Regularity of Szegö Projectors and Cauchy Integrals
On Test Functions with Cutoffs

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Preliminary Notes

1. Introduction

Let $B \subset \mathbb{C}^n$ be the open unit ball, and let

$$\mathcal{C} : L^2(\partial B) \longrightarrow H^2(B) \tag{1.1}$$

be the Cauchy integral, so if $f \in L^2(\partial B)$, then $\mathcal{C}f$ is holomorphic on B. In [AC], an examination is made of $\mathcal{C}f$ in case

$$f = \chi_\Omega g. \tag{1.2}$$

The authors took $\Omega \subset \partial B$ to be a domain with C^2 boundary $\partial\Omega$. Let

$$E(\partial\Omega) \subset \partial\Omega \tag{1.3}$$

denote the set of points where the contact line bundle $L \subset T^*\partial B$ is normal to $\partial\Omega$. It is shown in [AC] that, for f as in (1.2),

$$g \in C^1(\partial B) \Longrightarrow \mathcal{C}f \text{ is continuous on } \overline{B} \setminus E(\partial\Omega). \tag{1.4}$$

We desire to establish a number of related results in the following more general setting.

Let $\mathcal{O} \subset \mathbb{C}^n$ be a bounded, strongly pseudoconvex domain, with C^∞ boundary $\partial\mathcal{O}$, and let

$$\mathcal{C} : L^2(\partial\mathcal{O}) \longrightarrow H^2(\mathcal{O}) \tag{1.5}$$

be given by

$$\mathcal{C}f = \mathrm{PI}(Sf), \tag{1.6}$$

where $S : L^2(\partial\mathcal{O}) \to L^2(\partial\mathcal{O})$ is the Szegö projector, i.e., the orthogonal projection of $L^2(\partial\mathcal{O})$ onto the space of L^2 boundary values of holomorphic functions on $\mathcal{O}$, and $\mathrm{PI}\,\varphi$ gives the harmonic function on $\mathcal{O}$ with boundary value φ. The first variant of (1.4) we establish is the following.

Proposition 1.1. *Let $\Omega \subset \partial\mathcal{O}$ have C^∞ boundary, and let f have the form (1.2). Then*

$$g \in C^\infty(\partial\mathcal{O}) \Longrightarrow \mathcal{C}f \in C^\infty(\overline{\mathcal{O}} \setminus E(\partial\Omega)). \tag{1.7}$$

We recall some known results on the Szegö projector. For example,

$$S \in OPS^0_{1/2,1/2}(\partial\mathcal{O}). \tag{1.8}$$

Its microlocal properties include

$$\mathrm{WF}(Sf) \subset \mathrm{WF}(f) \cap (L \setminus 0), \tag{1.9}$$

where $L \subset T^*\partial\mathcal{O}$ is the contact line bundle mentioned above. From this, it follows that

$$\begin{aligned} g \in C^\infty(\partial\mathcal{O}),\ f = \chi_\Omega g &\Rightarrow \mathrm{WF}(f) \subset N^*\partial\Omega \setminus 0 \\ &\Rightarrow Sf \in C^\infty(\partial\mathcal{O} \setminus E(\partial\Omega)). \end{aligned} \tag{1.10}$$

The implication that

$$\mathrm{PI}(Sf) \in C^\infty(\overline{\mathcal{O}} \setminus E(\partial\Omega)) \tag{1.11}$$

when (1.10) holds is an elementary consequence of basic properties of PI (local elliptic regularity for the Dirichlet problem on $\mathcal{O}$). This proves Proposition 1.1.

