Curvature, Conformal Mapping, and 2D Stationary Fluid Flows

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1. Introduction

Let $\overline{\Omega}$ be a 2D Riemannian manifold (possibly with boundary). Assume $\overline{\Omega}$ is oriented, with J denoting counterclockwise rotation by 90°. As is well known, we can take a real-valued function (a stream function) $\psi : \overline{\Omega} \to \mathbb{R}$ (assume ψ is constant on each component γ_j of $\partial\Omega$) and then

$$v = J\,\nabla\psi \tag{1.1}$$

is a vector field on $\overline{\Omega}$ (tangent to $\partial\Omega$, if this is nonempty), defining a stationary solution to the Euler equation for incompressible (inviscid) fluid flow, provided ψ has the property

$$\nabla(\Delta\psi)(z) \parallel \nabla\psi(z), \quad \forall\, z \in \Omega. \tag{1.2}$$

In particular, this holds provided ψ satisfies any nonlinear PDE of the form

$$\Delta\psi = \Phi(\psi) \tag{1.3}$$

on Ω, with $\psi = c_j$ on γ_j. Constructing such solutions provides many stationary inviscid 2D fluid flows. (Remark: sometimes one sees the statement that (1.2) and (1.3) are equivalent, but actually (1.2) is more general than (1.3).)

An example of (1.3) with particular geometrical import is

$$\Delta\psi = -K\,e^{2\psi}, \tag{1.4}$$

with K a constant (typically $K = \pm 1$). The contact with geometry arises as follows. Suppose Ω has a flat metric (e.g., $\overline{\Omega}$ could be a planar domain, or covered by a planar domain). Then multiplying its metric tensor by the factor $e^{2\psi}$ produces a metric of Gauss curvature K if and only if (1.4) holds. We will use this observation and conformal mapping techniques to construct solutions to (1.4), which then yield interesting stationary solutions to the Euler equation, via (1.1). We note that the vorticity of such a flow is given by

$$\omega = \Delta\psi, \tag{1.5}$$

so our solutions will typically have nontrivial vorticity. Thus the use made here of conformal mapping contrasts with the well known use of conformal mappings to produce irrotational stationary planar flows.

2. Cat's eye flows

Our first example takes $\Omega = \mathbb{C}/\Gamma$, where $\Gamma = \{2\pi ik : k \in \mathbb{Z}\}$. We produce a 2-parameter family of stationary flows on Ω, containing the 1-parameter family of "Kelvin-Stuart cat's-eye flows" given on p. 53 of [MB], following [S]. These solutions arise from (1.4) with $K = 1$. We take the conformal diffeomorphism

$$E : \Omega \longrightarrow \mathbb{C} \setminus \{0\}, \quad E(z) = e^z, \tag{2.1}$$

and pull back to Ω a metric of curvature 1 on $\mathbb{C} \setminus \{0\}$. In fact, we take a family of metrics on $\mathbb{C}$ (conformally equivalent to the standard flat metric) of curvature 1, coming from conformal equivalence of $\mathbb{C} \cup \{\infty\}$ with S^2, the unit sphere in $\mathbb{R}^3$, with its standard metric (of curvature 1).

A conformal transformation of $\mathbb{C} \cup \{\infty\}$ to S^2 is given by

$$F : \mathbb{C} \longrightarrow S^2 \tag{2.2}$$

where $z = x + iy \in \mathbb{C}$ is mapped to $(u, v, t) \in S^2 \subset \mathbb{R}^3$ ($u^2 + v^2 + t^2 = 1$), via

$$u = \frac{2x}{x^2 + y^2 + 1}, \quad v = \frac{2y}{x^2 + y^2 + 1}, \quad t = \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}. \tag{2.3}$$

Under this map, $du^2 + dv^2 + dt^2$ pulls back to

$$4\,\frac{dx^2 + dy^2}{(x^2 + y^2 + 1)^2} = 4\,\frac{|dz|^2}{(|z|^2 + 1)^2}. \tag{2.4}$$

If we pull back the metric (2.4) to Ω via (2.1) we get a conformal multiple $e^{2\psi}$ of the standard metric, yielding a stream function. We get a more general class of stream functions by making the following construction. Given

