2(\Omega, \mu)$ is a random field on the discrete lattice $\mathbb{Z}^n$. We construct $(\mathcal{O}, \nu)$ and $\tau_y : \mathcal{O} \to \mathcal{O}$ as in §1. If Z is stationary and ergodic, then (2.1) holds, for ν-almost every $\eta \in \mathcal{O}$. If φ_1 and $\varphi_2 \in L^1(\mathcal{O}, \nu)$ are defined as in (1.6), then (1.8)–(1.9) and (1.13), in concert with (2.1), give, for each $x_1, x_2 \in \mathbb{Z}^n$,

$$\langle Z(x_1) \rangle = \lim_{R\to\infty} \frac{1}{V(R)} \sum_{y\in\mathbb{Z}^n\cap B_R} \eta(x_1 + y), \tag{2.3}$$

and

$$\langle Z(x_1)Z(x_2) \rangle = \lim_{R\to\infty} \frac{1}{V(R)} \sum_{y\in\mathbb{Z}^n\cap B_R} \eta(x_1 + y)\eta(x_2 + y), \tag{2.4}$$

for ν-almost every $\eta \in \mathcal{O}$.

In case $\mathbb{F} = \mathbb{R}$, matters are not so simple, because the requirement that $\varphi \circ \tau_y \in L^1(\mathcal{O}, \nu)$ be continuous in y for $\varphi \in L^1(\mathcal{O}, \nu)$ can fail to hold. If this continuity did hold it would apply to φ_1 and φ_2, given by (1.6). In such a case, (2.2) would yield, for each $x_1, x_2 \in \mathbb{R}^n$,

$$\langle Z(x_1) \rangle = \lim_{R\to\infty} \frac{1}{V(R)} \int_{B_R} \eta(x_1 + y)\, dy, \tag{2.5}$$

and

$$\langle Z(x_1)Z(x_2) \rangle = \lim_{R\to\infty} \frac{1}{V(R)} \int_{B_R} \eta(x_1 + y)\eta(x_2 + y)\, dy, \tag{2.6}$$

for ν-almost every $\eta \in \mathcal{O}$, provided the random field Z is stationary and ergodic.

Suppose for example that the random variables $Z(x)$ are identically distributed and *independent*, as x runs over $\mathbb{R}^n$, and that the distribution of $Z(0)$ is not concentrated at a single point. Then ν is a product measure on $\mathcal{O}$, an *uncountable* product measure. With φ_1 as above, we have

$$\|\varphi_1 \circ \tau_y - \varphi_1\|_{L^1(\mathcal{O},\nu)} = \int_{\mathcal{O}} |\eta(x_1 + y) - \eta(x_1)|\, d\nu(\eta) \tag{2.7}$$

equal to 0 for $y = 0$, and to a nonzero constant *independent of* y if $y \neq 0$. It follows that $\varphi_1 \circ \tau_y$ is an everywhere discontinuous function of y, with values in $L^1(\mathcal{O}, \nu)$. Furthermore, we expect that, for ν-almost every $\eta \in \mathcal{O}$, the function $y \mapsto \eta(y)$ is not Lebesgue measurable, so the right sides of (2.5) and (2.6) are not well defined.

On the other hand, for many important random fields on $\mathbb{R}^n$, matters are more tractable.

Proposition 2.2. *If $Z : \mathbb{R}^n \to L^2(\Omega, \mu)$ is stationary and continuous, then the action of $\{\tau_y : y \in \mathbb{R}^n\}$ on $L^1(\mathcal{O}, \nu)$ is strongly continuous.*

Proof. Since $\{\tau_y\}$ is a group of isometries of $L^1(\mathcal{O}, \nu)$, it suffices to show that $y \mapsto \varphi \circ \tau_y$ is continuous from $\mathbb{R}^n$ to $L^1(\mathcal{O}, \nu)$, for φ in a dense subspace of $L^1(\mathcal{O}, \nu)$. We consider functions φ of the form

$$\varphi(\eta) = \psi(\eta(x_1), \dots, \eta(x_k)), \tag{2.8}$$

where $x_1, \dots, x_k \in \mathbb{R}^n$ and $\psi : \mathbb{R}^k \to \mathbb{R}$ is globally Lipschitz.

We have

$$\begin{aligned} &|\varphi \circ \tau_y(\eta) - \varphi(\eta)| \\ &= |\psi(\eta(x_1 + y), \dots, \eta(x_k + y)) - \psi(\eta(x_1), \dots, \eta(x_k))| \\ &\le C \sum_{j=1}^{k} |\eta(x_j + y) - \eta(x_j)|. \end{aligned} \tag{2.9}$$

Hence

$$\begin{aligned}\|\varphi\circ\tau_y-\varphi\|_{L^1(\mathcal{O},\nu)} &\le C\sum_j\int_{\mathcal{O}}|\eta(x_j+y)-\eta(x_j)|\,d\nu(\eta)\\ &= C\sum_j\int_{\mathcal{O}}|\mathcal{Z}(x_j+y)-\mathcal{Z}(x_j)|\,d\nu\\ &= C\sum_j\int_{\Omega}|Z(x_j+y)-Z(x_j)|\,d\mu,\end{aligned}\tag{2.10}$$

the first identity by (1.22) and the second by (1.21). The last line is bounded by $C\sum_j\|Z(x_j+y)-Z(x_j)\|_{L^2(\Omega,\mu)}$, so Proposition 2.2 is proven.

Note that if $Z:\mathbb{R}^n\to L^2(\Omega,\mu)$ is stationary, then

$$\begin{aligned}\|Z(x+y)-Z(x)\|_{L^2}^2 &= \|Z(x+y)\|_{L^2}^2+\|Z(x)\|_{L^2}^2-2\langle Z(x)Z(x+y)\rangle\\ &= 2\|Z(0)\|_{L^2}^2-2\langle Z(0)Z(y)\rangle,\end{aligned}\tag{2.11}$$

so Z is continuous if and only if

$$\lim_{y\to 0}\langle Z(0)Z(y)\rangle=\|Z(0)\|_{L^2}^2.\tag{2.12}$$

When dealing with $\mathbb{F}=\mathbb{R}$, we will henceforth assume Z is continuous. However, we note that [R] emphasizes the importance of such discontinuous examples as described above to stochastic hydrogeology. This might point to some mathematical problems that need further study.

3. Stationary Gaussian fields

A random field $Z : \mathbb{F}^n \to L^2(\Omega, \mu)$ is said to be a Gaussian field if the following holds. For each $x_j \in \mathbb{F}^n,\ a_j \in \mathbb{R},\ k \in \mathbb{N}$,

$$\sum_{j=1}^{k} a_j Z(x_j) \ \text{ is a Gaussian random variable.} \tag{3.1}$$

The following important property is special to Gaussian fields.

Proposition 3.1. *If Z is a Gaussian field, then Z is stationary provided*

$$\langle Z(x)\rangle = \langle Z(0)\rangle, \quad \text{and} \ \ \langle Z(x)Z(x+y)\rangle = \langle Z(0), Z(y)\rangle, \quad \forall\, x, y, \in \mathbb{F}^n. \tag{3.2}$$

The proof uses the Gaussian property to obtain that, for each $k \geq 1$,

$$\langle Z(x_1+y)\cdots Z(x_k+y)\rangle \ \text{ is independent of } \ y \in \mathbb{F}^n,\ \forall\, x_1,\dots,x_k \in \mathbb{F}^n. \tag{3.3}$$

This follows from the fact that the data

$$\{\langle Z(x_1)\rangle, \langle Z(x_1)Z(x_2)\rangle : x_1, x_2 \in \mathbb{F}^n\} \tag{3.4}$$

uniquely determine the data

$$\{\langle Z(x_1)\cdots Z(x_k)\rangle : x_j \in \mathbb{F}^n,\ k \in \mathbb{N}\}, \tag{3.5}$$

under the hypothesis (3.1). In fact, the data (3.4) determine the data

$$\{\langle e^{i\sum \lambda_j Z(x_j)}\rangle : x_j \in \mathbb{F}^n,\ \lambda_j \in \mathbb{R}\}, \tag{3.6}$$

which in turn determine (3.5). See (3.11A) below for more on (3.6).

The fact that (3.3) implies the stationarity asserted in Proposition 3.1 is a special case of the following.

Lemma 3.2. *Let $Z : \mathbb{F}^n \to \cap_{p<\infty} L^p(\Omega, \mu)$. If (3.3) holds for each $k \in \mathbb{N}$, then Z is stationary.*

Sketch of proof. For a k-tuple $\overline{x} = (x_1, \dots, x_k)$, define $\varphi_{\overline{x}} \in L^1(\mathcal{O}, \nu)$ by

$$\varphi_{\overline{x}}(\eta) = \eta(x_1)\dots\eta(x_k). \tag{3.7}$$

Then, via (1.21), (3.3) implies

$$\int_{\mathcal{O}} \varphi_{\overline{x}} \circ \tau_y \, d\nu = \int_{\mathcal{O}} \varphi_{\overline{x}} \, d\nu, \quad \forall\, y \in \mathbb{F}^n. \tag{3.8}$$

Now one can show that, in this situation, the set of functions of the form (3.7) has dense linear span in $L^1(\mathcal{O}, \nu)$. This implies the desired stationarity.

We next consider existence of Gaussian fields with given first and second moments. The following is proven in [D], p. 72.

Theorem 3.3. *Let* $M : \mathbb{F}^n \to \mathbb{R}$ *and* $R : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{R}$. *Assume*

$$R(x,y) = R(y,x), \tag{3.9}$$

and, for all $k \geq 1$, $x_1, \dots, x_k \in \mathbb{F}^n$, *and* $a_1, \dots, a_k \in \mathbb{C}$,

$$\sum_{i,j} R(x_i, x_j) a_i \bar{a}_j \geq 0. \tag{3.10}$$

Then there exists a Gaussian field $Z : \mathbb{F}^n \to L^2(\Omega, \mu)$ *such that, for all* $x_1, x_2 \in \mathbb{F}^n$,

$$\langle Z(x_1) \rangle = M(x_1), \quad \langle (Z(x_1) - M(x_1))(Z(x_2) - M(x_2)) \rangle = R(x_1, x_2). \tag{3.11}$$

We refer to [D] for the proof, but remark that one ingredient is the formula

$$\Big\langle e^{i \sum \lambda_j Z(x_j)} \Big\rangle = \mathrm{Exp}\Big\{ -\frac{1}{2} \sum_{j,k} R(x_j, x_k) \lambda_j \lambda_k + i \sum_j M(x_j) \lambda_j \Big\}, \tag{3.11A}$$

given $x_1, \dots, x_\ell \in \mathbb{F}^n$, $\lambda_1, \dots, \lambda_\ell \in \mathbb{R}$, and $\ell \geq 1$.

Remark. The conditions (3.9)–(3.10) are necessary, as well as sufficient, for the existence of such a field Z.

In concert with Proposition 3.1, Theorem 3.3 yields the following.

Corollary 3.4. *Let* $M \in \mathbb{R}$ *and let* $C : \mathbb{F}^n \to \mathbb{R}$ *satisfy*

$$C(x) = C(-x), \tag{3.12}$$

and, for each $k \geq 1$, $x_1, \dots, x_k \in \mathbb{F}^n$, *and* $a_1, \dots, a_k \in \mathbb{C}$,

$$\sum_{i,j} C(x_i - x_j) a_i \bar{a}_j \geq 0. \tag{3.13}$$

Then there exists a stationary Gaussian field $Z : \mathbb{F}^n \to L^2(\Omega, \mu)$ *such that, for each* $x_1, x_2 \in \mathbb{F}^n$,

$$\langle Z(x_1) \rangle = M, \tag{3.14}$$

and

$$\langle (Z(x_1) - M)(Z(x_2) - M)) \rangle = C(x_1 - x_2). \tag{3.15}$$

Remark. Given (3.14), the condition (3.15) is equivalent to

$$\langle Z(x_1)Z(x_2)\rangle - M^2 = C(x_1 - x_2).$$

Also, (2.11) implies

$$\frac{1}{2}\|Z(x+y) - Z(x)\|_{L^2}^2 = C(0) - C(y). \tag{3.16}$$

One example of a function $C : \mathbb{R}^n \to \mathbb{R}$ satisfying (3.12)–(3.13) is

$$\begin{aligned} C(x) = 1 \quad &\text{if} \quad x = 0, \\ 0 \quad &\text{if} \quad x \neq 0, \end{aligned} \tag{3.17}$$

which yields a special case of the class of discontinuous random fields discussed in the paragraph following (2.6). If $C : \mathbb{R}^n \to \mathbb{R}$ is continuous and satisfies (3.12)–(3.13), then the stationary Gaussian field Z arising in Corollary 3.4 is continuous, by (3.16).

The search for continuous functions $C : \mathbb{R}^n \to \mathbb{R}$ satisfying (3.12)–(3.13) is aided by the Fourier transform, as we now discuss. (Note that, for such C, the restriction to $\mathbb{Z}^n$ also satisfies (3.12)–(3.13).) The Fourier transform of a function $F \in L^1(\mathbb{R}^n)$ is given by

$$\widehat{F}(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} F(\xi) e^{-ix\cdot\xi} \, d\xi. \tag{3.18}$$

In such a case, $C = \widehat{F}$ is continuous on $\mathbb{R}^n$. If F is even (i.e., $F(\xi) = F(-\xi)$) and real valued, so is $C = \widehat{F}$, so (3.12) holds. Also,

$$\sum_{j,k} \widehat{F}(x_j - x_k) a_j \overline{a}_k = (2\pi)^{-n/2} \int_{\mathbb{R}^n} F(\xi) B(\xi) \, d\xi, \tag{3.19}$$

where

$$\begin{aligned} B(\xi) &= \sum_{j,k} a_j \overline{a}_k e^{-i(x_j - x_k)\cdot\xi} \\ &= \Big|\sum_j a_j e^{-ix_j\cdot\xi}\Big|^2 \\ &\geq 0, \end{aligned} \tag{3.20}$$

so we have the following.

Proposition 3.5. *Let $F \in L^1(\mathbb{R}^n)$ be even and real valued. If*

$$F(\xi) \geq 0, \quad \forall\, \xi \in \mathbb{R}^n, \tag{3.21}$$

then $C(x) = \widehat{F}(x)$ is a continuous function satisfying (3.12)–(3.13).

If C is also integrable, the Fourier inversion formula gives $\tilde{C}(\xi) = F(\xi)$, where

$$\tilde{C}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} C(x) e^{ix\cdot\xi}\, dx. \tag{3.22}$$

If C is even, then $\tilde{C} = \widehat{C}$, so we have the following.

Corollary 3.6. *Assume $C : \mathbb{R}^n \to \mathbb{R}$ is even, continuous, and integrable. Then (3.13) holds provided*

$$\widehat{C}(\xi) \geq 0, \quad \forall\, \xi \in \mathbb{R}^n. \tag{3.23}$$

Remark. If C is real, even, continuous, and integrable, (3.23) is known to be necessary, as well as sufficient, for the validity of (3.13). When C satisfies all these conditions, including (3.23), it can be shown that $\widehat{C} \in L^1(\mathbb{R}^n)$. In fact, $\|\widehat{C}\|_{L^1} = 2^{n/2} C(0)$.

