The Schrödinger Equation on Cones

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1. Introduction

Here we study the solution operator $e^{it\Delta}$ to the Schrödinger equation on a cone $C(N)$ over a compact Riemannian manifold M. As a set, $C(N) = \mathbb{R}^+ \times N/\sim$, where $(0,\omega_1) \sim (0,\omega_2)$. The metric tensor on $C(N)$ is given by

$$ds^2 = dr^2 + r^2 g_N, \tag{1.1}$$

where g_N is the metric tensor on N. Then the Laplace-Beltrami operator Δ on $C(N)$ has the form

$$\Delta = \partial_r^2 + \frac{n-1}{r}\partial_r + \frac{1}{r^2}\Delta_N, \tag{1.2}$$

where $n = \dim C(N)$ and Δ_N is the Laplace operator on N. The approach to functions of Δ taken in [CT] made use of the Hankel transform to write

$$\varphi(\sqrt{-\Delta})g(r_1,\omega) = \int_0^\infty K_\varphi(r_1,r_2,A)g(r_2,\omega)r_2^{n-1}\,dr_2, \tag{1.3}$$

where

$$A = (-\Delta_N + \gamma^2)^{1/2}, \quad \gamma = \frac{n-2}{2}, \tag{1.4}$$

and $K_\varphi(r_1,r_2,A)$ is a family of operators on $L^2(N)$, given by

$$K_\varphi(r_1,r_2,A) = (r_1r_2)^{-\gamma}\int_0^\infty \varphi(\lambda)J_A(\lambda r_1)J_A(\lambda r_2)\,\lambda\,d\lambda. \tag{1.5}$$

(Cf. also [T], Chapter 8, §8.) Here J_ν is the Bessel function, defined by

$$J_\nu(r) = \frac{1}{\Gamma(1/2)\Gamma(\nu+1/2)} \Big(\frac{r}{2}\Big)^\nu \int_{-1}^{1} (1-t^2)^{\nu-1/2}\, e^{irt}\, dt, \tag{1.6}$$

and for each $r > 0$, $J_A(r)$ is defined by the spectral theorem. Equivalently,

$$J_A(r)f(\omega) = \sum J_{\nu_k}(r)\,(f,u_k)\,u_k(\omega), \tag{1.7}$$

where $\{u_k\}$ is an orthonormal basis of $L^2(N)$, consisting of eigenfunctions of A, with $Au_k = \nu_k u_k$. Note that each $\nu_k \ge \gamma$.

One useful identity exploited in [CT] is the Weber integral

$$\int_0^\infty e^{-t\lambda^2} J_\nu(r_1\lambda) J_\nu(r_2\lambda)\,\lambda\,d\lambda = \frac{1}{2t} e^{-(r_1^2+r_2^2)/4t}\, I_\nu\Big(\frac{r_1 r_2}{2t}\Big), \tag{1.8}$$

valid for $r_1, r_2, t > 0$, where

$$I_\nu(y) = e^{-\pi i\nu/2}\, J_\nu(iy), \quad y > 0. \tag{1.9}$$

Applying (1.8) in (1.5) yields the following formula for the solution to the heat equation on $C(N)$:

$$e^{t\Delta} g(r_1,\omega) = \int_0^\infty H_t(r_1,r_2,A) g(r_2,\omega) r_2^{n-1}\,dr_2,$$

where

$$H_t(r_1,r_2,A) = \frac{(r_1r_2)^{-\gamma}}{2t} e^{-(r_1^2+r_2^2)/4t} I_A\Big(\frac{r_1r_2}{2t}\Big).$$

One can proceed via analytic continuation to obtain

$$e^{it\Delta} g(r_1,\omega) = \int_0^\infty S_t(r_1,r_2,A)\, g(r_2,\omega) r_2^{n-1}\,dr_2, \tag{1.10}$$

with

$$S_t(r_1,r_2,A) = \frac{(r_1r_2)^{-\gamma}}{2it} e^{-(r_1^2+r_2^2)/4it} J_A\Big(\frac{r_1r_2}{2t}\Big) e^{-\pi iA/2}. \tag{1.11}$$

One of our goals here is to analyze the family of operators $J_A(r)$ on $L^2(N)$. In particular, we want to understand the integral kernel $\kappa_N(r,\omega_1,\omega_2)$, defined by

$$e^{-\pi iA/2} J_A(r) f(\omega_1) = \int_N \kappa_N(r,\omega_1,\omega_2) f(\omega_2)\,dS(\omega_2), \tag{1.12}$$

where dS denotes Lebesgue measure on N. Note that, with $u_k,\ \nu_k$ as in (1.7),

$$\kappa_N(r,\omega_1,\omega_2)=\sum_k e^{-\pi i\nu_k/2}J_{\nu_k}(r)u_k(\omega_1)\overline{u_k(\omega_2)}. \tag{1.13}$$

Analysis of (1.12) yields information on the integral kernel of $e^{it\Delta}$, defined by

$$\begin{aligned} e^{it\Delta}g(r_1,\omega_1)&=\int\limits_{C(N)} E_t(r_1,\omega_1,r_2,\omega_2)g(r_2,\omega_2)\,dV(r_2,\omega_2)\\ &=\int\limits_N\int_0^\infty E_t(r_1,\omega_1,r_2,\omega_2)g(r_2,\omega_2)r_2^{n-1}\,dr_2\,dS(\omega_2), \end{aligned} \tag{1.14}$$

where $dV(r,\omega)=r^{n-1}\,dr\,dS(\omega)$ is Lebesgue measure on $C(N)$. In fact, by (1.10)–(1.11),

$$E_t(r_1,\omega_1,r_2,\omega_2)=\frac{1}{2it(r_1r_2)^{(n-2)/2}}e^{-(r_1^2+r_2^2)/4it}\kappa_N\Big(\frac{r_1r_2}{2t},\omega_1,\omega_2\Big). \tag{1.15}$$

