Stirling's Formula and the Schrödinger Equation

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Contents

1. Introduction

Our goal is to derive Stirling's formula for the asymptotic behavior of $\Gamma(z)$ as $|z| \to \infty$. We will first do this for $\operatorname{Re} z \geq 0$, using a strong form of the Laplace asymptotic method. Then we treat $\operatorname{Re} z \leq 0$, using a functional equation.

We recall that $\Gamma(z)$ can be defined for $\operatorname{Re} z > 0$ by the integral

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt. \tag{1.1}$$

More convenient for us is

$$\begin{aligned} \Gamma(z+1) &= \int_0^\infty e^{-t} t^z \, dt \\ &= \int_0^\infty e^{-t+z\log t}\, dt \\ &= z \int_0^\infty e^{-zs+z(\log s+\log z)}\, ds, \end{aligned} \tag{1.2}$$

the last line via the substitution $t = zs$. Together with $\Gamma(z+1) = z\Gamma(z)$, this yields the identity

$$\begin{aligned} \Gamma(z) &= e^{z\log z} \int_0^\infty e^{-z(s-\log s)}\, ds \\ &= e^{z\log z - z} \int_{-1}^\infty e^{-z(\tau - \log(1+\tau))}\, d\tau, \end{aligned} \tag{1.3}$$

valid for $\operatorname{Re} z > 0$.

In §2 we will deduce from (1.3) that

$$\Gamma(z) = \Bigl(\frac{z}{e}\Bigr)^z \sqrt{\frac{2\pi}{z}}\; V_0\Bigl(\frac{1}{4z}\Bigr), \quad \operatorname{Re} z \geq 0,\ z \neq 0, \tag{1.4}$$

where

$$V_0(\zeta) \text{ is holomorphic on } \operatorname{Re}\zeta > 0,\ C^\infty \text{ on } \operatorname{Re}\zeta \ge 0, \quad V_0(0) = 1. \tag{1.5}$$

In fact, we obtain V_0 in the form $V_0(\zeta) = \sqrt{2}V(\zeta, 0)$, where $V(\zeta, u)$ solves a Schrödinger equation in $\{\operatorname{Re}\zeta \ge 0\}$, with a certain initial condition $V(0, u) = F(u)$.

The approach used in §2 follows [W] to some extent, except for our making a connection with the Schrödinger equation. In §3 we discuss an alternative approach, taken in [WW] and also in [Leb]. We follow [Leb] more closely than [WW], and supply a few additional arguments.

In §4 (also following [Leb]) we examine the asymptotic behavior of $\Gamma(z)$ on $\operatorname{Re} z \le 0$, via the identity

$$\Gamma(-z) \sin \pi z = -\frac{\pi}{z\Gamma(z)}. \tag{1.6}$$

2. Asymptotic behavior on $\operatorname{Re} z \geq 0$ via the Schrödinger equation

From (1.3) we have

$$\Gamma(z) = e^{z \log z - z} \int_{-1}^{\infty} e^{-z\psi(\tau)}\, d\tau, \tag{2.1}$$

for $\operatorname{Re} z > 0$, with

$$\psi(\tau) = \tau - \log(1+\tau), \quad -1 < \tau < \infty. \tag{2.2}$$

Note that

$$\psi'(\tau) = 1 - \frac{1}{1+\tau}, \quad \psi''(\tau) = \Big(\frac{1}{1+\tau}\Big)^2, \tag{2.3}$$

so ψ is convex, with a unique minimum at $\tau = 0$, and $\psi(0) = 0$. We can hence write

$$u^2 = \psi(\tau) = \tau - \log(1+\tau), \tag{2.4}$$

where $u : (-1,\infty) \to (-\infty,\infty)$ is a diffeomorphism, satisfying

$$\begin{gathered} u \to -\infty \ \text{ as } \ \tau \to -1, \quad u \to +\infty \ \text{ as } \ \tau \to +\infty, \\ 2u\, du = \frac{\tau}{1+\tau}\, d\tau. \end{gathered} \tag{2.5}$$

