ELECTROSTATIC SCREENING

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ABSTRACT. Using the methods of partial differential equatiions and functional analysis, we investigate the electromagnetic field in the presence of a screen composed of wires of radius r, spaced at distance R, spread over a surface S. In the limit as r and R converge to zero, if $(R \log r)^{-1} \to -\infty$ the field in the presence of the screen converges to the field with a conducting sheet spread over S. If $(R \log r)^{-1} \to 0$ the field converges to the field with no conductors.

1. Introduction

It is well known that a region enclosed by a mesh of conducting wire is shielded from external static electric fields. In this sense the mesh acts like a solid sheet of conductor. On the other hand, it is clear that if the wires of the mesh are sufficiently narrow (for a fixed mesh width) then they will have a negligible effect on the electric field. In this note we will study the problem of determining what range of physical parameters correspond to these two types of behavior. If the screen consists of wires of radius r whode axes are spaced at approximately distance R from each other the critical parameter is $(-R \log r)^{-1} = \gamma$. We consider screens spread over a surface S, in the limit as r and R approach zero and prove that for a charge distribution if $\gamma \to \infty$ then the field in the presence of the screens converges to the field in the presence of a sheet of conductor spread over S (Theorem 2). In the opposite extreme case if $\gamma \to 0$ then the field converges to the field without any conductors present, that is, the screen becomes negligible.

Remark. This is a TeXed version of the paper [RT], originally produced on an old-fashioned typewriter. Further work appears in [CHT].

2. Variational formulation of the basic boundary problem

We seek the electrostatic potential u in the exterior of a finite number of conductore $\kappa_1, \kappa_2, \dots, \kappa_j$, arising from a charge distribution with density $\rho(x)$. For convenience we suppose that the whole system lies inside a very large but bounded region $\mathcal{R}$ whose boundary is kept at potential zero and is assumed to be smooth.

With a little extra effort the problem in unbounded regions can also be handled by our methods. The boundary value problem for u is

$$\Delta u = -4\pi\rho, \quad \text{in } \mathcal{R} \setminus \bigcup \kappa_i, \tag{1}$$

$$u = \text{ constant on each } \kappa_i, \quad i = 1, \ldots, j, \tag{2}$$

$$\int_{\partial\kappa_i} \frac{\partial u}{\partial \nu} = 0, \quad i = 1, \ldots, j, \tag{3}$$

$$u = 0 \quad \text{on} \quad \partial\mathcal{R}. \tag{4}$$

From a mathematical standpoint the condition (3), which asserts that the conductors carry no charge, is the most troublesome, and we will give a weak or variational formulation in which (3) becomes a natural boundary condition.

Let $K = \cup\kappa_i$, $\Omega = \mathcal{R} \setminus K$, and $H^1(\Omega)$ the Sobolev space of functions on Ω which are square integrable, together with their partial derivatives of order one.

Definition 1. $\mathcal{B}$ is the closed subspace of $H^1(\Omega)$ consisting of functions u that vanish on $\partial\mathcal{R}$ and in addition are constant on each $\partial\kappa_i$, for $i = 1, 2, \ldots, j$. For $u, v \in H^1(\Omega)$, let

$$a(u, v) = -\int_\Omega \nabla u \cdot \nabla v.$$

It is not hard to show that u is a solution of (1)–(4) if and only if $u \in \mathcal{B}$ and

$$a(u, v) = 4\pi \int_\Omega \rho(x) v(x)\, dx, \quad \forall\, v \in \mathcal{B}. \tag{5}$$

Equation (5) is just the Euler-Lagrange equation associated with Thompson's principle: u minimizes $-a(u, u)/2 + \int_\Omega \rho u$ over all $u \in \mathcal{B}$. Note that (3) is a natural boundary condition. It is useful to notice that if $u \in \mathcal{B}$ satisfies (5) then $\Delta u = -4\pi\rho$ in the sense of distributions and u is constant on each $\partial\kappa_i$, so the regularity theorems for the Dirichlet problem can be applied to show that u is smooth provided that ρ and each $\partial\kappa_i$ are smooth, which we will assume henceforth.