In the special case when N is the standard sphere S^{n-1}, one has $C(N)=\mathbb{R}^n$. In such a case one has the well known integral kernel

$$E_t(x_1,x_2)=(4\pi it)^{-n/2}e^{-|x_1-x_2|^2/4it}, \tag{1.16}$$

for $e^{it\Delta}$. It is instructive to compute $\kappa_N(r,\omega_1,\omega_2)$ for $N=S^{n-1}$, by comparing (1.10)–(1.11) and (1.16). We get

$$\begin{aligned} &\frac{(r_1r_2)^{-\gamma}}{2it}e^{-(r_1^2+r_2^2)/4it}\kappa_{S^{n-1}}\Big(\frac{r_1r_2}{2t},\omega_1,\omega_2\Big)\\ &=(4\pi it)^{-n/2}\,e^{-|r_1\omega_1-r_2\omega_2|^2/4it}, \end{aligned} \tag{1.17}$$

or equivalently

$$\kappa_{S^{n-1}}(r,\omega_1,\omega_2)=C_nr^\gamma e^{-ir\omega_1\cdot\omega_2}. \tag{1.18}$$

In particular we have $|\kappa_{S^{n-1}}(r,\omega_1,\omega_2)|\le Cr^\gamma$, which is seen to be equivalent to the estimate $|E_t(x_1,x_2)|\le Ct^{-n/2}$ on the integral kernel given by (1.16).

Note however that, with A acting on functions of ω_1,

$$\Big|e^{i\sigma A}e^{-ir\omega_1\cdot\omega_2}\big|_{\omega_1=\omega_2}\Big|\sim C_\sigma\,r^{(n-2)/2},\quad \sigma\notin\pi\mathbb{C},\quad \text{as}\quad r\to\infty, \tag{1.19}$$

and hence, if $\kappa_N^s(r,\omega_1,\omega_2)$ denotes the integral kernel of $e^{-isA}J_A(r)$, then

$$\sup_{\omega_1,\omega_2}|\kappa^s_{S^{n-1}}(r,\omega_1,\omega_2)|\sim C_s\,r^{2\gamma},\quad s-\frac{\pi}{2}\notin\pi\mathbb{C},\quad \text{as}\quad r\to\infty. \tag{1.20}$$

In this note we establish the estimate

$$
(1.21)\qquad |\kappa_N^s(r,\omega_1,\omega_2)| \le \begin{cases} C\, r^\gamma, & 0 < r \le 1, \\ C r^{2\gamma+1/2}, & r \ge 1, \end{cases}
$$

for a general compact Riemannian manifold N, of dimension $n-1$. This result is sharp for $r \le 1$ and just a bit weaker than (1.20) for $r \ge 1$. In light of (1.15), this leads to estimates on $E_t(r_1,\omega_1,r_2)$. There are two regions to consider:

Region 1. Here $r_1 r_2 \le 2t$, and we get

$$
(1.22)\qquad |E_t(r_1,\omega_1,r_2,\omega_2)| \le \frac{C}{t(r_1r_2)^\gamma}\Big(\frac{r_1r_2}{2t}\Big)^\gamma = Ct^{-n/2}.
$$

Region 2. Here $r_1 r_2 \ge 2t$, and we get

$$
(1.23)\qquad |E_t(r_1,\omega_1,r_2,\omega_2)| \le \frac{C}{t(r_1r_2)^\gamma}\Big(\frac{r_1r_2}{2t}\Big)^{2\gamma+1/2} = C\Big(\frac{r_1r_2}{t}\Big)^{(n-1)/2} t^{-n/2}.
$$

To get (1.21) we recall in §2 various classical estimates on $J_\nu(r)$, which are then exploited in §3 to obtain (1.21). In §3 we also estimate the L^2-operator norm $\|J_A(r)\|_{\mathcal{L}(L^2)}$ and the Hilbert-Schmidt norm $\|J_A(r)\|_{HS}$, obtaining

$$
(1.24)\qquad \|J_A(r)\|_{\mathcal{L}(L^2)} \le C \min(r^\gamma, r^{-1/3}),
$$

$$
(1.25)\qquad \|J_A(r)\|_{HS} \le Cr^\gamma.
$$

In light of (1.18), the Hilbert-Schmidt norm estimate (1.25) is seen to be sharp. We will see that (1.24) is also sharp.

We have three appendices. In Appendix A we give a proof of the Weber identity (1.8). This result is classical; [W] gives a proof and references to several other proofs, but such a central result in the theory of Bessel functions can use still more proofs. We mention that yet another approach to (1.8) is given in [Ch]; cf. Theorem 2.4.1. In Appendix B we establish the Lipschitz-Hankel identity, a variant of (1.8) in which $e^{-t\lambda^2}$ is replaced by e^{-yA}. This provides a Poisson integral formula for e^{-yA}, used in [CT] in concert with analytic continuation to analyze the wave equation on $C(N)$. The Lipschitz-Hankel identity is also classical, and one can find a proof (rather different from ours) in [W]. Our approach is to deduce it from (1.8) via the subordination identity. Appendix C records some consequences of the Schläfli integral representation of $J_\nu(r)$, of use in some estimates in §2.

2. Estimates on $J_\nu(r)$

In this section we record various results on the behavior of $J_\nu(r)$ on

$$Q = \{(\nu, r) : \nu \ge \gamma, r \ge 0\}. \tag{2.1}$$

We will derive some of these results, though we quote other sources for the most delicate of these. We consider separately the following subsets of Q, which together cover Q:

$$\begin{aligned} Q_1 &= \{(\nu, r) \in Q : r \le 1\}, \\ Q_2 &= \{(\nu, r) \in Q : 1 \le r \le \nu/2\}, \\ Q_3 &= \{(\nu, r) \in Q : \nu \le 1,\ r \ge 1\}, \\ Q_4 &= \{(\nu, r) \in Q : \nu/4 \le r \le 4\nu\}, \\ Q_5 &= \{(\nu, r) \in Q : \nu \ge 1,\ r \ge 4\nu\}. \end{aligned} \tag{2.2}$$