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Gl(2, \mathbb{C}), \tag{2.5}$$

the map

$$T_A(w) = \frac{aw + b}{cw + d} = z \tag{2.6}$$

defines a conformal automorphism of $\mathbb{C} \cup \{\infty\}$. The metric (2.4) pulls back to

$$\frac{4|ad - bc|^2}{(|aw + b|^2 + |cw + d|^2)^2}\,|dw|^2. \tag{2.7}$$

Via (2.1), this pulls back to the following metric on Ω:

$$\begin{aligned}&\frac{4|ad-bc|^2|e^z|^2}{(|ae^z+b|^2+|ce^z+d|^2)^2}\,|dz|^2\\ &=\frac{4|ad-bc|^2}{(|ae^{z/2}+be^{-z/2}|^2+|ce^{z/2}+de^{-z/2}|^2)^2}\,|dz|^2.\end{aligned} \tag{2.8}$$

With $z=x+iy$, note that

$$|ae^{z/2}+be^{-z/2}|^2=|a|^2e^x+|b|^2e^{-x}+2\,\mathrm{Re}(a\bar{b}\,e^{iy}). \tag{2.9}$$

Thus we can write the metric (2.8) on Ω as

$$\frac{4|ad-bc|^2}{(|\alpha|^2e^x+|\beta|^2e^{-x}+2\,\mathrm{Re}(\alpha,\beta)e^{iy})^2}\,(dx^2+dy^2), \tag{2.10}$$

where

$$\alpha=\begin{pmatrix}a\\c\end{pmatrix},\quad \beta=\begin{pmatrix}b\\d\end{pmatrix}\in\mathbb{C}^2, \tag{2.11}$$

and

$$|\alpha|^2=|a|^2+|c|^2,\quad |\beta|^2=|b|^2+|d|^2,\quad (\alpha,\beta)=a\bar{b}+c\bar{d}. \tag{2.12}$$

The coefficient of dx^2+dy^2 in (2.10) defines $e^{2\psi}$. An alternative formulation is

$$\begin{aligned}e^{-\psi}&=\frac{1}{2}|\det A|^{-1}\left|A\begin{pmatrix}e^{z/2}\\e^{-z/2}\end{pmatrix}\right|^2\\ &=\frac{|\alpha|^2e^x+|\beta|^2e^{-x}+2\,\mathrm{Re}\,(\alpha,\beta)e^{iy}}{2|\det(\alpha,\beta)|}.\end{aligned} \tag{2.13}$$

This class of metrics is parametrized by $A=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in Gl(2,\mathbb{C})$, but in fact by reducing the numerator and denominator in (2.6) we can take $A\in Sl(2,\mathbb{C})$. Furthermore, multiplication of A on the left by a unitary matrix has a trivial effect, so the set of metrics in (2.10) (or stream functions (2.13)) is effectively parametrized by $SU(2)\backslash Sl(2,\mathbb{C})$, equivalent to 3D hyperbolic space.

To pick out the 1-parameter family of Kelvin-Stuart flows from (2.13), we specialize to

$$\alpha=\begin{pmatrix}a\\c\end{pmatrix},\quad \beta=\begin{pmatrix}b\\d\end{pmatrix}\in\mathbb{R}^2,\quad |\alpha|^2=|\beta|^2. \tag{2.14}$$

Then (2.13) specializes to

$$e^{-\psi} = \frac{|\alpha|^2 \cosh x + (\alpha \cdot \beta) \cos y}{|\alpha \times \beta|}. \tag{2.15}$$

If we set

$$\gamma = \frac{|\alpha|^2}{|\alpha \times \beta|}, \quad \sigma = \frac{\alpha \cdot \beta}{|\alpha \times \beta|}, \tag{2.16}$$

then, given (2.14), we have

$$\gamma \geq 1, \quad \sigma = \pm\sqrt{\gamma^2 - 1}. \tag{2.17}$$

This recovers the Kelvin-Stuart stream function:

$$\psi(z) = -\log(\gamma \cosh x + \sqrt{\gamma^2 - 1} \cos y), \quad \gamma \geq 1. \tag{2.18}$$

Back to the more general case (2.13), we can take

$$A = \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix}, \quad b \in \mathbb{C},\ d > 0. \tag{2.19}$$

Then (2.13) becomes (with $b = b_1 + ib_2$)

$$e^{-\psi} = \frac{e^x + (b_1^2 + b_2^2 + d^2)e^{-x}}{2d} + \frac{b_1 \cos y + b_2 \sin y}{d}, \tag{2.20}$$

which reduces to (2.15) when $b_2 = 0$ and $b_1^2 + d^2 = 1$ (with $\gamma = 1/d$). Note that if $b = |b|e^{i\theta}$ then $\operatorname{Re} \overline{b} e^{iy} = |b| \cos(y - \theta)$, so another way to write (2.20) is as

$$e^{-\psi} = \frac{e^x + (|b|^2 + d^2)e^{-x}}{2d} + \frac{|b|}{d} \cos(y - \theta). \tag{2.21}$$

The phase θ is not physically significant, so we see we have a 2-parameter family of stream functions. We can just take $b \geq 0$ in (2.19). We get

$$\psi = -\log\Big(\frac{\gamma}{2}\, e^x + \frac{\sigma^2 + 1}{2\gamma}\, e^{-x} + \sigma \cos y\Big), \tag{2.22}$$

where $\gamma = 1/d$ and $\sigma = b/d$ are independent positive numbers. The special case $\sigma = \sqrt{\gamma^2 - 1}$ ($\gamma \geq 1$) is (2.18).