Here are some examples to which Corollary 3.6 applies.

$$C(x) = e^{-|x|^2/2} \Longrightarrow \widehat{C}(\xi) = e^{-|\xi|^2/2}, \tag{3.24}$$

$$C(x) = e^{-|x|} \Longrightarrow \widehat{C}(\xi) = c_n(|\xi|^2 + 1)^{-(n+1)/2}, \tag{3.25}$$

where $c_n = 2^{n/2}\pi^{-1/2}\Gamma((n+1)/2)$. These calculations can be found in many places, e.g., Chapter 3 of [T]. Note that applying the Fourier inversion formula to (3.25) gives

$$C(x) = (|x|^2 + 1)^{-(n+1)/2} \Longrightarrow \widehat{C}(\xi) = c_n^{-1} e^{-|\xi|}. \tag{3.26}$$

Here is an example where $C(x)$ is not ≥ 0 everywhere. Let $\chi_B(\xi) = 1$ for $|\xi| \leq 1,\ 0$ for $|\xi| > 1$. Then

$$F(\xi) = \chi_B(\xi) \Longrightarrow C(x) = \widehat{F}(x) = c_n \frac{J_{n/2}(|x|)}{|x|^{n/2}}, \tag{3.27}$$

where c_n are positive constants, and J_ν is the Bessel function of order ν.

Variations on these examples can be obtained by linear changes of variables. If $b_j > 0$, then

$$C_b(x) = C(b_1 x_1, \dots, b_n x_n) \Longrightarrow \widehat{C}_b(\xi) = (b_1 \cdots b_n)^{-1} \widehat{C}(b_1^{-1}\xi_1, \dots, b_n^{-1}\xi_n). \tag{3.28}$$

More generally, if T is an $n \times n$ real matrix and $\det T \neq 0$,

$$C_T(x) = C(Tx) \Longrightarrow \widehat{C}_T(\xi) = |\det T|^{-1} \widehat{C}(T^{-1}\xi). \tag{3.29}$$

The following extension of Proposition 3.5 yields more general covariance functions for stationary Gaussian fields.

Proposition 3.7. *If σ is a finite, positive measure on $\mathbb{R}^n$, invariant under $x \mapsto -x$, then*

$$C(x) = \widehat{\sigma}(x) = (2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} e^{-ix\cdot\xi}\, d\sigma(\xi) \tag{3.30}$$

is a continuous function satisfying (3.12)–(3.13).

The proof is a slight variant of that of Proposition 3.5. In place of (3.19), we have

$$\sum_{j,k} C(x_j - x_k) a_j \overline{a}_k = (2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} B(\xi)\, d\sigma(\xi), \tag{3.31}$$

with $B(\xi)$ as in (3.20).

The Bochner-Herglotz theorem implies that, conversely, if C is a continuous function satisfying (3.12)–(3.13), then there exists a finite, positive measure σ such that $\widehat{\sigma} = C$.

If $p \in \mathbb{R}^n \setminus 0$ and δ_p is the point mass concentrated at p, then

$$\sigma = \delta_p + \delta_{-p} \Longrightarrow C(x) = \widehat{\sigma}(x) = \Big(\frac{2}{\pi}\Big)^{1/2} \cos(p \cdot x). \tag{3.32}$$

If $\delta(|\xi| - 1)$ denotes the surface measure of the unit sphere $S^{n-1} = \{\xi \in \mathbb{R}^n : |\xi| = 1\}$, then

$$\sigma = \delta(|\xi| - 1) \Longrightarrow C(x) = \widehat{\sigma}(x) = |x|^{1-n/2} J_{n/2-1}(|x|), \tag{3.33}$$

where, as in (3.27), J_ν is the Bessel function of order ν. The cases $n = 1$ and $n = 3$ yield

$$|x|^{1/2} J_{-1/2}(|x|) = \Big(\frac{2}{\pi}\Big)^{1/2} \cos|x|, \quad |x|^{-1/2} J_{1/2}(|x|) = \Big(\frac{2}{\pi}\Big)^{1/2} \frac{\sin|x|}{|x|}. \tag{3.34}$$

Of course, the $n = 1$ case agrees with (3.32), with $p = 1$.

4. Ergodic Gaussian fields

Here we discuss conditions under which a stationary Gaussian field $Z : \mathbb{F}^n \to L^2(\Omega,\mu)$ is ergodic, i.e., the action $\{\tau_y\}$ on $(\mathcal{O},\nu)$ is ergodic. The first result gives a condition that implies this action is *mixing*, i.e.,

$$\lim_{|y|\to\infty} \langle \varphi\circ\tau_y\,\psi\rangle = \langle\varphi\rangle\,\langle\psi\rangle, \quad \forall\,\varphi,\psi\in L^2(\mathcal{O},\nu). \tag{4.1}$$

This condition implies ergodicity. Compare (4.10) below, and see Appendix A for further discussion.

Proposition 4.1. *Let $Z : \mathbb{F}^n \to L^2(\mathcal{O},\nu)$ be a stationary Gaussian field, with mean $\langle Z(x)\rangle = M$ and covariance*

$$C(y) = \langle Z(x)Z(x+y)\rangle - M^2. \tag{4.2}$$

If $\mathbb{F}=\mathbb{R}$, assume $C:\mathbb{F}^n\to\mathbb{R}$ is continuous. If

$$\lim_{|y|\to\infty} C(y) = 0, \tag{4.3}$$

then this field is mixing, i.e., (4.1) holds.

For the proof, there is no loss of generality to assume $M=0$ (and it simplifies some formulas). Also, it suffices to show that (4.1) holds for φ,ψ in some dense subspace of $L^2(\mathcal{O},\nu)$. We prove it for φ and ψ of the form

$$\varphi(\eta) = f(\eta(x_1),\dots,\eta(x_\ell)), \quad \psi(\eta) = g(\eta(x_1),\dots,\eta(x_\ell)), \tag{4.3A}$$

where $\ell\in\mathbb{N}$, $x_1,\dots,x_\ell\in\mathbb{F}^n$, and $f,g\in\mathcal{S}(\mathbb{R}^\ell)$. Applying the Fourier inversion formula to f and g, we get

$$\varphi\circ\tau_y(\eta) = \int \widehat{f}(v)e^{i\sum\eta(x_j+y)v_j}\,dv, \tag{4.4}$$

and a similar formula for $\psi(\eta)$, hence

$$\langle\varphi\circ\tau_y\,\psi\rangle = \iint \widehat{f}(v)\widehat{g}(w)\Bigl\langle e^{i\sum\eta(x_j+y)v_j}e^{i\sum\eta(x_j)w_j}\Bigr\rangle\,dv\,dw. \tag{4.5}$$

Now (3.11A), with $R(x_j,x_k)=C(x_j-x_k)$ amd $M(x_j)\equiv 0$, yields

$$\Bigl\langle e^{i\sum\eta(x_j)\lambda_j}\Bigr\rangle = \mathrm{Exp}\Bigl\{-\frac{1}{2}\sum_{j,k} C(x_j-x_k)\lambda_j\lambda_k\Bigr\}, \tag{4.6}$$

hence

$$
\begin{aligned}
(4.7)\qquad &\Big\langle e^{i\sum \eta(x_j+y)v_j} e^{i\sum \eta(x_j)w_j}\Big\rangle \\
&= e^{-\sum C(x_j-x_k)v_jw_k/2} e^{-\sum C(x_j-x_k)w_jw_k/2} \\
&\quad \times e^{-\sum C(x_j+y-x_k)v_jw_k}.
\end{aligned}
$$

In this setting, $x_1, \dots, x_\ell \in \mathbb{F}^n$ are fixed. The hypothesis (4.3) implies that, as $|y| \to \infty$, the last factor on the right side of (4.7) tends to 1, so

$$
\begin{aligned}
(4.8)\qquad &\Big\langle e^{i\sum \eta(x_j+y)v_j} e^{i\sum \eta(x_j)w_j}\Big\rangle \\
&\quad \longrightarrow \Big\langle e^{i\sum \eta(x_j)v_j}\Big\rangle \Big\langle e^{i\sum \eta(x_j)w_j}\Big\rangle,
\end{aligned}
$$

pointwise in $v, w \in \mathbb{R}^\ell$. These quantities are ≤ 1 in absolute value, so the dominated convergence theorem applies to (4.5), giving

$$
\begin{aligned}
(4.9)\qquad &\lim_{|y|\to\infty} \langle \varphi \circ \tau_y\, \psi\rangle \\
&= \iint \widehat{f}(v)\widehat{g}(w)\Big\langle e^{i\sum \eta(x_j)v_j}\Big\rangle \Big\langle e^{i\sum \eta(x_j)w_j}\Big\rangle\, dv\, dw \\
&= \langle \varphi\rangle\, \langle \psi\rangle,
\end{aligned}
$$

completing the proof of Proposition 4.1.

We move on to more general conditions on C that imply ergodicity. We will work with $\mathbb{F} = \mathbb{R}$. The action of $\{\tau_y\}$ on $(\mathcal{O}, \nu)$ is ergodic provided

$$
(4.10)\qquad \lim_{R\to\infty} \frac{1}{V(R)} \int\limits_{|y|\le R} \langle \varphi \circ \tau_y\, \psi\rangle\, dy = \langle \varphi\rangle\, \langle \psi\rangle, \quad \forall\, \varphi, \psi \in L^2(\mathcal{O}, \nu).
$$

See Appendix A. Note that (4.1) implies (4.10). To establish (4.10), it suffices to check it for φ, ψ in a dense subspace of $L^2(\mathcal{O}, \nu)$, such as functions of the form (4.3A). Via (4.5) and the dominated convergence theorem, we see that (4.10) holds for such functions provided

$$
\begin{aligned}
(4.11)\qquad &\lim_{R\to\infty} \frac{1}{V(R)} \int\limits_{|y|\le R} \Big\langle e^{i\sum \eta(x_j+y)v_j)} e^{i\sum \eta(x_j)w_j}\Big\rangle\, dy \\
&\qquad = \Big\langle e^{i\sum \eta(x_j)v_j}\Big\rangle \Big\langle e^{i\sum \eta(x_j)w_j}\Big\rangle,
\end{aligned}
$$

for each $x_1, \dots, x_\ell \in \mathbb{R}^n,\ v, w \in \mathbb{R}^\ell,\ \ell \in \mathbb{N}$. In turn, by (4.6)–(4.7), we see that (4.11) holds provided

$$
(4.12)\qquad \lim_{R\to\infty} \frac{1}{V(R)} \int\limits_{|y|\le R} e^{-\sum C(x_j+y-x_k)v_jw_k}\, dy = 1,
$$

for all such x_j, v, w. This holds provided

$$\lim_{R\to\infty} \frac{1}{V(R)} \int\limits_{|y|\le R} \operatorname{Exp}\Bigl\{-\sum_{j=1}^{\ell} \lambda_j C(x_j+y)\Bigr\}\, dy = 1, \tag{4.13}$$

for each $\ell \in \mathbb{N},\ x_1,\dots,x_\ell \in \mathbb{R}^n,\ \lambda_1,\dots,\lambda_\ell \in \mathbb{R}$. Note that

$$|C(x_j+y)| \le C(0), \quad \forall\, x_j, y \in \mathbb{R}^n. \tag{4.14}$$

so the integrand in (4.13) is bounded by $e^{C(0)\sum|\lambda_j|}$. Now, for $s \in \mathbb{R}$,

$$e^s = 1 + \rho(s), \tag{4.15}$$

with $\rho(0) = 0,\ \rho'(s) = e^s$, hence

$$|s| \le C(0)L \Longrightarrow |\rho(s)| \le |s| e^{C(0)L}. \tag{4.16}$$

We have the following.

Proposition 4.2. *Let $Z : \mathbb{R}^n \to L^2(\Omega,\mu)$ be a stationary Gaussian field, with continuous covariance. If*

$$\lim_{R\to\infty} \frac{1}{V(R)} \int\limits_{|y|\le R} |C(y)|\, dy = 0, \tag{4.17}$$

then (4.10) holds, and Z is ergodic.

We remark that (4.17) is equivalent to

$$\lim_{R\to\infty} \frac{1}{V(R)} \int\limits_{|y|\le R} |C(y)|^2\, dy = 0, \tag{4.18}$$

one implication by Cauchy's inequality and the other by the bound (4.14). In turn, (4.18) is equivalent to the assertion that the measure $\sigma = \widehat{C}$ has no atoms. See Appendix B. In such a case, we say Z has continuous spectrum. Thus we have that Z is ergodic provided it has continuous spectrum. The converse is also true. A stationary Gaussian field on $\mathbb{R}^n$ with continuous covariance is ergodic if and only if it has continuous spectrum. This was proved in [M] and [G] when $n = 1$ and in [BE] when $n > 1$.

In light of these results, we see that stationary Gaussian fields with covariances given by (3.24), (3.25), (3.26), (3.27), or (when $n > 1$) by (3.33) are ergodic (in fact, mixing), while those with covariance given by (3.32) are not ergodic.

There is a straightforward analogue of Proposition 4.2 for $\mathbb{F} = \mathbb{Z}$.

5. Stationary random fields on Lie groups

We consider a random field on a Lie group G,

$$Z : G \longrightarrow L^2(\Omega, \mu), \tag{5.1}$$

where (Ω, μ) is a probability space. We assume that the random variables $Z(x)$ are real-valued, and that Z is continuous. Parallel to (1.2), we have

$$F : \Omega \longrightarrow \mathcal{O} = \mathbb{R}^G, \tag{5.2}$$

given by

$$F(\xi)(x) = Z(x)(\xi), \quad \xi \in \Omega, \ x \in G. \tag{5.3}$$

Then we get a probability measure ν on $\mathcal{O}$:

$$\nu(S) = \mu(F^{-1}(S)), \tag{5.4}$$

so

$$\int_{\mathcal{O}} \varphi(\eta)\, d\nu(\eta) = \int_{\Omega} \varphi(F(\xi))\, d\mu(\xi). \tag{5.5}$$

Formulas parallel to (1.6)–(1.9) hold. Parallel to (1.10)–(1.11), we have a g-action on $\mathcal{O}$:

$$\tau_g : \mathcal{O} \to \mathcal{O}, \quad \tau_g \eta(x) = \eta(gx), \quad x, g \in G. \tag{5.6}$$

We say Z is stationary if this G-action preserves ν, i.e.,

$$\int_{\mathcal{O}} \varphi(\tau_g \eta)\, d\nu(\eta) = \int_{\mathcal{O}} \varphi(\eta)\, d\nu(\eta), \quad \forall\, \varphi \in L^1(\mathcal{O}, \nu), \ g \in G. \tag{5.7}$$

Parallel to (1.16), we see that stationarity implies

$$\langle Z(gx_1) \rangle = \langle Z(x_1) \rangle, \quad \langle Z(gx_1) Z(gx_2) \rangle = \langle Z(x_1) Z(x_2) \rangle, \tag{5.8}$$

for all $g, x_1, x_2 \in G$. Consequently

$$\langle Z(x) \rangle = M \tag{5.9}$$

is independent of $x \in G$ and the covariance, given by

$$R(x,y) = \langle Z(x)Z(y)\rangle - M^2, \tag{5.10}$$

satisfies

$$R(x,y) = R(y,x), \quad R(gx,gy) = R(x,y), \tag{5.11}$$

hence there exists $C : G \to \mathbb{R}$ such that

$$R(x,y) = C(x^{-1}y), \quad C(x) = C(x^{-1}), \quad x,y \in G. \tag{5.12}$$

There is a positivity condition parallel to (3.10), which translates, for stationary fields, to

$$\sum_{i,j} C(x_i^{-1}x_j)a_i\overline{a}_j \geq 0. \tag{5.13}$$

Note that if $e \in G$ is the identity element,

$$\begin{aligned} \|Z(x) - Z(y)\|_{L^2}^2 &= \|Z(e) - Z(x^{-1}y)\|_{L^2}^2 \\ &= 2C(e) - 2C(x^{-1}y), \end{aligned} \tag{5.14}$$

so the continuity of a stationary field $Z : G \to L^2(\Omega, \mu)$ is equivalent to the continuity of $C : G \to \mathbb{R}$ at e (and implies the continuity of C on G). As mentioned above, we work under this continuity hypothesis.