Then (2.1) gives

$$\Gamma(z) = 2e^{z \log z - z} \int_{-\infty}^{\infty} e^{-zu^2} F(u)\, du, \tag{2.6}$$

with

$$F(u) = \frac{u}{\tau}(1+\tau), \tag{2.7}$$

where $\tau = \tau(u)$ is defined implicitly by (2.4). We have

$$F \in C^\infty(\mathbb{R}), \quad F(0) = \frac{1}{\sqrt{2}}, \tag{2.8}$$

and

$$\begin{aligned} F(u) &\sim u + \sum_{j\geq 1} a_j u^{1-j}, && u \to +\infty, \\ &\quad\ 0, && u \to -\infty. \end{aligned} \tag{2.9}$$

Also the derivatives $F^{(k)}(u)$ have asymptotic expansions as $u \to \pm\infty$ consistent with formal differentiation of (2.9), so

$$F \in S^1_{\mathrm{cl}}(\mathbb{R}). \tag{2.10}$$

If we set

$$z = \frac{1}{4\zeta}, \tag{2.11}$$

we have

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-zu^2} F(u)\, du &= \int_{-\infty}^{\infty} e^{-u^2/4\zeta} F(u)\, du \\ &= \sqrt{4\pi\zeta}\, e^{\zeta\Delta} F(0), \end{aligned} \tag{2.12}$$

for

$$\operatorname{Re}\zeta \ge 0, \quad \text{hence} \quad \operatorname{Re} z \ge 0 \ (z \neq 0). \tag{2.13}$$

Here $e^{\zeta\Delta}$ is the solution operator for the evolution equation

$$\frac{\partial V}{\partial \zeta} = \Delta V, \quad \Delta = \frac{\partial^2}{\partial u^2}. \tag{2.14}$$

Consequently, for $\operatorname{Re} z > 0$,

$$\Gamma(z) = 2\Big(\frac{z}{e}\Big)^z \sqrt{\frac{\pi}{z}}\, e^{\zeta\Delta} F(0)\Big|_{\zeta=1/4z}. \tag{2.15}$$

The following implies a version of Stirling's formula.

Proposition 2.1. *For $F(u)$ given by (2.7), hence satisfying (2.8)–(2.10), the function*

$$V(\zeta, u) = e^{\zeta\Delta} F(u) \tag{2.16}$$

is C^∞ on

$$\{\zeta \in \mathbb{C} : \operatorname{Re}\zeta \ge 0\} \times \{u \in \mathbb{R}\}. \tag{2.17}$$

Thus we have (1.4) with

$$V_0(\zeta) = \sqrt{2}\, V(\zeta, 0).$$

Proof. As is well known (cf. [T], Chapter 3, Proposition 8.2) if F satisfies (2.10), its Fourier transform $\widehat{F} \in \mathcal{S}'(\mathbb{R})$ is C^∞ on $\mathbb{R} \setminus 0$, and we can write

$$\widehat{F} = \widehat{F}_0 + \widehat{F}_1, \quad \widehat{F}_0 \in \mathcal{E}'(\mathbb{R}), \quad \widehat{F}_1 \in \mathcal{S}(\mathbb{R}). \tag{2.18}$$

That is, $\widehat{F}_0$ is a distribution with compact support and $\widehat{F}_1$ is smooth and rapidly decreasing (together with all its derivatives). Thus

$$e^{\zeta\Delta}F = \mathcal{F}^*\Big(e^{-\zeta|\xi|^2}\widehat{F}_0\Big) + \mathcal{F}^*\Big(e^{-\zeta|\xi|^2}\widehat{F}_1\Big), \tag{2.19}$$

where $\mathcal{F}^*$ is the inverse Fourier transform, having the mapping properties

$$\mathcal{F}^* : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}), \quad \mathcal{F}^* : \mathcal{E}'(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R}). \tag{2.20}$$

Note that

$$\begin{aligned} e^{-\zeta|\xi|^2}\widehat{F}_0 \ &\text{ is an entire function of } \zeta \in \mathbb{C} \\ &\text{with values in } \mathcal{E}'(\mathbb{R}), \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} e^{-\zeta|\xi|^2}\widehat{F}_1 \ &\text{ is a } C^\infty \text{ function of } \zeta \in \{\operatorname{Im}\zeta \ge 0\} \\ &\text{with values in } \mathcal{S}(\mathbb{R}). \end{aligned} \tag{2.22}$$

The conclusion that V is C^∞ on (2.17) is an immediate consequence of (2.18)–(2.22).

