Smooth Operators for Principal Series Representations
Microlocal Properties

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1. Introduction

Suppose we have a Lie group G acting smoothly on a compact Riemannian manifold X. This gives rise to a representation of G on $L^2(X)$, by

$$U(g)f(x) = f(g^{-1}x). \tag{1.1}$$

One might throw in a Jacobian factor to make the representation unitary, but this is not so important. We are interested in characterizing the linear operators $A \in \mathcal{L}(L^2(X))$ with the property that

$$A(g) = U(g)AU(g)^{-1} \tag{1.2}$$

is a C^∞ function on G with values in $\mathcal{L}(L^2(X))$. We call A a G-smooth operator, or if G is understood simply a smooth operator. We denote the space of G-smooth operators on $L^2(X)$ by

$$OP\mathcal{S}^0_G(X). \tag{1.3}$$

It was shown in [T] that if

$$G = SO_e(n+1,1), \quad X = S^n, \tag{1.4}$$

with G acting as the group of conformal transformations on S^n, then the set of G-smooth operators coincides with the algebra $OPS^0_{1,0}(X)$ of pseudodifferential operators.

Observe that such an action yields a principal series representation of $SO_e(n+1,1)$. It is tempting to look at other principal series representations. So we might consider more generally

$$X = G/MAN = K/M, \tag{1.5}$$

where G is a semisimple Lie group, with maximal compact K and Iwasawa decomposition $G = KAN$, and M is the centralizer of A in K.

For example, one can look at

$$G = S\ell(n,\mathbb{R}), \quad K = SO(n). \tag{1.6}$$

Then M is the group (of order 2^{n-1}) of diagonal $n \times n$ matrices, with ± 1s on the diagonal (and determinant 1), and MAN consists of upper triangular $n \times n$

matrices, of determinant 1. In such a case, as we will see, the set of G-smooth operators will be bigger than $OPS^0_{1,0}(X)$, but it will have an intriguing microlocal structure.

2. Distribution kernels of G-smooth operators

Note that if $A \in \mathcal{L}(L^2(X))$ has Schwartz kernel $k_A \in \mathcal{D}'(X \times X)$, and if $U(g)$ has the form (1.1), then $A(g) = U(g)AU(g)^{-1}$ has Schwartz kernel

$$J(g,y)\, k_A(g^{-1}x, g^{-1}y). \tag{2.1}$$

Here $J(g,y)$ is a Jacobian factor, belonging to $C^\infty(G \times X)$. To proceed, let $\mathcal{L} \subset \mathcal{D}'(X \times X)$ denote the space of Schwartz kernels of bounded linear operators on $L^2(X)$. Then $\mathcal{L}$ is a Banach space of distributions, with the following important property.

Lemma 2.1. *The space $\mathcal{L}$ is a module over $C^\infty(X \times X)$.*

Proof. Given $\varphi \in C^\infty(X \times X)$, we can write

$$\varphi(x,y) = \sum_{j,\ell} a_{j\ell}\, \psi_j(x)\psi_\ell(y), \tag{2.2}$$

with

$$\|\psi_j\|_{L^\infty} \le 1, \quad \sum_{j,\ell} |a_{j\ell}| < \infty. \tag{2.3}$$

One could use eigenfunction expansions or (easier and better) localize to boxes, reduce consideration to $X = \mathbb{T}^n$, and use Fourier series. Note that a distribution $k \in \mathcal{D}'(X \times X)$ belongs to $\mathcal{L}$ if and only if

$$\big|\langle k, u(x)v(y)\rangle\big| \le C\|u\|_{L^2}\|v\|_{L^2}. \tag{2.4}$$

Given this, we have

$$\begin{aligned}\big|\langle \varphi k, u(x)v(y)\rangle\big| &= \Big|\sum_{j,\ell} a_{j\ell}\,\langle k, \psi_j u(x)\psi_\ell v(y)\rangle\Big| \\ &\le C\Big(\sum |a_{j\ell}|\Big)\|u\|_{L^2}\|v\|_{L^2}.\end{aligned} \tag{2.5}$$

Remark. Consideration of singular integral operators shows that $\mathcal{L}$ is *not* a module over $C(X \times X)$. The proof above can be extended to show that $\mathcal{L}$ is a module over $H^s(X \times X)$ whenever $s > \dim X$.

Returning to (2.1), we have the following.

