## A Model for Harmonics on Stringed Instruments

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### 1. Introduction

We propose a model to explain the playing of "harmonics" on stringed instruments. The interesting phenomenon here is that placing a finger lightly at one of the nodes of the low frequency harmonics seems to force the string to play a note that sounds like a superposition of those normal modes with nodes at the location of the finger. For example, if the finger is placed 1/4 the way down the string, the note that is heard is two octaves above the fundamental. Pressing hard at that place on the string would yield a note with fundamental much lower. With care, one can play harmonics 4 or 5 octaves above the fundamental. It is very striking, for example, to hear a fat bass string play these shrill high tones vibrating along their entire length. A second aspect is that if the finger is lightly placed at a point which is not a node of a low frequency normal mode, the observed sound is a rapidly dying thud.

The problem we propose is to construct a model for the lightly placed finger which explains these observations. It turns out that a strong frictional resistance which is localized in a very small region has the desired properties. More precisely, the model we propose for the string occupying the interval $0 \le x \le \pi$ and fixed at the endpoints $x = 0$ and $\pi$ is

$$u_{tt} + b(x)u_t = u_{xx}; \quad 0 \le x \le \pi, \ t \ge 0, \tag{1.1}$$

$$u(t,0) = u(t,\pi) = 0; \quad t \ge 0. \tag{1.2}$$

Here the frictional resistance $b(x)$, which models the finger friction, is assumed to be $\ge 0$ and strongly localized near a point $a \in (0,\pi)$. Existence, uniqueness, and qualitative behavior of solutions of the problem are developed from the law of energy decay

$$\frac{d}{dt}\int_0^\pi (u_t^2 + u_x^2)\,dx = -\int_0^\pi b(x)u_t(t,x)^2\,dx \le 0.$$

The analysis proceeds in two steps. First, we show that for highly localized $b$, the behavior of (1.1) is approximated by that of a singular equation

$$u_{tt} + \alpha\delta(x-a)u_t = u_{xx}, \quad \alpha = \int_0^\pi b(x)\,dx. \tag{1.3}$$

This formal limit is equivalent to the wave equation for $x \ne a$, supplemented by a transmission condition at $x = a$. The second step is a fairly precise analysis of this limiting equation. Among other things, we show that if $a/\pi$ is irrational, then all solutions tend to zero as $t \to \infty$, while if $a/\pi$ is rational, the components of $u$ in the span of the nodes that vanish at $x = a$ propagate as if there were no friction while the components orthogonal (in the natural scalar product given by the energy) to these decay exponentially. These results mirror the observed phenomena described above.

In §4 we summarize our findings and bring up the phenomenon of using the "correct touch" to produce harmonics. We also discuss the appropriateness of our model.

Remark. This paper appeared as reference [10] in Arch. Rat., in 1982. At the time, the manuscript was prepared on an old fashioned typewriter. Here the paper has been typed in TeX. More recently, additional work on this phenomenon has appeared in [11].

## 2. The limiting transmission problem

To study the behavior of (1.1)–(1.2) with highly localized friction, we investigate the limiting behavior as $b(x)$ becomes more and more localized. It turns out that the limiting behavior corresponds quite closely to the observations above concerning stringed instruments. In this section, we will show that the limiting behavior is given by solving a specific transmission problem, and an analysis of this transmission problem is given in the next section.

First, we cast the basic problem in the framework of the theory of semigroups of operators. Consider the pair $U = (u_t, u)$ as an element of the Hilbert space $\mathcal{H} = L^2([0,\pi]) \oplus H_0^1([0,\pi])$, where $H_0^1([0,\pi])$ is the completion of $C_0^\infty((0,\pi))$ in the norm $\left(\int_0^\pi (u^2 + u_x^2)\,dx\right)^{1/2}$. The mixed problem (1.1)–(1.2) is equivalent to the evolution equation

$$U_t = \Gamma U, \tag{2.1}$$

where

$$\Gamma = \begin{pmatrix} -b & D^2 \\ I & 0 \end{pmatrix}, \quad D = \frac{\partial}{\partial x},$$
$$\mathcal{D}(\Gamma) = H_0^1([0,\pi]) \oplus \left(H^2([0,\pi]) \cap H_0^1([0,\pi])\right).$$

It is a simple matter, using the theory of ordinary differential equations, to show that $\Gamma$ so defined is a maximal dissipative operator. We will present a similar but slightly harder proof for the operator $G_\alpha$, which occurs further on, and therefore omit the details of the present argument. The theory of semigroups provides a solution of the differential equation (2.1) with initial condition

$$U(0) = (g, f) \in \mathcal{H}, \tag{2.2}$$

the solution $U$ being a continuous function of $t$ with values in $\mathcal{H}$. In addition, if $(g, f) \in \mathcal{D}(\Gamma^3)$, it is not hard to show that the associated function $u(t,x)$ is a classical solution of the mixed problem (1.1)–(1.2), with $u(0,x) = f$ and $u_t(0,x) = g$. There are, of course, other ways to treat this mixed problem, using for example the method of characteristics [1, Chap. 5], or the theory of symmetric positive systems, [2,8]. We have chosen the present approach because it seems to yield the strongest results when we consider the limiting behavior as $b$ becomes more localized.

For future use, we record one more fact about $\Gamma$. The equation $(I - \Gamma)U = F$ for $U = (u, v)$ and $F = (F_1, F_2)$ in $\mathcal{H}$ is equivalent to the following equations:

$$w - v = F_2, \tag{2.3}$$

and

$$\int_0^\pi (w\varphi + bv\varphi + w_x\varphi_x)\,dx = \int_0^\pi F_1\varphi\,dx, \quad \forall\,\varphi \in H_0^1([0,\pi]). \tag{2.4}$$

This weak formulation is ideally suited to our needs. We consider a sequence of non-negative friction coefficients $b_n \in C^\infty([0,\pi])$ with the property that

$$\lim_n \int_0^\pi b_n(x)\psi(x)\,dx = \alpha\psi(a), \quad \forall\,\psi \in C([0,\pi]), \tag{2.5}$$

where $0 \le \alpha < \infty$. If $b_n$ satisfies (2.5) and $u_n$ is the solution of the mixed problem (1.1)–(1.2) with initial data $(g,f)$, then, as $n \to \infty$, the functions $u_n$ will be shown to converge to the solution $u$ of the transmission problem

$$u_{tt} - u_{xx} = 0 \quad \text{for} \quad x \in [0,\pi] \setminus \{a\},\ t \ge 0, \tag{2.6}$$

$$[u_t] = 0 \quad \text{and} \quad \alpha u_t = [u_x] \quad \text{at} \quad x = a,\ t \ge 0, \tag{2.7}$$

$$u = 0 \quad \text{at} \quad x = 0,\ \pi, \tag{2.8}$$

with $(u_t(0,x), u(0,x)) = (g,f)$. The quantity $[h]$ at $x=a$ is the jump $h(a+0) - h(a-0)$ in $h$ at the point $a$.

There is also a semigroup formulation of the problem (2.6)–(2.8). For $U \in \mathcal{H}$ we get the equation

$$U_t = G_\alpha U,$$

where

$$G_\alpha = \begin{pmatrix} 0 & D^2 \\ I & 0 \end{pmatrix} \quad \text{for} \quad x \ne a,$$

and

$$\mathcal{D}(G_\alpha) = \Big\{(v,w) \in H_0^1([0,\pi]) \oplus \big(H^2([0,a]) \cap\ H^2([a,\pi])\big) : \alpha v = [w_x] \text{ at } x = a\Big\}.$$

This is reasonable, since formally $\Gamma$ approaches $\begin{pmatrix} -\alpha\delta_a & D^2 \\ I & 0 \end{pmatrix}$, while the equation $D^2w - \alpha\delta_a v = F_1$ translates to $D^2w = F_1$ for $x \ne a$ and $\alpha v = [w_x]$ at $x = a$.