Since V is C^∞ on (2.17), it follows that $V(\zeta,0)$ is C^∞ on $\{\zeta \in \mathbb{C} : \operatorname{Re}\zeta \ge 0\}$, and we have the asymptotic formula

$$V(\zeta,0) \sim \sum_{k\ge 0} \frac{1}{k!} F^{(2k)}(0)\,\zeta^k, \quad \zeta \to 0,\ \operatorname{Re}\zeta \ge 0. \tag{2.27}$$

Thus (2.15) yields

$$\Gamma(z) \sim \Big(\frac{z}{e}\Big)^z \sqrt{\frac{2\pi}{z}}\Big(1 + \sum_{k\ge 1} \frac{\sqrt{2}}{k!} F^{(2k)}(0)\Big(\frac{1}{4z}\Big)^k\Big), \tag{2.28}$$

as $z \to \infty,\ \operatorname{Re} z \ge 0$.

3. Classical approach via the Laplace transform

Another approach to Stirling's formula involves writing

$$\log \Gamma(z) = \Bigl(z - \frac{1}{2}\Bigr) \log z - z + \frac{1}{2} \log 2\pi + \omega(z), \tag{3.1}$$

with a convenient integral formula for $\omega(z)$. This is done in §12.33 of [WW], and, to even better effect, in §1.4 of [Leb], which obtains

$$\omega(z) = \int_0^\infty f(t) e^{-tz}\, dt, \tag{3.2}$$

with

$$f(t) = \Bigl(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\Bigr) \frac{1}{t}, \tag{3.3}$$

which is an entire function of t, asymptotic to $1/2t$ as $t \nearrow +\infty$. In [Leb] it is observed that $f'(t) < 0$ for $t > 0$ and deduced that

$$|\omega(z)| \le \frac{2f(0)}{|z|} = \frac{1}{6|z|}, \quad \text{for} \quad \operatorname{Re} z \ge 0. \tag{3.4}$$

Exponentiating (3.1) gives

$$\Gamma(z) = \Bigl(\frac{z}{e}\Bigr)^z \sqrt{\frac{2\pi}{z}}\, e^{\omega(z)}, \tag{3.5}$$

and then (3.4) gives the leading term in (2.24), valid for $z \to \infty, \operatorname{Re} z \ge 0$. In §12.33 of [WW], the formula derived for $\omega(z)$ is

$$\omega(z) = \int_0^\infty \frac{\tan^{-1}(t/z)}{e^{2\pi t} - 1}\, dt, \tag{3.6}$$

with which the entire asymptotic series (2.24) is derived, in a more explicit form, but its validity is demonstrated only for

$$z \longrightarrow \infty, \quad |\arg z| \le \frac{\pi}{2} - \delta, \quad \delta > 0.$$

The validity is established in §13.6 of [WW] in the larger domain $|\arg z| \le \pi - \delta$. We discuss that in §4.

The complete asymptotic expansion is mentioned in (1.4.24) of [Leb], but without a derivation. We next point out how to derive a complete asymptotic expansion of the Laplace transform (3.2), valid for $z \to \infty$, $\operatorname{Re} z \geq 0$, just given that $f \in C^\infty([0,\infty))$ and that $f^{(j)}$ is integrable on $[0,\infty)$ for each $j \geq 1$. In fact, integration by parts yields

$$\begin{aligned}\int_0^\infty f(t)e^{-zt}\,dt &= -\frac{1}{z}\int_0^\infty f(t)\,\frac{d}{dt}e^{-zt}\,dt \\ &= \frac{1}{z}\int_0^\infty f'(t)e^{-zt}\,dt + \frac{f(0)}{z},\end{aligned}$$

valid for $\operatorname{Re} z \geq 1$. We can iterate this argument to obtain

$$\omega(z) = \sum_{k=1}^{N} \frac{f^{(k-1)}(0)}{z^k} + \frac{1}{z^N}\int_0^\infty f^{(N)}(t)e^{-zt}\,dt, \tag{3.7}$$

and

$$\Bigl|\int_0^\infty f^{(N)}(t)e^{-zt}\,dt\Bigr| \leq \int_0^\infty |f^{(N)}(t)|\,dt < \infty, \quad \text{for} \quad N \geq 1,\ \operatorname{Re} z \geq 0. \tag{3.8}$$