Proposition 2.2. *A distribution $k \in \mathcal{D}'(X \times X)$ is the Schwartz kernel of a G-smooth operator if and only if*

$$k(g^{-1}x, g^{-1}y) \ \text{ is a } \ C^\infty \ \text{ function of } \ g \ \text{ with values in } \ \mathcal{L}. \tag{2.6}$$

To proceed, given $Y_j \in \mathfrak{g}$, the Lie algebra of G, denote by Y_j^b the vector field on X given by

$$Y_j^b f(x) = \frac{d}{dt} f\big(\exp(-tY_j)x\big)\big|_{t=0}, \tag{2.7}$$

and by $Y_j^\#$ the vector field on $X \times X$ given by

$$Y_j^\# f(x,y) = \frac{d}{dt} f\big(\exp(-tY_j)x, \exp(-tY_j)y\big)\big|_{t=0}. \tag{2.8}$$

Also denote by $\mathfrak{g}^b$ the associated Lie algebra of vector fields on X and by $\mathfrak{g}^\#$ the associated Lie algebra of vector fields on $X \times X$. Note that

$$\mathfrak{g}^\# = \{L_x + L_y : L \in \mathfrak{g}^b\}. \tag{2.9}$$

The smoothness condition (2.6) is equivalent to

$$Y_j^\# \in \mathfrak{g}^\#, \ m \in \mathbb{Z}^+ \Longrightarrow Y_m^\# \cdots Y_1^\# k \in \mathcal{L}. \tag{2.10}$$

We are hence motivated to study the following class of spaces of distributions. Let M be a compact smooth manifold, and let

$$E \subset \mathcal{D}'(M) \tag{2.11}$$

be a Banach space of distributions. Assume

$$E \ \text{ is a module over } \ C^\infty(M). \tag{2.12}$$

Let V be a set of smooth vector fields on M. We set

$$\mathcal{I}_E(M,V) = \{f \in E : X_m \cdots X_1 f \in E, \ \forall\, X_j \in V, \, m \in \mathbb{Z}^+\}. \tag{2.13}$$

The following is easily established.

Proposition 2.3. *Let $\mathcal{V}$ be the Lie algebra over $C^\infty(M)$ generated by V. If (2.12) holds,*

$$\mathcal{I}_E(M,V) = \mathcal{I}_E(M,\mathcal{V}). \tag{2.14}$$

Returning to the case $M = X \times X,\ E = \mathcal{L}$, we can phrase the Beals-Cordes characterization of $OPS^0_{1,0}(X)$ as follows. Let $\Psi^0(X)$ denote the space of Schwartz kernels of operators in $OPS^0_{1,0}(X)$. Let $\mathfrak{X}(X)$ denote the space of smooth vector fields on X, and consider the set

$$\mathfrak{X}^{\#} = \{L_x + L_y : L \in \mathfrak{X}(X)\} \tag{2.15}$$

of vector fields on $X \times X$. Then

$$\Psi^0(X) = \mathcal{I}_{\mathcal{L}}(X \times X, \mathfrak{X}^{\#}). \tag{2.16}$$

Note that the Lie algebra over $C^\infty(X \times X)$ generated by $\mathfrak{X}^{\#}$ is equal to

$$\mathfrak{X}_\Delta = \{Y \in \mathfrak{X}(X \times X) : Y \text{ is tangent to } \Delta\}, \tag{2.17}$$

where

$$\Delta = \{(x,x) : x \in X\} \subset X \times X \tag{2.18}$$

is the diagonal. Hence the Beals-Cordes result is equivalent to

$$\Psi^0(X) = \mathcal{I}_{\mathcal{L}}(X \times X, \mathfrak{X}_\Delta). \tag{2.19}$$

Looking at Proposition 2.2 and (2.10), we have:

Proposition 2.4. *Give a G-action on X, the set of G-smooth operators coincides with $OPS^0_{1,0}(X)$ provided the Lie algebra $\mathcal{G}^{\#}$ over $C^\infty(X \times X)$ generated by $\mathfrak{g}^{\#}$ coincides with $\mathfrak{X}_\Delta$.*

Proof. The analysis given above shows that the space of Schwartz kernels of G-smooth operators is equal to

$$\mathcal{I}_{\mathcal{L}}(X \times X, \mathcal{G}^{\#}). \tag{2.20}$$

It is clear that in general $\mathcal{G}^{\#} \subset \mathfrak{X}_\Delta$. We will show in §3 that $\mathcal{G}^{\#} = \mathfrak{X}_\Delta$ when $G = SO_e(n+1,1),\ X = S^n$, as in (1.4). In §4 we will see that $\mathcal{G}^{\#}$ is somewhat smaller than $\mathfrak{X}_\Delta$ when $G = S\ell(3,\mathbb{R}),\ X = SO(3)/M$.

Remark. Proposition 2.4 can be compared with the basic general result of [T], which is that $OPS^0_G(X) = OPS^0_{1,0}(X)$ provided the moment map

$$T^*X \setminus 0 \longrightarrow \mathfrak{g}^* \setminus 0 \tag{2.21}$$

is an embedding.

