Special Case of Seeley's Trace Theorem via Green's Formula

Michael Taylor

The trace theorem for L^2 Sobolev spaces states that, if $\overline{\Omega}$ is a smooth compact manifold with boundary, then the trace $u \mapsto \operatorname{Tr} u = u|_{\partial\Omega}$ yields a bounded map

$$\operatorname{Tr} : H^s(\Omega) \longrightarrow H^{s-1/2}(\partial\Omega), \quad \forall\, s > \frac{1}{2}. \tag{1}$$

The following extension was proven in [Se]. Let P be an mth order elliptic differential operator, say $P : C^\infty(\overline{\Omega}, E_0) \to C^\infty(\overline{\Omega}, E_1)$, where E_0 and E_1 are smooth vector bundles of the same fiber dimension. Then

$$\operatorname{Tr} : H^s(\Omega) \cap \operatorname{Ker} P \longrightarrow H^{s-1/2}(\partial\Omega), \quad \forall\, s \in \mathbb{R}. \tag{2}$$

Here we provide an elementary proof of a special case. Namely, let $D : C^\infty(\overline{\Omega}, E_0) \to C^\infty(\overline{\Omega}, E_1)$ be a first order elliptic differential operator. We show that

$$\operatorname{Tr} : L^2(\Omega) \cap \operatorname{Ker} D \longrightarrow H^{-1/2}(\partial\Omega). \tag{3}$$

In fact, we establish a slightly more general result, namely

$$\operatorname{Tr} : \{u \in L^2(\Omega) : Du \in L^2(\Omega)\} \longrightarrow H^{-1/2}(\partial\Omega). \tag{3A}$$

The argument is based on Green's formula:

$$(Du, v)_{L^2(\Omega)} - (u, D^* v)_{L^2(\Omega)} = \frac{1}{i} \int_{\partial\Omega} \langle \sigma_D(x, \nu) u, v \rangle \, dS, \tag{4}$$

cf. [T], Chapter 2, (9.17), where it is established for $u, v \in C^\infty(\overline{\Omega})$. By (1), this extends in an elementary fashion to $u, v \in H^1(\Omega)$. We will extend its scope to include u as in (3A). To begin, note that the left side of (4) is well defined for

$$u \in L^2(\Omega), \quad Du \in L^2(\Omega), \quad v \in H^1(\Omega). \tag{5}$$

Thus we can define Tr in (3A) as follows. Pick $\varphi \in H^{1/2}(\partial\Omega)$. Then pick $v \in H^1(\Omega)$ such that $v|_{\partial\Omega} = \varphi$. (Such v exists; Tr in (1) is surjective.) Then we propose to define

$$\psi = \operatorname{Tr} u \in H^{-1/2}(\partial\Omega) \tag{6}$$

by

$$\frac{1}{i}\langle \sigma_D(x,\nu)\psi, \varphi\rangle = (Du, v)_{L^2(\Omega)} - (u, D^*v)_{L^2(\Omega)}. \tag{7}$$

Note that $\sigma_D(x,\nu)$ is smooth and invertible (Since D is elliptic) so $\sigma_D(x,\nu) : H^\sigma(\partial\Omega) \to H^\sigma(\partial\Omega)$ is an isomorphism for all $\sigma \in \mathbb{R}$. In order for (7) to actually define ψ, we need to know that the right side of (7) is independent of the choice of $v \in H^1(\Omega)$ satisfying $v|_{\partial\Omega} = \varphi$. So take $v_1 \in H^1(\Omega)$ such that $v_1|_{\partial\Omega} = \varphi$. We need to know that, if u satisfies (5), then

$$(Du, v)_{L^2(\Omega)} - (u, D^*v)_{L^2(\Omega)} = (Du, v_1)_{L^2(\Omega)} - (u, D^*v_1)_{L^2(\Omega)}. \tag{8}$$

In other words, we need to know that

$$w = v - v_1 \in H^1_0(\Omega) \Longrightarrow (Du, w)_{L^2(\Omega)} = (u, D^*w)_{L^2(\Omega)}, \tag{9}$$

as long as u satisfies (5). Indeed, the identity is clear for $w \in C_0^\infty(\Omega)$, and then it follows by an approximation argument, since $C_0^\infty(\Omega)$ is dense in $H^1_0(\Omega)$.

