The Riemann zeta function

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Abstract. This material is excerpted from Section 19 of [T].

The Riemann zeta function is defined by

$$\zeta(s)=\sum_{n=1}^{\infty} n^{-s}, \quad \operatorname{Re} s>1. \tag{19.1}$$

Some special cases of this arose in §13, namely

$$\zeta(2)=\frac{\pi^2}{6}, \quad \zeta(4)=\frac{\pi^4}{90}. \tag{19.2}$$

This function is of great interest in number theory, due to the following result.

Proposition 19.1. *Let* $\{p_j : j\ge 1\}=\{2,3,5,7,11,\dots\}$ *denote the set of prime numbers in* $\mathbb{N}$. *Then, for* $\operatorname{Re} s>1$,

$$\zeta(s)=\prod_{j=1}^{\infty}(1-p_j^{-s})^{-1}. \tag{19.3}$$

Proof. Write the right side of (19.3) as

$$\begin{aligned}&\prod_{j=1}^{\infty}(1+p_j^{-s}+p_j^{-2s}+p_j^{-3s}+\cdots)\\ &=1+\sum_j p_j^{-s}+\sum_{j_1\le j_2}(p_{j_1}p_{j_2})^{-s}+\sum_{j_1\le j_2\le j_3}(p_{j_1}p_{j_2}p_{j_3})^{-s}+\cdots .\end{aligned} \tag{19.4}$$

That this is identical to the right side of (19.1) follows from the fundamental theorem of arithmetic, which says that each integer $n\ge 2$ has a unique factorization into a product of primes.

From (19.1) we see that

$$s\searrow 1 \Longrightarrow \zeta(s)\nearrow +\infty. \tag{19.5}$$

Hence

$$\prod_{j=1}^{\infty}(1-p_j^{-1})=0. \tag{19.6}$$

Applying (18.59), we deduce that

$$\sum_{j=1}^{\infty} \frac{1}{p_j} = \infty, \tag{19.7}$$

which is a quantitative strengthening of the result that there are infinitely many primes. Of course, comparison with $\sum_{n\ge 1} n^{-s}$ implies

$$\sum_{j=1}^{\infty} \frac{1}{|p_j^s|} < \infty, \quad \text{for} \quad \operatorname{Re} s > 1. \tag{19.8}$$

Another application of (18.59) gives

$$\zeta(s) \neq 0, \quad \text{for} \quad \operatorname{Re} s > 1. \tag{19.9}$$

Our next goal is to establish the following.

Proposition 19.2. *The function $\zeta(s)$ extends to a meromorphic function on $\mathbb{C}$, with one simple pole, at $s = 1$.*

To start the demonstration, we relate the Riemann zeta function to the function

$$g(t) = \sum_{n=1}^{\infty} e^{-n^2 \pi t}. \tag{19.10}$$

Indeed, we have

$$\begin{aligned} \int_0^\infty g(t) t^{s-1}\, dt &= \sum_{n=1}^{\infty} n^{-2s} \pi^{-s} \int_0^\infty e^{-t} t^{s-1}\, dt \\ &= \zeta(2s) \pi^{-s} \Gamma(s). \end{aligned} \tag{19.11}$$

This gives rise to further useful identities, via the Jacobi identity (14.43), i.e.,

$$\sum_{\ell=-\infty}^{\infty} e^{-\pi \ell^2 t} = \sqrt{\frac{1}{t}} \sum_{k=-\infty}^{\infty} e^{-\pi k^2/t}, \tag{19.12}$$

which implies

$$g(t) = -\frac{1}{2} + \frac{1}{2} t^{-1/2} + t^{-1/2} g\Big(\frac{1}{t}\Big). \tag{19.13}$$

To use this, we first note from (19.11) that, for Re $s > 1$,

$$\begin{aligned} \Gamma\Big(\frac{s}{2}\Big) \pi^{-s/2} \zeta(s) &= \int_0^\infty g(t) t^{s/2-1}\, dt \\ &= \int_0^1 g(t) t^{s/2-1}\, dt + \int_1^\infty g(t) t^{s/2-1}\, dt. \end{aligned} \tag{19.14}$$

Into the integral over $[0,1]$ we substitute the right side of (19.13) for $g(t)$, to obtain

$$\begin{aligned}\Gamma\Bigl(\frac{s}{2}\Bigr)\pi^{-s/2}\zeta(s) = &\int_0^1 \Bigl(-\frac{1}{2}+\frac{1}{2}t^{-1/2}\Bigr)t^{s/2-1}\,dt \\ &+\int_0^1 g(t^{-1})t^{s/2-3/2}\,dt+\int_1^\infty g(t)t^{s/2-1}\,dt.\end{aligned} \tag{19.15}$$

We evaluate the first integral on the right, and replace t by $1/t$ in the second integral, to obtain, for Re $s>1$,

$$\Gamma\Bigl(\frac{s}{2}\Bigr)\pi^{-s/2}\zeta(s) = \frac{1}{s-1}-\frac{1}{s}+\int_1^\infty \bigl[t^{s/2}+t^{(1-s)/2}\bigr]g(t)t^{-1}dt. \tag{19.16}$$

Note that $g(t)\le Ce^{-\pi t}$ for $t\in[1,\infty)$, so the integral on the right side of (19.16) defines an entire function of s. Since $1/\Gamma(s/2)$ is entire, with simple zeros at $s=0,-2,-4,\dots$, as seen in §18, this implies that $\zeta(s)$ is continued as a meromorphic function on $\mathbb{C}$, with one simple pole, at $s=1$. This finishes the proof of Proposition 19.2.