The quadratic form a on $L^2(\Omega)$ with domain $D(a) = \mathcal{B}$ is closed, symmetric, and nonpositive. It is well known [1] that there is a self-adjoint operator Δ, defined by the recipe:

$$D(\Delta) = \{u \in \mathcal{B} : \exists f \in L^2(\Omega) \text{ such that } a(u, v) = (f, v)_{L^2(\Omega)}\},$$
$$\Delta u = f \quad \text{for} \quad u \in D(\Delta).$$

With the aid of the regularity theorems mentioned above, one can show that

$$D(\Delta) = \Big\{u \in H^2(\Omega) : u \in \mathcal{B} \text{ and } \int_{\partial\kappa_i} \frac{\partial u}{\partial \nu} = 0,\ i = 1, \ldots, j\Big\},$$

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}, \quad \text{for} \quad u \in D(\Delta).$$

The solution to the electrostatic problem (1)–(4) is therefore $\Delta^{-1}(-4\pi\rho)$, the inverse of Δ applied to $-4\pi\rho$.

3. A theorem on vanishing screens

We now pose the basic problem. For each integer n we consider the electrostatics problem in the presence of conductors $\kappa_1^n, \kappa_2^n, \ldots, \kappa_{i_n}^n$, and we ask whether the effect of the conductors has some limiting behavior as $n \to \infty$. In this section we prove a theorem which asserts that the effect of the conductors disappears as $n \to \infty$ provided they are sufficiently small. As an application we obtain the result on vanishing screens mentioned in the introduction.

The appropriate measure of smallness turns out to be electrostatic capacity. Recall that for reasonable subsets Λ of $\mathbb{R}^3$, $\mathrm{cap}(\Lambda)$ is defined as follows. Let v be the solution of the boundary value problem

$$\begin{aligned} \Delta v &= 0 \quad \text{on} \quad \mathbb{R}^3 \setminus \Lambda, \\ v &= O(|x|^{-1}) \quad \text{as} \quad |x| \to \infty, \\ v &= 1 \quad \text{on} \quad \partial\Lambda. \end{aligned}$$

Then $-\int_{|x|=L} \partial v/\partial r$ is independent of L for L large and is the total charge on a conductor occupying the region Λ and raised to potential one. This quantity is $\mathrm{cap}(\Lambda)$, the capacity of Λ.

Notations. Let Δ_n, a_n, $\mathcal{B}_n$ be the operator, form, and form domain on $\Omega_n = \mathcal{R} \setminus \cup_{i=1}^{j_n} \kappa_i^n$, as defined in §2. In addition, for $v \in L^2(\mathcal{R})$, let $P_n v \in L^2(\Omega_n)$ be the restriction of v to Ω_n. Any element of $L^2(\Omega_n)$ is considered as an element of $L^2(\mathcal{R})$ by extending it to vanish on the union of the κ_i^n. Let $K(n) = \cup_i \kappa_i^n$ denote this union. We suppose all $K(n)$ are contained in some compact set $\Gamma \subset \mathcal{R}$.

The main tool we use to show that $K(n)$ vanishes is Theorem 3.1 of [2]. This asserts that $f(\Delta_n)P_n u \to f(\Delta)u$ in $L^2(\mathcal{R})$ for all $u \in L^2(\mathcal{R})$ and any f bounded and continuous on $(-\infty, 0]$ provided that Ω_n satisfy mild regularity conditions, that the quadratic form $a(u,u)$ satisfies the coerciveness hypothesis

$$-a(u,u) \geq \int_{\Omega_n} |\nabla u|^2, \quad \forall\, u \in \mathcal{B}_n,$$

and the following two special assumptions:

(A) There exist extension operators $E_n : \mathcal{B}_n \to \mathcal{B}$ (the domain of the form $a(u,v)$ on $\mathcal{R}$ without conductors) with the properties

(i) $E_n u = u$ on Ω_n for all $u \in \mathcal{B}_n$.