We show that $G_\alpha$ as defined above is a maximal dissipative operator. First, for all $U = (v,w) \in \mathcal{D}(G_\alpha)$ we have

$$(GU,U)_{\mathcal{H}} = -\alpha v(a)^2 \le 0,$$

so $G_\alpha$ is dissipative. To see that the range $\mathcal{R}(I - G_\alpha) = \mathcal{H}$, suppose that $F = (F_1,F_2) \in \mathcal{H}$. Then the equation $(I - G_\alpha)U = F$ is equivalent to

$$(v,w) \in \mathcal{D}(G_\alpha), \quad w - v = F_2,$$
$$v - D^2w = F_1 \quad \text{on} \quad [0,\pi] \setminus \{a\}.$$

Using the first equation to eliminate $v$ from the second, we get

$$-D^2 w + w = F_1 - F_2, \quad x \neq a. \tag{2.9}$$

There is a two parameter family of solutions of this equation which, in addition, satisfies $w = 0$ at $x = 0, \pi$. In fact, we may take $w'(0), w'(\pi)$ as the parameters. In addition to (2.9), $w$ must satisfy

$$[w] = 0 \;\text{ and }\; \alpha w - [w_x] = F_2 \;\text{ at }\; x = a, \tag{2.10}$$

where the second condition comes from eliminating $v$ from the transmission condition. To see that these restrictions uniquely determine $w'(0), w'(\pi)$, we need only show that the map

$$(w'(0), w'(\pi)) \mapsto ([w], \alpha w(a+) - [w_x]_a)$$

is a nonsingular linear transformation from $\mathbb{R}^2$ to itself. For this, it suffices to show that the map is injective. If $(w'(0), w'(\pi)) \mapsto (0, 0)$, then the associated function $w$ satisfies

$$\begin{gathered}(I - D^2)w = 0 \;\text{ for }\; x \neq a,\\ [w] = \alpha w - [w_x] = 0 \;\text{ at }\; x = a,\\ w = 0 \;\text{ at }\; x = 0,\ \pi.\end{gathered}$$

Integrating by parts in the identity

$$\int_0^a w(I - D^2)w\,dx + \int_a^\pi w(I - D^2)w\,dx = 0$$

then yields

$$\int_0^\pi ((Dw)^2 + w^2)\,dx + \alpha w(a)^2 = 0,$$

so $w = 0$. Thus $w$ is uniquely determined by (2.9)–(2.10). Setting $v = w - F_2$ gives a pair $U = (v, w)$ that satisfies $(I - G_\alpha)U = F$, and the proof of maximality is complete.

The next theorem asserts the convergence of the solutions of the mixed problem (1.1)–(1.2), with friction coefficients $b_n$ satisfying (2.5), to the solution of the transmission problem defined above.

**Theorem 2.1.** *If the non-negative friction coefficients $b_n$ satisfy (2.5), and if $\Gamma_n$ are the associated maximal dissipative operators, then for each $t \geq 0$,*

$$s - \lim_{n\to\infty} e^{t\Gamma_n} = e^{tG_\alpha},$$

*and the convergence is uniform on compact time intervals.*

*Proof.* We apply the Trotter-Kato theorem [4, Chapter 9, Thm. 2.16], thereby reducing the problem to showing that for every $F \in \mathcal{H}$,

$$(I - \Gamma_n)^{-1} F \longrightarrow (I - G_\alpha)^{-1} F, \quad \text{in } \mathcal{H}.$$

Let $U_n = (v_n, w_n) = (I - \Gamma_n)^{-1} F$. We first show that $U_n$ converges weakly in $\mathcal{H}$ to $(I - G_\alpha)^{-1} F$. Notice that

$$\|U_n\|_{\mathcal{H}} \leq \|(I - \Gamma_n)^{-1}\| \cdot \|F\|_{\mathcal{H}} \leq \|F\|_{\mathcal{H}},$$

so $\{U_n\}$ is weakly compact in $\mathcal{H}$. Let $U = (v, w)$ be a weak limit point, and choose a subsequence $U_{n_k}$ converging weakly to $U$. We claim that $U = (I - G_\alpha)^{-1} F$.

From (2.3), we have $w_{n_k} - v_{n_k} = F_2$. Hence, passing to the limit $k \to \infty$, we have $w - v = F_2$. Similarly, (2.4) holds for $w_{n_k}, v_{n_k}$ provided $b$ is replaced by $b_{n_k}$. Passing to the limit, we obtain

$$\int_0^\pi (w\varphi + Dw \cdot D\varphi)\, dx + \lim \int_0^\pi b_{n_k} v_{n_k} \varphi\, dx = \int_0^\pi F_1 \varphi\, dx, \quad \forall\, \varphi \in H_0^1([0, \pi]).$$

By (2.5), if we consider $b_n \varphi$ as an element of the dual of $C([0, \pi])$, the sequence $b_n \varphi$ converges weak* to $\alpha \varphi(a) \delta_a$. However, $v_{n_k} = w_{n_k} - F_2$ converges weakly to $v$ in $H_0^1([0, \pi])$ and therefore uniformly. Thus the limit above is $\alpha v(a) \varphi(a)$, and we have

$$\int_0^\pi (w\varphi + Dw \cdot D\varphi)\, dx + \alpha v(a) \varphi(a) = \int_0^\pi F_1 \varphi\, dx, \tag{2.11}$$

for all $\varphi \in H_0^1([0, \pi])$. With the help of the identity $w - v = F_2$, the condition (2.11) is easily shown to be equivalent to the equation $(I - G_\alpha) U = F$. This proves the weak convergence of $U_n$ to $U$.

To prove the strong convergence, we investigate the sequence $\|U_n\|_{\mathcal{H}}$. To do this, set $\varphi = w_n$ in (2.4) to obtain

$$\int_0^\pi (w_n^2 + (Dw_n)^2)\, dx = \int_0^\pi F_1 w_n\, dx - \int_0^\pi b_n v_n w_n\, dx.$$

The limit of the right hand side as $n \to \infty$ is

$$\int_0^\pi F_1 w\, dx - \alpha w(a) v(a),$$

which is precisely $\int_0^\pi (w^2 + (Dw)^2)\, dx$, as can be seen by choosing $\varphi = w$ in (2.11). This takes care of the $w$ component; that is, $w_n \to w$ in $H_0^1([0, \pi])$. It follows then that $v_n = w_n - F_2$ converges to $v$ in $L^2([0, \pi])$. The proof of Theorem 2.1 is complete.

If one considers $b_n(x)$ satisfying $b_n \geq 0$, Theorem 2.1 shows that $\int_0^\pi b_n\,dx$ is an appropriate measure of the strength of the frictional force, since if $\int b_n\,dx \to \alpha \geq 0$, the limiting behavior is given by (2.6)–(2.8). We now turn our attention to the case of extremly large friction. Suppose $b_n \geq 0$ and

$$\begin{aligned} &\int_0^\pi b_n(x)\,dx = \beta_n \nearrow \infty, \quad \text{as} \quad n \to \infty, \\ &\limsup_{x \in K, n \in \mathbb{Z}^+} b_n(x) = 0, \quad \forall \text{ compact } K \subset [0,\pi] \setminus \{a\}. \end{aligned} \tag{2.12}$$

The second hypothesis asserts that, away from the point $a$, the functions $b_n$ concentrate near $a$, as $n \to \infty$.

Suppose that $u_n$ is the solution of (1.1)–(1.2) with Cauchy data $\partial_t u_n(0) = \psi,\ u_n(0) = \varphi$, and energy at time $t$ given by

$$\|U_n(t)\|_{\mathcal{H}}^2 = \int_0^\pi (u_t(t,x)^2 + u_x(t,x)^2)\,dx, \quad U_n = (\partial_t u_n, u_n).$$

The formula for energy decay is

$$\|U_n(T)\|_{\mathcal{H}}^2 = \|U_n(0)\|_{\mathcal{H}}^2 - \int_0^T \int_0^\pi b_n \Big(\frac{\partial u_n}{\partial t}\Big)^2\,dx\,dt.$$

In particular, for all $n$,

$$\int_0^T \int_0^\pi b_n \Big(\frac{\partial u_n}{\partial t}\Big)^2\,dx\,dt \leq \int_0^\pi (\psi(x)^2 + \varphi'(x)^2)\,dx.$$

This implies that if we consider $\partial_t u_n(\cdot,x) \in L^2([0,T])$ for each $x$, the weighted averages satisfy

$$\frac{1}{\beta_n} \int_0^\pi b_n(x) \|\partial_t u_n(\cdot,x)\|_{L^2([0,T])}^2\,dx \longrightarrow 0, \quad \text{as} \quad n \to \infty. \tag{2.13}$$