The formula (19.16) does more than establish the meromorphic continuation of the zeta function. Note that the right side of (19.16) is *invariant* under replacing s by $1-s$. Thus we have an identity known as Riemann's functional equation:

$$\Gamma\Bigl(\frac{s}{2}\Bigr)\pi^{-s/2}\zeta(s) = \Gamma\Bigl(\frac{1-s}{2}\Bigr)\pi^{-(1-s)/2}\zeta(1-s). \tag{19.17}$$

The meromorphic continuation of $\zeta(s)$ can be used to obtain facts about the set of primes deeper than (19.7). It is possible to strengthen (19.9) to

$$\zeta(s)\neq 0 \quad \text{for} \quad \operatorname{Re} s\ge 1. \tag{19.18}$$

This plays a role in the following result, known as the

Prime number theorem.

$$\lim_{j\to\infty}\frac{p_j}{j\log j}=1. \tag{19.19}$$

A proof can be found in [BN], and in [Ed]. Of course, (19.19) is much more precise than (19.7).

There has been a great deal of work on determining where $\zeta(s)$ can vanish. By (19.16), it must vanish at all the poles of $\Gamma(s/2)$, other than $s=0$, i.e.,

$$\zeta(s)=0 \quad \text{on} \quad \{-2,-4,-6,\dots\}. \tag{19.20}$$

These are known as the "trivial zeros" of $\zeta(s)$. It follows from (19.18) and the functional equation (19.16) that all the other zeros of $\zeta(s)$ are contained in the "critical strip"

$$\Omega=\{s\in\mathbb{C}:0<\operatorname{Re} s<1\}. \tag{19.21}$$

Concerning where in Ω these zeros can be, there is the following famous conjecture.

The Riemann hypothesis. *All zeros in* Ω *of* $\zeta(s)$ *lie on the critical line*

$$\Big\{\frac{1}{2}+i\sigma:\sigma\in\mathbb{R}\Big\}. \tag{19.22}$$

Many zeros of $\zeta(s)$ have been computed and shown to lie on this line, but after over a century, a proof (or refutation) of the Riemann hypothesis eludes the mathematics community. The reader can consult [Ed] for more on the zeta function.

Exercises

1. Use the functional equation (19.17) together with (18.6) and the Legendre duplication formula (18.47) to show that

$$\zeta(1-s)=2^{1-s}\pi^{-s}\Big(\cos\frac{\pi s}{2}\Big)\Gamma(s)\zeta(s).$$

2. Sum the identity

$$\Gamma(s)n^{-s}=\int_0^\infty e^{-nt}t^{s-1}\,dt$$

over $n\in\mathbb{Z}^+$ to show that

$$\Gamma(s)\zeta(s)=\int_0^\infty\frac{t^{s-1}}{e^t-1}\,dt=\int_0^1\frac{t^{s-1}}{e^t-1}\,dt+\int_1^\infty\frac{t^{s-1}}{e^t-1}\,dt=A(s)+B(s).$$

Show that $B(s)$ continues as an entire function. Use a Laurent series

$$\frac{1}{e^t-1}=\frac{1}{t}+a_0+a_1t+a_2t^2+\cdots$$

to show that

$$\int_0^1\frac{t^{s-1}}{e^t-1}\,dt=\frac{1}{s-1}+\frac{a_0}{s}+\frac{a_1}{s+1}+\cdots$$

provides a meromorphic continuation of $A(s)$, with poles at $\{1,0,-1,\dots\}$. Use this to give a second proof that $\zeta(s)$ has a meromorphic continuation with one simple pole, at $s=1$.

3. Show that, for $\mathrm{Re}\, s > 1$, the following identities hold:

(a)
$$\zeta(s)^2 = \sum_{n=1}^{\infty} \frac{d(n)}{n^s},$$

(b)
$$\zeta(s)\zeta(s-1) = \sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s},$$

(c)
$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s},$$

(d)
$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s},$$

(e)
$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s},$$

where
$$\begin{aligned} d(n) &= \#\ \text{divisors of}\ \ n, \\ \sigma(n) &= \text{sum of divisors of}\ \ n, \\ \varphi(n) &= \#\ \text{positive integers}\ \ \le n,\ \text{relatively prime to}\ n, \\ \mu(n) &= (-1)^{\#\ \text{prime factors}},\ \text{if}\ n\ \text{is square-free},\ 0\ \text{otherwise}, \\ \Lambda(n) &= \log p\ \text{if}\ n = p^m\ \text{for some prime}\ p,\ 0\ \text{otherwise}. \end{aligned}$$

Reference

[BN] J. Bak and D. J. Newman, Complex Analysis, Springer-Verlag, New York 1982.

[Ed] H. Edwards, Riemann's Zeta Function, Dover, New York, 2001.

[T] M. Taylor, Introduction to Complex Analysis. Notes, available this website.