To carry on, we note that, for $|t| < 2\pi$,

$$\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k)!} B_k t^{2k-1}, \tag{3.9}$$

where B_k are the *Bernoulli numbers* (cf. [T], §12, Exercises 6–8), so, for $|t| < 2\pi$,

$$f(t) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(2\ell+2)!} B_{\ell+1} t^{2\ell}. \tag{3.10}$$

Thus

$$f^{(j)}(0) = \begin{cases} 0 & j \text{ odd}, \\ \dfrac{(-1)^\ell B_{\ell+1}}{(2\ell+1)(2\ell+2)} & j = 2\ell, \end{cases} \tag{3.11}$$

so

$$\omega(z) \sim \sum_{\ell \geq 0} \frac{(-1)^\ell B_{\ell+1}}{(2\ell+1)(2\ell+2)}\,\frac{1}{z^{2\ell+1}}, \quad z \to \infty,\ \operatorname{Re} z \geq 0. \tag{3.12}$$

Thus there are $A_k \in \mathbb{R}$ such that

$$e^{\omega(z)} \sim 1 + \sum_{k \geq 1} \frac{A_k}{z^k}, \quad z \to \infty,\ \operatorname{Re} z \geq 0, \tag{3.13}$$

consistent with (2.24), but arguably more straightforward to compute.

4. Asymptotic behavior on $\operatorname{Re} z \le 0$

Following [Leb], we use the identity

$$\Gamma(-z)\sin \pi z = -\frac{\pi}{z\Gamma(z)} \tag{4.1}$$

to extend (3.5), i.e.,

$$\Gamma(z) = \left(\frac{z}{e}\right)^z \sqrt{\frac{2\pi}{z}}\, e^{\omega(z)}, \quad \text{for} \quad \operatorname{Re} z \ge 0,\ z \ne 0, \tag{4.2}$$

to the rest of $\mathbb{C} \setminus \mathbb{R}^-$. If we define z^z and $\sqrt{z}$ in the standard fashion for $z \in (0, \infty)$ and to be holomorphic on $\mathbb{C} \setminus \mathbb{R}^-$, we get

$$\Gamma(z) = \frac{1}{1 - e^{2\pi i z}} \left(\frac{z}{e}\right)^z \sqrt{\frac{2\pi}{z}}\, e^{-\omega(-z)}, \quad \text{for} \quad \operatorname{Re} z \le 0,\ \operatorname{Im} z > 0, \tag{4.3}$$

and

$$\Gamma(z) = \frac{1}{1 - e^{-2\pi i z}} \left(\frac{z}{e}\right)^z \sqrt{\frac{2\pi}{z}}\, e^{-\omega(-z)}, \quad \text{for} \quad \operatorname{Re} z \le 0,\ \operatorname{Im} z < 0. \tag{4.4}$$

Comparing (4.2) and (4.3) for $z = iy,\ y > 0$, we see that

$$e^{-\omega(-iy)} = (1 - e^{-2\pi y}) e^{\omega(iy)}, \quad y > 0. \tag{4.5}$$

That $e^{-\omega(-iy)}$ and $e^{\omega(iy)}$ have the same asymptotic behavior as $y \to +\infty$ also follows from the fact that only odd powers of z^{-1} appear in (3.12). On the other hand, such a result is not so apparent from (2.24).

In [WW], §13.6, the validity of the asymptotic expansion arising from (4.2) was extended to $|\arg z| \le \pi - \delta$ (for each $\delta > 0$) by a different method. Since §12.3 of [WW] obtained (3.12) only for $|\arg z| \le \pi/2 - \delta$, use of (4.1) on that result would miss the asymptotic expansion of $\Gamma(z)$ near the imaginary axis.

References

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