Remark. If we replace $\mathcal{L}$ by $L^2 = L^2(X \times X)$, we see that $\mathcal{I}_{L^2}(X \times X, \mathcal{G}^{\#})$ is the space of Schwartz kernels of Hilbert-Schmidt operators A on $L^2(X)$ for which $A(g)$ in (1.2) is a C^∞ function on G with values in $\mathrm{HS}(L^2(X))$.

3. The case of $SO_e(n+1,1)$ acting on S^n

Here we consider $G = SO_e(n+1,1)$, $X = S^n = G/P$. We consider the Lie algebra $\mathfrak{g}^\#$ of vector fields on $X \times X$ generated by the G-action

$$g \cdot (x,y) = (gx, gy). \tag{3.1}$$

We claim that $\mathcal{G}^\#$, the Lie algebra over $C^\infty(X \times X)$ generated by $\mathfrak{g}^\#$, is equal to $\mathfrak{X}_\Delta$, the Lie algebra of C^∞ vector fields on $X \times X$ that are tangent to the diagonal $\Delta \subset X \times X$. In particular, we claim that there are just two G-orbits in $X \times X$, namely Δ and $X \times X \setminus \Delta$.

Lemma 3.1. *Given $x, y, x', y' \in S^n$, with $x \neq y$ and $x' \neq y'$, there exists $g \in SO_e(n+1,1)$ such that $gx = x'$ and $gy = y'$.*

Proof. Since S^n is a homogeneous G-space, we readily reduce to the case $x = x'$. So, given $y \neq x$ and $y' \neq x$, we claim there exists $g \in G$, fixing x, such that $gy = y'$. Considering rotations that fix x and conformal maps that push points in S^n away from x and toward its antipode makes the statement clear when neither y nor y' is antipodal to x. The case where y (or y') is antipodal to x is taken care of by a simple additional argument.

4. The case of $S\ell(3,\mathbb{R})$ acting on $SO(3)/M$

Here we consider $G = S\ell(3,\mathbb{R})$, acting on $X = G/MAN = SO(3)/M$, where M is the group of order 4 consisting of 3×3 diagonal matrices, with ± 1s on the diagonal (and determinant 1). We consider the G-action on $X \times X$ given by

$$g \cdot (x,y) = (gx, gy). \tag{4.1}$$

Proposition 4.1. *The G-action on $X \times X$ has five (or is it four?) orbits:*

(i) Δ, *the diagonal in $X \times X$, of codimension 3,*

(ii) $\mathcal{A}$, $\mathcal{B}$, *orbits of codimension 2,*

(iii) $\mathcal{C}$, *orbit of codimension 1,*

(iv) $\mathcal{D}$, *a dense orbit.*

Furthermore, $\overline{\mathcal{A}} = \mathcal{A} \cup \Delta$, $\overline{\mathcal{B}} = \mathcal{B} \cup \Delta$, and $\overline{\mathcal{C}} = \mathcal{C} \cup \mathcal{B} \cup \mathcal{A} \cup \Delta$.

Fix $x_0 \in X$ and for each G-orbit $\mathcal{O}$, let $\mathcal{O}_{x_0} = \{y \in X : (x_0, y) \in \mathcal{O}\}$. If P_{x_0} is the subgroup of G fixing x_0, and if $y_0 \in \mathcal{O}_{x_0}$ is some point, then

$$\mathcal{O}_{x_0} = \{gy_0 : g \in P_{x_0}\}. \tag{4.2}$$

Let us take x_0 to be the image of the identity $I \in G$ under $G \to G/P$, and look at the orbits of P acting on G/P. There are 5 such orbits:

(i) $\Delta_{x_0} = \{x_0\}$,

(ii) $\mathcal{A}_{x_0},\ \mathcal{B}_{x_0}$, orbits of dimension 1,

(iii) $\mathcal{C}_{x_0}$, orbit of dimension 2,

(iv) $\mathcal{D}_{x_0}$, an orbit dense in X.

Furthermore, $\overline{\mathcal{A}}_{x_0} = \mathcal{A}_{x_0} \cup \Delta_{x_0}$, $\overline{\mathcal{B}}_{x_0} = \mathcal{B}_{x_0} \cup \Delta_{x_0}$, and $\overline{\mathcal{C}}_{x_0} = \mathcal{C}_{x_0} \cup \mathcal{B}_{x_0} \cup \mathcal{A}_{x_0} \cup \Delta_{x_0}$.