(ii) There is a constant M such that for all n and $u \in \mathcal{B}_n$,

$$\|E_n u\|_{H^1(\mathcal{R})} \le M \|u\|_{H^1(\Omega_n)}.$$

Next, either

(B) $\mathrm{Meas}(K(n)) \to 0$, and, if $u \in \mathcal{B}$, there exist $u_j \to u$ in $\mathcal{B}$ such that $u_j|_{\Omega_j} \in \mathcal{B}_j$,

or

(B$'$) $\mathrm{cap}(K(n)) \to 0$ as $n \to \infty$.

That (A) and (B$'$) imply operator convergence is stated in Theorem 4.2 of [2]. Alternatively, condition (B$'$) implies condition (B).

Theorem 1. *Suppose there is a compact set $\Gamma \subset \mathcal{R}$ with $K(n) \subset \Gamma$ for all n and that $\mathrm{cap}(K(n)) \to 0$ as $n \to \infty$. Then for any continuous function f on $(-\infty, 0)$ bounded at $-\infty$ and any $u \in L^2(\mathcal{R})$, we have*

$$f(\Delta_n) P_n u \longrightarrow f(\Delta) u \quad \text{in} \quad L^2(\mathcal{R}),$$

where Δ is the operator on $\mathcal{R}$ without any conductors.

As a particular example for $\rho \in L^2(\mathcal{R})$ with ρ supported in the exterior of all conductors we can take $f(x) = 1/x$ to get $\Delta_n^{-1}\rho \to \Delta^{-1}\rho$ in $L^2(\mathcal{R})$. Thus the solutions of the electrostatics problem converge to the solution to the problem with no conductors at all.

Proof. Note that the spectra $\sigma(\Delta)$ and $\sigma(\Delta_n)$ are all contained in $(-\infty, \delta)$ for some $\delta < 0$, so f can be altered to be bounded and continuous on $(-\infty, 0]$ without changing $f(\Delta_n)$ or $f(\Delta)$. To complete the proof, it is only necessary to verify hypothesis (A).

To describe E_n, notice that if $u \in \mathcal{B}_n$ then u is constant on $\partial\kappa_i^n$, $i = 1, 2, \ldots, j_n$, say $u = c_i$ on $\partial\kappa_i^n$. Define $E_n u = c_i$ on κ_i^n. It is clear that $\int_{\mathcal{R}} |\nabla E_n u|^2 = \int_{\Omega_n} |\nabla u|^2$. Furthermore, since $E_n u = 0$ on $\partial\mathcal{R}$, we have

$$\int_{\mathcal{R}} |E_n u|^2 \le \frac{1}{\lambda} \int_{\mathcal{R}} |\nabla E_n u|^2,$$

where $-\lambda$ is the largest eigenvalue of the Laplacian on $\mathcal{R}$ with the Dirichlet boundary condition on $\partial\mathcal{R}$. Thus (ii) is satisfied with $M = 1 + \lambda^{-1}$, and the proof is complete.

It is easy to apply this result to screens. The basic fact that is needed is that the capacity of a solid circular cylinder of length L and radius r is proportional to $-L/\log r$. Similarly a not excessively curved piece of wire of length L and radius r has capacity $\approx -L/\log r$. In addition, capacity is a subadditive set function, that is, $\mathrm{cap}(\cup A_i) \leq \sum_i \mathrm{cap}(A_i)$ for any countable union of sets. Thus the capacity of a curved screen of fixed area with wires of radius r and spacing R between axes of wires is $O(-1/R\log r)$. Thus if $K(n)$ is a screen as above with r and R approaching zero as $n \to \infty$ in such a way that $1/R\log r \to 0$, then the effect of the screen is negligible for n large.