Good estimates for (ν, r) in Q_1 and Q_2 follow readily from the integral formula (1.6), which we repeat here:

$$J_\nu(r) = \frac{1}{\Gamma(1/2)\Gamma(\nu + 1/2)} \Big(\frac{r}{2}\Big)^\nu \int_{-1}^{1} (1 - t^2)^{\nu - 1/2} e^{irt}\, dt. \tag{2.3}$$

Note that whenever $\nu \ge 0$ the integral is bounded in absolute value by π, so we have

$$|J_\nu(r)| \le \frac{\sqrt{\pi}}{\Gamma(\nu + 1/2)} \Big(\frac{r}{2}\Big)^\nu. \tag{2.4}$$

In particular, since $\nu \ge \gamma$ on Q,

$$(\nu, r) \in Q_1 \Rightarrow |J_\nu(r)| \le \frac{\sqrt{\pi}}{\Gamma(\nu + 1/2) 2^\nu}\, r^\gamma. \tag{2.5}$$

Also, Stirling's formula gives

$$\Gamma\Big(\nu + \frac{1}{2}\Big) = \sqrt{\frac{2\pi}{e}} \Big(\frac{\nu + 1/2}{e}\Big)^\nu A(\nu), \quad A(\nu) = 1 + O(\langle \nu \rangle^{-1}), \tag{2.6}$$

and hence

$$(\nu, r) \in Q_2 \Rightarrow |J_\nu(r)| \le C \Big(\frac{e}{4}\Big)^\nu \le C 2^{-r}. \tag{2.7}$$

If $\gamma > 1$, Q_3 is empty. Otherwise, we can examine the integral in (2.3) as the Fourier transform of a function with simple singularities as $t = \pm 1$ and produce an asymptotic expansion

$$J_\nu(r) \sim a(\nu) r^{-1/2} \cos r + O(r^{-3/2}), \quad r \to +\infty, \tag{2.8}$$

uniformly for ν in a compact subset of $[0,\infty)$. In particular,

$$(\nu,r)\in Q_3 \Rightarrow |J_\nu(r)| \le Cr^{-1/2}. \tag{2.9}$$

The behavior of $J_\nu(r)$ for $(\nu,r)\in Q_4$ is subtle. It is given by the following asymptotic expansion:

$$\begin{aligned} J_\nu(\nu z) \sim \frac{1}{\nu^{1/3}}\Big(\frac{4\zeta}{1-z^2}\Big)^{1/4}\Big\{ & Ai(\nu^{2/3}\zeta)\sum_{k\ge 0}\frac{A_k(\zeta)}{\nu^{2k}} \\ & + \frac{Ai'(\nu^{2/3}\zeta)}{\nu^{4/3}}\sum_{k\ge 0}\frac{B_k(\zeta)}{\nu^{2k}}\Big\}. \end{aligned} \tag{2.10}$$

Here Ai is the Airy function and $\zeta=\zeta(z)$ is given by

$$\frac{2}{3}\zeta^{3/2} = -\int_1^z \frac{(1-t^2)}{t}\,dt. \tag{2.11}$$

See [Olv], pp. 423–425 for a derivation. We mention that $\zeta(z)$ is analytic in $\{z : \operatorname{Re} z > 0\}$ and satisfies $\zeta(1)=0,\ \zeta'(1)<0$. The expansion (2.11) is valid as $\nu\to+\infty$, uniformly for z in any compact neighborhood of 1 in $(0,\infty)$. The Airy function has the following asymptotic behavior as $s\to+\infty$:

$$Ai(s) = O(s^{-\infty}), \quad Ai(-s) \sim \pi^{-1/2}s^{-1/4}\cos\Big(\frac{2}{3}s^{3/2}-\frac{\pi}{4}\Big) + O(s^{-1/4-3/2}). \tag{2.12}$$

In particular,

$$|Ai(s)| \le C(1+|s|)^{-1/4}. \tag{2.13}$$

Consequently,

$$(\nu,r)\in Q_4 \Rightarrow |J_\nu(r)| \le C\nu^{-1/3}\Big(1+\nu^{2/3}\Big|1-\frac{\nu}{r}\Big|\Big)^{-1/4}. \tag{2.14}$$

Note in particular the behavior on the boundary ray $r=4\nu$:

$$|J_\nu(4\nu)| \le C\nu^{-1/2}. \tag{2.15}$$

For one estimate on $J_\nu(r)$ for $(\nu,r)\in Q_5$, we use (2.15) and the differential equation

$$\Big[\partial_r^2 + \frac{1}{r}\partial_r + \Big(1-\frac{\nu^2}{r^2}\Big)\Big]J_\nu(r) = 0. \tag{2.16}$$

We also need an estimate on $J_\nu'(r)$ for $r = 4\nu$. This comes from

$$J_\nu'(r) = -\frac{\nu}{r} J_\nu(r) + J_{\nu-1}(r), \tag{2.17}$$

which with (2.14)–(2.15) yields

$$|J_\nu'(4\nu)| \le C\nu^{-1/2}. \tag{2.18}$$

To proceed further, consider

$$\big[\partial_r^2 + r^{-1}\partial_r + p(r)\big]u(r) = 0, \quad p(r) = 1 - \frac{\nu^2}{r^2}, \tag{2.19}$$

and set

$$w(r) = \frac{1}{2}\big[p(r)u(r)^2 + u'(r)^2\big]. \tag{2.20}$$

We have

$$w'(r) = \frac{1}{2}p'(r)u(r)^2 - \frac{1}{r}u'(r)^2 \le \frac{\nu^2}{r^3}w(r), \tag{2.21}$$

when $(\nu, r) \in Q_5$, and hence $w' - (\nu^2/r^3)w = e^{-\nu^2/2r^2}\partial_r(e^{\nu^2/2r^2}w) \le 0$, which implies

$$e^{\nu^2/2r^2}w(r) \searrow, \tag{2.22}$$

as r increases, when $(\nu, r) \in Q_5$. Applying this to $w(r) = J_\nu(r)$ and using (2.15) and (2.18), we have