Given such continuity, the condition (5.13) is equivalent to

$$\int_G \int_G C(x^{-1}y)f(x)\overline{f(y)}\, dx\, dy \geq 0, \tag{5.15}$$

for all $f \in C_0^\infty(G)$, where dx denotes Haar measure on G. We henceforth assume

$$G \text{ is unimodular}, \tag{5.16}$$

so left Haar measure and right Haar measure coincide. Note that, by (5.12), we can replace $C(x^{-1}y)$ by $C(y^{-1}x)$ in (5.15). Now the *convolution* is defined by

$$f * C(x) = \int_G f(y)C(y^{-1}x)\, dy, \tag{5.17}$$

so the condition (5.15) is equivalent to

$$(f, f * C)_{L^2} \geq 0, \tag{5.18}$$

for all $f \in C_0^\infty(G)$.

From here on in this section we will assume G is *compact*, which of course implies unimodularity. The Peter-Weyl theorem yields a unitary isomorphism

$$\mathcal{F} : L^2(G) \longrightarrow \bigoplus_{\pi \in \widehat{G}} \mathrm{End}(V^\pi), \tag{5.19}$$

where $\widehat{G}$ consists of (equivalence classes of) the irreducible unitary representations of G. This is given by

$$\mathcal{F}(f)(\pi) = \pi(f) = \int_G f(x)\pi(x)\,dx, \tag{5.20}$$

with Plancherel formula

$$(f, g)_{L^2} = \sum_{\pi \in \widehat{G}} d_\pi \ \mathrm{Tr}(\pi(f)\pi(g)^*), \tag{5.21}$$

and inversion formula

$$f(x) = \sum_{\pi \in \widehat{G}} d_\pi \ \mathrm{Tr}(\pi(f)\pi(x)^*). \tag{5.22}$$

Since $\pi(f * g) = \pi(f)\pi(g)$, we get

$$(f, f * C)_{L^2} = \sum_{\pi \in \widehat{G}} d_\pi \ \mathrm{Tr}(\pi(f)\pi(C)^*\pi(f)^*), \tag{5.23}$$

and the condition (5.18) on $C \in C(G)$ becomes

$$\mathrm{Tr}(A\pi(C)^*A^*) \ge 0, \quad \forall\, \pi \in \widehat{G},\ A \in \mathrm{End}(V^\pi). \tag{5.24}$$

Note that if A is an orthogonal projection, $Av = (v, w)w,\ \|w\| = 1$, then $A\pi(C)^*A^*v = (v, w)A\pi(C)^*w = (v, w)(\pi(C)^*w, w)w$, so $\mathrm{Tr}(A\pi(X)^*A^*) = (\pi(C)^*w, w)$. Thus (5.24) implies

$$\pi(C) \ge 0, \quad \forall\, \pi \in \widehat{G}. \tag{5.25}$$

The reverse implication is also readily established.

For compact G, a stationary random field $Z : G \to L^2(\Omega, \mu)$ yields random variables

$$Z^\pi_{ij} = \int_G Z(x)\pi_{ij}(x)\,dx \in L^2(\Omega, \mu), \tag{5.26}$$

where $\pi_{ij}(x)$ are the matrix entries of $\pi(x)$, with respect to some given orthonormal basis of V^π. These entries fit together to produce

$$Z^\pi = \int_G Z(x)\pi(x)\,dx \in L^2(\Omega,\mu,\mathrm{End}(V^\pi)). \tag{5.27}$$

Let us assume

$$\langle Z(x)\rangle \equiv M = 0. \tag{5.28}$$

Then

$$\begin{aligned}
\langle Z^\pi_{ij}\overline{Z}^\pi_{k\ell}\rangle &= \int_G\int_G C(y^{-1}x)\pi_{ij}(x)\overline{\pi}_{k\ell}(y)\,dx\,dy \\
&= \int_G\int_G C(z)\pi_{ij}(yz)\overline{\pi}_{k\ell}(y)\,dz\,dy \\
&= \sum_m \int_G\int_G C(z)\pi_{im}(y)\pi_{mj}(z)\overline{\pi}_{k\ell}(y)\,dz\,dy \\
&= \frac{1}{d_\pi}\sum_m \delta_{ik}\delta_{m\ell}\int_G C(z)\pi_{mj}(z)\,dz \\
&= \frac{1}{d_\pi}\delta_{ik}\int_G C(z)\pi_{\ell j}(z)\,dz \\
&= \frac{1}{d_\pi}\delta_{ik}\big(\pi(C)\big)_{\ell j}.
\end{aligned} \tag{5.29}$$

If π and λ are distinct elements of $\widehat{G}$,

$$\langle Z^\pi_{ij}\overline{Z}^\lambda_{k\ell}\rangle \equiv 0. \tag{5.30}$$

The Peter-Weyl theorem gives

$$Z(x) = \sum_{\pi\in\widehat{G}} Z^\pi(x), \tag{5.31}$$

with

$$Z^\pi(x) = d_\pi \sum_{i,j} Z^\pi_{ij}\overline{\pi}_{ij}(x) = d_\pi\,\mathrm{Tr}(Z^\pi\pi(x)^*). \tag{5.32}$$

We have

$$\langle Z^\pi(x)\overline{Z}^\pi(y)\rangle = d_\pi^2 \sum_{i,j}\sum_{k,\ell}\langle Z^\pi_{ij}\overline{Z}^\pi_{k\ell}\rangle \overline{\pi}_{ij}(x)\pi_{k\ell}(y)$$
$$= d_\pi \sum_{i,j}\sum_{k,\ell}\delta_{ik}\big(\pi(C)\big)_{\ell j}\overline{\pi}_{ij}(x)\pi_{k\ell}(y)$$
$$= d_\pi \sum_{j,k,\ell}\big(\pi(C)\big)_{\ell j}\overline{\pi}_{kj}(x)\pi_{k\ell}(y) \tag{5.33}$$
$$= d_\pi \sum_{j,k}\big(\pi(y)\pi(C)\big)_{kj}\overline{\pi}_{kj}(x)$$
$$= d_\pi \ \mathrm{Tr}\big(\pi(x)^*\pi(y)\pi(C)\big)$$
$$= d_\pi \ \mathrm{Tr}\big(\pi(x^{-1}y)\pi(C)\big),$$

and $\langle Z^\pi(x)\overline{Z}^\lambda(y)\rangle \equiv 0$ if π and λ are distinct elements of $\widehat{G}$.

Note that, since $Z(x)$ is real valued, $\overline{Z}^\pi = Z^{\overline{\pi}}$, so

$$\langle Z^\pi_{ij}Z^\pi_{k\ell}\rangle = \begin{cases}\langle Z^\pi_{ij}\overline{Z}^\pi_{k\ell}\rangle & \text{if } \ \pi = \overline{\pi}, \\ 0 & \text{if } \ \pi \neq \overline{\pi}.\end{cases} \tag{5.34}$$

Hence

$$\langle Z^\pi(x)Z^\pi(y)\rangle = \begin{cases}\langle Z^\pi(x)\overline{Z}^\pi(y)\rangle & \text{if } \ \pi = \overline{\pi}, \\ 0 & \text{if } \ \pi \neq \overline{\pi}.\end{cases} \tag{5.35}$$

Furthermore, $\langle Z^\pi(x)Z^\lambda(y)\rangle \equiv 0$ if π and $\overline{\lambda}$ are distinct elements of $\widehat{G}$.

Remark. The condition (5.8) is the condition that the field Z is "2-weakly stationary." This condition is weaker than stationarity, but it suffices for all the results in (5.9)–(5.35).

We now give a result that follows from stationarity but not from 2-weak stationarity. First, some notation. Let Y^σ and $\widetilde{Y}^\sigma$ denote two families of elements of $L^1(\Omega,\mu)$, indexed by $\sigma \in \Sigma$. We write

$$Y^\sigma \leftrightarrow_\sigma \widetilde{Y}^\sigma \tag{5.36}$$

provided that, for arbitrary $\sigma_1,\dots,\sigma_N \in \Sigma,\ N \in \mathbb{N}$, the random variables

$$\{Y^{\sigma_1},\dots,Y^{\sigma_N}\} \ \text{ and } \ \{\widetilde{Y}^{\sigma_1},\dots,\widetilde{Y}^{\sigma_N}\} \tag{5.37}$$

have the same joint distribution. Note that a field $Z : G \to L^2(\Omega,\mu)$ is stationary if and only if

$$Z(gx) \leftrightarrow_g Z(x), \quad \forall\, x \in X. \tag{5.38}$$

Now if Z^π is defined by (5.27), then

$$\pi(g)Z^\pi = \int_G Z(x)\pi(gx)\,dx = \int_G Z(g^{-1}y)\pi(y)\,dy, \tag{5.39}$$

so (5.38) gives

$$\pi(g)Z^\pi \leftrightarrow_\pi Z^\pi, \quad \forall\, g \in G, \tag{5.40}$$

provided Z is stationary. This result does not follow from 2-weak stationarity.

6. Stationary random fields on homogeneous spaces

Let X be a Riemannian manifold with a transitive group G of isometries. If $K \subset G$ is the subgroup fixing a point $p_0 \in X$, then K is compact and $X \approx G/K$. We have

$$\gamma : G \longrightarrow X, \quad \gamma(g) = g \cdot p_0. \tag{6.1}$$

Given a continuous random field

$$Y : X \longrightarrow L^2(\Omega, \mu), \tag{6.2}$$

we have

$$Z = Y \circ \gamma : G \longrightarrow L^2(X, \mu). \tag{6.3}$$

We say Y is stationary if Z is stationary. Note that a field $Z : G \to L^2(\Omega, \mu)$ has the form (6.3) if and only if

$$Z(xk) = Z(x), \quad \forall\, x \in G,\ k \in K. \tag{6.4}$$

In such a case, $R(x, y) = \langle Z(x)Z(y)\rangle - M^2$ satisfies

$$R(xk_1, yk_2) = R(x, y), \quad \forall\, k_j \in K,\ x, y \in G, \tag{6.5}$$

so, given stationarity, with $R(x, y) = C(x^{-1}y)$, we have

$$C((xk_1)^{-1}yk_2) = C(x^{-1}y), \quad \forall\, x, y, \in G,\ k_j \in K, \tag{6.6}$$

or equivalently

$$C(k_2xk_1) = C(x), \quad \forall\, x \in G,\ k_j \in K. \tag{6.7}$$

In particular, we have a function

$$\mathcal{C} : X \longrightarrow \mathbb{R}, \quad C = \mathcal{C} \circ \gamma, \tag{6.8}$$

and

$$\mathcal{C}(kp) = \mathcal{C}(p), \quad \forall\, k \in K,\ p \in X. \tag{6.9}$$

Conversely, given a continuous $\mathcal{C} : X \to \mathbb{R}$ satisfying (6.9), $C = \mathcal{C} \circ \gamma$ satisfies (6.7).

A Riemannian manifold X might have more than one group of isometries acting transitively, so one might use the phrase "G-stationary" to be more precise (though we will not). For example, if $X = \mathbb{R}^n$, one has $G = \mathbb{R}^n$ acting by translations, and also the larger group $G = E(n)$ of rigid motions, a semidirect product of $\mathbb{R}^n$ and $SO(n)$. For an $\mathbb{R}^n$-stationary field on $X = \mathbb{R}^n$ to be $E(n)$-stationary ([MP] prefers the term "isotropic"), one needs $\mathcal{C} : \mathbb{R}^n \to \mathbb{R}$ to be radial. (And this condition would suffice for Gaussian fields.) For another example, if $X = S^{n-1}$ is the unit sphere in $\mathbb{R}^n$, one has $SO(n)$ acting transitively as a group of isometries, and if $n = 2k$, one has the subgroups $U(k)$ and $SU(k)$ also acting transitively.

For the rest of this section, we assume X is *compact*, hence G is compact. As shown in [Z], p. 80, the regular representation R of G on $L^2(X)$,

$$R(g)f(p) = f(g^{-1}p), \tag{6.10}$$

decomposes into a family of finite-dimensional representations,

$$L^2(X) = \bigoplus_{\pi \in \widehat{G}_0} L^2_\pi(X). \tag{6.11}$$

Here $\widehat{G}_0 \subset \widehat{G}$ is defined by

$$\pi \in \widehat{G}_0 \Leftrightarrow V_0^\pi = \{\varphi \in V^\pi : \pi(k)\varphi = \varphi,\ \forall\, k \in K\} \neq 0, \tag{6.12}$$

and we have isomorphisms

$$\begin{aligned} &\Psi_\pi : V_0^\pi \otimes V^\pi \longrightarrow L^2_\pi(X), \\ &\Psi_\pi(\varphi \otimes \psi)(g \cdot p_0) = (\pi(g)\varphi, \psi). \end{aligned} \tag{6.13}$$

Note that, given $h \in G,\ \varphi \in V_0^\pi,\ \psi \in V^\pi$,

$$\begin{aligned} R(h)\Psi_\pi(\varphi \otimes \psi)(g \cdot p_0) &= (\pi(h^{-1}g)\varphi, \psi) \\ &= (\pi(g)\varphi, \pi(h)\psi) \\ &= \Psi_\pi(\varphi \otimes \pi(h)\psi)(g \cdot p_0). \end{aligned} \tag{6.14}$$

In other words, for each $\varphi \in V_0^\pi$, we have

$$\Phi_\varphi : V^\pi \to L^2(X), \quad \Phi_\varphi(\psi)(g \cdot p_0) = (\pi(g)\varphi, \psi), \tag{6.15}$$

and

$$R(h)\Phi_\varphi(\psi) = \Phi_\varphi(\pi(h)\psi). \tag{6.16}$$

The case $X = S^2,\ G = SO(3)$ was emphasized in [MP]. Then (6.11) becomes

$$L^2(S^2) = \bigoplus_{j \geq 0} V_j, \tag{6.17}$$

where V_j is an eigenspace of the Laplace-Beltrami operator on S^2, of dimension $2j+1$, and $SO(3)$ acts on V_j by the representation denoted D_j. Elements of V_j are called spherical harmonics. In this case, $V_{j,0}$ is one dimensional, spanned by the zonal harmonic in V_j. It is desired to understand the behavior of the spherical harmonic expansion of the continuous function $\mathcal{C} : S^2 \to \mathbb{R}$ arising from a stationary field $Y : S^2 \to L^2(\Omega, \mu)$, via (6.2)–(6.8).