For the electrostatic problem, $\mathrm{cap}\, K(n) \to 0$ is by no means a necessary condition for the $K(n)$ to have a negligible effect. Suppose for example that $K(n)$ consists of n balls, of radius r_n, and say their centers are spaced at a distance at least $4r_n$. Defining extension operators E_n as in the proof of Theorem 1, it is easy to see that hypothesis (A) is satisfied. We show that hypothesis (B) is verified, assuming vol $K(n) = (4\pi/3)nr_n^3 \to 0$.

Define a continuous linear map $Q : H^1(B_2) \to H^1(B_2)$, with $B_2 = \{x : |x| \leq 2\}$, such that

(i) $$Qu(x) = u(x) \quad \text{for} \quad 3/2 \leq |x| \leq 2,$$

(ii) $$Qu(x) \quad \text{is constant for} \quad |x| \leq 1,$$

(iii) $$\int_{B_2} |Qu|^2 \leq C_0 \int_{B_2} |u|^2,$$

(iv) $$\int_{B_2} |\nabla Qu|^2 \leq C_0 \int_{B_2} |\nabla u|^2.$$

This is easy to arrange. Given this, you can scale B_2 to $B_{2r_n}(\xi_{jn}) = \{x : |x-\xi_{jn}| \leq 2r_n\}$ and get maps with the same properties as (i)–(iv), with the same constant C_0. Thus you get maps $Q_n : \mathcal{B}_n \to \mathcal{B}$ such that

(i) $$Q_n u(x) = u(x), \qquad x \notin \bigcup_j B_{2r_n}(\xi_{jn}),$$

(ii) $$\|Q_n u\|^2_{H^1(B_{2r_n}(\xi_{jn}))} \leq C_0 \|u\|^2_{H^1(B_{2r_n}(\xi_{jn}))},$$

(iii) $$Q_n u\big|_{\Omega_n} \in \mathcal{B}_n.$$

Now with $u_n = Q_n u$ you get

$$\begin{aligned} \|u_n - u\|^2_{H^1(\mathcal{R})} &= \sum_j \|u_n - u\|^2_{H^1(B_{2r_n}(\xi_{jn}))} \\ &\leq 4C_0 \sum_j \|u\|^2_{H^1(B_{2r_n}(\xi_{jn}))} \\ &\longrightarrow 0 \quad \text{as } n \to \infty, \end{aligned}$$

since meas $\cup_j B_{2r_n}(\xi_{jn}) \to 0$. This verifies hypothesis (B).

The conclusion is that if $K(n)$ consists of n "well spaced" balls of radius r_n, then $K(n)$ disappears as $n \to \infty$, assuming only that vol $K(n) \to 0$.

4. The case of electromagnetic screening

In this section we will investigate the observed phenomenon of screens behaving like solid barriers. To be more precise, suppose that $K(n)$ is a conducting screen, with wires of radius r and spacing R, spread out over the surface S, and that r and R tend to zero as $n \to \infty$. If $(-R\,\log r)^{-1} \to +\infty$ as $n \to \infty$, then for any charge distribution ρ on $\mathcal{R}$ the solutions, $\Delta_n^{-1}\rho$, of the electrostatic problems in $\mathcal{R} \setminus K(n)$ converge to the solution, u, of the problem where S is covered by a sheet of perfect conductor, that is,

$$\Delta u = -4\pi\rho \quad \text{in} \quad \mathcal{R} \setminus S, \tag{6}$$

$$u = \ \text{constant on} \quad S, \tag{7}$$

$$\int_S \left[\frac{\partial u}{\partial \nu}\right] = 0 \quad ([\ ] \text{ denotes the jump on crossing } S), \tag{8}$$

$$u = 0 \quad \text{on} \quad \partial\mathcal{R}. \tag{9}$$

This result complements the result of §3 and confirms the idea that the parameter $(-R\,\log r)^{-1}$ is a reasonable measure of the solidity of a screen. It is interesting to note that the same parameter occurs in the clever special problem treated in §203 of Maxwell's treatise [3]. In addition, as Maxwell observed, a complete screen is not needed, just one family of parallel wires which are connected to each other in any way at all will suffice.