Note that the argument of the logarithm in (2.22) has the form $f(x) + \sigma \cos y$. We see that f is convex, $f(\pm\infty) = +\infty$, and

$$f'(x) = 0 \Leftrightarrow e^{2x} = \frac{\sigma^2 + 1}{\gamma^2}, \quad f_{\min} = \sqrt{\sigma^2 + 1}. \tag{2.23}$$

Thus $\psi(x, y)$ has local maxima where (2.23) holds and $y = 2k\pi,\ k \in \mathbb{Z}$, and saddle points where (2.23) holds and $y = 2k\pi + \pi,\ k \in \mathbb{Z}$. The qualitative behavior of such stream functions and their associated flows are much the same as those of the special cases given by (2.18).

3. The case $K = -1$

Solutions to (1.4) with $K = -1$ are also stream functions, yielding steady solutions to the Euler equations on domains $\Omega \subset \mathbb{R}^2$. In particular, if $\mathbb{R}^2 \setminus \Omega$ has at least two points, Ω has a canonical Poincaré metric, of the form $e^{2\psi}(dx^2 + dy^2)$, ψ satisfying (1.4) with $K = -1$. Examples include

$$e^{2\psi} = \frac{4}{(1 - |z|^2)^2}, \quad \text{on} \ \ D_1 = \{z \in \mathbb{R}^2 : |z| < 1\}, \tag{3.1}$$

and

$$e^{2\psi} = \Bigl(|z| \log \frac{1}{|z|}\Bigr)^{-2}, \quad \text{on} \ \ D_1 \setminus \{0\}, \tag{3.2}$$

which is a limiting case of

$$e^{2\psi} = \Bigl(\frac{b}{|z| \sin\bigl(b \log \frac{1}{|z|}\bigr)}\Bigr)^2, \quad \text{on} \ \ \{z : e^{-\pi/b} < |z| < 1\}, \tag{3.3}$$

given $b > 0$. The formulas (3.2) and (3.3) can be derived from (3.1) via various conformal covering maps. (See, e.g., the introduction to [MT].)

These radial stream functions are not particularly interesting from the point of view of studying fluid flows, but one can pull back Poincaré metrics to other domains, via conformal mappings, and thereby obtain explicit formulas for other stream functions. In these cases, ψ blows up at $\partial\Omega$, but one can restrict to $\Omega_a = \{z \in \Omega : \psi(z) < a\}$ to obtain smooth steady flows.

For example, if we use the transformation (2.6) to pull back the metric tensor on $D_1 \setminus \{0\}$ given by (3.2), we obtain a metric tensor $e^{2\psi}\,|dz|^2$ with

$$e^{2\psi} = \frac{|ad - bc|^2}{|az + b|^2 |cz + d|^2} \Bigl(\log\Bigl|\frac{cz + d}{az + b}\Bigr|\Bigr)^{-2}. \tag{3.4}$$

If we take

$$A = \begin{pmatrix} 1 & b \\ \bar{b} & 1 \end{pmatrix}, \quad |b| < 1, \tag{3.5}$$

then (3.4) is defined on $D_1 \setminus \{-b\}$ (and also on $\{z : |z| > 1, z \neq -1/\bar{b}\}$). The attached Mathematica notebook (Figure2A.nb) shows some streamlines when we take $b = 0.2$ in (3.5). As one can see, the vector field $v = J\nabla\psi$ has a critical point of saddle type at $x \approx -0.45,\ y = 0$, and a critical point of center type at $x \approx 0.25,\ y = 0$. These features are also present in the cat's eye flows, discussed in §2. Compare Figure 2.4 in [MB].

If we pull back this metric on $D_1 \setminus \{0\}$ to the upper half plane, minus $(0, 1)$, via $z \mapsto (z - i)/(z + i)$, we obtain the stream function given by

$$e^{2\psi} = \frac{4}{|z^2+1|^2}\Bigl(\log \frac{|z+i|}{|z-i|}\Bigr)^{-2}, \tag{3.6}$$

whose streamlines are depicted in Figure2B.nb. Note the saddle point at $x = 0,\ y \approx 0.65$.