More generally, for a compact homogeneous space $X = G/K$, we want to understand the behavior of $\pi(C)$, as π runs over $\widehat{G}_0$. Results of §5 apply, of course, particularly (5.25). Further structure arises from (6.7), which implies

$$\pi(C) = \pi(k_1)\pi(C)\pi(k_2), \quad \forall\, k_j \in K,\ \pi \in \widehat{G}. \tag{6.18}$$

Note that

$$P_0 = \int_K \pi(k)\, dk \Longrightarrow P_0 : V^\pi \to V_0^\pi, \ \text{ orthogonal projection.} \tag{6.19}$$

Integrating (6.18) yields

$$\pi(C) = P_0 \pi(C) P_0. \tag{6.20}$$

Conversely, (6.20) ⇒ (6.18). Note that if (6.18) holds, then

$$C(x) = \sum_{\pi \in \widehat{G}} d_\pi \ \mathrm{Tr}\big(\pi(C)\pi(x)^*\big) \tag{6.21}$$

satisfies

$$C(k_2 x k_1) = \sum_{\pi \in \widehat{G}} d_\pi \ \mathrm{Tr}\big(\pi(C)\pi(k_1)^*\pi(x)^*\pi(k_2)^*\big) = C(x), \quad \forall\, k_j \in K. \tag{6.22}$$

We also have

$$\begin{aligned} C(x^{-1}) &= \sum_{\pi \in \widehat{G}} d_\pi \ \mathrm{Tr}\big(\pi(C)\pi(x)\big) \\ &= \sum_{\pi} d_\pi \ \mathrm{Tr}\big(\pi(x)^*\pi(C)^*\big) \\ &= C(x), \end{aligned} \tag{6.23}$$

given that C is real valued and that $\pi(C)$, as a consequence of (5.25), is self-adjoint.

As we have mentioned, if $X = S^2$, $G = SO(3)$, then

$$\dim V_0^\pi = 1, \tag{6.24}$$

for all $\pi \in \widehat{G}_0$. This holds more generally for $X = S^{n-1},\ G = SO(n)$, but it does not hold for $X = S^3,\ G = SU(2)$. When (6.24) holds, we can write (6.20) as

$$\pi(C) = \tau_\pi(C) P_0, \tag{6.25}$$

with

$$\tau_\pi(C) = \operatorname{Tr} \pi(C) = \int_G C(x)\chi_\pi(x)\, dx, \tag{6.26}$$

where $\chi_\pi(x) = \operatorname{Tr} \pi(x)$ is the character of the representation π. The positivity condition (5.25) becomes

$$\tau_\pi(C) \geq 0, \quad \forall\, \pi \in \widehat{G}_0. \tag{6.26A}$$

Recall from (5.27) the construction of

$$Z^\pi = \int_G Z(x)\pi(x)\, dx \in L^2(\Omega, \mu, \operatorname{End}(V^\pi)). \tag{6.27}$$

If (6.4) holds, then

$$Z^\pi = Z^\pi \pi(k), \quad \forall\, k \in K, \tag{6.28}$$

and integration over $k \in K$ gives

$$Z^\pi = Z^\pi P_0, \tag{6.29}$$

with P_0 as in (6.19). Conversely, (6.29) $\Rightarrow$ (6.28), which in turn implies that

$$Z(x) = \sum_{\pi \in \widehat{G}} d_\pi \ \operatorname{Tr}\big(Z^\pi \pi(x)^*\big) \tag{6.30}$$

(note that the sum is actually over $\pi \in \widehat{G}_0$) satisfies

$$Z(xk) = \sum_\pi d_\pi \ \operatorname{Tr}\big(Z^\pi \pi(k)^* \pi(x)^*\big) = Z(x), \quad \forall\, k \in K. \tag{6.31}$$

If (6.24) holds and $M = 0$, we deduce from (5.29) and (6.25) that

$$\langle Z^\pi_{ij} \overline{Z}^\pi_{k\ell} \rangle = \frac{\tau_\pi(C)}{d_\pi} \delta_{ik} (P_0)_{\ell j} \quad (\text{if } \dim V^\pi_0 = 1). \tag{6.32}$$

It would be typical to pick an orthonormal basis of V^π such that $(P_0)_{\ell j} = \delta_{\ell 0}\delta_{j0}$, so

$$\langle Z^\pi_{ij}\overline{Z}^\pi_{k\ell}\rangle = \frac{\tau_\pi(C)}{d_\pi}\delta_{\ell 0}\delta_{j0}\delta_{ik}. \tag{6.33}$$

We also have from (5.33) and (6.25) that

$$\begin{aligned}\langle Z^\pi(x)\overline{Z}^\pi(y)\rangle &= d_\pi\tau_\pi(C)\ \mathrm{Tr}(\pi(x^{-1}y)P_0)\\ &= d_\pi\tau_\pi(C)\ \mathrm{Tr}(P_0\pi(x^{-1}y)P_0).\end{aligned} \tag{6.34}$$

Note that (5.34)–(5.35) apply to $\langle Z^\pi_{ij}Z^\pi_{k\ell}\rangle$ and $\langle Z^\pi(x)Z^\pi(y)\rangle$. We mention that, for $X = S^2$, $G = SO(3)$ (more generally, for $X = S^{n-1}$, $G = SO(n)$) all the representations $\pi \in \widehat{G}_0$ are real, and $Z^\pi(x) = \overline{Z}^\pi(x)$, for all x.

Remark. Parallel to the remark following (5.35), we mention that the results (6.5)–(6.34) hold for Z as in (6.3), whenever Z is 2-weakly stationary (we then say Y is 2-weakly stationary). This condition is weaker than the assumption that Y is stationary. See the remarks at the end of §7 for more on this.

7. The inverse problem: constructing $Z(x)$ from spectral data

As in the latter part of §6, X will be a compact Riemannian manifold, G a transitive group of isometries of X, $K \subset G$ the subgroup fixing a given point $p_0 \in X$. We are given data

$$C^\pi \in \mathrm{End}(V^\pi), \quad Z^\pi \in L^2(\Omega, \mu, \mathrm{End}(V^\pi)), \tag{7.1}$$

for $\pi \in \widehat{G}_0$, defined in (6.12). We want to specify the conditions on this data that guarantee the existence of a continuous, real valued, $Y : X \to L^2(\Omega, \mu)$ such that, with $Z = Y \circ \gamma$, as in (6.3), we have

$$\begin{gathered} Z^\pi = \int_G Z(x)\pi(x)\, dx, \quad \pi(C) = C^\pi, \\ \langle Z(x) \rangle = 0, \quad \langle Z(x)Z(y) \rangle = C(x^{-1}y). \end{gathered} \tag{7.2}$$

Necessary conditions follow from results of §§5–6. Here we want to show they are sufficient. We start out in the general setting described above, but later on we will make some simplifying assumptions, which are satisfied when $X = S^{n-1}$, $G = SO(n)$, $n \geq 3$.

We begin by seeing what condition on $\{C^\pi\}$ gives rise to a continuous, positive-definite function C on G satisfying

$$C(k_2 x k_1) = C(x), \quad \forall\, x \in G,\ k_j \in K. \tag{7.3}$$

As seen in §§5–6, a necessary condition is

$$C^\pi = P_0 C^\pi P_0 \geq 0, \quad \forall\, \pi \in \widehat{G}_0, \tag{7.4}$$

where $P_0 : V^\pi \to V_0^\pi$ is the orthogonal projection. (For notational simplicity, we do not record the dependence of P_0 on π.) We now show that (7.4) is sufficient for the existence of the desired function C (given appropriate decay of C^π as $\pi \to \infty$). In fact, taking a cue from (5.22), we set

$$C(x) = \sum_{\pi \in \widehat{G}_0} d_\pi\ \mathrm{Tr}(C^\pi \pi(x)^*). \tag{7.5}$$

Given sufficient decay (cf. (7.34) below), this converges to $C \in C(G)$, and

$$\pi(C) = C^\pi, \tag{7.6}$$

for all π. Furthermore, given $k_j \in K$,

$$
\begin{aligned}
C(k_2 x k_1) &= \sum_\pi d_\pi \ \mathrm{Tr}(C^\pi \pi(k_2)^* \pi(x)^* \pi(k_1)^*) \\
&= \sum_\pi d_\pi \ \mathrm{Tr}(\pi(k_1^{-1}) C^\pi \pi(k_2^{-1}) \pi(x)^*),
\end{aligned} \tag{7.7}
$$

and (7.4) implies

$$
\pi(k_1^{-1}) C^\pi \pi(k_2^{-1}) = C^\pi, \quad \forall\, k_j \in K, \tag{7.8}
$$

so (7.3) holds. By (7.6) and the argument around (5.25), the function C in (7.5) is positive definite. Let us also note that if $C(x)$ is given by (7.5), then

$$
\begin{aligned}
C(x^{-1}) &= \sum_\pi d_\pi \ \mathrm{Tr}(C^\pi \pi(x)) \\
&= \sum_\pi d_\pi \ \overline{\mathrm{Tr}(\pi(x)^* C^\pi)} \\
&= \overline{C(x)},
\end{aligned} \tag{7.9}
$$

where we have used self-adjointness of C^π. We want $C(x)$ to be real valued, so we will impose the following restriction:

$$
\text{Each representation } \pi \in \widehat{G}_0 \text{ is real,} \tag{7.10}
$$

with respect to some orthonormal basis of V^π. As mentioned in §6, this holds when $X = S^{n-1},\ G = SO(n), n \geq 3$. We also complement (7.4) with the condition that

$$
C^\pi \ \text{ is real,} \tag{7.11}
$$

with respect to such a basis of V^π.

We move on to Z^π, with matrix entries $Z^\pi_{ij} \in L^2(\Omega, \mu)$. We set

$$
Z(x) = \sum_\pi d_\pi \ \mathrm{Tr}(Z^\pi \pi(x)^*), \tag{7.12}
$$

which yields a continuous function $Z : G \to L^2(\Omega, \mu)$, given appropriate decay of $\{Z^\pi\}$ as $\pi \to \infty$, and we have

$$
Z^\pi = \int_G Z(x) \pi(x)\, dx. \tag{7.13}
$$

Now

$$
\langle Z(x) \rangle = \sum_\pi d_\pi \ \mathrm{Tr}(\langle Z^\pi \rangle \pi(x)^*) = 0, \tag{7.14}
$$

provided

$$\langle Z^\pi_{ij}\rangle = 0, \quad \forall\, i,j,\pi. \tag{7.15}$$

As seen in §6, a necessary condition on Z^π is

$$Z^\pi = Z^\pi P_0. \tag{7.16}$$

This implies $Z^\pi = Z^\pi \pi(k)$ for all $k \in K$, hence

$$Z(xk) = \sum_\pi d_\pi \ \mathrm{Tr}(Z^\pi \pi(k)^* \pi(x)^*) = Z(x), \tag{7.17}$$

for all $x \in G,\ k \in K$.

At this point we bring in the following simplifying assumption.

$$\dim V_0^\pi = 1, \quad \forall\, \pi \in \widehat{G}_0. \tag{7.18}$$

As mentioned in in §6, this holds for $X = S^{n-1},\ G = SO(n),\ n \geq 3$. Given (7.18), (7.4) becomes

$$C^\pi = \tau_\pi P_0, \quad \tau_\pi \in [0,\infty). \tag{7.19}$$

It is common to take an orthonormal basis $\{v_j\}$ of V^π for which (7.10) holds and $V_0^\pi = \mathrm{Span}(v_0)$, so $(P_0)_{ij} = \delta_{i0}\delta_{j0}$. Then $Z^\pi_{ij} = 0$ unless $j = 0$, so

$$Z^\pi_{ij} = \zeta^\pi_i \delta_{j0}, \quad \zeta^\pi_i \in L^2(\Omega,\mu) \ \ \text{(real valued)}. \tag{7.20}$$

Finally, we need to make an appropriate hypothesis on ζ^π_i. The condition (7.15) gives

$$\langle \zeta^\pi_i \rangle = 0, \quad \forall\, i,\pi, \tag{7.21}$$

and the formula (6.33) (plus (5.30)) yields the necessary condition

$$\langle \zeta^\pi_i \zeta^\lambda_k \rangle = \frac{\tau_\pi}{d_\pi} \delta_{ik}\delta_{\pi\lambda}, \quad \pi,\lambda \in \widehat{G}_0. \tag{7.22}$$

It remains to check the covariance identity in (7.2). To break this down, we write (7.12) as

$$Z(x) = \sum_\pi Z^\pi(x), \tag{7.23}$$

with

$$(7.24)\qquad \begin{aligned} Z^\pi(z) &= d_\pi \ \mathrm{Tr}(Z^\pi \pi(x)^*) \\ &= d_\pi \sum_{i,j} Z^\pi_{ij} \overline{\pi}_{ij}(x) \\ &= d_\pi \sum_i \zeta^\pi_i \overline{\pi}_{i0}(x), \end{aligned}$$

the last identity by (7.20). From here, we get the following. (We keep the bar, but recall that in this setting $\overline{Z}^\pi(y) = Z^\pi(y)$.)

$$(7.25)\qquad \begin{aligned} \langle Z^\pi(x)\overline{Z}^\pi(y)\rangle &= d_\pi^2 \sum_{i,k} \langle \zeta^\pi_i \overline{\zeta}^\pi_k \rangle \overline{\pi}_{i0}(x)\pi_{k0}(y) \\ &= d_\pi \tau_\pi \sum_i \overline{\pi}_{i0}(x)\pi_{i0}(y) \\ &= d_\pi \tau_\pi \pi(x^{-1}y)_{00}. \end{aligned}$$

Hence (celebrating reality and dropping the bars),

$$(7.26)\qquad \begin{aligned} \langle Z(x)Z(y)\rangle &= \sum_{\pi,\lambda} \langle Z^\pi(x) Z^\lambda(y)\rangle \\ &= \sum_\pi \langle Z^\pi(x) Z^\pi(y)\rangle \\ &= \sum_\pi d_\pi \tau_\pi \pi(x^{-1}y)_{00}, \end{aligned}$$

the second identity by (7.22). Meanwhile,

$$(7.27)\qquad \begin{aligned} C(x) &= \sum_\pi d_\pi \ \mathrm{Tr}(C^\pi \pi(x)^*) \\ &= \sum_\pi d_\pi \tau_\pi \ \mathrm{Tr}(P_0 \pi(x)^*) \\ &= \sum_\pi d_\pi \tau_\pi \ \mathrm{Tr}(P_0 \pi(x) P_0) \\ &= \sum_\pi d_\pi \tau_\pi \pi(x)_{00}, \end{aligned}$$

so

$$(7.28)\qquad \langle Z(x)Z(y)\rangle = C(x^{-1}y).$$

We formulate our result.