We must make precise the notion of a screen spread smoothly over S, where S is an open subset of a smooth surface in the interior of $\mathcal{R}$. The intuitive idea is to take a piece of planar screen and give a mapping of the planar region to the surface. Precisely, if $s \in S$ and $\mathcal{O}$ is an open neighborhood of s in $\mathbb{R}^3$, then a mapping $\varphi : \mathcal{U} \to \mathcal{O}$ is called a δ-bending if

(i) $\mathcal{U}$ is a cube $|x_i| < \alpha, \quad i = 1, 2, 3,$

(ii) $\psi\big(\mathcal{U} \cap \{x_3 = 0\}\big) = S \cap \mathcal{O},$

(iii) ψ is a diffeomorphism with $\|J_\psi\|$ and $\|J_{\psi^{-1}}\|$ less than δ, where $\|J\|$ is the norm of the Jacobian matrix.

Screens are laid on S by placing a screen in the $x_3 = 0$ plane of $\mathcal{U}$ and carrying it to S by the map ψ.

Definition 2. A *patch of δ-bent screen* on S consisting of wires of radius r and spacing R is the set $\psi(\Sigma)$, where $\psi : \mathcal{U} \to \mathcal{O}$ is δ-bending and

$$\Sigma = \{x \in \mathcal{U} : (x_1 - jR)^2 + x_2^2 \le r^2, \text{ for some } j\}.$$

In addition, we require $R > 3r$.

To form a picture, notice that the wires in Σ are parallel to the x_2-axis. The only interesting case of screening is when the screen has large gaps, that is, $R >> r$.

Definition 3. A sequence of systems of conductors will be called *screens smoothly covering* S if there is a $\delta > 0$, an $\alpha > 0$, and an integer M such that (1) each system consists of at most M patches of δ bent screen on S, (2) the sets $\psi(\mathcal{U}_i),\ i = 1, \dots, M$ cover S for each system, and (3) the lengths of the sides of the cubes are all greater than α.

It is important that the electromagnetic potential be constant on the screen, not just on the individual wires from which it is constructed (for which, see §5). There are two ways we could arrange this. In one approach, we could suppose that a few wires are added to the screen so that it becomes a connected set. In the second, we just prescribe the constancy of the potential on the screen as a boundary condition. Both methods yield the same results, and we will adapt the second, so that the basic boundary value problem becomes (1)–(4) with $j = 1$ and κ_1 the screen on S.

As in §2, the boundary value problem (6)–(9) can be given a variational formulation in which $u = -4\pi\Delta_\infty^{-1}\rho$, where Δ_∞ is the operator on $L^2(\mathcal{R})$ defined by the quadratic form

$$a_\infty(u, v) = \int\limits_{\mathcal{R}} \nabla u \cdot \nabla v,$$
$$D(a_\infty) = \{u \in H^1(\mathcal{R}) : u = 0 \text{ on } \partial\mathcal{R},\ u \text{ constant on } S\}.$$

Theorem 2. *Suppose that $K(n),\ n = 1, 2, \dots,$ are screens smoothly placed on S, where $K(n)$ consists of wires of radius r_n and spacing R_n. Let Δ_n be the operator on $L^2(\mathcal{R} \setminus K(n))$ as in §2, and $P_n : L^2(\mathcal{R}) \to L^2(\mathcal{R} \setminus K(n))$ be the restriction mapping. If $(-R_n \log r_n)^{-1} \to \infty$, then for any continuous function f on $(-\infty, 0)$ bounded at $-\infty$,*

$$f(\Delta_n)P_n\rho \longrightarrow f(\Delta_\infty)\rho \quad \text{in} \quad L^2(\mathcal{R}),$$

for each $\rho \in L^2(\mathcal{R})$.