Here is one way to describe these orbits in $X = SO(3)/M$. The space $SO(3)$ consisting of orthogonal matrices

$$\begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix}, \tag{4.3}$$

with $w = u \times v$, is naturally identified with the unit tangent bundle T_1S^2 of S^2, with $u \in S^2,\ v \in T_uS^2$. The action of elements of M can take u to $\pm u$ and v to $\pm v$. So antipodal elements of S^2 identified, as are unit tangent vectors pointing in opposite directions. We describe the pre-images in $SO(3) = T_1S^2$ of the P-orbits, using the same labels as for the orbits themselves. Recall that we are taking x_0 to be the class of P in G/P. We have

$$\begin{aligned}
\Delta_{x_0} &= \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \right\}, \\
\mathcal{A}_{x_0} &= \left\{ \begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & a & * \end{pmatrix} \in SO(3) : a \neq 0 \right\}, \\
\mathcal{B}_{x_0} &= \left\{ \begin{pmatrix} * & * & * \\ a & * & * \\ 0 & 0 & * \end{pmatrix} \in SO(3) : a \neq 0 \right\}, \\
\mathcal{C}_{x_0} &= \left\{ \begin{pmatrix} * & * & * \\ a & * & * \\ 0 & b & * \end{pmatrix} \in SO(3) : a \neq 0,\ b \neq 0 \right\}, \\
\mathcal{D}_{x_0} &= \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ a & * & * \end{pmatrix} \in SO(3) : a \neq 0 \right\}.
\end{aligned} \tag{4.4}$$

The last four sets are (pre-images of) P-orbits containing, respectively, the following points in $X = G/P$:

$$
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \bmod P, \quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \bmod P,
$$
$$
\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \bmod P, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \bmod P. \tag{4.5}
$$

Recall that P consists of upper triangular matrices in $S\ell(3, \mathbb{R})$.

A way to picture these pre-images in T_1S^2 is as follows. We define the equator $\mathcal{E} \subset S^2$ to consist of $\{u \in S^2 : u_3 = 0\}$, and we set $e_1 = (1, 0, 0)^t \in S^2$. Then

$$
\begin{aligned}
&\Delta_{x_0} \text{ consists of points in } T_1S^2 \text{ lying over } e_1, \text{ tangent to the equator } \mathcal{E}, \\
&\mathcal{A}_{x_0} \text{ consists of points lying over } e_1, \text{ not tangent to } \mathcal{E}, \\
&\mathcal{B}_{x_0} \text{ consists of points lying over } \mathcal{E} \setminus \{e_1\}, \text{ tangent to } \mathcal{E}, \\
&\mathcal{C}_{x_0} \text{ consists of points lying over } \mathcal{E} \setminus \{e_1\}, \text{ not tangent to } \mathcal{E}, \\
&\mathcal{D}_{x_0} \text{ consists of points lying over } S^2 \setminus \mathcal{E}.
\end{aligned} \tag{4.6}
$$

Note that some of these sets are disconnected, but once you mod out by the action of M the resulting orbits are seen to be connected.

Remark. While there are five different orbits described in (4.6), it is conceivable that $\mathcal{A}_{x_0}$ and $\mathcal{B}_{x_0}$ twist about as x_0 varies in such a fashion that $\mathcal{A} = \mathcal{B}$. This needs to be checked.

Assertion. $\mathcal{G}^{\#}$ consists of all C^∞ vector fields on $X \times X$ that are tangent to Δ, $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$.

Recall that the space of Schwartz kernels of G-smooth operators is equal to

$$
\mathcal{I}_{\mathcal{L}}(X \times X, \mathcal{G}^{\#}). \tag{4.7}
$$

5. Further musings

As we have noted, the Beals-Cordes characterization of $OPS^0_{1,0}(X)$ gives

$$
\Psi^0(X) = \mathcal{I}_{\mathcal{L}}(X \times X, \mathfrak{X}_\Delta). \tag{5.1}
$$

Meanwhile, according to the development of Lagrangian distributions in Chapter 25 of [H], there is a parallel result with $\mathcal{L}$ replaced by a Besov space:

$$\Psi^0(X) = \mathcal{I}_{\mathcal{B}}(X \times X, \mathfrak{X}_\Delta), \quad \mathcal{B} = B^{-n/2}_{2,\infty}(X \times X), \tag{5.2}$$

where $n = \dim X$. Thus the right sides of (5.1) and (5.2) coincide, though certainly $\mathcal{L} \neq \mathcal{B}$. One might wonder if there is a short demonstration of this, which does not proceed through the demonstration that each such space is equal to $\Psi^0(X)$. One might also wonder about the following.

Question 1. For what other Banach spaces of distributions $E \subset \mathcal{D}'(X \times X)$ can one say

$$\Psi^0(X) = \mathcal{I}_E(X \times X, \mathfrak{X}_\Delta)? \tag{5.3}$$

Question 2. Given a Lie algebra $\mathcal{G}^\# \subset \mathfrak{X}_\Delta$, when can one say

$$\mathcal{I}_{\mathcal{L}}(X \times X, \mathcal{G}^\#) = \mathcal{I}_E(X \times X, \mathcal{G}^\#)? \tag{5.4}$$

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