In Figure3.nb we show streamlines of a stream function obtained as follows. Pick $b \in (0, 1)$ (here $b = 0.15$), take the Poincaré metric on $D_1 \setminus \{-b\}$ arising from (3.4)–(3.5), and pull it back to $D_1 \setminus \{\pm i\sqrt{b}\}$ via the branched covering map $z \mapsto z^2$. The resulting metric tensor is $e^{2\psi}\,|dz|^2$ with

$$e^{2\psi} = \frac{4(1-b^2)|z|^2}{|z^2+b|^2|bz^2+1|^2}\ \Bigl(\log\Bigl|\frac{bz^2+1}{z^2+b}\Bigr|\Bigr)^{-2}. \tag{3.7}$$

This metric tensor is degenerate at $z = 0$. In fact we have $\psi(z) \to -\infty$ as $z \to 0$, while $\psi(z) \to +\infty$ as $z \to \pm i\sqrt{b}$ and as $|z| \to 1$. We can regard some closed level curve of ψ around the origin as the boundary of an obstacle, and Figure3.nb shows streamlines of a flow in a region that is approximately a disk, with three obstacles.

In Figure4.nb we show streamlines of a stream function obtained as follows. Take the Poincaré metric on $\{z : \operatorname{Im} z > 0,\ z \neq i\}$ given by (3.6) and pull it back to the strip $\{z : 0 < \operatorname{Im}\, z < \pi\}$, with the point $z = \pi i/2$ deleted, via the map $z \mapsto e^z$. The resulting metric tensor is $e^{2\psi}\,|dz|^2$ with

$$e^{2\psi} = \frac{1}{|\cosh z|^2}\ \Bigl(\log\Bigl|\frac{e^z+i}{e^z-i}\Bigr|\Bigr)^{-2}. \tag{3.8}$$

This models flow in a (not quite straight) pipe, with an obstacle, the flow moving to the left at the top and to the right at the bottom.

While we have dwelt on Poincaré metrics here (with one sort of generalization for the case of $D_1 \setminus \{\pm i\sqrt{b}\}$), more general metrics of the form $e^{2\psi}\,|dz|^2$ with curvature -1 describe stream functions. We note that such a metric does not exist on the space $\mathbb{C}/\Gamma$ discussed in §2. (Recall we had $K = +1$ there.) Indeed, $\mathbb{C}/\Gamma$ is holomorphically covered by $\mathbb{C}$, not by D_1, so it does not have a Poincaré metric. Now if we had any metric of the form $e^{2\psi}\,|dz|^2$ on $\mathbb{C}/\Gamma$ with Gauss curvature $K = -1$, then this would provide a barrier function allowing the construction of a Poincaré metric (cf. [MT], Proposition 7.1), giving a contradiction.

4. Applications to Beltrami flows

A Beltrami flow on $\mathcal{O} \subset \mathbb{R}^3$ is a vector field v on $\mathcal{O}$ (tangent to the boundary, if nonempty), satisfying

$$\operatorname{curl} v(p) \parallel v(p), \quad \forall\, p \in \mathcal{O}. \tag{4.1}$$

Such v is a steady solution to the 3D Euler equation. If ψ is a function on $\Omega \subset \mathbb{R}^2$ satisfying (1.3), then the vector field on $\mathcal{O} = \Omega \times \mathbb{R}$ defined by

$$v = (-\psi_y, \psi_x, W(\psi)) \tag{4.2}$$

gives a Beltrami flow, provided $W(\psi)$ satisfies

$$W'(\psi)W(\psi) = -\Phi(\psi); \tag{4.3}$$

cf. [MB], §2.3.2. As pointed out there, if ψ satisfies (1.4) with $K = +1$, one can take $W(\psi) = e^{\psi}$, so if ψ defines the cat's eye flow, i.e., if ψ is given by (2.18), or more generally if ψ is given by (2.22), then $(-\psi_y, \psi_x, e^{\psi})$ yields a Beltrami flow.

If ψ satisfies (1.4) with $K = -1$, then (4.3) becomes

$$W'(\psi)W(\psi) = -e^{2\psi}, \tag{4.4}$$

which has solutions of the form

$$W(\psi) = \sqrt{A - e^{2\psi}}, \tag{4.5}$$

for a given constant $A > 0$, as long as $e^{2\psi} < A$ on Ω. This gives Beltrami flows on regions $\Omega \times \mathbb{R} \subset \mathbb{R}^3$ from stream functions constructed as indicated in §3.

References

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[S] J. Stuart, Stability problems in fluids, pp. 139–155 in *Mathematical Problems in the Geophysical Sciences*, AMS, Providence, R.I., 1971.