In subsequent sections, we establish variants of Proposition 1.1, including some results that imply (1.4). In all cases, the main task will be to establish appropriate regularity of $S(\chi_\Omega g)$. The following further results on S will be useful. First,

$$S : L^p(\partial\mathcal{O}) \longrightarrow L^p(\partial\mathcal{O}), \quad 1 < p < \infty. \tag{1.12}$$

More generally, we have the L^p-Sobolev space mappings

$$S : H^{s,p}(\partial\mathcal{O}) \longrightarrow H^{s,p}(\partial\mathcal{O}), \quad 1 < p < \infty,\ s \in \mathbb{R}. \tag{1.13}$$

For $p = 2$, (1.13) follows from (1.8), but for other $p \in (1, \infty)$, one needs finer results on S. See, e.g., [T1]. Another result is the following:

$$\begin{aligned} &A \in OPS^0(\partial\mathcal{O}),\ \sigma_A = 1 \text{ on a conic neighborhood of } L \setminus 0 \text{ in } T^*\partial\mathcal{O} \setminus 0 \\ &\Longrightarrow SA = AS = S \bmod OPS^{-\infty}(\partial\mathcal{O}). \end{aligned} \tag{1.14}$$

Here σ_A stands for the *complete* symbol of A, in local coordinates.

2. Further results when $\partial\Omega$ is C^∞

Here we assume $\Omega \subset \partial\mathcal{O}$ has C^∞ boundary. We aim to prove the following.

Proposition 2.1. *Let $p \in (1,\infty)$ and $s \geq 1/p$. Let $\varphi \in C^\infty(\partial\mathcal{O})$ satisfy*

$$\varphi = 0 \text{ on a neighborhood of } E(\partial\Omega). \tag{2.1}$$

Then

$$g \in H^{s,p}(\partial\mathcal{O}),\ f = \chi_\Omega g \Longrightarrow \varphi S f \in H^{s,p}(\partial\mathcal{O}). \tag{2.2}$$

Remark. The conclusion (2.2) is also valid for $s \in [0, 1/p)$, but in such a case we actually have

$$h \in H^{s,p}(\partial\mathcal{O}),\ f = \chi_\Omega g \Longrightarrow f \in H^{s,p}(\partial\mathcal{O}),$$

which by (1.13) implies the conclusion in (2.2). Similar comments also apply to Propositions 3.1–3.2.

Before tackling the proof, we note the following consequence.

Corollary 2.2. *Assume $p \in (1,\infty)$ and $sp > \dim \partial\mathcal{O}$. Then*

$$g \in H^{s,p}(\partial\mathcal{O}),\ f = \chi_\Omega g \Longrightarrow \mathcal{C} f \in C(\overline{\mathcal{O}} \setminus E(\partial\Omega)). \tag{2.3}$$

Therefore

$$g \in C^\alpha(\partial\mathcal{O}),\ \alpha > 0,\ f = \chi_\Omega g \Longrightarrow \mathcal{C} f \in C(\overline{\mathcal{O}} \setminus E(\partial\Omega)). \tag{2.4}$$

In light of the results (1.12)–(1.14), Proposition 2.1 is a consequence of the following.

Proposition 2.3. *Let $p \in (1,\infty)$ and $s \geq 0$. Take $P \in OPS^0(\partial\mathcal{O})$, and assume its complete symbol σ_P satisfies*

$$\sigma_P = 0 \text{ on a conic neighborhood of } N^*\partial\Omega \setminus 0 \text{ in } T^*\partial\mathcal{O} \setminus 0. \tag{2.5}$$

Then

$$g \in H^{s,p}(\partial\mathcal{O}),\ f = \chi_\Omega g \Longrightarrow P f \in H^{s,p}(\partial\mathcal{O}). \tag{2.6}$$

Another statement of Proposition 2.3 is that if $\Gamma \subset T^*\partial\mathcal{O} \setminus 0$ is an open conic set and

$$\Gamma \cap N^*\partial\Omega = \emptyset, \tag{2.7}$$

then

$$g \in H^{s,p}(\partial\mathcal{O}),\ f = \chi_\Omega g \Longrightarrow f \in H^{s,p}_{\mathrm{mcl}}(\Gamma), \tag{2.8}$$

provided $s \geq 0$. See [T2], §3.1. Another formulation is that, for $s \geq 0,\ p \in (1,\infty)$,

$$g \in H^{s,p}(\partial\mathcal{O}) \Longrightarrow \mathrm{WF}_{H^{s,p}}(\chi_\Omega g) \subset N^*\partial\Omega. \tag{2.8A}$$