Proof. We describe the modifications that are required to adopt the methods of our paper [2] on wild perturbations to this setting. First we define uniformly bounded extension operators $E_n : \mathcal{B}_n \to \mathcal{B}_\infty \equiv D(A_\infty)$ by extending functions to be constant inside κ_i^n. As in our previous work (see the proof of Theorem 1.2 in [2]) it suffices to prove the result for $f(x) = (1-x)^{-1}$. Imitating the proof of Theorem 4.4 of [2], we notice that for $g \in L^2(\mathcal{R})$,

$$\begin{aligned}\|(I-\Delta_n)^{-1}P_n g\|^2_{H^1(\mathcal{R}\setminus K(n))} &= ((I-\Delta_n)(I-\Delta_n)^{-1}P_n g, (I-\Delta_n)^{-1}P_n g)_{\mathcal{R}\setminus K(n)} \\ &= (P_n g, (I-\Delta_n)^{-1}P_n g)_{\mathcal{R}\setminus K(n)} \\ &\leq \|g\|^2_{L^2(\mathcal{R})},\end{aligned}$$

so that $w_n \equiv E_n(I-\Delta_n)^{-1}P_n g$ is a bounded sequence in $H^1(\mathcal{R})$. Using equation (5) on $\Omega = \mathcal{R}\setminus K(n)$ for the function w_n, it is easy to show that if w is a limit point of the sequence $\{w_n\}$ in the weak topology of $H^1(\mathcal{R})$, then

$$\int_{\mathcal{R}} (wu - \nabla w \cdot \nabla u) = \int_{\mathcal{R}} gu, \tag{10}$$

for all $u \in H^1(\mathcal{R})$ such that u is constant on a neighborhood of S. Since these u are dense in $\mathcal{B}_\infty = D(a_\infty)$, (10) holds for all $u \in \mathcal{B}_\infty$. To show that $w = (I-\Delta_\infty)^{-1}g$, it therefore suffices to prove that $w \in \mathcal{B}_\infty$, that is, w is constant on S and $w = 0$ on $\partial\mathcal{R}$. The latter is true since $\{v \in H^1(\mathcal{R}) : v = 0 \text{ on } \partial\mathcal{R}\}$ is a closed linear subspace, hence weakly closed. That w is constant on S lies considerably deeper. The crucial step is the following.

Lemma 3. *Let $\mathcal{U}, \Sigma, r$, and R be as in Definition 2, and let $\mathcal{U}_H = \mathcal{U} \cap \{|x_3| \leq H\}$. Then there is a constant b, independent of H, such that for all $v \in H^1(\mathcal{U})$ with $v|_\Sigma = 0$,*

$$\frac{\int_{\mathcal{U}_H} |\nabla v|^2}{\int_{\mathcal{U}_H} |v|^2} \geq \frac{b}{H^2 - HR\log(r/R)},$$

provided $H > R > 3r$.

We postpone the proof of this lemma to the end of this section. To proceed, let $\mathcal{U}_1^n, \ldots, \mathcal{U}_{j_n}^n$ be cubes with

$$\bigcup_i \psi_i^n(\mathcal{U}_i^n) \supset S,$$

the screen $K(n) = \cup_i \psi_i^n(\sigma_i^n)$. Let $c_n = w_n|_{\text{screen}}$, and apply (11) to $w_n \circ \psi_i^n - c_n$. In this case, $R_n \log(r_n/R_n) \to 0$ as $n \to \infty$, so the right side of (11) behaves like $\text{const} \cdot H^{-2}$ for n large. Letting $S_H^n = \cup_i \psi_i^n(\mathcal{U}_H)$, we get