Now $\{u_n\}$ is bounded in $H^1((0,T)\times(0,\pi))$, and

$$H^1((0,T)\times(0,\pi)) \hookrightarrow C([0,\pi], L^2((0,T)) \text{ compactly.}$$

Thus $\{\partial_t u_n\}$ lies in a compact subset of $C([0,\pi], H^{-1}((0,T)))$. Since (2.13) implies that

$$\frac{1}{\beta_n} \int_0^\pi b_n(x) \|\partial_t u_n(\cdot,x)\|_{H^{-1}((0,T))}^2\,dx \longrightarrow 0 \quad \text{as} \quad n \to \infty,$$

it follows that

$$\frac{\partial u_n}{\partial t}(\cdot,a) \longrightarrow 0 \quad \text{in} \quad H^{-1}((0,T)), \quad \text{as} \quad n \to \infty. \tag{2.14}$$

It is therefore reasonable to expect that $u_n$ tends in the limit to the unique solution $u \in H^1((0,T)\times(0,\pi))$ of the mixed problem

$$u_{tt} - u_{xx} = 0 \quad \text{in} \quad (0,T)\times((0,\pi)\setminus\{a\}), \tag{2.15}$$

$$u(\cdot,0) = u(\cdot,\pi) \quad \text{on} \quad (0,T), \tag{2.16}$$

$$\frac{\partial u}{\partial t}(\cdot,a) = 0 \quad \text{on} \quad (0,T), \tag{2.17}$$

$$u(0,\cdot) = \varphi \quad \text{on} \quad (0,\pi), \tag{2.18}$$

$$\frac{\partial u}{\partial t}(0,\cdot) = \psi \quad \text{on} \quad (0,\pi). \tag{2.19}$$

Note that this limiting problem is energy conserving. Here is our next result.

**Theorem 2.2.** *If (2.12) holds and the Cauchy data $(\psi,\varphi)$ is in $\mathcal{H}$, then, for each $t \geq 0$,*

$$(\partial_t u_n(t), u_n(t)) \longrightarrow (\partial_t u(t), u(t)) \quad \text{in} \quad \mathcal{H} \quad \text{as} \quad n \to \infty,$$

*where $u \in C([0,T],\mathcal{H})$ is the solution of the mixed problem (2.15)–(2.19).*

*Proof.* Because the energy is a decreasing function of time for each $U_n$, it suffices to prove the theorem for a set of Cauchy data that is dense in $\mathcal{H}$. Thus we may assume there is an $\eta > 0$ and $C \in \mathbb{R}$ such that

$$\psi = 0 \quad \text{on} \quad [a-\eta, a+\eta],$$

and

$$\varphi = C \quad \text{on} \quad [a-\eta, a+\eta].$$

It follows from finite propagation speed that, for all $n$,

$$u_n = C \quad \text{on} \quad \{(t,x) : |t| + |x-a| < \eta\}. \tag{2.20}$$

To proceed, we begin by showing that $u_n \to u$ weakly in $H^1((0,T)\times(0,\pi))$. To do tis, we show that every subsequence of $(u_n)$ has a further subsequence converging weakly to $u$. The crucial observations are

$$\{u_n\} \quad \text{is bounded in} \quad C([0,T], H_0^1([0,\pi])) \tag{2.21}$$

and

$$\left\{\frac{\partial u_n}{\partial t}\right\} \quad \text{is bounded in} \quad C([0,T], L^2([0,\pi])), \tag{2.22}$$

both consequences of energy decay. From (1.1) and (2.12), we see that, for any compact interval $K$ in $[0,\pi]\setminus\{a\}$,

$$\left\{\frac{\partial^2 u_n}{\partial t^2}\right\} \quad \text{is bounded in} \quad C([0,T], H^{-1}(K)).$$

It follows that, given any subsequence of $(u_n)$, we may choose a further subsequence $(u_{n_j})$ such that in the weak* topologies,

$$u_{n_j} \to w \quad \text{in } L^\infty((0,T), H_0^1(0,\pi)), \tag{2.23}$$

$$\frac{\partial u_{n_j}}{\partial t} \to \frac{\partial w}{\partial t} \quad \text{in } \ L^\infty((0,T), L^2([0,\pi]), \tag{2.24}$$

$$\frac{\partial^2 u_{n_j}}{\partial t^2} \to \frac{\partial^2 w}{\partial t^2} \quad \text{in } \ L^\infty((0,T), H^{-1}_{\text{loc}}((0,\pi) \setminus \{a\})). \tag{2.25}$$

It follows immediately that

$$\begin{gathered} w \in H^1((0,T) \times (0,\pi)), \\ w_{tt} - w_{xx} = 0 \ \text{ in } \ (0,T) \times ((0,\pi) \setminus \{a\}), \\ w = 0 \ \text{ in } \ (0,T) \times \{0\} \ \text{ and } (0,T) \times \{\pi\}, \\ w = \varphi \ \text{ on } \ \{0\} \times (0,\pi). \end{gathered}$$

From (2.14) we see that, as an element of $H^{-1}([0,T])$,

$$\frac{\partial w}{\partial t} = 0 \ \text{ on } \ (0,T) \times \{a\}.$$

From (2.20) we get $\partial_t w = 0$ on $(0,\eta) \times \{a\}$, and therefore

$$\frac{\partial w}{\partial t} = 0 \ \text{ on } \ (0,T) \times \{a\}. \tag{2.26}$$

We claim that $w_t(0) = \psi$. From (2.23)–(2.25), it follows that

$$\partial_t u_{n_j}(0) \to \partial_t w(0) \ \text{ weakly in } \ L^2_{\text{loc}}((0,\pi) \setminus \{a\}),$$

so that

$$\frac{\partial w}{\partial t}\Big|_{t=0} = \psi \ \text{ on } \ (0,\pi) \setminus \{a\}. \tag{2.27}$$

However, from (2.21),

$$\psi = \frac{\partial w}{\partial t} = 0 \ \text{ on } \ \{0\} \times (a-\eta, a+\eta),$$

which together with (2.27) yields the desired result.

At this point, we have proved that $u_n \to u$ weakly in $H^1((0,T) \times (0,\pi))$. The functions $u_n$ and $u$ lie in the closed subspace of $\mathcal{H}$ consisting of functions vanishing at $(0,T) \times \{0\}$. On this subspace the quantity

$$\|u\|^2_{\mathcal{E}} = \int_0^T \int_0^\pi (u_t^2 + u_x^2)\, dx\, dt$$

furnishes a norm equivalent to that of $H^1((0,T)\times(0,\pi))$. Clearly

$$\|u_n\|_{\mathcal{E}}^2 \le T\|(\psi,\varphi)\|_{\mathcal{H}}^2 = \|u\|_{\mathcal{E}}^2. \tag{2.28}$$

This norm inequality together with weak convergence implies that $u_n \to u$ strongly in $H^1((0,T)\times(0,\pi))$.

It remains to show that, for each $t \in [0,T]$,

$$(\partial_t u_n(t), u_n(t)) \longrightarrow (\partial_t u(t), u(t)) \ \text{ in } \ \mathcal{H}.$$

From the laws of energy decrease, we know that $\{u_n(t)\}$ is bounded in $H^1((0,\pi))$ and from the convergence in $H^1((0,T)\times(0,\pi))$ we know that $u_n(t) \to u(t)$ in $L^2([0,\pi])$. It follows that $u_n(t) \to u(t)$ weakly in $H^1((0,\pi))$. Similarly, $\{\partial_t u_n(t)\}$ is bounded in $L^2([0,\pi])$, and (2.23) and (2.25) imply that

$$\frac{\partial u_n}{\partial t}(t) \longrightarrow \frac{\partial u}{\partial t}(t) \ \text{ weakly in } \ L^2_{\mathrm{loc}}((0,\pi)\setminus\{a\}).$$

Consequently

$$\frac{\partial u_n}{\partial t}(t) \longrightarrow \frac{\partial u}{\partial t}(t) \ \text{ weakly in } \ L^2([0,\pi]).$$

Thus

$$(\partial_t u_n(t), u_n(t)) \longrightarrow (\partial_t u(t), u(t)) \ \text{ weakly in } \ \mathcal{H}.$$

In addition,

$$\|(\partial_t u_n(t), u_n(t))\|_{\mathcal{H}} \le \|(\psi,\varphi)\|_{\mathcal{H}} = \|\partial_t u(t), u(t)\|_{\mathcal{H}},$$

and the strong convergence in $\mathcal{H}$ follows.