Proof of Proposition 2.3. We set

$$Tg = P(\chi_\Omega g), \tag{2.9}$$

and desire to show that

$$T : H^{s,p}(\partial\mathcal{O}) \longrightarrow H^{s,p}(\partial\mathcal{O}), \quad \forall\, p \in (1,\infty),\ s \geq 0. \tag{2.10}$$

Clearly this works for $s = 0$. If we show that

$$T : H^{1,p}(\partial\mathcal{O}) \longrightarrow H^{1,p}(\partial\mathcal{O}), \quad 1 < p < \infty, \tag{2.11}$$

then (2.10) follows for all $s \in [0,1]$, by interpolation. Now, if

$$X \text{ is a smooth vector field on } \partial\mathcal{O}, \text{ tangent to } \partial\Omega, \tag{2.12}$$

then

$$g \in H^{1,p}(\partial\mathcal{O}) \Longrightarrow X(\chi_\Omega g) \in L^p(\partial\mathcal{O}). \tag{2.13}$$

Microlocal elliptic regularity for the differential operator X yields

$$\overline{\Gamma} \cap \operatorname{char} X = \emptyset,\ f, Xf \in L^p(\partial\mathcal{O}) \Longrightarrow f \in H^{1,p}_{\mathrm{mcl}}(\Gamma). \tag{2.14}$$

This gives (2.11). A similar argument, applied to the observation

$$g \in H^{k,p}(\partial\mathcal{O}) \Longrightarrow X_1 \cdots X_k(\chi_\Omega g) \in L^p(\partial\mathcal{O}) \tag{2.15}$$

for all smooth vector fields X_ν tangent to $\partial\Omega$, gives

$$T : H^{k,p}(\partial\mathcal{O}) \longrightarrow H^{k,p}(\partial\mathcal{O}), \quad 1 < p < \infty,\ k \in \mathbb{N}, \tag{2.16}$$

yielding (2.10), by interpolation.

3. Results for $\partial\Omega$ smoother than C^2

Here we assume $\Omega \subset \partial\mathcal{O}$ has boundary smooth of class $C^{2+\sigma}$, $\sigma > 0$. We aim to prove the following.

Proposition 3.1. *Let $p \in (1,\infty)$ and $s \in [1/p, 1]$. Let $\varphi \in C^\infty(\partial\mathcal{O})$ satisfy (2.1). Then*

$$g \in H^{s,p}(\partial\mathcal{O}),\ f = \chi_\Omega g \Longrightarrow \varphi S f \in H^{s,p}(\partial\mathcal{O}). \tag{3.1}$$

If, in addition, $sp > \dim\partial\mathcal{O}$ (which now requires $p > \dim\partial\mathcal{O}$), then

$$g \in H^{s,p}(\partial\mathcal{O}),\ f = \chi_\Omega g \Longrightarrow \mathcal{C} f \in C(\overline{\mathcal{O}} \setminus E(\partial\Omega)). \tag{3.2}$$

Consequently, the implication (2.4) is valid in this setting.

As in §2, Proposition 3.1 is a consequence of the following.