Thus the map in (3A) can be specified as follows. Choose an extension operator

$$E : H^{1/2}(\partial\Omega) \longrightarrow H^1(\Omega), \tag{10}$$

so that $\mathrm{Tr}(E\varphi) = \varphi$. Many exist. Then Tr in (3A) is given by

$$\frac{1}{i}\langle \sigma_D(x,\nu)\,\mathrm{Tr}\,u, \varphi\rangle = (Du, E\varphi)_{L^2(\Omega)} - (u, D^*E\varphi)_{L^2(\Omega)}. \tag{11}$$

Example. Let Ω be a smoothly bounded domain in $\mathbb{C}^n$ For $n = 1$, we take $D = \partial/\partial\bar{z}$. If $n > 1$, $\overline{\partial}$ is overdetermined elliptic. For (3A) to apply to holomorphic functions in $L^2(\Omega)$, when $n > 1$, one can take

$$D = \overline{\partial} + \overline{\partial}^*, \tag{12}$$

acting on forms of type $(0, q)$, $0 \le q \le n$.

We next consider another sense in which Tr is well defined in (3A). Let X be a smooth vector field on $\overline{\Omega}$, transverse to $\partial\Omega$, and pointing into Ω on $\partial\Omega$. Let $\mathcal{F}_t$ denote the flow it generates. Let

$$\overline{\Omega}_t = \mathcal{F}_t(\overline{\Omega}), \quad t \ge 0, \tag{13}$$

so, for $0 < t_1 < t_2$, $\overline{\Omega}_{t_2} \subset \overline{\Omega}_{t_1} \subset \overline{\Omega}_0 = \overline{\Omega}$. If u satisfies the hypotheses of (3A), then, by local elliptic regularlty, $u \in H^1_{loc}(\Omega)$, so $u|_{\Omega_t}$ belongs to $H^1(\Omega_t)$ for each $t > 0$, and we have

$$(Du, v)_{L^2(\Omega_t)} - (u, D^*v)_{L^2(\Omega_t)} = \frac{1}{i}\int\limits_{\partial\Omega_t} \langle \sigma_D(x,\nu)u, v\rangle\, dS, \tag{14}$$

for each $v \in H^1(\Omega)$. Let us set, for $t > 0$,

$$u_t = \mathcal{F}_t^*\Bigl(u\bigr|_{\Omega_t}\Bigr) \in H^1(\Omega), \quad \psi_t = u_t\bigr|_{\partial\Omega} \in H^{1/2}(\partial\Omega). \tag{15}$$

From (14) we deduce that

$$\|\psi_t\|_{H^{-1/2}(\partial\Omega)} \le C\bigl(\|u\|_{L^2(\Omega)} + \|Du\|_{L^2(\Omega)}\bigr), \tag{16}$$

with C independent of $t \in (0,1]$. Also, as $t \searrow 0$, the left side of (14) converges to

$$(Du, v)_{L^2(\Omega)} - (u, D^*v)_{L^2(\Omega)}, \tag{17}$$

which, by (7), is equal to

$$\frac{1}{i}\langle \sigma_D(x,\nu)\psi, \varphi\rangle, \quad \psi = \operatorname{Tr} u, \tag{18}$$

for each $\varphi \in H^{1/2}(\partial\Omega)$, $\varphi = v|_{\partial\Omega}$. Putting together (14), (16), and (18), we deduce that

$$\psi_t = \mathcal{F}_t^*\Bigl(u\bigr|_{\partial\Omega_t}\Bigr) \text{ converges weak}^* \text{ in } H^{-1/2}(\partial\Omega) \text{ to } \operatorname{Tr} u, \tag{19}$$

as $t \searrow 0$.

References

[Se] R. Seeley, Singular integrals and boundary value problems, Amer. J. Math. 88 (1966), 781–809.

[T] M. Taylor, Partial Differential Equations, Vol. 1, Springer-Verlag NY 1996 (2nd ed. 2011).