## 3. Analysis of the transmission problem

In this section we make a qualitative analysis of the transmission problems associated with localized friction. A crucial role is played by the position $a$ of the friction and in particular the rationality or irrationality of $a/\pi$. The main results describe the spectrum of $G_\alpha$ and the asymptotic behavior of $e^{tG_\alpha}$ as $t \to +\infty$. First, we present some results that do not depend on the value of $a$.

In §2, we tacitly assumed elements of $\mathcal{H}$ had real valued components. Here, it is convenient to take the complexification.

**Theorem 3.1.** *The family $G_\alpha$ is holomorphic in $\alpha \in \mathbb{C}$. The operators $G_\alpha$ have compact resolvents and the eigenvalues of $G_\alpha$ are simple. The eigenvalues and eigenprojections are analytic functions of $a \in (0, \pi)$. For $\alpha \geq 0$, the spectrum of $G_\alpha$ is contained in $\{z \in \mathbb{C} : \operatorname{Re} z \leq 0\}$.*

*Proof.* That $G_\alpha$ is holomorphic in $\alpha$ follows from Theorem VII.1.14 of [4], by precisely the argument in Example 1.15, which follows the proof of that theorem. The details are omitted.

That $G_\alpha$ has a nonempty resolvent set follows from the observation that for $\operatorname{Re}\alpha \geq 0$, $G_\alpha$ is maximal dissipative and for $\operatorname{Re}\alpha \leq 0$, $-G_\alpha$ is maximal dissipative. All the $G_\alpha$ are restrictions of order one ($\dim \mathcal{D}(T)/\mathcal{D}(G_\alpha) = 1$) of the operator $T$, defined by

$$\mathcal{D}(T) = H_0^1([0,\pi]) \oplus \Big(H_0^1([0,\pi]) \cap H^2([0,a]) \cap H^2([a,\pi])\Big),$$

$$T = \begin{pmatrix} 0 & D^2 \\ I & 0 \end{pmatrix}, \quad \text{for} \quad x \neq a.$$

By Corollary III.6.14 of [4], it follows that $G_\alpha$ has compact resolvent either for all values of $\alpha$ or for none. For $\alpha = 0$, $G_0$ is skew-adjoint with a complete system of eigenvectors

$$(\pm in \sin nx, \sin nx), \quad n = 1, 2, \ldots,$$

with eigenvalues $\pm in$. Thus $G_0^{-1}$ is compact and $G_\alpha$ has compact resolvent for all $\alpha$.

To prove the simplicity of the eigenvalues, notice that the equation $G_\alpha U = i\lambda U$ for $U = (v, w)$ is equivalent to

$$\begin{aligned} D^2 w &= i\lambda v, \quad x \neq a, \\ v &= i\lambda w, \\ [w] = 0, &\quad \alpha v = [Dw] \quad \text{at} \quad x = a. \end{aligned}$$

Eliminating $v$ yields

$$D^2 w + \lambda^2 w = 0 \quad \text{for} \quad x \neq a, \tag{3.1}$$

$$i\lambda\alpha w = [Dw], \quad [w] = 0 \quad \text{at} \quad x = a. \tag{3.2}$$

Given two eigenvectors with the eigenvalue $i\lambda$, we may form a linear combination $(v, w)$ such that $w'(0) = 0$. Then, since $w(0) = 0$, the differential equation (3.1) implies $w \equiv 0$ on $[0, a]$. The transmission conditions (3.2) then show that $Dw(a+) = w(a+) = 0$, and equation (3.1) yields $w \equiv 0$ on $[a, \pi]$. Hence $w = 0$ and therefore $v = i\lambda w = 0$. Thus $(v, w) = 0$. This shows that any two eigenvectors with the same eigenvalue are linearly dependent.

Remark. The argument above shows that the eigenvalues of $G_\alpha$ have geometric multiplicity one. The paper [11] treats the issue of their having *algebraic* multiplicity one.

The analytic dependence on $\alpha$ is now a consequence of the fact that $G_\alpha$ is holomorphic in $\alpha$ (cf. [4, Th. VII.1.7]).

The analyticity in $a$ is a little harder since the location of the transmission condition is changing. Fortunately, a standard method takes care of this. Let

$$y_a(x) = \begin{cases} \dfrac{\pi}{2a} x, & \text{if} \ \ 0 \le x \le a, \\[2mm] \dfrac{\pi}{2(\pi - a)}(x - \pi) + \pi, & \text{if} \ \ a \le x \le \pi. \end{cases}$$

The map $x \mapsto y_a(x)$ is a homeomorphism of $[0, \pi]$ onto itself which takes $a$ to $\pi/2$. Denote the inverse mapping by $x_a(y)$, and define maps

$$S_a : \mathcal{H} \longrightarrow \mathcal{H}$$

by

$$(S_a U)(y) = U(x_a(y)).$$

The mapping $S_a$ is invertible, and $\widetilde{G}_\alpha = S_a^{-1} G_\alpha S_a$ is the operator given by the following procedure.

$$\mathcal{D}(\widetilde{G}_\alpha) = \Big\{ (v, w) \in H_0^1([0, \pi]) \oplus \Big( H_0^1([0, \pi]) \cap H^2([0, a]) \cap H^2([a, \pi]) \Big) : \\ \text{at } x = a, \ \alpha v = \frac{\pi}{2(\pi - a)} Dw\Big(\frac{\pi}{2}+\Big) - \frac{\pi}{2a} Dw\Big(\frac{\pi}{2}-\Big) \Big\},$$

$$\widetilde{G}_\alpha = \begin{cases} \begin{pmatrix} 0 & (\pi/2a)^2 D^2 \\ I & 0 \end{pmatrix}, & 0 < y < \dfrac{\pi}{2}, \\[4mm] \begin{pmatrix} 0 & (\pi/2(a - \pi))^2 D^2 \\ I & 0 \end{pmatrix}, & \dfrac{\pi}{2} < y < \pi. \end{cases}$$

The operator $\widetilde{G}_\alpha$ has coefficients depending analytically on $a$, and the transmission condition at $y = \pi/2$ depends analytically on $a$. Thus the family $\widetilde{G}_\alpha$ is analytic in $a$ (in the sense of [4, Chap. III, §1]). Consequently the eigenvalaues of $\widetilde{G}_\alpha$ depend analytically on $a$, and these coincide with the eigenvalues of $G_\alpha$.

For $\operatorname{Re}\alpha \geq 0$, $G_\alpha$ is dissipative. Accordingly, its spectrum is in the half plane $\{z \in \mathbb{C} : \operatorname{Re} z \leq 0\}$, and the proof of Theorem 3.1 is complete.

**Theorem 3.2.** *If $a/\pi$ is irrational and $\alpha > 0$, then the spectrum of $G_\alpha$ is contained in the open half plane $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$. All solutions of the transmission problem $U_t = G_\alpha U$ decay to zero as $t \to +\infty$. More precisely,*

$$s - \lim_{t \to +\infty} e^{tG_\alpha} = 0.$$

*However, $G_\alpha$ has eigenvalues with arbitrarily small real part, so $\|e^{tG_\alpha}\| = 1$ for all $t \geq 0$.*

*Proof.* We analyze more closely the conditions (3.1)–(3.2) that must be satisfied by solutions of $G_\alpha U = i\lambda U$. In the proof of the simplicity of the eigenvalues we showed that $w'(0) \neq 0$ if $w \neq 0$. Thus the differential equation (3.1) and Dirichlet conditions at $x = 0, \pi$ imply that up to a scalar multiple $w$ must be given by

$$w(x) = \begin{array}{ll} \sin \lambda x, & 0 \leq x \leq a, \\ b \sin \lambda(x - \pi), & a \leq x \leq \pi. \end{array} \tag{3.3}$$

Since $[w] = 0$ at $x = a$, we must have

$$\sin \lambda a = b \sin \lambda(a - \pi), \tag{3.4}$$

and the transmission condition $i\lambda\alpha w = [Dw]$ at $x = a$ yields

$$-i\alpha \sin \lambda a = \cos \lambda a - b \cos \lambda(a - \pi). \tag{3.5}$$

Now, purely imaginary eigenvalues correspond to real values of $\lambda$ for which the right hand side of (3.5) is real, so we must have $\sin \lambda a = 0$. From (3.4) it follows that either $\sin \lambda(a - \pi) = 0$ or $b = 0$. However, if $b = 0$, then the right hand side of (3.5) is $\cos \lambda a = \pm 1 \neq 0$, so (3.5) cannot hold. Thus $\sin \lambda a = \sin \lambda(\pi - a) = 0$. Hence $\lambda a = n\pi$ and $\lambda(a - \pi) = m\pi$ for integers $m, n \neq 0$ (for $\lambda = 0$, (3.3) yields $w \equiv 0$), so

$$\frac{a}{\pi} = \Big(\frac{1}{m} - \frac{1}{n}\Big)^{-1} \frac{1}{m},$$

a rational number. Thus, since $a/\pi$ is irrational, the assertion about the spectrum of $G_\alpha$ lying in $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$ is proved.