Proposition 7.1. *Assume on $\widehat{G}_0$ that (7.10) and (7.18) hold. Take $\tau_\pi \in [0,\infty)$, decreasing sufficiently rapidly as $\pi \to \infty$, and define C^π by (7.19). Let $\zeta_j^\pi \in L^2(\Omega,\mu)$ satisfy (7.21)–(7.22), and define $Z(x)$ by (7.23)–(7.24) and $C(x)$ by (7.5). Then*

$$C : G \to \mathbb{R}, \quad Z : G \to L^2(\Omega,\mu) \tag{7.29}$$

are continuous, $Z(xk) = Z(k)$ for all $x \in G$, $k \in K$, and the identities in (7.2) hold.

Let us record just what decay is required on $\{\tau_\pi : \pi \in \widehat{G}_0\}$. We have

$$Z(x) = \sum_{\pi\in\widehat{G}_0} d_\pi \sum_i \zeta_i^\pi \overline{\pi}_{i0}(x), \tag{7.30}$$

and $\{\zeta_i^\pi\}$ consists of mutually orthogonal elements of $L^2(\Omega,\mu)$, with square norm τ_π/d_π. Hence

$$\begin{aligned} \|Z(x)\|^2_{L^2(\Omega)} &= \sum_\pi d_\pi^2 \sum_i \frac{\tau_\pi}{d_\pi} |\pi_{i0}(x)|^2 \\ &= \sum_\pi d_\pi \tau_\pi, \end{aligned} \tag{7.31}$$

since, by unitarity,

$$\sum_i |\pi_{i0}(x)|^2 \equiv 1. \tag{7.32}$$

Hence, as long as

$$\sum_{\pi\in\widehat{G}_0} d_\pi \tau_\pi < \infty, \tag{7.33}$$

the infinite series (7.30) converges uniformly on G to a continuous function with values in $L^2(\Omega,\mu)$.

Remark. The random field $Z : G \to L^2(\Omega,\mu)$ constructed in Proposition 7.1 satisfies

$$\langle Z(x)\rangle = 0, \quad \langle Z(gx)Z(gy)\rangle = \langle Z(x)Z(y)\rangle, \quad \forall\, x,y,g \in G. \tag{7.34}$$

As mentioned in §5, one says such a random field is "2-weakly stationary." If $\{\zeta_j^\pi\}$ are mutually independent Gaussian random variables satisfying (7.22), then Z is a Gaussian field, and arguments mentioned in §3 show that (7.34) implies stationarity. In the non-Gaussian case, 2-weak stationarity does not imply stationarity.

Here is a result that follows from stationarity but not from 2-weak stationarity. Namely, with respect to the orthonormal basis of V^π mentioned below (7.19), the elements $\zeta_i^\pi \in L^2(\Omega, \mu)$ introduced in (7.20) are the components of

$$\zeta^\pi \in L^2(\Omega, \mu, V^\pi). \tag{7.35}$$

Then (5.40) implies

$$\pi(g)\zeta^\pi \leftrightarrow_\pi \zeta^\pi, \quad \forall\, g \in G, \tag{7.36}$$

provided G is stationary. (See (5.36)–(5.37) for the notation used in (7.36).) It follows from (7.36) that

$$S_{\pi_1 \cdots \pi_N} = \langle \zeta^{\pi_1} \otimes \cdots \otimes \zeta^{\pi_N} \rangle \in V^{\pi_1} \otimes \cdots \otimes V^{\pi_N} \tag{7.37}$$

satisfies

$$\pi_1(g) \otimes \cdots \otimes \pi_N(g) S_{\pi_1 \cdots \pi_N} = S_{\pi_1 \cdots \pi_N}, \quad \forall\, g \in G. \tag{7.38}$$

Let us note, parenthetically, that (7.20) is equivalent to

$$Z^\pi = \zeta^\pi \otimes v_0^\pi, \quad V_0^\pi = \operatorname{Span}(v_0^\pi), \ \|v_0^\pi\| = 1, \tag{7.39}$$

that is,

$$Z^\pi v = (v, v_0^\pi)\zeta^\pi, \quad v \in V^\pi. \tag{7.40}$$

In Chapter 6 of [MP] the following result is established, in the case $X = S^2$, $G = SO(3)$. Assume $Y : S^2 \to L^2(X, \mu)$ is stationary. Take $\pi \in \widehat{G}$ and assume the elements $\zeta_i^\pi \in L^2(\Omega, \mu)$ (known to be mutually orthogonal, as i varies, by (7.22)) are actually independent. Then these random variables must be Gaussian. The proof makes use of (7.36). This analysis is extended to general compact homogeneous spaces $X = G/K$ in [BMV].

Now we can find non-Gaussian ζ_i^π, that are mutually independent and satisfy the hypotheses of Proposition 7.1. Then this proposition yields a continuous field $Z : G \to L^2(\Omega, \mu)$ that is 2-weakly stationary, but (by the result of [MP] stated above) not stationary.

Non-Gaussian stationary fields $Z : G \to L^2(\Omega, \mu)$ can be obtained from a Gaussian stationary field Z_G by taking

$$Z(x) = F(Z_G(x)), \tag{7.41}$$

where $F : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies moderate bounds. Such a class of stationary fields, called Gaussian-subordinated stationary fields, are studied in [MP] (with $X = S^2$, $G = SO(3)$).

8. V-valued random fields

To start, let X be a Riemannian manifold with a transitive group G of isometries. Let V be a finite dimensional inner product space (over $\mathbb{R}$) and π an orthogonal representation of G on V. We take a continuous function

$$Z : X \longrightarrow L^2(\Omega, \mu), \tag{8.1}$$

where (Ω, μ) is a probability space. We induce a measure ν on $\mathcal{O} = V^X$ as follows. We have a map

$$F : \Omega \longrightarrow \mathcal{O}, \quad F(\xi)(x) = Z(x)(\xi), \quad \xi \in \Omega,\ x \in X, \tag{8.2}$$

giving rise to

$$\nu(S) = \mu(F^{-1}(S)). \tag{8.3}$$

Parallel to (1.10)–(1.11) and to (5.6), we have a G-action on $\mathcal{O}$:

$$\tau_g : \mathcal{O} \longrightarrow \mathcal{O}, \quad (\tau_g \eta)(x) = \pi(g)^{-1}\eta(gx), \quad x \in X,\ g \in G,\ \eta \in \mathcal{O}. \tag{8.4}$$

With this convention, $\tau_{gh} = \tau_h \tau_g$. We say that Z is stationary (G-stationary, for clarity, when needed) provided the action $\{\tau_g\}$ preserves ν. We say Z is ergodic if in addition this action is ergodic on $(\mathcal{O}, \nu)$.

We now specialize to $X = \mathbb{R}^n$, and consider two cases of G:

$$\mathbb{R}^n, \quad E(n) = SO(n) \times_\varphi \mathbb{R}^n. \tag{8.5}$$

The group operation on $E(n)$ is given by

$$(g, x) \cdot (h, y) = (x + \varphi(g)y, gh), \quad x, y \in \mathbb{R}^n,\ g, h \in SO(n), \tag{8.6}$$

where φ is the standard action of $SO(n)$ on $\mathbb{R}^n$, i.e, $\varphi(g)y = gy$. In case $G = \mathbb{R}^n$, we take the trivial representation on V. In case $G = E(n)$, we consider representations of the form

$$\lambda(g, x)v = \pi(g)v, \tag{8.7}$$

where π is a unitary representation of $SO(n)$ on V. The resulting actions on $\mathcal{O} = V^{\mathbb{R}^n}$ are

$$\tau_y \eta(x) = \eta(x + y), \quad \tau_{(g,y)}\eta(x) = \pi(g)^{-1}\eta(\varphi(g)x + y). \tag{8.8}$$

Given a continuous $Z : \mathbb{R}^n \to L^2(\Omega, \mu, V)$, in case $G = \mathbb{R}^n$ and Z is G-stationary, we say Z is a homogeneous random field. In case $G = E(n)$ and Z is G-stationary, we say Z is an isotropic random field. A case of central importance is

$$V = \mathbb{R}^n, \quad \pi = \varphi, \tag{8.9}$$

the standard action of $SO(n)$ on $\mathbb{R}^n$. Then we say Z is a random vector field. For $G = \mathbb{R}^n$ or $E(n)$, respectively, we say a G-stationary Z is a homogeneous random vector field or, respectively, an isotropic random vector field.

Let us return to the general setting (8.1) and note that we have expectations and correlations,

$$\langle Z(x) \rangle \in V, \quad \langle Z(x) \otimes Z(y) \rangle = \mathcal{R}(x, y) \in V \otimes V, \quad x, y \in X. \tag{8.10}$$

G-stationarity implies, in the language of (5.36)–(5.37),

$$Z(gx) \leftrightarrow_g \pi(g) Z(x), \quad \forall\, x \in X. \tag{8.11}$$

Hence G-stationarity implies

$$\begin{aligned} \langle Z(gx) \rangle &= \pi(g) \langle Z(x) \rangle, \\ \mathcal{R}(gx, gy) &= (\pi(g) \otimes \pi(g)) \mathcal{R}(x, y), \end{aligned} \tag{8.12}$$

for all $x, y \in X,\ g \in G$.

The inner product on V gives rise to an isomorphism,

$$j : V \otimes V \xrightarrow{\approx} \operatorname{End}(V), \quad j(v \otimes w) u = (u, w) v. \tag{8.13}$$

We can also write $j(v \otimes w) = v w^t$, with $w^t(u) = (u, w)$. We then have

$$R(x, y) = j \mathcal{R}(x, y) \in \operatorname{End}(V), \tag{8.14}$$

for $x, y \in X$. A useful alternative notation is

$$R(x, y) = \langle Z(x) Z(y)^t \rangle. \tag{8.14A}$$

Note that, for $A \in \operatorname{End}(V)$,

$$j(\pi(g) \otimes \pi(h)) j^{-1} A = \pi(g) A \pi(h)^{-1}. \tag{8.15}$$

Hence (8.12) implies

$$R(gx, gy) = \pi(g) R(x, y) \pi(g)^{-1}, \quad \forall\, x, y \in X,\ g \in G. \tag{8.16}$$

Specializing to $X = \mathbb{R}^n,\ G = \mathbb{R}^n$, we have, for G-stationary Z,

$$\begin{aligned} \langle Z(x)\rangle &= \langle Z(y)\rangle, \quad \forall\, x, y \in \mathbb{R}^n, \\ R(x,y) &= C(x-y), \quad C : \mathbb{R}^n \to \mathrm{End}(V). \end{aligned} \tag{8.17}$$

If $G = E(n)$, we also have (8.17), and in addition

$$\begin{gathered} \langle Z(gx)\rangle = \pi(g)\langle Z(x)\rangle, \quad \text{hence} \ \ \pi(g)\langle Z(x)\rangle = \langle Z(x)\rangle, \\ C(gx) = \pi(g)C(x)\pi(g)^{-1}, \quad \forall\, x \in \mathbb{R}^n,\ g \in SO(n). \end{gathered} \tag{8.18}$$

NOTE. Our definition of $C(x-y)$ as $\langle Z(x)Z(y)\rangle$ differs slightly from that in (4.2) and (5.10), but the definitions coincide when $\langle Z(x)\rangle \equiv 0$, which is the typical situation.

Another symmetry property is the following. By (8.14A), $R(y,x) = R(x,y)^t$ (the adjoint in $\mathrm{End}(V)$), hence

$$C(-x) = C(x)^t, \quad \forall\, x \in \mathbb{R}^n. \tag{8.19}$$

We have a positivity property parallel to (3.10). Let $k \geq 1$, $x_1, \dots, x_k \in \mathbb{R}^n$, and $a_1, \dots, a_k \in \mathbb{C}$. Then

$$\begin{gathered} \sum_{i,j} R(x_i, x_j) a_i \overline{a}_j = \langle WW^*\rangle \geq 0 \ \text{ in } \ \mathrm{End}(V), \\ W = \sum_i a_i Z(x_i) \in L^2(\Omega, \mu, V). \end{gathered} \tag{8.20}$$

Hence, in the setting of (8.17),

$$\sum_{i,j} C(x_i - x_j) a_i \overline{a}_j \geq 0 \ \text{ in } \ \mathrm{End}(V). \tag{8.21}$$

Given that $C : \mathbb{R}^n \to \mathrm{End}(V)$ is continuous, (8.21) is equivalent to

$$\iint C(x-y) f(x) \overline{f(y)}\, dx\, dy \geq 0, \quad \forall\, f \in \mathcal{S}(\mathbb{R}^n), \tag{8.22}$$

and also, via Bochner-Herglotz, to

$$\widehat{C} \text{ is a (finite) positive } \mathrm{End}(V_{\mathbb{C}})\text{-valued measure on } \mathbb{R}^n, \tag{8.23}$$

given (8.19), which implies

$$\widehat{C}^* = \widehat{C} \ \text{ in } \ \mathcal{S}'(\mathbb{R}^n, \mathrm{End}(V_{\mathbb{C}})). \tag{8.24}$$

Also (8.19) implies

$$\widehat{C}(-\xi) = \widehat{C}(\xi)^t. \tag{8.25}$$

(Given $A \in \mathrm{End}(V_{\mathbb{C}}),\ A^* = \overline{A}^t$.) Note that the Fourier transform of $C_g(x) = C(gx)$ is

$$\widehat{C}_g(\xi) = (2\pi)^{-n/2} \int C(gx) e^{-ix\cdot\xi}\, dx = \widehat{C}(g\xi), \tag{8.26}$$

so (8.18) implies

$$\widehat{C}(g\xi) = \pi(g)\widehat{C}(\xi)\pi(g)^{-1}. \tag{8.27}$$

It follows from (8.18) that if $Z : \mathbb{R}^n \to L^2(\Omega, \mu, V)$ is isotropic, then the hypothesis

(8.28) π does not contain a trivial representation of $SO(n)$

implies

$$\langle Z(x)\rangle \equiv 0, \tag{8.29}$$

and the hypothesis

(8.30) π acts irreducibly on $V_{\mathbb{C}}$

implies

$$C(0) = \alpha I, \quad \alpha \in \mathbb{R}^+. \tag{8.31}$$

Also, if $\widehat{C}$ is continuous in a neighborhood of $0 \in \mathbb{R}^n$, (8.30) implies

$$\widehat{C}(0) = \beta I, \quad \beta \in \mathbb{R}^+ \tag{8.32}$$

(inclusion in $\mathbb{R}^+$ by (8.23)).

If π is the standard representation of $SO(n)$ on $\mathbb{R}^n$, then (8.28) holds whenever $n \geq 2$, and (8.30) holds whenever $n \geq 3$. The hypothesis (8.30) fails for $n = 2$, but nevertheless (8.31) continues to hold. In fact, if $n = 2$, (8.18) implies $C(0)$ must be a scalar multiple of a rotation on $\mathbb{R}^2$. Since also $C(0) = \langle Z(0)Z(0)^t\rangle \geq 0$ in $\mathrm{End}(\mathbb{R}^2)$, (8.31) follows. A similar argument applies to (8.32).

We continue to take $X = \mathbb{R}^n$, $G = E(n)$, $V = \mathbb{R}^n$, and π the standard representation of $SO(n)$ on $\mathbb{R}^n$. The result (8.18) on C implies that it is uniquely specified by $C(re_n)$, $r \in [0, \infty)$, where $\{e_1, \dots, e_n\}$ is the standard basis of $\mathbb{R}^n$.