$$\int_{S_H^n} |w_n - c_n|^2 \leq \text{ const.} \cdot H^2. \tag{12}$$

Let $S_H = \{x : \operatorname{dist}(x, S) \leq H\}$. Then since $|c_n - c_m| \leq |w_n - c_n| + |w_n - w_m| + |w_m - c_m|$, and since, by virtue of (w_n) converging weakly in $H^1(\mathcal{R})$, we have norm convergence in $L^2(\mathcal{R})$, it follows that, for large m and n,

$$\int_{S_H} |c_n - c_m|^2 \leq \text{ const.} \cdot H^2,$$

provided $\delta H > R_n, R_m$. Since $\operatorname{Vol}(S_H)$ approaches zero like a multiple of H, we get $|c_n - c_m|^2 = O(H)$ for $H > R_n, R_m$. Letting n, m tend to infinity, we see that $\{c_n\}$ is a Cauchy sequence, so $c_n \to c$ for some real c. Passing to the limit in (12) yields

$$\frac{1}{H} \int_{S_H} |w - c|^2 = O(H), \tag{13}$$

and it follows that $w = c$ on S, since

$$\int_S |w - c|^2 \leq \text{ const. } \lim_{H \to 0} \frac{1}{H} \int_{S_H} |w - c|^2.$$

We have now shown that w_n converges weakly in $H^1(\mathcal{R})$ to $w = (I - \Delta_\infty)^{-1} g$. Since $\|w_n\|_{H^1(\mathcal{R})}$ is bounded independent of n, it follows by the Rellich compactness theorem that $\{w_n\}$ is precompact in $L^2(\mathcal{R})$. Since w_n converges weakly to w in $H^1(\mathcal{R})$, it follows that $w_n \to w$ in norm in $L^2(\mathcal{R})$, which is the desired result.

It remains to prove Lemma 3. The proof of (11) is reduced to a two dimensional problem by considering the x_2 = constant cross sections of $\mathcal{U}_H$. For these cross sections $\mathfrak{X}$, we prove that

$$\int_{\mathfrak{X}} \left[\left(\frac{\partial v}{\partial x_1} \right)^2 + \left(\frac{\partial v}{\partial x_2} \right)^2 \right] dx_1 \, dx_2 \geq \frac{\text{const.}}{H^2 - HR \log(r/R)} \int_{\mathfrak{X}} v^2, \tag{14}$$

for v that vanish on Σ. This in turn can be proved by chopping the cross section $\mathfrak{X}$ into punctured rectangles, of the following form. The rectangles lie in the x_1, x_3-plane. They are symmetric about the x_3-axis, of height $2H$, and width R (the spacing between wires). Each one has the cross section of a wire (of radius r) at its center. Thus each rectangle has height $2H$, width R, and a puncture at its center of radius r.

It suffices to prove (14) when the integration is over just one of these punctured rectangles. The lower bound for such an integral as arises on the left side of (14) is proved exactly as the inequality (4.1) of [2], so we do not reproduce the argument here.

This completes the proof of Theorem 2.

5. Screens whose wires are not connected

The phenomenon just considered in §4 has a great deal in common with the behavior of the Dirichlet problem, although the proof in the case of the electrostatic boundary problem is a little more involved. It is interesting to note that the electrostatic problem can exhibit behavior markedly different from that of the Dirichlet problem. For example, suppose the wire screen described above consists of wires that are not connected, that is, not at a common potential. If the wires are parallel to a vector field X on the surface S, and if $(-R_n \log r_n)^{-1} \to \infty$, then we claim that the u_n converge to a solution to the problem

$$\Delta u = -4\pi\rho, \quad \text{on } \mathcal{R} \setminus S, \tag{14}$$

$$[u] = 0, \quad \text{on } S, \tag{15}$$

$$Xu = 0, \quad \text{on } S, \tag{16}$$

$$\int_S \Big[\frac{\partial u}{\partial \nu}\Big] v = 0, \quad \forall\, v \in C^\infty(S) \text{ with } Xv = 0, \tag{17}$$