The decay properties now follow by applying the next result, which is implicit in the work of §9 of [5].

**Abstract Decay Theorem.** *If $A$ is a maximal dissipative operator on a Hilbert space $\mathcal{H}$ such that*

1. *$A$ has no purely imaginary eigenvalues, and*
2. *$A$ has compact resolvent,*

*then*

$$s-\lim_{t\to+\infty} e^{tA} = 0.$$

Taking $A = G_\alpha$, we obtain the required decay condition.

To continue with Theorem 3.2, next we show that $G_\alpha$ has eigenvalues arbitrarily close to the imaginary axis. For $\lambda$ not real, it is clear that $\sin \lambda a \neq 0$ and $\sin \lambda(a-\pi) \neq 0$. Hence we may divide (3.5) by the product of these sines to obtain

$$i\alpha = \cot \lambda(a-\pi) - \cot \lambda a. \tag{3.6}$$

Since $a/\pi$ is irrational, we may choose ([3, Theorem 36]) fractions $p/q$ (in lowest terms) with $q$ arbitrarily large such that

$$\left|\frac{p}{q} - \frac{a}{\pi}\right| < \frac{1}{q^2}. \tag{3.7}$$

The spectrum of $G_0$ consists of the numbers $\pm in,\ n = 1, 2, \dots$, with corresponding eigenfunctions $(\pm in \sin nx, \sin nx)$. Consider the eigenvalue $i\lambda(\alpha)$, which starts at $\alpha = 0$ from the point $iq$ (that is, $\lambda(0) = q$). We claim that $\lambda(\alpha)$ remains close to $q$ even for rather large $\alpha$, provided $q$ is large enough.

To make this precise, choose $p/q$ satisfying (3.7) and let $\alpha_c > 0$ be the smallest value of $\alpha$ such that $|\lambda(\alpha) - q| = 1/q$. Let

$$E_1 = qa - p\pi \quad (\text{so } 0 < |E_1| < 1/q),$$

and

$$E_2 = \lambda(\alpha_c) - q \quad (\text{so } |E_2| = 1/q,\ \operatorname{Im} E_2 > 0).$$

If $\equiv$ denotes equality in $\mathbb{C}/(\pi\mathbb{Z})$, we have

$$\begin{aligned} \lambda(\alpha_c)a &= qa + E_2 a \equiv E_1 + E_2 a, \\ \lambda(\alpha_c)(a-\pi) &\equiv E_1 + E_2 a - E_2 \pi. \end{aligned}$$

The cotangent function is periodic of period $\pi$. Thus (3.6) implies that

$$i\alpha_c = \cot(E_1 + E_2 a - E_2\pi) - \cot(E_1 + E_2 a). \tag{3.8}$$

Since the singular part of the Laurent expansion of $\cot z$ about $z = 0$ is $1/z$, we conclude that there is a positive constant $C$ such that, for $|z_1| < 1,\ |z_2| < 1$,

$$|\cot z_1 - \cot z_2| \geq \left|\frac{1}{z_1} - \frac{1}{z_2}\right| - C.$$

Using this estimate in (3.8) yields

$$|\alpha_c| \geq \frac{|E_2\pi|}{|E_1 + (a-\pi)E_2| \cdot |E_1 + E_2 a|} - C \geq \frac{\pi}{(\pi+1)^2} q - C.$$

Thus for $q$ sufficiently large, satisfying (3.7), we have $|\alpha_c| \geq q/10$. Consequently, if $\alpha < q/10$, then $|\lambda(\alpha) - q| < 1/q$. For fixed $\alpha$ this shows that there are eigenvalues $i\lambda(\alpha)$ arbitrarily close to the imaginary axis. The proof of Theorem 3.2 is complete.

**Theorem 3.3.** *Assume $\alpha > 0$, $q/\pi$ is rational, and $M$ is the closed linear span of the eigenvectors of $G_\alpha$ with purely imaginary eigenvalues. Then the following hold.*

*1. $M$ is precisely the closed linear span of the eigenvectors of $G_\alpha$ that vanish at $a$, and $G_\alpha = G_0$ for such eigenvectors.*

*2. $e^{tG_\alpha} = e^{tG_0}$ on $M$. In particular, $e^{tG_\alpha}$ is unitary on $M$.*

*3. $M^\perp$ is invariant under the semigroup $e^{tG_\alpha}$. Furthermore, there exist $C_1, C_2 \in (0,\infty)$, depending only on $\alpha$ and $a$, such that*

$$\|e^{tG_\alpha} U\|_{\mathcal{H}} \leq C_1 e^{-C_2 t} \|U\|_{\mathcal{H}}, \quad \forall U \in M^\perp, \ t \geq 0.$$

*Proof.* Suppose that $G_\alpha U = i\lambda U$ with $\alpha > 0$ and $\lambda \in \mathbb{R}$. Then $U = (v,w)$, $v = i\lambda w$, and $w$ is given by (3.3) up to a scalar multiple, where $b$ satisfies (3.4)–(3.5). For $\lambda = 0$, (3.3) shows that $w \equiv 0$. For $\lambda \in \mathbb{R} \setminus 0$, the imaginary part of (3.5) yields $\sin \lambda a = 0$, which shows that $w(a) = 0$. Thus $[Dw](a) = 0$ and $w$ satisfies $D^2 w + \lambda^2 w = 0$ on the entire interval $[0,\pi]$. It follows that $U = (v,w) \in \mathcal{D}(G_0)$, $G_0 U = i\lambda U$, and $v(a) = (1/\alpha)[Dw] = 0$. Conversely, if $G_0 U = i\lambda U$ and $U(a) = 0$, then $U = (v,w) \in \mathcal{D}(G_\alpha)$ and $G_\alpha U = i\lambda U$. Note here that the transmission condition $\alpha v = [Dw]$ at $x = a$ is automatically satisfied, since both sides vanish. This establishes point (1).

That $e^{tG_\alpha} = e^{tG_0}$ on $M$ follows from the fact that the two semigroups agree on finite linear combinations of the eigenvectors of $G_0$ and that this is dense in $M$. That $M^\perp$ is invariant under $e^{tG_\alpha}$ follows by applying the following simple lemma to $C = e^{tG_\alpha}$.

**Lemma 3.4.** *If $C$ is a linear contraction on the Hilbert space $\mathcal{H}$ and $M \subset \mathcal{H}$ is a closed invariant subspace such that $C : M \to M$ is unitary, then $M^\perp$ is invariant under $C$.*

*Proof.* Suppose $m \in M$ and $n \in M^\perp$. Then for all $\varepsilon \in \mathbb{R}$ we have

$$\|C(m + \varepsilon n)\|^2 \leq \|m + \varepsilon n\|^2.$$

Expanding both sides and using the relations $\|Cm\|^2 = \|m\|^2$ and $(m,n) = 0$, we get

$$2\varepsilon(Cm, Cn) + \varepsilon^2\|Cn\|^2 \le \varepsilon^2\|n\|^2.$$

Thus $(Cm, Cn) = 0$ for all $m \in M$. Since $C$ maps $M$ to $M$, we have $Cn \in M^\perp$, proving the lemma.

Now $e^{tG_\alpha}|_{M^\perp}$ is a semigroup of contractions whose generator has no purely imaginary eigenvalues. Thus the abstract decay theorem implies

$$s - \lim_{t\to\infty} e^{tG_\alpha}\big|_{M^\perp} = 0.$$

The exponential decay asserted in point (3) of Theorem 3.3 lies deeper. The idea of our demonstration is that for $a/\pi = p/q$ with $p$ and $q$ relatively prime integers we can find a simple and explicit formula for $e^{q^{-1}G_\alpha}$. Let $h = 1/q$ and suppose that the underlying space $\mathcal{H}$ is the *complex* Hilbert space $L^2 \oplus H^1_0$. The form of the explicit solution is described in the following lemma. In the proof, the mappings $\Lambda$ and $D$ are described explicitly.