Proposition 3.2. *Let $p \in (1,\infty)$ and $s \in [1/p, 1]$. If $\Gamma \subset T^*\partial\mathcal{O} \setminus 0$ is an open conic set satisfying (2.7), then*

$$g \in H^{s,p}(\partial\mathcal{O}),\ f = \chi_\Omega g \Longrightarrow f \in H^{s,p}_{\text{mcl}}(\Gamma). \tag{3.3}$$

Also, parallel to §2, it suffices to show that if $P \in OPS^0(\partial\mathcal{O})$ satisfies (2.5), and

$$Tg = P(\chi_\Omega g), \tag{3.4}$$

then

$$T : H^{s,p}(\partial\mathcal{O}) \longrightarrow H^{s,p}(\partial\mathcal{O}), \quad 1 < p < \infty, \quad 0 \le s \le 1, \tag{3.5}$$

and it suffices to show that this holds for $s = 1$. The proof of this also parallels the argument in §2, but with an extra complication in the current setting.

In more detail, given that $\partial\Omega$ is smooth of class $C^{2+\sigma}$, we have vector fields on $\partial\mathcal{O}$,

$$X, \text{ smooth of class } C^r, \text{ tangent to } \partial\Omega,\ r = 1 + \sigma > 1, \tag{3.6}$$

and, for any such vector field,

$$g \in H^{1,p}(\partial\mathcal{O}) \Longrightarrow X(\chi_\Omega g) \in L^p(\partial\mathcal{O}). \tag{3.7}$$

Again we want to draw the implication (2.14), but since the coefficients of X are not C^∞, we need to examine this more carefully. We are given

$$\Gamma \cap \operatorname{char} X = \emptyset, \quad f, Xf \in L^p(\partial\mathcal{O}). \tag{3.8}$$

To obtain microlocal regularity results for f, apply symbol smoothing to X (cf. [T2], or [T3], Chapter 13, §9), as follows. We pick $\delta \in (0,1)$ and write

$$X = X^{\#} + X^{b}, \quad X^{\#} \in OPS^{1}_{1,\delta}, \quad X^{b} \in OPC^{r} S^{1-\delta r}_{1,\delta}. \tag{3.9}$$

Furthermore, $X^{\#}$ is microlocally elliptic on $T^{*}\partial\mathcal{O} \setminus \operatorname{char} X$, so, given $\overline{\Gamma} \cap \operatorname{char} X = \emptyset$, there exists

$$E \in OPS^{-1}_{1,\delta}, \quad \sigma_{EX^{\#}} = 1 \text{ on } \Gamma, \tag{3.10}$$

and

$$EX^{\#} f = E(Xf) - E(X^{b} f). \tag{3.11}$$

Clearly

$$Xf \in L^{p} \Longrightarrow E(Xf) \in H^{1,p}. \tag{3.12}$$

Furthermore, given $X^{b} \in OPC^{r} S^{1-\delta r}_{1,\delta}$, we have (cf. [T3], Chapter 13, Proposition 9.10)

$$f \in L^{p} \Longrightarrow X^{b} f \in H^{-(1-\delta r),p}, \tag{3.13}$$

provided $-(1-\delta)r < -(1-\delta r)$, i.e.,

$$r > 1. \tag{3.14}$$

In such a case,

$$E(X^{b} f) \in H^{\delta r,p} \subset H^{1,p}, \tag{3.15}$$

provided $\delta \in (0,1)$ is picked close enough to 1. Then we have $EX^{\#} f \in H^{1,p}$ in (3.11), so the hypotheses (3.8) imply $f \in H^{s,p}_{\mathrm{mcl}}(\Gamma)$, recovering (2.14) in this setting. This completes the proof of (3.5), hence of Proposition 3.2, hence of Proposition 3.1.