$$(\nu, r) \in Q_5 \Rightarrow |J_\nu(r)| \le C\nu^{-1/2}. \tag{2.23}$$

While the estimate (2.23) will prove adequate for our estimate in §3 of the Hilbert-Schmidt norm of $J_A(r)$, it does not provide a sharp bound on the operator norm. To get that, we will improve (2.23) to the estimate

$$(\nu, r) \in Q_5 \Longrightarrow |J_\nu(r)| \le Cr^{-1/2}. \tag{2.24}$$

To do this we make use of the Schläfli integral representation (C.4), which we rewrite as

$$J_\nu(\nu \sec a) = F_1(\nu, \nu \sec a) + F_2(\nu, \nu \sec a), \tag{2.25}$$

for $0 \le a < \pi/2$, where

$$(2.26)\qquad \begin{aligned} F_1(\nu,\nu\sec a) &= \frac{e^{-i\nu a}}{2\pi}\int_{-\pi}^{\pi} e^{-i\nu(t-\sin t-\tan a\cos t)}\,dt \\ F_2(\nu,\nu\sec a) &= -\frac{\sin\pi\nu}{\pi}e^{-i\nu a}\int_0^\infty e^{-\nu(t+\sinh t+i\tan a\,\cosh t)}\,dt. \end{aligned}$$

We first estimate $F_1(\nu,r)$, which we write as $(2\pi)^{-1}e^{-i\nu a}B(\nu,\rho)$, where

$$(2.27)\qquad B(\nu,\rho)=\int_{-\pi}^{\pi} e^{i\varphi(\nu,\rho,t)}\,dt,$$

with

$$(2.28)\qquad \varphi(\nu,\rho,t)=\nu(\sin t-t)+\rho\cos t,\quad \rho=r\sin a.$$

To estimate (2.27) we use the van der Corput lemma, which states that if $\psi(t)$ is real valued and

$$(2.29)\qquad F=\int_a^b e^{i\psi(t)}\,dt,$$

then

$$(2.30)\qquad \begin{aligned} \psi' \ \text{ monotone, } |\psi'|\ge R &\Longrightarrow |F|\le 4R^{-1}, \\ |\psi''|\ge R &\Longrightarrow |F|\le 8R^{-1/2}. \end{aligned}$$

Cf. [Duo], p. 183.

Before implementing (2.30), we note that the asymptotic expansion (2.10), or (C.6), applies to $J_\nu(\nu\sec a)$ uniformly on $a\in[a_0,a_1]$, given $0<a_0<a_1<\pi/2$, so we merely have to estimate (2.26) for $a\in(0,\pi/2)$ close to $\pi/2$, hence $\sin a\approx 1$ and $\rho\approx r,\ r/\nu >> 1$. To proceed, implementing (2.30) for (2.27), we compute

$$(2.31)\qquad \begin{aligned} \partial_t\varphi(\nu,\rho,t) &= \nu(\cos t-1)-\rho\sin t, \\ \partial_t^2\varphi(\nu,\rho,t) &= -\nu\sin t-\rho\cos t. \end{aligned}$$

Note that $\partial_t\varphi(\nu,\rho,t)=0$ at $t_0=0$ and at $t_1=-\pi+\delta$, where $\rho/\nu>>1\Rightarrow\delta<<1$. Also $\partial_t^2\varphi(\nu,\rho,t)=0$ at $t_i=-\pi/2+\varepsilon$, where $\rho/\nu>>1\Rightarrow|\varepsilon|<<1$. With these facts in mind, we divide the interval $(-\pi,\pi)$ into four pieces:

$$(2.32)\quad I_1=(-\pi,-3\pi/4],\quad I_2=(-3\pi/4,t_i],\quad I_3=(t_i,-\pi/4],\quad I_4=(-\pi/4,\pi).$$

Setting $B_j(\nu,\rho)=\int_{I_j} e^{i\varphi(\nu,\rho,t)}\,dt$ and using (2.30), we have

$$(2.33)\qquad \begin{aligned} &|B_1(\nu,\rho)|\le C\rho^{-1/2},\quad |B_2(\nu,\rho)|\le C\rho^{-1}, \\ &|B_3(\nu,\rho)|\le C\rho^{-1},\quad |B_4(\nu,\rho)|\le C\rho^{-1/2}. \end{aligned}$$

Hence

$$|F_1(\nu,r)| \le Cr^{-1/2}, \quad (\nu,r)\in Q_5. \tag{2.34}$$

To estimate $F_2(\nu,r)$, it remains to estimate

$$G(\nu,\rho) = \int_0^\infty e^{-\psi(\nu,\rho,t)}\,dt, \tag{2.35}$$

where

$$\psi(\nu,\rho,t) = \nu(t+\sinh t) + i\rho\cosh t, \tag{2.36}$$

and again $\rho = r\sin a \approx r$ since we need merely check this estimate for $a \approx \pi/2$. Writing

$$e^{-\psi} = -\frac{1}{\psi_t}\,\partial_t e^{-\psi} \tag{2.37}$$

and integrating by parts over $[\delta,\infty)$, where $\delta > 0$ will be specified below, we have

$$|G(\nu,\rho)| \le \Bigl|\int_0^\delta e^{-\psi}\,dt\Bigr| + |\psi_t(\nu,\rho,\delta)|^{-1} + \int_\delta^\infty \Bigl|\partial_t\frac{1}{\psi_t}\Bigr|\,e^{-\nu(t+\sinh t)}\,dt. \tag{2.38}$$

Note that the first integral on the right side of (2.38) is $\le \delta$. Also,

$$\partial_t\psi(\nu,\rho,t) = \nu(1+\cosh t) + i\rho\sinh t, \tag{2.39}$$

so, given $\delta\in(0,1)$,

$$|\psi_1(\nu,\rho,\delta)|^{-1} \le \frac{1}{\rho\delta}. \tag{2.40}$$

Next, if $t>0,\ 1\le\nu\le\rho/2$,

$$\begin{aligned}\Bigl|\partial_t\frac{1}{\psi_t(\nu,\rho,t)}\Bigr| &= \Bigl|\frac{\nu\sinh t + i\rho\cosh t}{(\nu(1+\cosh t)+i\rho\sinh t)^2}\Bigr| \\ &\le \frac{\nu}{\rho^2\sinh t} + \frac{\cosh t}{\rho\sinh^2 t}.\end{aligned} \tag{2.41}$$