**Lemma 3.5.** *Let $h = 1/q$ and let $\alpha \ge 0$. Then there are Hilbert spaces $\mathcal{K}$ and $\mathcal{E}$, with dim $\mathcal{E} = 4q - 1$, and a unitary map*

$$\Lambda : \mathcal{H} \longrightarrow \mathcal{K} \oplus L^2((0, h/2), \mathcal{E}),$$

*such that*

1. *$\mathcal{K}$ and $L^2((0,h/2),\mathcal{E})$ are invariant under $\Lambda e^{hG_\alpha}\Lambda^{-1}$,*

2. *$\Lambda e^{hG_\alpha}\Lambda^{-1}\big|_{\mathcal{K}} = Id_{\mathcal{K}}$,*

3. *There is a $D : \mathcal{E} \to \mathcal{E}$ such that*

$$\Lambda e^{hG_\alpha}\Lambda^{-1}\big|_{L^2((0,h/2),\mathcal{E})}$$

*is multiplication by $D$, that is,*

$$(\Lambda e^{hG_\alpha}\Lambda^{-1}V)(x) = DV(x), \quad \forall V \in L^2((0,h/2),\mathcal{E}).$$

Before proving Lemma 3.5, we use it to complete the proof of Theorem 3.3. Since $\Lambda e^{hG_\alpha}\Lambda^{-1}$ is a contraction, the same must be true of $D$. Write $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1$, where $\mathcal{E}_0$ is the span of eigenvectors of $D$ with eigenvalues of modus one. Then $D$ is unitary on $\mathcal{E}_0$, and there are constants $c > 0$ and $\rho \in (0,1)$ such that

$$\big\|D^n\big|_{\mathcal{E}_1}\big\| \le C\rho^n, \quad n = 0, 1, 2, \ldots.$$

Corresponding to this decomposition of $\mathcal{E}$, we have

$$L^2((0,h/2),\mathcal{E}) = L^2((0,h/2),\mathcal{E}_0) \oplus L^2((0,h/2)),\mathcal{E}_1).$$

Define $M_0 \subset \mathcal{H}$ by

$$\Lambda M_0 = \mathcal{K} \oplus L^2((0,h/2),\mathcal{E}_0).$$

Then $M_0$ has the following properties:

$$\begin{aligned} e^{hG_\alpha} &: M_0 \longrightarrow M_0 \ \text{ is unitary,} \\ e^{hG_\alpha} &: M_0^\perp \longrightarrow M_0^\perp, \end{aligned}$$

and

$$\big\| e^{nhG_\alpha}\big|_{M_0^\perp} \big\| \le C\rho^n, \quad n = 0,1,2,\dots. \tag{3.9}$$

As a consequence, we must have $M_0 = M$. Hence (3.9) yields part (3) of Theorem 3.3.

We turn to the proof of Lemma 3.5. Introduce the characteristic coordinates

$$\xi = \frac{1}{\sqrt{2}}(t+x), \quad \eta = \frac{1}{\sqrt{2}}(t-x),$$

and the characteristic derivatives

$$\begin{aligned} u_\xi &= \frac{\partial u}{\partial \xi} = \frac{1}{\sqrt{2}}\Big(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\Big), \\ u_\eta &= \frac{\partial u}{\partial \eta} = \frac{1}{\sqrt{2}}\Big(\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x}\Big). \end{aligned}$$

The wave equation (2.6) becomes

$$\frac{\partial^2 u}{\partial \xi\,\partial \eta} = 0 \ \text{ for } \ x \ne a. \tag{3.10}$$

Equivalently, $u_\xi$ is constant on the characteristics of speed $-1$, and $u_\eta$ is constant on the characteristics of speed $+1$.

The operator $\Lambda$ will be a product of unitary maps, the first one being the map that passes from the variables $(u_t, u)$ to the variables$(u_\xi, u_\eta)$. More precisely, we define $\Lambda_1 : \mathcal{H} \to \mathcal{H}_1$ by

$$\begin{aligned} \mathcal{H}_1 &= \Big\{(u,v) \in L^2([0,\pi])^2 : \int_0^\pi (v-w)\,dx = 0\Big\}, \\ \Lambda_1(\varphi,\psi) &= \Big(\varphi + \frac{\partial \psi}{\partial x}, \varphi - \frac{\partial \psi}{\partial x}\Big). \end{aligned}$$

Hence if

$$(u_t(t), u(t)) = e^{tG_\alpha}(\varphi, \psi),$$

then

$$\Lambda_1(u_t(t), u(t)) = (u_\xi(t), u_\eta(t)).$$

The inverse of $\Lambda_1$ is given by

$$\Lambda_1^{-1}(v, w) = \Big(\frac{1}{\sqrt{2}}(v + w), \frac{1}{\sqrt{2}} \int_0^x (v - w)(s)\, ds\Big).$$

The condition $\int_0^\pi (v - w) = 0$ in the definition of $\mathcal{H}_1$ reflects the fact that $u \in H_0^1$. The fact that $\mathcal{H}_1$ is not quite all of $L^2([0, \pi])^2$ will cause some small problems later on. The operator $\Lambda_1 e^{hG_\alpha} \Lambda_1^{-1}$ gives the operator "evolution by $h$ units of time in the $(u_\xi, u_\eta)$ variables." Because of (3.10), the evolution of $(u_\xi, u_\eta)$ is particularly simple. To study this evolution, we decompose the interval $(0, \pi)$ into $q$ intervals of length $h$, the $j$th interval being

$$((j - 1)h, jh), \quad j = 1, 2, \ldots, q.$$

We define $u_\xi^j, u_\eta^j$ to be the restrictions of $u_\xi, u_\eta$ to the $j$th interval, translated to the reference position $(0, h)$, that is, $u_\xi^j, u_\eta^j \in L^2([0, h])$,

$$u_\xi^j(t, x) = u_\xi(t, (j - 1)h + x), \tag{3.11}$$

$$u_\eta^j(t, x) = u_\eta(t, (j - 1)h + x), \tag{3.12}$$

for $0 < x < h$. Equation (3.10) yields

$$\begin{aligned} u_\xi^j(t + h) &= u_\xi^{j+1}(t), \quad j = 1, 2, \ldots, p - 1, p + 1, \ldots, q - 1, \\ u_\eta^j(t + h) &= u_\eta^{j-1}(t), \quad j = 2, 3, \ldots, p, p + 2, \ldots, q. \end{aligned} \tag{3.13}$$

To complete the description of the evolution, we must give rules for determining

$$u_\eta^1(t + h),\ u_\xi^q((t + h),\ u_\xi^p(t + h),\ \text{and } u_\eta^{p+1}(t + h),$$

from $u_\xi(t)$ and $u_\eta(t)$. The first two of these are determined with the aid of the boundary condition

$$u_\xi + u_\eta = 0 \ \text{ at } \ x = 0, \pi.$$

One finds

$$u_\eta^1(t + h) = -Ru_\xi^1(t), \quad u_\xi^q(t + h) = -Ru_\eta^q(t), \tag{3.14}$$

where $R : L^2([0, h]) \to L^2([0, h])$ denotes reflection about $x = h/2$, that is

$$(R\varphi)(x) = \varphi(h - x), \quad 0 < x < h.$$

The transmission condition (2.7) provides the values of $u_\xi^p(t+h)$ and $u_\eta^{p+1}(t+h)$. Let

$$u_\xi^-(a,t) = \lim_{x\nearrow a} u_\xi(x,t), \quad u_\xi^+(a,t) = \lim_{x\searrow a} u_\xi(x,t),$$
$$u_\eta^-(a,t) = \lim_{x\nearrow a} u_\eta(x,t), \quad u_\eta^+(a,t) = \lim_{x\searrow a} u_\eta(x,t),$$