If $SO(n-1)$ acts on $\mathbb{R}^n$, fixing e_n and taking the standard $SO(n-1)$ action on $\text{Span}(e_1, \dots, e_{n-1}) = \mathbb{R}^{n-1}$, then C is well defined on $\mathbb{R}^n \setminus 0$ if and only if

$$\pi(g)C(re_n)\pi(g)^{-1} = C(re_n), \quad \forall\, g \in SO(n-1), \tag{8.33}$$

the case of $C(0)$ having been discussed above. Now $\mathbb{C}^n$ splits into two factors, $\mathbb{C}e_n$ and $\mathbb{C}$-$\text{Span}(e_1, \dots, e_{n-1})$, on each of which $SO(n-1)$ acts irreducibly. Hence (8.33) is equivalent to

$$C(re_n) = A(r)P_{e_n} + B(r)(I - P_{e_n}), \tag{8.34}$$

where, for $x \in \mathbb{R}^n$,

$$P_x = \ \text{orthogonal projection of } \mathbb{R}^n \text{ onto } \ \text{Span}(x), \tag{8.34A}$$

and A and B are scalar. Now,

$$g \in SO(n) \Longrightarrow \pi(g)P_{e_n}\pi(g)^{-1} = P_{ge_n}, \tag{8.35}$$

so we get

$$C(x) = A(|x|)P_x + B(|x|)(I - P_x). \tag{8.36}$$

From (8.31), we have $A(0) = B(0) = \alpha$. In view of (8.27), a similar analysis holds for $\widehat{C}$. Assuming $\widehat{C}$ is continuous on $\mathbb{R}^n \setminus 0$, we have

$$\widehat{C}(\xi) = A^{\#}(|\xi|)P_\xi + B^{\#}(|\xi|)(I - P_\xi), \tag{8.37}$$

with $A^{\#}$ and $B^{\#}$ scalar. If in addition $\widehat{C}$ is continuous in a neighborhood of 0, we have $A^{\#}(0) = B^{\#}(0) = \beta$. To celebrate the positivity result (8.23), we also write

$$\widehat{C} = A^{\#}P_\xi + B^{\#}(I - P_\xi), \tag{8.38}$$

where

$$A^{\#} \text{ and } B^{\#} \text{ are finite, positive (scalar) radial measures on } \mathbb{R}^n. \tag{8.39}$$

Since P_ξ is not continuous at $\xi = 0$, we elaborate on (8.38). We have

$$\widehat{C} = A^b P_\xi + B^b(I - P_\xi) + \gamma I\delta, \tag{8.40}$$

where A^b and B^b are finite, positive, scalar, rotationally invariant measures on $\mathbb{R}^n$ with no atom at 0, and $\gamma \ge 0$.

Note that (8.38)–(8.40) imply a reality condition, sharpening (8.24) to

$$\widehat{C}^* = \widehat{C}^t = \widehat{C} \ \text{ in } \ \mathcal{S}'(\mathbb{R}^n, \text{End}(\mathbb{R}^n)), \tag{8.41}$$

in the isotropic case, and converting (8.25) to

$$\widehat{C}(-\xi) = \widehat{C}(\xi). \tag{8.42}$$

9. Random divergence-free vector fields

We take the setting of §8, with $X = \mathbb{R}^n$, $G = \mathbb{R}^n$ or $E(n)$, and $V = \mathbb{R}^n$. As usual, (Ω, μ) is a probability space. A continuous function

$$Z : \mathbb{R}^n \longrightarrow L^2(\Omega, \mu) \tag{9.1}$$

is divergence-free provided $\operatorname{div} Z = 0$, i.e., with $Z = (Z_1, \dots, Z_n)^t$,

$$\sum_j \partial_j Z_j = 0, \tag{9.2}$$

considered as an element of $\mathcal{S}'(\mathbb{R}^n, L^2(\Omega, \mu))$. Equivalently,

$$\int_{\mathbb{R}^n} Z(x) \cdot \nabla f(x) = 0, \quad \forall\, f \in C_0^\infty(\mathbb{R}^n). \tag{9.3}$$

Recall that if Z is homogeneous (i.e., $\mathbb{R}^n$-stationary) and $\langle Z(x) \rangle \equiv 0$, we have

$$C(x) = \langle Z(x) Z(0)^t \rangle \in \operatorname{End}(\mathbb{R}^n), \quad \text{i.e.,} \ \ C_{ij}(x) = \langle Z_i(x) Z_j(0) \rangle. \tag{9.4}$$

Then (9.2) implies

$$\sum_i \partial_i C_{ij} = 0, \quad \text{hence} \ \sum_j \partial_j C_{ij} = 0, \tag{9.5}$$

the latter identity following since $C_{ji}(x) = C_{ij}(-x)$.

Applying the Fourier transform (cf. Appendix C) to (9.2) gives

$$\sum_j \xi_j \widehat{Z}_j = 0 \ \text{ in } \ \mathcal{S}'(\mathbb{R}^n, L^2(\Omega, \mu)), \tag{9.6}$$

and, in case Z is homogeneous, applying the Fourier transform to (9.5) gives

$$\sum_i \xi_i \widehat{C}_{ij}(\xi) = 0, \quad \sum_j \xi_j \widehat{C}_{ij}(\xi) = 0. \tag{9.7}$$

If Z is isotropic, then (8.40), i.e.,

$$\widehat{C} = A^b P_\xi + B^b (I - P_\xi) + \gamma I \delta, \tag{9.8}$$

plus (9.7) gives

$$\widehat{C} = B^b(I - P_\xi) + \gamma I\delta, \tag{9.9}$$

where B^b is a finite, positive (scalar), rotationally invariant measure on $\mathbb{R}^n$, with no atom at 0 (hence no atoms at all), and $\gamma \ge 0$. As shown in Appendix C, mild decay conditions on $C(x)$ as $|x| \to \infty$ imply no atoms for $\widehat{C}$, hence $\gamma = 0$.

We next discuss the existence of nontrivial homogeneous (or isotropic) divergence-free vector fields. Take a continuous $C : \mathbb{R}^n \to \mathrm{End}(\mathbb{R}^n)$ satisfying $C(x) = C(-x)^t$ and $\widehat{C}$ positive, e.g., $\widehat{C}$ as in (9.8). Parallel to Theorem 3.3 and Corollary 3.4, there is (no doubt) an existence result for a Gaussian random field $Y : \mathbb{R}^n \to L^2(\Omega, \mu, \mathbb{R}^n)$ such that

$$\langle Y(x)\rangle \equiv 0, \quad \langle Y(x)Y(y)^t\rangle = C_Y(x-y), \tag{9.10}$$

and such a Gaussian field will be homogeneous ($\mathbb{R}^n$-stationary). If $C_Y(x)$ satisfies (8.18), e.g., if $\widehat{C}_Y$ is given by (9.8), then (no doubt) such a Gaussian field Y will be isotropic ($E(n)$-stationary). (Justifying this would involve establishing a generalization of Proposition 3.1.)

To get a random field satisfying (9.2), we might need to alter Y. Consider

$$Z(x) = f * Y(x) = \int f(x-y)Y(y)\,dy, \tag{9.11}$$

with

$$f \in L^1(\mathbb{R}^n, \mathrm{End}(\mathbb{R}^n)). \tag{9.12}$$

Note that

$$\widehat{Z} = \widehat{f}\,\widehat{Y}. \tag{9.12A}$$

It easily follows from (9.11) that Z is homogeneous if Y is; compare similar results in Appendix C. We next investigate when Z can be said to be isotropic, given that Y is isotropic, i.e., Y is homogeneous, and, in the terminology of (5.36)–(5.37), with g running over $SO(n)$ and π the standard action of $SO(n)$ on $\mathbb{R}^n$,

$$Y(gx) \leftrightarrow_g \pi(g)Y(x), \quad \forall\, x \in \mathbb{R}^n. \tag{9.13}$$

Note that

$$\begin{aligned} Z(gx) &= \int f(gx-y)Y(y)\,dy \\ &= \int f(g(x-y))Y(gy)\,dy \\ &\leftrightarrow_g \int f(g(x-y))\pi(g)Y(y)\,dy. \end{aligned} \tag{9.14}$$

Thus, to achieve

$$Z(gx) \leftrightarrow_g \pi(g)Z(x), \quad \forall\, x \in \mathbb{R}^n, \tag{9.19}$$

we need f to satisfy

$$f(gx) = \pi(g)f(x)\pi(g)^{-1}, \tag{9.16}$$

or equivalently

$$\widehat{f}(g\xi) = \pi(g)\widehat{f}(\xi)\pi(g)^{-1}. \tag{9.17}$$

Note that

$$P_{g\xi} = \pi(g)P_\xi\pi(g)^{-1}, \quad \forall\, g \in SO(n),\ \xi \in \mathbb{R}^n \setminus 0. \tag{9.18}$$

Hence, we take $f \in L^1(\mathbb{R}^n, \mathrm{End}(\mathbb{R}^n))$ such that

$$\widehat{f}(\xi) = a(\xi)(I - P_\xi), \tag{9.19}$$

with $a : \mathbb{R}^n \to \mathbb{R}$ radial, and sufficiently regular, and vanishing sufficiently as $\xi \to 0$ and as $|\xi| \to \infty$ to ensure that f is integrable. With such a choice of f, Z, defined by (9.11), will be divergence free. It will be homogeneous if Y is, and it will be isotropic if Y is.

We remark that if Y and Z are related by (9.11), then $\langle Y(x)\rangle \equiv 0 \Rightarrow \langle Z(x)\rangle \equiv 0$ and, with $C_Y(x-y)$ as in (9.10) and

$$C_Z(x-y) = \langle Z(x)Z(y)^t\rangle, \tag{9.20}$$

a calculation gives

$$C_Z = f * C_Y * f^\#, \quad f^\#(x) = f(-x)^t, \tag{9.21}$$

hence

$$\begin{aligned} \widehat{C}_Z &= (2\pi)^n \widehat{f}(\xi)\widehat{C}_Y\widehat{f}(-\xi)^t \\ &= (2\pi)^n \widehat{f}(\xi)\widehat{C}_Y\widehat{f}(\xi)^*. \end{aligned} \tag{9.22}$$

If $\widehat{f}$ is given by (9.19), we obtain

$$\widehat{C}_Z = (2\pi)^n a(\xi)(I - P_\xi)\widehat{C}_Y(I - P_\xi)a(\xi). \tag{9.23}$$

Recall from (8.31)–(8.32) that if Z is an isotropic random vector field on $\mathbb{R}^n$ and $\langle Z(x)\rangle \equiv 0$ and $C = C_Z$ is given by (9.20), then $C(0)$ is a scalar multiple of the identity. If $\widehat{C}$ is continuous on a neighborhood of 0, $\widehat{C}(0)$ is also a scalar multiple of the identity. We note that if Z is also divergence free, then

$$\widehat{C}(\xi)\xi = 0 \Longrightarrow \widehat{C}(0) = 0, \tag{9.24}$$

given such continuity. Indeed, fixing $\omega \in S^{n-1} \subset \mathbb{R}^n$, we have $\widehat{C}(r\omega)\omega = 0$, for all $r > 0$, and letting $r \to 0$ yields $\widehat{C}(0)\omega = 0$, for all $\omega \in S^{n-1}$. (In fact, this argument applies more generally to homogeneous, divergence-free random vector fields.)

On the other hand, there exist isotropic, divergence-free random fields on $\mathbb{R}^n$ for which $\widehat{C}$ is continuous on $\mathbb{R}^n \setminus 0$ and does not tend to 0 at the origin. Examples can be obtained as follows. Set

$$\widehat{C}_Y(\xi) = |\xi|^{-a} e^{-|\xi|} I, \quad a \in (0, n), \tag{9.25}$$

which is positive and integrable. Then there exists a homogeneous Gaussian random vector field Y such that $\langle Y(x)\rangle \equiv 0$ and (9.10) holds. Then form Z as in (9.11), with

$$\widehat{f}(\xi) = |\xi|^{a/2} e^{-|\xi|} (I - P_\xi). \tag{9.26}$$

Then $f \in C^\infty(\mathbb{R}^n)$ and $|f(x)| \le C(1 + |x|)^{-n-\alpha/2}$, so $f \in L^1(\mathbb{R}^n, \mathrm{End}(\mathbb{R}^n))$. We have an isotropic, divergence-free random vector field Z, and, by (9.23),

$$\widehat{C}_Z(\xi) = (2\pi)^n e^{-3|\xi|} (I - P_\xi). \tag{9.27}$$

This is bounded, continuous on $\mathbb{R}^n \setminus 0$, and has no limit as $\xi \to 0$. If instead of (9.26) we took $\widehat{f}(\xi) = |\xi|^{b/2} e^{-|\xi|} (I - P_\xi)$, with $b \in (0, a)$, we would get an isotropic, divergence-free random vector field Z for which $\widehat{C}_Z$ blows up at the origin.

10. Generalized random fields

As before, we fix a probability space (Ω, μ). Let us explicitly assume that $L^2(\Omega, \mu)$ is separable. A generalized random field on $\mathbb{R}^n$ is an $L^2(\Omega, \mu)$-valued distribution, $Z \in \mathcal{D}'(\mathbb{R}^n, L^2(\Omega, \mu))$, i.e., a continuous linear map

$$Z : C_0^\infty(\mathbb{R}^n) \longrightarrow L^2(\Omega, \mu). \tag{10.1}$$

More generally, we can take $L^2(\Omega, \mu, V)$, as in §8, but for now we drop the V. Given $f \in C_0^\infty(\mathbb{R}^n)$, we have the convolution

$$\begin{aligned} &f * Z \in C^\infty(\mathbb{R}^n, L^2(\Omega, \mu)), \\ &f * Z(x) = Z(\check{f}_x), \quad \check{f}_x = f(x - y). \end{aligned} \tag{10.2}$$

Definition. A generalized random field Z is stationary if and only if $f * Z$ is stationary (as a continuous random field) for all $f \in C_0^\infty(\mathbb{R}^n)$.