Hence (as long as $\nu\le\rho$)

$$\begin{aligned}\Bigl|\partial_t\frac{1}{\psi_t}\Bigr| \le\ &\frac{C}{\rho t^2}, \quad 0<t<1, \\ &\frac{C}{\rho e^t}, \quad 1\le t<\infty.\end{aligned} \tag{2.42}$$

Thus (as long as $\nu \geq 1$),

$$\int_\delta^\infty \Bigl|\partial_t \frac{1}{\psi_t}\Bigr| e^{-\nu(t+\sinh t)}\, dt \leq \frac{C}{\rho} \int_\delta^1 \frac{1}{t^2}\, dt + \frac{C}{\rho} \int_1^\infty e^{-t}\, dt \leq \frac{C}{\rho\delta}. \tag{2.43}$$

Thus we pick

$$\delta = \rho^{-1/2} \tag{2.44}$$

and get $|G(\nu,\rho)| \leq C\rho^{-1/2}$, hence

$$|F_2(\nu,r)| \leq Cr^{-1/2}, \quad (\nu,r) \in Q_5. \tag{2.45}$$

In concert with (2.34), this proves the asserted estimate (2.24).

3. Estimates on $J_A(r)$

Here we make use of the estimates on the Bessel function $J_\nu(r)$ from §2 to investigate properties of the operators $J_A(r)$ and $e^{-\pi i A/2} J_A(r)$, acting on functions on N. First we estimate the L^2-operator norm, given by the spectral theorem as

$$\|J_A(r)\|_{\mathcal{L}(L^2)} = \sup_{\nu \in \operatorname{Spec} A} |J_\nu(r)| \leq \sup_{\nu \geq 0} |J_\nu(r)|. \tag{3.1}$$

If we set

$$q_\ell(r) = \sup_{\{\nu:(\nu,r)\in Q_\ell\}} |J_\nu(r)|, \tag{3.2}$$

we see from §2 that

$$\begin{gathered} q_1(r) \leq Cr^\gamma, \quad q_2(r) \leq C2^{-r}, \quad q_3(r) \leq Cr^{-1/2}, \\ q_4(r) \leq Cr^{-1/3}, \quad q_5(r) \leq Cr^{-1/2}. \end{gathered} \tag{3.3}$$

Hence

$$\|J_A(r)\|_{\mathcal{L}(L^2)} \leq C \min(r^\gamma, r^{-1/3}). \tag{3.4}$$

We next estimate the Hilbert-Schmidt norm $\|J_A(r)\|_{HS}$, defined by

$$\|J_A(r)\|_{HS}^2 = \sum_{\nu_k \in \operatorname{Spec} A} |J_{\nu_k}(r)|^2. \tag{3.5}$$

To estimate (3.5), it is convenient to set

$$\sigma_\ell(r) = \{\nu \in \operatorname{Spec} A : (\nu, r) \in Q_\ell\}. \tag{3.6}$$

We also have from [Ho] the following estimate on Spec A:

$$\#\{\nu_k \in \operatorname{Spec} A : \nu \le \nu_k \le \nu + 1\} \le C\nu^{n-2}, \tag{3.7}$$

given that $n - 1 = \dim N$.

The estimate on (3.5) is easy if $r \le 1$. Applying (3.7) and (2.5) yields

$$0 \le r \le 1 \Rightarrow \|J_A(r)\|_{HS}^2 \le Cr^{2\gamma}. \tag{3.8}$$

When $r \ge 1$, we consider the sum of $|J_{\nu_k}(r)|^2$ over $\nu_k \in \sigma_\ell(r)$, for $2 \le \ell \le 5$. By (3.7) and (2.7),

$$\sum_{\nu_k \in \sigma_2(r)} |J_{\nu_k}(r)|^2 \le C_K \int_r^\infty \nu^{-K} \nu^{n-2}\, d\nu \le C'_K r^{-K+n-1}, \tag{3.9}$$

for each $K < \infty$. By (2.9),

$$\sum_{\nu_k \in \sigma_3(r)} |J_{\nu_k}(r)|^2 \le Cr^{-1}. \tag{3.10}$$

By (2.14),

$$\begin{aligned} \sum_{\nu_k \in \sigma_4(r)} |J_{\nu_k}(r)|^2 &\le C \int_{r/4}^{4r} \nu^{-2/3} \Bigl(1 + \nu^{2/3}\Bigl|1 - \frac{\nu}{r}\Bigr|\Bigr)^{-1/2} \nu^{n-2}\, d\nu \\ &\le Cr^{n-2}. \end{aligned} \tag{3.11}$$

By (2.23),

$$\sum_{\nu_k \in \sigma_5(r)} |J_{\nu_k}(r)|^2 \le C \int_1^r \nu^{-1} \nu^{n-2}\, d\nu \le Cr^{n-2}. \tag{3.12}$$

Summing (3.9)–(3.12) yields

$$r \ge 1 \Rightarrow \|J_A(r)\|_{HS}^2 \le Cr^{2\gamma}. \tag{3.13}$$

Comparison with (3.6) then gives

$$\|J_A(r)\|_{HS} \le Cr^{\gamma}, \quad r \in [0, \infty). \tag{3.14}$$

In view of the computation (1.18) for the example $N = S^{n-1}$, we see that such an estimate is sharp.