These quantities exist provided $(u_t(0), u(0)) \in \mathcal{D}(G_\alpha)$. In fact, the entire calculation of $\Lambda_1 e^{tG_\alpha}\Lambda_1^{-1}$ should be considered for such $u$ and then extended by continuity to $u$ with $(u_t(0), u(0)) \in \mathcal{H}$. The wave equation (3.10) yields

$$u_\xi^+(t+s,a) = u_\xi(t, a+s), \quad 0 < s < h,$$
$$u_\eta^-(t+s,a) = u_\eta(t, a-s), \quad 0 < s < h.$$

In terms of the $u_\xi, u_\eta$ variables, the transmission conditions (2.7) become

$$u_\xi^+ + u_\eta^+ = u_\xi^- + u_\eta^-, \quad ([u_t] = 0),$$
$$(u_\xi^+ - u_\eta^+) - (u_\xi^- - u_\eta^-) = \alpha(u_\xi^+ + u_\eta^+), \quad ([u_x] = \alpha u_t).$$

Solving these equations for $u_\xi^-$ and $u_\eta^+$ yields, for $\alpha \neq -2$,

$$u_\xi^- = \frac{2}{2+\alpha}u_\xi^+ - \frac{\alpha}{2+\alpha}u_\eta^-, \quad u_\eta^+ = \frac{-\alpha}{2+\alpha}u_\xi^+ + \frac{2}{2+\alpha}u_\eta^-.$$

Using the previous expressions for $u_\xi^+$ and $u_\eta^-$, we find

$$\begin{aligned} u_\xi^p(t+h) &= \frac{2}{2+\alpha}u_\xi^{p+1}(t) - \frac{\alpha}{2+\alpha}Ru_\eta^p, \\ u_\eta^{p+1}(t+h) &= \frac{-\alpha}{2+\alpha}Ru_\xi^{p+1}(t) + \frac{2}{2+\alpha}u_\eta^p(t). \end{aligned} \tag{3.15}$$

The formulas (3.13)–(3.15) give a simple expression for the time evolution of $(u_\xi, u_\eta)$. The reflections $R$ can be removed from these formulas by splitting

$$U(t) = (u_\xi^1, u_\eta^1, u_\xi^2, u_\eta^2, \ldots, u_\xi^q, u_\eta^q) \in L^2([0,h], \mathbb{C}^{2q})$$

into its even and odd part,

$$U = U_{\text{even}} + U_{\text{odd}},$$
$$(U_{\text{even}}(t))(x) = \frac{1}{2}\big[U(t)(x) + U(t)(h-x)\big],$$
$$(U_{\text{odd}}(t))(x) = \frac{1}{2}\big[U(t)(x) - U(t)(h-x)\big].$$

From formulas (3.13)–(3.15) we see that the even and odd parts are preserved by the evolution, that is

$$(\Lambda_1 e^{hG_\alpha}\Lambda_1^{-1}U)_{\text{even}} = \Lambda_1 e^{hG_\alpha}\Lambda_1^{-1}(U_{\text{even}}),$$

with a similar formula for the odd part.

Furthermore, on the even (respectively odd) parts $R$ acts as multiplication by 1 (respectively, $-1$). Thus, if we let $\Lambda_2$ be defined by

$$\begin{gathered}\Lambda_2 : L^2([0,h],\mathbb{C}^{2q}) \longrightarrow L^2([0,h/2],\mathbb{C}^{4q}),\\ \Lambda_2 U = \frac{1}{2}(U_{\mathrm{even}}, U_{\mathrm{odd}})\big|_{[0,h/2]},\end{gathered}$$

and let

$$\mathcal{H}_2 = \Lambda_2 \mathcal{H}_1 \subset L^2([0,h/2],\mathbb{C}^{4q}),$$

then

$$\begin{gathered}\Lambda_2 : \mathcal{H}_1 \longrightarrow \mathcal{H}_2 \ \text{ is unitary},\\ \Lambda_2\Lambda_1 e^{hG_\alpha}(\Lambda_2\Lambda_1)^{-1} : \mathcal{H}_2 \longrightarrow \mathcal{H}_2,\end{gathered}$$

and

$$\Lambda_2\Lambda_1 e^{hG_\alpha}(\Lambda_2\Lambda_1)^{-1} = \text{ multiplication by a } D_1 \in \mathcal{L}(\mathbb{C}^{4q}).$$

From the definition of $\mathcal{H}_1$ and $\Lambda_2$ we find that, for

$$\zeta = (1,-1,1,-1,\ldots,1,-1,0,0,\ldots,0) \in \mathbb{C}^{4q},$$

with $q$ ones, $q$ minus ones, and $2q$ zeroes,

$$\mathcal{H}_2 = \Big\{\varphi \in L^2([0,h/2],\mathbb{C}^{4q}) : \int_0^{h/2} \langle \varphi(x), \zeta\rangle \, dx = 0\Big\}.$$

If $\mathcal{H}_2$ were all of $L^2([0,h/2],\mathbb{C}^{4q})$, the proof would be finished. To eliminate the orthogonality condition, we write

$$\mathbb{C}^{4q} = \mathbb{C}\zeta \oplus \mathcal{E}, \quad \mathcal{E} = \{\eta \in \mathbb{C}^{4q} : \langle \eta, \zeta \rangle = 0\}.$$

Corresponding to this decomposition, there is a canonical decomposition

$$\begin{gathered}L^2([0,h/2],\mathbb{C}^{4q}) = L^2([0,h/2],\mathbb{C}\zeta) \oplus L^2([0,h/2],\mathcal{E}),\\ \mathcal{H}_2 \approx \mathcal{K} \oplus L^2([0,h/2],\mathcal{E}),\end{gathered} \tag{3.17}$$

where

$$\mathcal{K} = \Big\{\psi \in L^2([0,h/2],\mathbb{C}\zeta) : \int_0^{h/2} \langle \psi(x), \zeta\rangle \, dx = 0\Big\}.$$

To finish the proof, we need two simple properties of the matrix $D_1$, namely

$$D_1\zeta = \zeta, \quad \text{and} \ \ D_1^*\zeta = \zeta, \tag{3.18}$$

where $D_1^*$ is the adjoint of $D_1$. The first of these properties is equivalent to the easily verified fact that the evolution defined by (3.13)–(3.15) leaves invariant the function defined by

$$u_\xi(t,x) = -u_\eta(t,x) = 1, \quad 0 < x < \pi.$$

The second property follows from the first. Indeed, (3.16) and the first part of (3.18) show that

$$\Lambda_2\Lambda_1 e^{hG_\alpha}(\Lambda_2\Lambda_1)^{-1}\big|_{\mathcal{K}} = Id_{\mathcal{K}}. \tag{3.19}$$

Since $\alpha \geq 0$, $\Lambda_2\Lambda_1 e^{hG_\alpha}(\Lambda_2\Lambda_1)^{-1}$ is a contraction. Hence Lemma 3.4 implies that $\mathcal{K}^\perp = L^2([0,h/2],\mathcal{E})$ is invariant. Thus we may form the adjoint of equation (3.19), to obtain

$$[\Lambda_2\Lambda_1 e^{hG_\alpha}(\Lambda_2\Lambda_1)^{-1}]^*\big|_{\mathcal{K}} = Id_{\mathcal{K}}.$$

Because of (3.16), this is equivalent to the second identity in (3.18).

The condition $D_1^*\zeta = \zeta$ implies $D_1(\mathcal{E}) \subset \mathcal{E}$. Consequently we may define $D \in \mathcal{L}(\mathcal{E})$ by $D = D_1|_{\mathcal{E}}$. If $\Lambda_3 : \mathcal{H}_2 \to \mathcal{K} \oplus L^2$ is the isomorphism in (3.17), then (3.16), (3.17) and (3.19) show that

$$\Lambda = \Lambda_3\Lambda_2\Lambda_1,$$

whence $\mathcal{K}$ and $D$ have the properties required in Lemma 3.5.

Theorem 3.3 shows that if $a/\pi$ is rational, the spectrum of $G_\alpha|_{M^\perp}$ must lie in a half plane $\operatorname{Re} z \leq -C_2$, while the spectrum of $G_\alpha|_M$ is on the imaginary axis. In contrast, Theorem 3.2 shows that if $a/\pi$ is irrational, the spectrum of $G_\alpha$ lies in the half plane $\operatorname{Re} z < 0$, but has points arbitrarily close to the imaginary axis.

These results all involve bounds on the spectrum, from the right. A complementary result, which is much more elementary, is the following.