If $Z \in \mathcal{D}'(\mathbb{R}^n, L^2(\Omega, \mu))$ is stationary, the continuous linear map

$$\begin{aligned} &K_Z : C_0^\infty(B) \longrightarrow C^\infty(\mathbb{R}^n, L^2(\Omega, \mu)), \\ &K_Z f(x) = f * Z(x) = Z(\check{f}_x), \end{aligned} \tag{10.3}$$

where $B = \{x \in \mathbb{R}^n : |x| \le 1\}$, has the property that

$$K_Z : C_0^\infty(B) \longrightarrow L^\infty(\mathbb{R}^n, L^2(\Omega, \mu)). \tag{10.4}$$

It follows that K_Z in (10.4) is a closed linear map from a Frechet space to a Banach space, hence continuous. Thus there exist $k \in \mathbb{N}$ and $C \in (0, \infty)$ (depending on Z) such that

$$\sup_x \|Z(\check{f}_x)\|_{L^2(\Omega,\mu)} \le C\|f\|_{C^k}, \quad \forall\, f \in C_0^\infty(B). \tag{10.5}$$

This estimate leads to an extension of the action of such Z, as follows. If $\mathcal{Q} = \{Q_\alpha\}$ denotes the tiling of $\mathbb{R}^n$ by n-dimensional cubes with vertices in $\mathbb{Z}^n$, we can define

$$f \in L^1C^k(\mathbb{R}^n) \Leftrightarrow f \in C^k(\mathbb{R}^n) \text{ and } \|f\|_{L^1C^k} = \sum_{Q_\alpha \in \mathcal{Q}} \|f\|_{C^k(Q_\alpha)} < \infty. \tag{10.6}$$

A partition of unity argument leads from (10.5) to

$$\|Z(f)\|_{L^2(\Omega,\mu)} \le C\|f\|_{L^1C^k}, \tag{10.7}$$

for all $f \in C_0^\infty(\mathbb{R}^n)$, and from there to a continuous extension

$$Z : L^1C^k(\mathbb{R}^n) \longrightarrow L^2(\Omega, \mu), \tag{10.8}$$

whenever $Z \in \mathcal{D}'(\mathbb{R}^n, L^2(\Omega, \mu))$ is stationary. In particular,

$$Z : \mathcal{S}(\mathbb{R}^n) \longrightarrow L^2(\Omega, \mu), \tag{10.9}$$

i.e.,

$$Z \in \mathcal{S}'(\mathbb{R}^n, L^2(\Omega, \mu)). \tag{10.9A}$$

Furthermore, K_z in (10.3) extends to a continuous linear map

$$K_Z : L^1C^k(\mathbb{R}^n) \longrightarrow L^\infty \cap C(\mathbb{R}^n, L^2(\Omega, \mu)). \tag{10.10}$$

The following "Tauberian theorem" provides a useful characterization of stationary generalized random fields.

Proposition 10.1. *Let $Z \in \mathcal{S}'(\mathbb{R}^n, L^2(\Omega, \mu))$, and assume there exists a single $f \in \mathcal{S}(\mathbb{R}^n)$ such that $\hat{f}(\xi)$ is nowhere vanishing and*

$$f * Z \quad \text{is stationary,} \tag{10.11}$$

as a continuous random field. Then (10.11) holds for all $f \in \mathcal{S}(\mathbb{R}^n)$, so Z is stationary. Furthermore, given $k \in \mathbb{N}$ such that (10.7) holds, the result (10.11) holds for all $f \in L^1C^k(\mathbb{R}^n)$.

Proof. Given that $Z \in \mathcal{S}'(\mathbb{R}^n, L^2(\Omega, \mu))$, it suffices to show that (10.11) holds for a set of functions with dense linear span in $\mathcal{S}(\mathbb{R}^n)$. Thus it suffices to note that if $f \in \mathcal{S}(\mathbb{R}^n)$ and $\hat{f}(\xi)$ is nowhere vanishing, then $\{\check{f}_x : x \in \mathbb{R}^n\}$ has dense linear span in $\mathcal{S}(\mathbb{R}^n)$, where $\check{f}_x(y) = f(x - y)$. The well known proof goes as follows. If $\omega \in \mathcal{S}'(\mathbb{R}^n)$ annihilates this span, then $f * \omega = 0$. This implies $\hat{f}\hat{\omega} = 0$, which implies $\omega = 0$, given that $\hat{f}(\xi)$ never vanishes. The asserted density then follows from the Hahn-Banach theorem.

We consider the following class of generalized random fields. Assume

$$Z : \mathbb{R}^n \longrightarrow L^2(\Omega, \mu) \ \text{ is weakly continuous.} \tag{10.12}$$

Assume Z is stationary, as an element of $\mathcal{D}'(\mathbb{R}^n, L^2(\Omega, \mu))$. Take $f_1 \in C_0^\infty(\mathbb{R}^n)$, satisfying $f_1 \geq 0$ and $\int f \, dx = 1$, let $f_k(x) = k^n f(kx)$, and set

$$Z_k = f_k * Z \in C^\infty(\mathbb{R}^n, L^2(\Omega, \mu)). \tag{10.13}$$

which are stationary as continuous random fields. We have $\langle Z(x) \rangle = \langle Z(x)\, 1 \rangle$, continuous in x, and

$$\langle Z_k \rangle = f_k * \langle Z \rangle \longrightarrow \langle Z \rangle, \quad \text{locally uniformly on } \mathbb{R}^n. \tag{10.14}$$

Since each $\langle Z_k(x) \rangle = M_k$ is constant, so is $\langle Z(x) \rangle \equiv M = \lim M_k$. Subtracting M, we assume $\langle Z(x) \rangle \equiv 0$, and then $M_k \equiv 0$.

Our aim is to prove the following.

Proposition 10.2. *If Z satisfies (10.12) and is statinary, as a generalized random field, then*

$$Z:\mathbb{R}^n\longrightarrow L^2(\Omega,\mu)\quad \textit{is norm-continuous,} \tag{10.15}$$

and Z is stationary, as a continuous random field.

To start the proof, using the constructions above, we define Z_k as in (10.13) and reduce to the case $\langle Z(x)\rangle\equiv 0$, so $\langle Z_k(x)\rangle\equiv 0$. We have

$$\langle Z_k(x)\,\varphi\rangle\longrightarrow\langle Z(x)\,\varphi\rangle,\quad \text{locally uniformly in } x,\quad \forall\,\varphi\in L^2(\Omega,\mu). \tag{10.16}$$

Stationarity of Z_k implies

$$\|Z_k(x)\|_{L^2}\equiv E_k \ \text{(independent of } x\text{)}. \tag{10.17}$$

Hence

$$\|Z(x)\|_{L^2}\le \liminf_{k\to\infty}\, E_k,\quad \forall\,x\in\mathbb{R}^n. \tag{10.18}$$

The Dunford-Pettis theorem implies

$$Z:\mathbb{R}^n\longrightarrow L^2(\Omega,\mu)\ \text{ is strongly measurable.} \tag{10.19}$$

Hence, for each $x\in\mathbb{R}^n$,

$$Z_k(x)=\int f_k(x-y)Z(y)\,dy \tag{10.20}$$

exists as a Bochner integral, and

$$E_k\equiv\|Z_k(x)\|_{L^2}\le\int f_k(x-y)\|Z(y)\|_{L^2}\,dy. \tag{10.21}$$

Also, since $y\mapsto\|Z(y)\|_{L^2}$ is bounded (by (10.18)) and measurable,

$$\int f_k(x-y)\|Z(y)\|_{L^2}\,dy\longrightarrow\|Z(x)\|_{L^2},\quad \text{for a.e. } x, \tag{10.22}$$

and hence

$$\limsup_{k\to\infty}\, E_k\le\|Z(x)\|_{L^2},\quad \text{for a.e. } x. \tag{10.23}$$

We are in a posiiton to establish the following.

Lemma 10.3. *There exists* $S \subset \mathbb{R}^n$ *such that* $m(\mathbb{R}^n \setminus S) = 0$ *and*

$$\|Z(x)\|_{L^2} = E = \lim_{k\to\infty} E_k, \quad \forall\, x \in S, \tag{10.24}$$

$$Z_k(x) \longrightarrow Z(x) \text{ in } L^2(\Omega,\mu)\text{-norm}, \quad \forall\, x \in S, \tag{10.25}$$

$$\langle Z_k(x) Z_k(y)\rangle \longrightarrow \langle Z(x)Z(y)\rangle, \quad \forall\, x \in S,\ y \in \mathbb{R}^n. \tag{10.26}$$

$$\langle Z_k(x-y) Z_k(0)\rangle \longrightarrow \langle Z(x-y)Z(0)\rangle, \quad \forall\, x-y \in S. \tag{10.27}$$

Proof. We get (10.24) from (10.18) and (10.23). Then (10.27) follows from (10.11) and (10.24). Next, (10.26) follows from (10.16) and (10.25), and then (10.27) follows from (10.26).

To proceed, we know that

$$\langle Z_k(x) Z_k(y)\rangle = C_k(x-y), \tag{10.28}$$

and $C_k : \mathbb{R}^n \to \mathbb{R}$ is continuous, for each k. Let us define

$$C(x) = \langle Z(x) Z(0)\rangle, \tag{10.29}$$

so $C : \mathbb{R}^n \to \mathbb{R}$ is continuous, by (10.12). We want to show that

$$\langle Z(x) Z(y)\rangle = C(x-y), \quad \forall\, x, y \in \mathbb{R}^n. \tag{10.30}$$

Note that

$$C_k(x) = \langle Z_k(x) Z_k(0)\rangle \longrightarrow C(x), \quad \forall\, x \in S, \tag{10.31}$$

by (10.26). As noted in (10.27), it follows that

$$\langle Z_k(x) Z_k(y)\rangle = \langle Z_k(x-y) Z_k(0)\rangle \to C(x-y), \quad \forall\, x-y \in S. \tag{10.36}$$

Comparison with (10.26) yields

$$\langle Z(x) Z(y)\rangle = C(x-y), \quad \text{provided } \ x,\ x-y \in S, \tag{10.33}$$

which is a special case of (10.30). Fixing $x \in S$ and using (10.12) and the continuity of C, we have

$$\langle Z(x) Z(y)\rangle = C(x-y), \quad \forall\, x \in S,\ y \in \mathbb{R}^n. \tag{10.34}$$

Then taking $y \in \mathbb{R}^n$ and applying a similar argument, we have (10.30).

From (10.30), the norm continuity (10.15) follows readily. We have

$$\begin{aligned} \|Z(x+y) - Z(y)\|_{L^2}^2 &= \langle (Z(x+y) - Z(y))(Z(x+y) - Z(y))\rangle \\ &= 2C(0) - 2C(y), \end{aligned} \tag{10.35}$$

which tends to 0 as $|y| \to 0$.

A. Multiparameter ergodic theory

We assume $\{\tau_y : y \in \mathbb{F}^n\}$ is a family of measure preserving transformations on the probability space $(\mathcal{O}, \nu)$, satisfying $\tau_{y_1+y_2} = \tau_{y_1} \circ \tau_{y_2}$. To be definite, we take $\mathbb{F} = \mathbb{R}$, and we assume the induced action on $L^p(\mathcal{O}, \nu)$,

$$U(y)\varphi(\eta) = \varphi(\tau_y(\eta)), \tag{A.1}$$

is strongly continuous in y, for each $p \in [1, \infty)$. Note that $U(y)$ is an invertible isometry on $L^p(\mathcal{O}, \nu)$ (unitary on $L^2(\mathcal{O}, \nu)$) and $U(y_1 + y_2) = U(y_1)U(y_2)$. We aim to discuss ergodic theorems, dealing with averages of the form

$$A_R\varphi = \frac{1}{V(R)} \int\limits_{|y|\le R} U(y)\varphi \, dy. \tag{A.2}$$

First, there is an abstract mean ergodic theorem, valid when $\{U(y) : y \in \mathbb{R}^n\}$ is a strongly continuous unitary group on a Hilbert space H. It starts as follows.

Lemma A.1. *We have the orthogonal direct sum $H = K \oplus \overline{R}$, where*

$$\begin{aligned} K &= \{\varphi \in H : U(y)\varphi = \varphi, \ \forall \, y\}, \\ R &= \bigoplus_y \mathit{Range}\,(I - U(y)). \end{aligned} \tag{A.3}$$

Proof. This follows from the observation that

$$R^\perp = \bigcap_y \operatorname{Ker}\,(I - U(y)^*) = \bigcap_y \operatorname{Ker}\,(I - U(y)). \tag{A.4}$$

Here is the resulting abstract mean ergodic theorem.

Proposition A.2. *For all $\varphi \in H$, $A_R\varphi \to P\varphi$ in H-norm, where P is the orthogonal projection of H onto K.*

Proof. Note that $\varphi \in K \Rightarrow A_R\varphi \equiv \varphi$. Next, if $\varphi = (I - U(y_0))\psi, \ \psi \in H$, then

$$\frac{1}{V(R)} \int\limits_{|y|\le R} U(y)(I - U(y_0))\psi \, dy \to 0, \quad \text{as} \quad R \to \infty. \tag{A.5}$$

In view of Lemma A.1, this yields the asserted result.

Proposition A.2 applies to the case $H = L^2(\mathcal{O}, \nu)$, with $U(y)$ given by (A.1). We record the following extension.

Proposition A.3. *Let $\{\tau_y : y \in \mathbb{R}^n\}$ satisfy the hypotheses given above, and take $U(y)$, A_R as in (A.1)–(A.2). Then, for $p \in [1,2]$, P extends to a continuous projection $P : L^p(\mathcal{O},\nu) \to L^p(\mathcal{O},\nu)$, and*

$$A_R\varphi \longrightarrow P\varphi \quad \text{in} \quad L^p\text{-norm}, \quad \forall\, \varphi \in L^p(\mathcal{O},\nu). \tag{A.6}$$

Proof. Note that $\|A_R\|_{\mathcal{L}(L^p)} \le 1$ and use denseness of $L^2(\mathcal{O},\nu)$ in such $L^p(\mathcal{O},\nu)$.

Using other arguments, one can extend the scope of Proposition A.3 to all $p \in [1,\infty)$, but we omit details here.

The action of $\{\tau_y\}$ on $(\mathcal{O},\nu)$ is ergodic precisely when only constant functions on $\mathcal{O}$ belong to K, defined in (A.3). Then, and only then,

$$P\varphi = \Bigl(\int_{\mathcal{O}} \varphi\, d\nu\Bigr)\, 1, \quad \forall\, \varphi \in L^2(\mathcal{O},\nu). \tag{A.7}$$

Note that this implies the criterion (4.10) for ergodicity.

It is of interest to extend Proposition A.2, replacing $\mathbb{R}^n$ by a broader class of Lie groups. Let G be a Lie group, endowed with a right-invariant Haar measure. Let $U : G \to \mathcal{L}(H)$ be a strongly continuous unitary representation of G on a Hilbert space H. Take, for $R \in \mathbb{R}^+$,

$$f_R \in L^1(G), \quad f_R \ge 0, \quad \int_G f_R(y)\, dy \equiv 1, \tag{A.8}$$

and set

$$A_R\varphi = \int_G f_R(y)\, U(y)\varphi\, dy. \tag{A.9}$$

We seek conditions that lead to a result of the form $A_R\varphi \to P\varphi$ as $R \to \infty$.

To start, we note that Lemma A.1 holds in this more general setting, with y running over G to define K and R as in (A.3). Again (A.4) provides the proof.

To proceed, clearly

$$\varphi \in K \Longrightarrow A_R\varphi \equiv \varphi. \tag{A.10}$$

Next, if $\varphi = (I - U(y_0))\psi,\ \psi \in H$, then

$$\begin{aligned} A_R\varphi &= \int_G f_R(y)\bigl(U(y) - U(y)U(y_0)\bigr)\psi\, dy \\ &= \int_G [f_R(y) - f_R(yy_0^{-1})]U(y)\psi\, dy. \end{aligned} \tag{A.11}$$

We can deduce that

$$A_R\varphi \longrightarrow 0 \ \text{ as } \ R \to \infty, \tag{A.12}$$

for all $\varphi \in R$, hence for all $\varphi \in \overline{R}$, provided $\{f_R\}$ satisfies (A.8) and also

$$\lim_{R\to\infty} \int\limits_G |f_R(y) - f_R(yy_0^{-1})|\, dy = 0, \quad \forall\, y_0 \in G. \tag{A.13}$$

We record the conclusion.