We now turn to an estimate of $\kappa_N(r, \omega_1, \omega_2)$, the integral kernel of $e^{-\pi i A/2} J_A(r)$. From (1.13) we have

$$|\kappa_N(r, \omega_1, \omega_2)| \le \sum |J_{\nu_k}(r)| \cdot |u_k(\omega_1) u_k(\omega_2)|. \tag{3.15}$$

To estimate this, we replace (3.7) by the estimate

$$\sum_{\nu_k \in \mathrm{Spec}\, A, \nu \le \nu_k \le \nu+1} |u_k(\omega)|^2 \le C\nu^{n-2}, \tag{3.16}$$

also due to [Ho]. Again for $r \in [0,1]$ we get an optimal estimate on $\kappa_N(r, \omega_1, \omega_2)$ by applying (3.16) and (2.5):

$$0 \le r \le 1 \Rightarrow |\kappa_N(r, \omega_1, \omega_2)| \le Cr^{\gamma}. \tag{3.17}$$

For $r \ge 1$, we do not meet with such neat success, but we proceed. We estimate the sum of the right side of (3.15) over ν_k in $\sigma_\ell(r)$, for $2 \le \ell \le 5$. By (3.16) and (2.7),

$$\sum_{\nu_k \in \sigma_2(r)} |J_{\nu_k}(r)| \cdot |u_k(\omega)|^2 \le C_K r^{-K}, \tag{3.18}$$

as in (3.9). By (2.9) we have, parallel to (3.10),

$$\sum_{\nu_k \in \sigma_3(r)} |J_{\nu_k}(r)| \cdot |u_k(\omega)|^2 \le Cr^{-1/2}. \tag{3.19}$$

By (2.14),

$$\begin{aligned} \sum_{\nu_k \in \sigma_4(r)} |J_{\nu_k}(r)| \cdot |u_k(\omega)|^2 &\le C \int_{r/4}^{4r} \nu^{-1/3} \Big(1 + \nu^{2/3} \Big|1 - \frac{\nu}{r}\Big|\Big)^{-1/4} \nu^{n-2}\, d\nu \\ &\le Cr^{n-3/2}. \end{aligned} \tag{3.20}$$

By (2.23),

$$\sum_{\nu_k \in \sigma_5(r)} |J_{\nu_k}(r)| \cdot |u_k(\omega)|^2 \le C \int_1^r \nu^{-1/2} \nu^{n-2}\, d\nu \le Cr^{n-3/2}. \tag{3.21}$$

The same estimate also follows from (2.24). In summary, we have established

$$r \ge 1 \Rightarrow |\kappa_N(r, \omega_1, \omega_2)| \le Cr^{n-3/2} = Cr^{2\gamma+1/2}. \tag{3.22}$$

Note that this is far from the estimate $|\kappa_N(r,\omega_1,\omega_2)| \le Cr^\gamma$, which holds for $N = S^{n-1}$ by (1.18). On the other hand, it is only slightly weaker than the estimate (1.20) on $|\kappa^s_{S^{n-1}}(r,\omega_1,\omega_2)|$.

A. Proof of the Weber integral formula

We desire to prove the identity

$$\int_0^\infty e^{-t\lambda^2} J_\nu(r_1\lambda)J_\nu(r_2\lambda)\lambda\,d\lambda = \frac{1}{2t}e^{-(r_1^2+r_2^2)/4t} I_\nu\Bigl(\frac{r_1r_2}{2t}\Bigr), \tag{A.1}$$

for $t, r_1, r_2 > 0$, where $J_\nu(z)$ is the standard Bessel function and $I_\nu(y) = e^{-\pi i\nu/2}J_\nu(iy),\ y > 0$, so

$$I_\nu(z) = \Bigl(\frac{z}{2}\Bigr)^\nu \sum_{k=0}^\infty \frac{1}{k!\Gamma(\nu+k+1)}\Bigl(\frac{z}{2}\Bigr)^{2k}. \tag{A.2}$$

To begin, one can expand $J_\nu(r_j\lambda)$ in power series (similar to (A.2)) and integrate term by term, to see that the left side of (A.1) is equal to

$$\frac{1}{2t}\Bigl(\frac{r_1r_2}{4t}\Bigr)^\nu \sum_{j,k\ge 0} \frac{\Gamma(\nu+j+k+1)}{\Gamma(\nu+j+1)\Gamma(\nu+k+1)}\,\frac{1}{j!k!}\Bigl(-\frac{r_1^2}{4t}\Bigr)^j\Bigl(-\frac{r_2^2}{4t}\Bigr)^k. \tag{A.3}$$

Meanwhile, by (A.2), the right side of (A.1) is equal to

$$\sum_{\ell,m\ge 0}\frac{1}{\ell!m!}\Bigl(-\frac{r_1^2}{4t}\Bigr)^\ell\Bigl(-\frac{r_2^2}{4t}\Bigr)^m \sum_{n=0}^\infty \frac{1}{n!\Gamma(\nu+n+1)}\Bigl(\frac{r_1r_2}{4t}\Bigr)^{2n}. \tag{A.4}$$

If we set $y_j = -r_j^2/4t$, we see that the asserted identity (A.1) is equivalent to the identity

$$\begin{aligned}&\sum_{j,k\ge 0}\frac{\Gamma(\nu+j+k+1)}{\Gamma(\nu+j+1)\Gamma(\nu+k+1)}\,\frac{1}{j!k!}\,y_1^jy_2^k \\ &\qquad = \sum_{\ell,m,n\ge 0}\frac{1}{\ell!m!}\,\frac{1}{n!\Gamma(\nu+n+1)}\,y_1^{\ell+n}y_2^{m+n}.\end{aligned} \tag{A.5}$$

This approach was taken in §8, Chapter 8 of [T], but no explicit proof of (A.5) was given. We fill in the details here.