$$u = 0, \quad \text{on } \partial\mathcal{R}. \tag{18}$$

Since this is not a straightforward application of previously stated results, we indicate a proof. Let $u_n = -4\pi\Delta_n^{-1}\rho$, where Δ_n is defined on $\mathcal{R}$ with electrostatic boundary conditions on the wires $K(n)$, and u_n is extended by a constant on each wire. As usual, $\{u_n\}$ is bounded in $H_0^1(\mathcal{R})$, so has a weak limit point $u \in H_0^1(\mathcal{R})$. Clearly u satisfies (14), (15), and (18) above, so we need to prove (16) and (17). Furthermore, we need only consider those ρ that vanish in a neighborhood of S, since these are dense in $L^2(\mathcal{R})$. If we prove

$$\begin{aligned} -\int \nabla u \cdot \nabla v = (-4\pi\rho, v), \quad \forall\, v \in B, \text{ where,} \\ B = \{v \in H^1(\mathcal{R}) : Xv = 0 \text{ on } S\}, \end{aligned} \tag{19}$$

then (17) will arise as a natural boundary condition.

To prove (19), we need only observe that for each $v \in B$ there exist $v_n \in B$ such that v_n is constant on each wire of $K(n)$ and $v_n \to v$ in B as $n \to \infty$. Then (19) holds for v_n and we can pass to the limit. The existence of such v_n is proved by constructing operators analogous to the Qs at the end of §3.

It remains to prove that $Xu =$ on S, i.e., that $u \in B$. Indeed, by previous calculations,

$$\frac{1}{H}\int_{S_H} |u_n - c_n|^2 \le b\Big(H - R_n \log \frac{r_n}{R_n}\Big).$$

This time, c_n is not a constant, but it is constant on each wire of $K(n)$, and in the direction normal to S. It merely varies from wire to wire. Thus $c_n \in L^2(S)$ and $Xc_n = 0$. A trivial estimate is

$$\frac{1}{H}\int\limits_{S_H} |u_n - \tilde{u}_n|^2 \le \beta(H) \longrightarrow 0, \quad \text{as} \quad H \to 0,$$

where $\tilde{u}_n = u_n|_S$, independent of the normal variable. Putting these together and letting $H \to 0$ yields

$$\int\limits_S |\tilde{u}_n - c_n|^2 \longrightarrow 0, \quad \text{as} \quad n \to \infty.$$

Since $\tilde{u}_n \in H^{1/2}(S)$ is bounded, passing to a subsequence yields $\tilde{u}_n \to u|_S$ in $L^2(S)$. Hence $c_n \to u|_S$ in $L^2(S)$, so $Xu = 0$ on S, as desired.

An even greater disparity is observed if $K(n)$ consists of n balls of radius r_n, with centers ξ_{jn} lying on S and spaced apart a distance at least $4r_n$ (or $K(n)$ could consist of disks, the intersection of S with these balls). If these obstacles are connected, say by arbitrarily thin wires, arguments as in the proof of Theorem 2 show that $K(n)$ behaves in the limit as a solid screen S, provided $nr_n \to \infty$. For this proof, Lemma 4.5 of [2] is needed in place of (11). On the other hand, surely Vol $K(n) \to 0$, so if these balls are not connected up, then, as we have seen at the end of §3, the obstacles disappear as $n \to \infty$.

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References

1. T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, New York, 1966.

2. J. Rauch and M. Taylor, Potential and scattering theory on wildly perturbed domains, J. Funct. Anal. 18 (1975), 27–59.

3. J. C. Maxwell, *A Treatise on Electricity and Magnetism*, Dover Publ., New York, 1954.

[RT] J. Rauch and M. Taylor, Electrostatic Screening, Jour. Math. Phys. 16 (1975), 284–288.

[CHT] S. Chapman, D. Hewett, and L. N. Trefethen, Mathematics of the Faraday cage, Preprint, 2014.