Proposition A.4. *Let $\{f_R : R \in \mathbb{R}^+\}$ satisfy (A.8) and define $A_R : H \to H$ by (A.9). For all $\varphi \in H$, $A_R\varphi \to P\varphi$ in H-norm, where P is the orthogonal projection of H on K, provided $\{f_R\}$ also satisfies (A.13).*

A Lie group G for which a family $\{f_R\}$ satisfying (A.8) and (A.13) exists is said to be *amenable.* For $G = \mathbb{R}^n$, one can pick $f_1 \geq 0$ such that $\int_{\mathbb{R}^n} f_1(y)\, dy = 1$ and set $f_R(y) = R^{-n} f(R^{-1}y)$. Many non-abelian Lie groups are amenable, but not all of them are.

So far, we have discussed mean ergodic theorems. The demonstrations given above are straightforward variants of the classical case of the real line. (Compare [T2], Chapter 14.) There are also pointwise a.e. results, known as Birkhoff ergodic theorems, that are classical for $\mathbb{R}^n$, having been extended from $n = 1$ to general n in [W]. See [L] for treatments of some other groups.

B. Atoms of $\widehat{C}$

Let σ be a finite (possibly complex) measure on $\mathbb{R}^n$. Its Fourier transform

$$C(x) = \widehat{\sigma}(x) = (2\pi)^{-n/2} \int e^{-ix\cdot\xi}\, d\sigma(\xi) \tag{B.1}$$

is a bounded, continuous function on $\mathbb{R}^n$. We say σ has an atom at $p \in \mathbb{R}^n$ if $\sigma(\{p\}) \neq 0$. The set $\mathcal{A}(\sigma)$ of such points is countable, and we can write

$$\sigma = \sigma_0 + \sum_{p_j \in \mathcal{A}(\sigma)} a_j \delta_{p_j}, \tag{B.2}$$

where σ_0 has no atoms (we say σ_0 is a continuous measure). Here we prove the following result (due to N. Wiener), of interest in §4.

Proposition B.1. *With σ and C as above,*

$$\lim_{R\to\infty} \frac{1}{V(R)} \int\limits_{|y|\le R} |C(y)|^2\, dy = (2\pi)^{-n} \sum |a_j|^2. \tag{B.3}$$

Proof. We show that, more generally, if $f \in L^1(\mathbb{R}^n)$, $\int f(y)\, dy = 1$, and $f_R(y) = f(y/R)$, then

$$R^{-n} \int f_R(y) |\widehat{\sigma}(y)|^2\, dy \longrightarrow (2\pi)^{-n} \sum |a_j|^2, \quad \text{as} \quad R \to \infty. \tag{B.4}$$

In fact, the left side of (B.4) is equal to

$$\begin{aligned} &(2\pi R)^{-n} \int f_R(y) \int e^{iy\cdot\xi}\, d\sigma(\xi) \int e^{-iy\cdot\eta}\, d\overline{\sigma(\eta)}\, dy \\ &= (2\pi)^{-n} \iint \Big\{ \int f(y) e^{iR(\xi-\eta)\cdot y}\, dy \Big\}\, d\sigma(\xi)\, d\overline{\sigma(\eta)}. \end{aligned} \tag{B.5}$$

Since the expression in brackets is bounded by $\|f\|_{L^1}$, we can pass to the limit under the integral sign. Now $\int f(y) e^{iR(\xi-\eta)\cdot y}\, dy$ tends to 0 as $R \to \infty$ if $\xi \neq \eta$, by the Riemann-Lebesgue lemma, while the expression is 1 at $\xi = \eta$. Thus

$$\lim_{R\to\infty} R^{-n} \int f_R(y) |\widehat{\sigma}(y)|^2\, dy = (2\pi)^{-n} \iint\limits_{\xi=\eta} d\sigma(\xi)\, d\overline{\sigma(\eta)} = (2\pi)^{-n} \sum |a_j|^2. \tag{B.6}$$

This completes the proof.

C. Fourier transform of a stationary field

If we have a real-valued, continuous, stationary field on the n-torus,

$$Z : \mathbb{T}^n \longrightarrow L^2(\Omega, \mu), \tag{C.1}$$

it has a representation

$$Z(x) = \sum_{k \in \mathbb{Z}^n} \widehat{Z}(k) e^{ik \cdot x}, \tag{C.2}$$

with

$$\widehat{Z} : \mathbb{Z}^n \longrightarrow L^2(\Omega, \mu), \quad \widehat{Z}(k) = (2\pi)^{-n} \int_{\mathbb{T}^n} Z(x) e^{-ik \cdot x} \, dx. \tag{C.3}$$

Note that $\widehat{Z}$ is not stationary. However, the various random variables $\widehat{Z}(k)$ are uncorrelated. In fact, in such a case,

$$\begin{aligned} \langle \widehat{Z}(k) \overline{\widehat{Z}(\ell)} \rangle &= (2\pi)^{-2n} \iint \langle Z(x) Z(y) \rangle e^{-ik \cdot x} e^{i\ell \cdot y} \, dx \, dy \\ &= (2\pi)^{-2n} \iint C(x-y) e^{-ik \cdot x} e^{i\ell \cdot y} \, dx \, dy \\ &= (2\pi)^{-n} \widehat{C}(k) \int e^{i(\ell - k) \cdot y} \, dy \\ &= \widehat{C}(k) \delta_{k\ell}. \end{aligned} \tag{C.4}$$

This is a special case of (5.29). (Note also that $\overline{\widehat{Z}(\ell)} = \widehat{Z}(-\ell)$.)

NOTE. The characterization of $C(x-y)$ as $\langle Z(x)Z(y) \rangle$ is equivalent to that in (4.2) and (5.10) if and only if $\langle Z(x) \rangle \equiv 0$. The same applies to (C.9).

Treating the Fourier transform of a real-valued, continuous, stationary field

$$Z : \mathbb{R}^n \longrightarrow L^2(\Omega, \mu) \tag{C.5}$$

will require the use of vector-valued tempered distributions, since Z is bounded but not integrable. We have

$$\widehat{Z} \in \mathcal{S}'(\mathbb{R}^n, L^2(\Omega, \mu)), \tag{C.6}$$

that is, $\widehat{Z} : \mathcal{S}(\mathbb{R}^n) \to L^2(\Omega, \mu)$, defined by

$$\widehat{Z}(f) = Z(\widehat{f}) = \int Z(x)\widehat{f}(x)\,dx, \quad f \in \mathcal{S}(\mathbb{R}^n), \tag{C.7}$$

where $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz space of rapidly decreasing functions (cf. [T], Chapter 3). Formally (i.e., informally),

$$\widehat{Z}(f) = \int \widehat{Z}(\xi)f(\xi)\,d\xi. \tag{C.8}$$

Now, parallel to (C.4), we have

$$\begin{aligned} \langle \widehat{Z}(f), \overline{\widehat{Z}(g)}\rangle &= \langle Z(\widehat{f})\overline{Z(\widehat{g})}\rangle \\ &= \iint \langle Z(x)Z(y)\rangle \widehat{f}(x)\overline{\widehat{g}(y)}\,dx\,dy \\ &= \iint C(x-y)\widehat{f}(x)\overline{\widehat{g}(y)}\,dx\,dy. \end{aligned} \tag{C.9}$$

Note that

$$\begin{aligned} \int C(x-y)\widehat{f}(x)\,dx &= (2\pi)^{-n/2} \iint f(\xi)e^{-ix\cdot\xi}C(x-y)\,d\xi\,dx \\ &= \int e^{-iy\cdot\xi}\widehat{C}(\xi)f(\xi)\,d\xi, \end{aligned} \tag{C.10}$$

and

$$\begin{aligned} \int e^{-iy\cdot\xi}\overline{\widehat{g}(y)}\,dy &= \Big(\int e^{iy\cdot\xi}\widehat{g}(y)\,dy\Big)^* \\ &= (2\pi)^{n/2}\overline{g(\xi)}, \end{aligned} \tag{C.11}$$

so

$$\langle \widehat{Z}(f)\overline{\widehat{Z}(g)}\rangle = (2\pi)^{n/2} \int \widehat{C}(\xi)f(\xi)\overline{g(\xi)}\,d\xi. \tag{C.12}$$

If we formally take $f = \delta_{\xi_1},\ g = \delta_{\xi_2}$ in (C.9), we get, formally,

$$\begin{aligned} \langle \widehat{Z}(\xi_1)\overline{\widehat{Z}(\xi_2)}\rangle &= (2\pi)^{-n} \iint C(x-y)e^{-ix\cdot\xi_1}e^{iy\cdot\xi_2}\,dx\,dy \\ &= (2\pi)^{-n/2} \int \widehat{C}(\xi_1)e^{iy\cdot(\xi_2-\xi_1)}\,dy \\ &= \widehat{C}(\xi_1)\delta(\xi_1-\xi_2), \end{aligned} \tag{C.13}$$

at least if $\widehat{C}$ is continuous (which holds if $C \in L^1(\mathbb{R}^n)$). We have stated in §3 that $\widehat{C}$ is always a finite positive measure; call it σ. In this setting, we would write (C.12) as

$$\langle \widehat{Z}(f), \overline{\widehat{Z}(g)} \rangle = (2\pi)^{n/2} \int f(\xi) \overline{g(\xi)} \, d\sigma(\xi). \tag{C.14}$$

In such a case, the bottom line of (C.13) can be interpreted as a finite positive measure on $\mathbb{R}^n \times \mathbb{R}^n$.

We also note that, in case $n = 1$, we can write

$$\widehat{Z} = \frac{d}{d\xi} F, \quad (1 + |\xi|)^{-2} F \in L^2(\mathbb{R}, L^2(\Omega, \mu)), \tag{C.15}$$

and hence

$$Z(x) = (2\pi)^{-1/2} \int e^{ix\xi} \, dF(\xi). \tag{C.16}$$

To get this, we first note that (for general n) since $Z : \mathbb{R}^n \to L^2(\Omega, \mu)$ is bounded and continuous,

$$g \in L^2(\mathbb{R}^n) \Longrightarrow \widehat{Z} * g \in L^2(\mathbb{R}^n, L^2(\Omega, \mu)). \tag{C.17}$$

In case $n = 1$, we can take

$$g(\xi) = \begin{cases} e^{-\xi}, & \xi > 0, \\ 0, & \xi < 0, \end{cases} \tag{C.18}$$

which satisfies $g' = \delta - g$, and then

$$W = \widehat{Z} * g \Longrightarrow \widehat{Z} = W' + W, \quad W \in L^2(\mathbb{R}, L^2(\Omega, \mu)), \tag{C.19}$$

so (C.15) holds with

$$F(\xi) = W(\xi) + \int_0^\xi W(\eta) \, d\eta. \tag{C.20}$$

It follows readily from (C.14) and a limiting argument that the increments $F(\xi') - F(\xi)$ are uncorrelated over non-overlapping intervals.

A representation alternative to (C.16) is

$$Z(x) = (2\pi)^{-1/2} (1 - ix) \int e^{ix\xi} W(\xi) \, d\xi, \quad W \in L^2(\mathbb{R}, L^2(\Omega, \mu)), \tag{C.21}$$

with W as in (C.19). This has n-dimensional variants. We can take

$$g(\xi) = (1-\Delta)^{-k}\delta(\xi), \quad k > \frac{n}{4}, \tag{C.22}$$

so $g \in L^2(\mathbb{R}^n)$ and $(1-\Delta)^k g = \delta$. Then

$$W = \widehat{Z} * g \Longrightarrow \widehat{Z} = (1-\Delta)^k W, \quad W \in L^2(\mathbb{R}^n, L^2(\Omega,\mu)), \tag{C.23}$$

and

$$Z(x) = (1+|x|^2)^k (2\pi)^{-n/2} \int e^{ix\cdot\xi} W(\xi)\, d\xi. \tag{C.24}$$

Returning to (C.6)–(C.7), we note that $\widehat{Z}$ has a more precise description than being in $\mathcal{S}'(\mathbb{R}^n, L^2(\Omega,\mu))$. It is useful to introduce some notation. We fix the probability space (Ω,μ) and associated Hilbert space $L^2(\Omega,\mu)$. We say

$$Z \in \Sigma(\mathbb{R}^n) \tag{C.25}$$

provided $Z : \mathbb{R}^n \to L^2(\Omega,\mu)$ is a continuous, stationary field. We then set

$$\mathcal{F}\Sigma(\mathbb{R}^n) = \{\widehat{Z} : Z \in \Sigma(\mathbb{R}^n)\} \subset \mathcal{S}'(\mathbb{R}^n, L^2(\Omega,\mu)). \tag{C.26}$$

One observation is that if $Z \in \Sigma(\mathbb{R}^n)$, then (C.7) extends to $\widehat{f} \in L^1(\mathbb{R}^n)$, i.e., to $f \in \mathcal{F}L^1(\mathbb{R}^n)$. More generally, given

$$\nu \in \mathcal{M}(\mathbb{R}^n), \quad \tilde{\nu} \in \mathcal{F}\mathcal{M}(\mathbb{R}^n), \tag{C.27}$$

where $\mathcal{M}(\mathbb{R}^n)$ is the space of finite Borel measures on $\mathbb{R}^n$, and $\tilde{\nu}(\xi) = (2\pi)^{-n/2}\int e^{ix\cdot\xi}\, d\nu(x)$ is the inverse Fourier transform, we have

$$\widehat{Z}(\tilde{\nu}) = Z(\nu) = \int Z(x)\, d\nu(x) \in L^2(\Omega,\mu), \tag{C.28}$$

so, extending (C.6), we have

$$\widehat{Z} : \mathcal{F}\mathcal{M}(\mathbb{R}^n) \longrightarrow L^2(\Omega,\mu), \tag{C.29}$$

if $Z \in \Sigma(\mathbb{R}^n)$.

Some related results arise as follows. First, given $f \in L^1(\mathbb{R}^n)$,

$$Z * f(x) = \int Z(x-y) f(y)\, dy \tag{C.30}$$

is well defined, and

$$f \in L^1(\mathbb{R}^n),\ Z \in \Sigma(\mathbb{R}^n) \Longrightarrow Z * f \in \Sigma(\mathbb{R}^n). \tag{C.31}$$

More generally, given $\nu \in \mathcal{M}(\mathbb{R}^n)$, we can set

$$Z * \nu(x) = \int Z(x-y)\, d\nu(y), \tag{C.32}$$

and then

$$\nu \in \mathcal{M}(\mathbb{R}^n),\ Z \in \Sigma(\mathbb{R}^n) \Longrightarrow Z * \nu \in \Sigma(\mathbb{R}^n). \tag{C.33}$$

Furthermore,

$$\mathcal{F}\Sigma(\mathbb{R}^n) \ \text{ is a module over } \ \mathcal{F}L^1(\mathbb{R}^n), \tag{C.34}$$

and, more generally,

$$\mathcal{F}\Sigma(\mathbb{R}^n) \ \text{ is a module over } \ \mathcal{F}\mathcal{M}(\mathbb{R}^n), \tag{C.35}$$

under pointwise multiplication, and

$$\widehat{Z * f} = \hat{f}\,\widehat{Z}, \quad \widehat{Z * \nu} = \hat{\nu}\,\widehat{Z}. \tag{C.36}$$

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