We compare coefficients of $y_1^jy_2^k$ in (A.5). Since both sides of (A.5) are symmetric in (y_1,y_2), it suffices to treat the case

$$j \le k, \tag{A.6}$$

which we assume henceforth. Then we take $\ell + n = j$, $m + n = k$ and sum over $n \in \{0, \dots, j\}$, to see that (A.5) is equivalent to the validity of

$$\sum_{n=0}^{j} \frac{1}{(j-n)!(k-n)!n!\Gamma(\nu+n+1)} = \frac{\Gamma(\nu+j+k+1)}{\Gamma(\nu+j+1)\Gamma(\nu+k+1)} \frac{1}{j!k!}, \tag{A.7}$$

whenever $0 \le j \le k$. Using the identity

$$\Gamma(\nu+j+1) = (\nu+j)\cdots(\nu+n+1)\Gamma(\nu+n+1)$$

and its analogues for the other Γ-factors in (A.7), we see that (A.7) is equivalent to the validity of

$$\sum_{n=0}^{j} \frac{j!k!}{(j-n)!(k-n)!n!} (\nu+j)\cdots(\nu+n+1) = (\nu+j+k)\cdots(\nu+k+1), \tag{A.8}$$

for $0 \le j \le k$. Note that the right side of (A.8) is a polynomial of degree j in ν, and the general term on the left side of (A.8) is a polynomial of degree $j - n$ in ν.

In order to establish (A.8), it is convenient to set

$$\mu = \nu + j \tag{A.9}$$

and consider the associated polynomial identity in μ. With

(A.10)

$$p_0(\mu) = 1, \quad p_1(\mu) = \mu, \quad p_2(\mu) = \mu(\mu-1), \quad \dots \quad , p_j(\mu) = \mu(\mu-1)\cdots(\mu-j+1),$$

we see that $\{p_0, p_1, \dots, p_j\}$ is a basis of the space $\mathcal{P}_j$ of polynomials of degree j in μ, and our task is to write

$$p_j(\mu+k) = (\mu+k)(\mu+k-1)\cdots(\mu+k-j+1) \tag{A.11}$$

as a linear combination of $p_0, \dots, p_j$. To this end, define

$$T : \mathcal{P}_j \longrightarrow \mathcal{P}_j, \quad Tp(\mu) = p(\mu+1). \tag{A.12}$$

By explicit calculation,

$$\begin{aligned} p_1(\mu+1) &= p_1(\mu) + p_0(\mu), \\ p_2(\mu+1) &= (\mu+1)\mu = \mu(\mu-1) + 2\mu = p_2(\mu) + 2p_1(\mu), \end{aligned} \tag{A.13}$$

and an inductive argument gives

$$Tp_i = p_i + ip_{i-1}. \tag{A.14}$$

By convention we set $p_i = 0$ for $i < 0$. Our goal is to compute $T^k p_j$. Note that

$$T = I + N, \quad N p_i = i p_{i-1}, \tag{A.15}$$

and

$$T^k = \sum_{n=0}^{j} \binom{k}{n} N^n, \tag{A.16}$$

if $j \le k$. By (A.15),

$$N^n p_i = i(i-1)\cdots(i-n+1)p_{i-n}, \tag{A.17}$$

so we have

$$\begin{aligned} T^k p_j &= \sum_{n=0}^{j} \binom{k}{n} j(j-1)\cdots(j-n+1)p_{j-n} \\ &= \sum_{n=0}^{j} \frac{k!}{(k-n)!n!} \frac{j!}{(j-n)!} p_{j-n}. \end{aligned} \tag{A.18}$$

This verifies (A.8) and completes the proof of (A.1).

B. Proof of the Lipschitz-Hankel integral formula

We desire to prove the identity

$$\int_0^\infty e^{-y\lambda} J_\nu(r_1\lambda) J_\nu(r_2\lambda)\, d\lambda = \frac{1}{\pi}(r_1 r_2)^{-1/2}\, Q_{\nu-1/2}\Big(\frac{r_1^2 + r_2^2 + y^2}{2 r_1 r_2}\Big), \tag{B.1}$$

due to Lipschitz and Hankel, of great use for analysis on cones (cf. [CT]). We derive (B.1) from the identity

$$\int_0^\infty e^{-t\lambda^2} J_\nu(r_1\lambda) J_\nu(r_2\lambda)\lambda\, d\lambda = \frac{1}{2t} e^{-(r_1^2+r_2^2)/4t}\, I_\nu\Big(\frac{r_1 r_2}{2t}\Big), \tag{B.2}$$

whose proof was just given in Appendix A. Here, as in (A.1),

$$I_\nu(y) = e^{-\pi i\nu/2}\, J_\nu(iy), \quad y > 0. \tag{B.3}$$

To work on (B.2), we use the subordination identity

$$\lambda^{-1} e^{-y\lambda} = \pi^{-1/2} \int_0^\infty e^{-y^2/4t} e^{-t\lambda^2} t^{-1/2}\, dt; \tag{B.4}$$

cf. [T], Chapter 3, (5.31) for a proof. Plugging this into the left side of (B.1), and using (B.2), we have

$$\text{(B.5)} \qquad LHS(B.1) = \frac{1}{2\sqrt{\pi}} \int_0^\infty e^{-(r_1^2+r_2^2+y^2)/4t}\, I_\nu\Big(\frac{r_1 r_2}{2t}\Big) t^{-3/2}\, dt.$$

The change of variable $s = r_1 r_2 / 2t$ gives

$$\text{(B.6)} \qquad LHS(B.1) = \sqrt{\frac{1}{2\pi}}\, (r_1 r_2)^{-1/2} \int_0^\infty e^{-s(r_1^2+r_2^2+y^2)/2r_1r_2}\, I_\nu(s) s^{-1/2}\, ds.$$

Thus the asserted identity (B.1) follows from the identity

$$\text{(B.7)} \qquad \int_0^\infty e^{-sz} I_\nu(s) s^{-1/2}\, ds = \sqrt{\frac{2}{\pi}} Q_{\nu-1/2}(z), \quad z > 0.$$

As for the validity of (B.7), we mention two identities. First, we have

$$\text{(B.8)} \qquad \begin{aligned} &\int_0^\infty e^{-sz} J_\nu(\lambda s) s^{\mu-1}\, ds \\ &= \Big(\frac{\lambda}{2}\Big)^\nu z^{-\mu-\nu} \frac{\Gamma(\mu+\nu)}{\Gamma(\nu+1)}\, {}_2F_1\Big(\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2}, \frac{\mu}{2}+\frac{\nu}{2}; \nu+1; -\frac{\lambda^2}{z^2}\Big). \end{aligned}$$