**Theorem 3.6.** *For $a \in (0,\pi)$ fixed and $\alpha \neq -2$, the spectrum of $G_\alpha$ is contained in a strip* $\operatorname{Re} z \geq C(\alpha)$, *where the function $C(\alpha)$ is bounded on compact subsets of $\mathbb{C} \setminus \{-2\}$.*

*Proof.* We need only consider eigenvalues $i\lambda$ with $\lambda \notin \mathbb{R}$. For these, equation (3.6) holds and the theorem follows from the observation that $\cot z \to \pm i$ as $\operatorname{Im} z \to \pm\infty$, the convergence being uniform in $\operatorname{Re} z$.

The exceptional value $\alpha = -2$ occurs in another (not unrelated) context. For $\alpha \neq -2$, $G_\alpha$ is the generator of a one parameter *group* on $\mathcal{H}$, while for $\alpha = -2$ it only generates a semigroup. It is interesting to note that the higher dimensional analogues of our transmission problem, for example localized friction on a membrane, are *never* reversible, that is, if $\operatorname{Re}\alpha \neq 0$ one gets a semigroup and not a group. The proofs of these facts are omitted.

In case $a = \pi/2$, $e^{\pi G_\alpha}$ can be computed without great effort. Though somewhat special, this result will play a role in our discussion of the significance of the model.

**Theorem 3.7.** *If $a = \pi/2$ and $\alpha \neq -2$, then*

$$e^{\pi G_\alpha}\big|_{M^\perp} = \frac{\alpha - 2}{\alpha + 2} Id_{M^\perp}.$$

*Proof.* We calculate, in somewhat more detail, the explicit solution constructed in the proof of Theorem 3.3. First of all, we observe that $M$ consists simply of those functions $\varphi, \psi$ that are even with respect to reflection about $x = \pi/2$, while $M^\perp$ consists of those functions that are odd. Thus, for time evolution it suffices to consider functions $u$ for which $u_t$ is even and $u_x$ is odd, that is

$$(u_\xi + u_\eta)(t, x) = (u_\xi + u_\eta)(t, \pi - x)$$

and

$$(u_\xi - u_\eta)(t, x) = -(u_\xi - u_\eta)(t, \pi - x).$$

These relations are equivalent to

$$R(u_\xi^1 + u_\eta^1) = u_\xi^2 + u_\eta^2$$

and

$$-R(u_\xi^1 - u_\eta^1) = u_\xi^2 - u_\eta^2,$$

where $R$ and $u_\xi^1, u_\xi^2, \ldots$ occur in (3.11)–(3.14); while also

$$u_\xi^2 = Ru_\eta^1, \quad \text{and} \quad u_\eta^2 = Ru_\xi^1.$$

With these equations in mind, we may write the time evolution in terms of $u_\xi^1, u_\eta^1$ only. Equation (3.15) yields

$$u_\xi^1\Big(t + \frac{\pi}{2}\Big) = \frac{2}{2+\alpha} Ru_\eta^1 - \frac{\alpha}{2+\alpha} Ru_\eta^1 = \frac{2-\alpha}{2+\alpha} Ru_\eta^1(t),$$

while (3.14) gives

$$u_\eta^1\Big(t + \frac{\pi}{2}\Big) = -Ru_\xi^1(t).$$

Iterating, we obtain

$$(u_\xi^1(t+\pi), u_\eta^1(t+\pi)) = \frac{\alpha - 2}{\alpha + 2}(u_\xi^1(t), u_\eta^1(t)),$$

which is the desired result.

## 4. Discussion

It is hard to quantify how well (1.1) with a highly localized friction describes the mechanism for the production of harmonics, since it seems hard to measure directly the effect of a musician's finger. We must therefore rely on qualitative predictions of the model. Fortunately, we have obtained many such in Sections 2 and 3. The principal results assert that, given an initial configuration (determined, for example, by plucking a string), the motion governed by (1.1) approximates the limiting transmission problem (2.6)–(2.8), provided the friction $b(x)$ is highly localized. This limiting problem has the following properties:

1. If $a/\pi$ is irrational, all solutions tend to zero.

2. If $a/\pi$ is rational, the modes that vanish at $a$ are unaffected by the friction, while those in the orthogonal complement decay exponentially.

These properties coincide with the observed fact that to play harmonics one touches the string with a finger at a point $a$ on the string (of length $L$) such that $a/L$ is rational with a small denominator. For other placements, one only hears a short lived thud. When playing harmonics, a musician removes his finger from the string after a short time. In view of the fact that for a friction $b(x)$ spread over a finite interval all solutions tend to zero, this seems wise. Presumably what is happening is that the rate of decay is much slower for the modes vanishing at $a$ (in fact, this is rigorously true in the limit $b(x)\,dx = \alpha\delta(x-a)$). Thus the musician leaves his finger in contact with the string just long enough to damp the components in $M^{\perp}$, but not those in $M$. As this description indicates, the playing of harmonics is a delicate matter, a fact that can easily be verified by anyone inexperienced in the art.

There is an additional sensitivity to the artistry of the player, clearly indicated in Theorem 3.7. The object of playing harmonics is to obtain as rapid decay as possible on $M^{\perp}$. For $a = \pi/2$ (here, $L = \pi$) we have

$$e^{\pi G_\alpha}\big|_{M^{\perp}} = \frac{\alpha - 2}{\alpha + 2}\, I.$$

Clearly the desirable value of $\alpha$ is $\alpha = 2$. The optimal strategy is to apply a friction with $b(x)\,dx \approx 2\delta(x - \pi/2)$ for approximately $\pi$ units of time ($L/c$ units of time for a string of length $L$, with $c$ the propagation speed of transverse vibrations). Too much friction ($\alpha >> 2$) or too little ($\alpha \approx 0$) yields a very slow damping on $M^{\perp}$.

We expect that similar phenomena occur for other rational values of $a/\pi$. The player must strive to achieve the "correct touch," which for $a = \pi/2$ corresponds to $b(x), dx \approx 2\delta(x - \pi/2)$. The idea of correct touch leads to the following.

**Problem.** *Given $a/\pi = p/q$, find the value of $\alpha$ which, in some sense, gives the most rapid decay for $e^{tG_\alpha}|_{M^\perp}$.*

Remark. Since this paper first appeared, in 1982, there has been a successful treatment of this problem, in [11].

The descriptions above correspond to common experience with harmonics. This is not to say that we consider (1.1) an exact model. What we believe is that localized frictional damping is a reasonable candidate for the primary mechanism in the playing of harmonics. To support this idea, one must know that similar qualitative behavior occurs when other effects are included, since it is more than likely that the finger introduces effects other than friction. For example, the finger might exert a spring force on the spring with a strongly localized spring constant $k(x)$. The basic equation of motion is then

$$u_{tt} + b(x)u_t = u_{xx} - k(x)u, \tag{4.1}$$

and the energy

$$\int_0^\pi \Big(u_t^2 + u_x^2 + k\frac{u^2}{2}\Big)\,dx$$

is a decreasing function of time. If, as before, we consider a sequence $b_n$ and $k_n$ becoming increasingly localized, that is $b_n \to \alpha\delta_a$ $k_n \to \beta\delta_a$ (perhaps $\beta = 0$), then the solutions to (4.1) will converge to solutions to the transmission problem

$$\begin{gathered} u_{tt} - u_{xx} = 0, \quad x \neq a, \\ [u_t] = 0, \quad \alpha u_t = [u_x] - \beta u \ \text{ at } \ x = a, \\ u = 0 \ \text{ at } \ x = 0,\ \pi. \end{gathered}$$

An analysis like that given in §3 shows that this problem behaves qualitatively like (2.6)–(2.8) provided $\alpha > 0$. For example, regardless of the value of $a$, the solutions yield a contraction semigroup on $\mathcal{H}$, with square-norm

$$\|(v,w)\|^2 = \int_0^\pi (v^2 + (Dw)^2)\,dx + \beta u(a)^2. \tag{4.2}$$

In addition, if $a/\pi$ is irrational, all solutions decay, while if $a/\pi$ is rational the motion on the pace $M$ of Theorem 3.3 is the same as the free motion. Finally, if $M^\perp$ is the orthogonal complement in the scalar product induced by (4.2), then $M^\perp$ is invariant and solutions in $M^\perp$ decay. Since the proofs are similar to those already presented, the details are omitted. The point is that the qualitative behavior of our model is somewhat stable under perturbation.

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