4. Results for $\partial\Omega$ rougher than C^2

Here we assume $\Omega \subset \partial\mathcal{O}$ has the property

$$\partial\Omega \text{ is of class } C^{1+r}, \quad 0 < r < 1. \tag{4.1}$$

We want to examine the action on $H^{s,p}(\partial\mathcal{O})$ of T, given by

$$Tg = P(\chi_\Omega g), \tag{4.2}$$

with $P \in OPS^0(\partial\mathcal{O})$ satisfying (2.5) and $s \in [0,1]$. Clearly

$$T : L^p(\partial\mathcal{O}) \longrightarrow L^p(\partial\mathcal{O}), \tag{4.3}$$

so we consider $g \in H^{1,p}(\partial\mathcal{O})$. In the current setting, we have vector fields on $\partial\mathcal{O}$,

$$X, \text{ smooth of class } C^r, \text{ tangent to } \partial\Omega, \quad r \in (0,1), \tag{4.4}$$

and, for any such vector field,

$$g \in H^{1,p}(\partial\mathcal{O}),\ f = \chi_\Omega g \Longrightarrow Xf \in L^p(\partial\mathcal{O}). \tag{4.5}$$

We apply symbol smoothing to X as in (3.9), obtaining (3.10)–(3.12). However, we cannot use (3.13) here, since we do not have (3.14).

To get around this, we note an improvement over the hypotheses

$$f, Xf \in L^p(\partial\mathcal{O}), \tag{4.6}$$

used in §3. In fact, given $f = \chi_\Omega g$ and $g \in H^{1,p}(\partial\mathcal{O})$, we have

$$f \in H^{\sigma,p}(\partial\mathcal{O}), \quad \forall\, \sigma < \frac{1}{p}. \tag{4.7}$$

In fact, since the spaces $H^{s,p}$ are invariant under C^1 diffeomorphisms for $s \in [0,1]$, the proof of (4.7) reduces to the case Ω is a half space; see [Str] for this. Thus, in place of (3.13), we get

$$X^b f \in H^{\sigma-(1-\delta r),p}(\partial\mathcal{O}), \tag{4.8}$$

provided $-(1-\delta)r < \sigma - (1-\delta r)$, i.e., $r > 1-\sigma$, which can be arranged if

$$r > 1 - \frac{1}{p}. \tag{4.9}$$

Then, as in (3.15), we get

$$E(X^b f) \in H^{\sigma+\delta r,p}(\partial\mathcal{O}) \subset H^{1,p}(\partial\mathcal{O}), \tag{4.10}$$

and hence

$$T : H^{1,p}(\partial\mathcal{O}) \longrightarrow H^{1,p}(\partial\mathcal{O}), \tag{4.11}$$

which can be interpolated with (4.3) to get

$$T : H^{s,p}(\partial\mathcal{O}) \longrightarrow H^{s,p}(\partial\mathcal{O}), \quad s \in [0,1], \tag{4.12}$$

provided $p \in (1,\infty)$ and r satisfies (4.9). Thus, in place of Proposition 3.1, we have the following.

Proposition 4.1. *Assume $\Omega \subset \partial\mathcal{O}$ satisfies (4.1), $p \in (1,\infty)$, and $r > 1 - 1/p$. Let $s \in [0,1]$. Let $\varphi \in C^\infty(\partial\mathcal{O})$ satisfy (2.1). Then*

$$g \in H^{s,p}(\partial\mathcal{O}),\ f = \chi_\Omega g \Longrightarrow \varphi S f \in H^{s,p}(\partial\mathcal{O}). \tag{4.13}$$

If, in addition, $sp > \dim \partial\mathcal{O}$, then

$$g \in H^{s,p}(\partial\mathcal{O}),\ f = \chi_\Omega g \Longrightarrow \mathcal{C} f \in C(\overline{\mathcal{O}} \setminus E(\partial\Omega)). \tag{4.14}$$

This gives the following extension of (1.4).

Corollary 4.2. *Given $\alpha > 0$, there exists $\beta = \beta(\alpha, n) > 0$ such that*

$$g \in C^\alpha(\partial\mathcal{O}),\ f = \chi_\Omega g \Longrightarrow \mathcal{C} f \in C(\overline{\mathcal{O}} \setminus E(\partial\Omega)), \tag{4.15}$$

whenever $\Omega \subset \partial\mathcal{O}$ has boundary $\partial\Omega$, smooth of class $C^{2-\beta}$.

References

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