This can be proven by expanding $J_\nu(\lambda s)$ in a power series in λs and integrating term by term. Cf. (8.42) of [T], Chapter 8. Next, there is the classical representation of the Legendre function $Q_{\nu-1/2}(z)$ as a hypergeometric function:

$$\text{(B.9)} \quad Q_{\nu-1/2}(z) = \frac{\Gamma(1/2)\Gamma(\nu+1/2)}{\Gamma(\nu+1)}\, (2z)^{-\nu-1/2}\, {}_2F_1\Big(\frac{\nu}{2}+\frac{3}{4}, \frac{\nu}{2}+\frac{1}{4}; \nu+1; \frac{1}{z^2}\Big);$$

cf. [Leb], (7.3.7). If we apply (B.8) with $\lambda = i,\ \mu = 1/2$, then (B.7) follows.

Remark. Formulas (B.1) and (B.2) are proven in the opposite order in [W].

C. Some integral formulas for $J_\nu(r)$

In addition to the integral formula (2.3) for $J_\nu(r)$, there are some others that are useful for asymptotic expansions and estimates of $J_\nu(r)$ on $Q = \{(\nu, r) : \nu \geq \gamma, r \geq 0\}$. In particular there is the Schläfli integral

$$\text{(C.1)} \qquad J_\nu(r) = \frac{1}{2\pi i} \int_{\infty-\pi i}^{\infty+\pi i} e^{r\,\sinh\tau - \nu\tau}\, d\tau,$$

the integral being taken along a path $\tau(t)$ asymptotic to the line $\operatorname{Im} z = -\pi$ as $t \to -\infty$ and asymptotic to the line $\operatorname{Im} z = \pi$ as $t \to +\infty$. Cf. [Olv], p. 58. There is flexibility in selecting γ (by Cauchy's integral theorem). In case $r = \nu \operatorname{sech} a,\ a \in \mathbb{R}^+$, it is convenient to take

$$\gamma = \partial\Omega_a, \quad \Omega_a = \{z \in \mathbb{C} : |\operatorname{Im} z| \le \pi,\ \operatorname{Re} z \ge a\}. \tag{C.2}$$

This gives

$$\begin{aligned} J_\nu(\nu \operatorname{sech} a) = {} & \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-\nu(a - \tanh a\ \cos t + it - i \sin t)}\, dt \\ & - \frac{\sin \pi\nu}{\pi}\, e^{-\nu a} \int_0^\infty e^{-\nu(t + \sinh t + \tanh a \cosh t)}\, dt. \end{aligned} \tag{C.3}$$

Noting that $\operatorname{sech} ia = \sec a$, we also have, by analytic continuation,

$$\begin{aligned} J_\nu(\nu \sec a) = {} & \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\nu(a - \tan a \cos t + t - \sin t)}\, dt \\ & - \frac{\sin \pi\nu}{\pi} e^{-i\nu a} \int_0^\infty e^{-\nu(t + \sinh t + i \tan a \cosh t)}\, dt, \end{aligned} \tag{C.4}$$

for $a \in [0, \pi/2)$. Note that (C.3) represents $J_\nu(r)$ for $0 < r \le \nu$ and (C.4) represents $J_\nu(r)$ for $r \ge \nu > 0$.

As described on p. 134 of [Olv], one has, as $\nu \to +\infty$,

$$J_\nu(\nu \operatorname{sech} a) \sim \frac{e^{-\nu(a - \tanh a)}}{\pi i} \sum_{k \ge 0} b_k(a)\, \nu^{-k-1/2}. \tag{C.5}$$

This is valid uniformly for $a \in [a_0, a_1]$, given $0 < a_0 < a_1 < \infty$. Similarly, one has

$$J_\nu(\nu \sec a) \sim \Bigl(\frac{\pi\nu}{2} \tan a\Bigr)^{-1/2} \cos\Bigl(\nu \tan a - \nu a - \frac{\pi}{4}\Bigr) + O(\nu^{-3/2}), \tag{C.6}$$

valid uniformly for $a \in [a_0, a_1]$, given $0 < a_0 < a_1 < \pi/2$. The results (C.5)–(C.6) also follow from the stronger result (2.10)–(2.12).

Our purpose in recording these results here is to provide material needed to establish the estimate

$$|J_\nu(r)| \le C r^{-1/2}, \quad (\nu, r) \in Q_5. \tag{C.7}$$

See the end of §2 for this.

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1A.

Remark. The various formulas given above for $E_t(r_1, \omega_1, r_2, \omega_2)$ and $S_t(r_1, r_2, A)$ hold for $t, r_1, r_2 \in (0, \infty)$, but there is no difficulty passing to the limit $r_1 \to 0$, using

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^\infty \frac{(-1)^k}{k!\Gamma(k+\nu+1)} \left(\frac{z}{2}\right)^{2k}. \tag{1.26}$$

Noting that $\nu \geq \gamma$ for $\nu \in \operatorname{Spec} A$, we obtain from (1.11) that

$$S_t(0, r_2, A) = \frac{2}{\Gamma(n/2)} \frac{1}{(4it)^{n/2}} e^{-r_2^2/4it} P_0, \tag{1.27}$$

where P_0 is the orthogonal projection of $L^2(N)$ onto $\operatorname{Ker} \Delta_N = \{\text{constants}\}$, whose integral kernel is $\kappa_{P_0}(\omega_1, \omega_2) = A(N)^{-1}$, $A(N)$ denoting the $(n-1)$-dimensional area of N. This leads to the formula

$$E_t(0, \omega_1, r_2, \omega_2) = \frac{A(S^{n-1})}{A(N)} \frac{1}{(4\pi it)^{n/2}} e^{-r_2^2/4it}, \tag{1.28}$$

where $A(S^{n-1}) = 2\pi^{n/2}/\Gamma(n/2)$ is the area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$. This identity can also be derived directly by separation of variables and comparison with the Euclidean case.