Sobolev Spaces

We now define spaces $H^{1,p}(\mathbb{R}^n)$, known as Sobolev spaces. For u to belong to $H^{1,p}(\mathbb{R}^n)$, we require that $u \in L^p(\mathbb{R}^n)$ and that u have *weak derivatives* of first order in $L^p(\mathbb{R}^n)$:

$$\partial_j u = f_j \in L^p(\mathbb{R}^n), \tag{10.1}$$

where (10.1) means

$$-\int \frac{\partial \varphi}{\partial x_j}\, u\, dx = \int \varphi f_j\, dx, \quad \forall\ \varphi \in C_0^\infty(\mathbb{R}^n). \tag{10.2}$$

If $u \in C_0^\infty(\mathbb{R}^n)$, we see that $\partial_j u = \partial u/\partial x_j$, by integrating by parts, using (7.67). We define a norm on $H^{1,p}(\mathbb{R}^n)$ by

$$\|u\|_{H^{1,p}} = \|u\|_{L^p} + \sum_j \|\partial_j u\|_{L^p}. \tag{10.3}$$

We claim that $H^{1,p}(\mathbb{R}^n)$ is complete, hence a Banach space. Indeed, let (u_ν) be a Cauchy sequence in $H^{1,p}(\mathbb{R}^n)$. Then (u_ν) is Cauchy in $L^p(\mathbb{R}^n)$; hence it has an L^p-norm limit $u \in L^p(\mathbb{R}^n)$. Also, for each j, $\partial_j u_\nu = f_{j\nu}$ is Cauchy in $L^p(\mathbb{R}^n)$, so there is a limit $f_j \in L^p(\mathbb{R}^n)$, and it is easily verified from (10.2) that $\partial_j u = f_j$.

We can consider convolutions and products of elements of $H^{1,p}(\mathbb{R}^n)$ with elements of $C_0^\infty(\mathbb{R}^n)$ and readily obtain identities

$$\begin{aligned} \partial_j(\varphi * u) &= (\partial_j \varphi) * u = \varphi * (\partial_j u), \\ \partial_j(\psi u) &= \frac{\partial \psi}{\partial x_j} u + \psi(\partial_j u), \end{aligned} \tag{10.4}$$

and estimates

$$\|\varphi * u\|_{H^{1,p}} \le \|\varphi\|_{L^1} \|u\|_{H^{1,p}},$$
$$\|\psi u\|_{H^{1,p}} \le \|\psi\|_{L^\infty} \|u\|_{H^{1,p}} + \sum \|\partial_j \psi\|_{L^\infty} \|u\|_{L^p}, \tag{10.5}$$

for $\varphi,\ \psi \in C_0^\infty(\mathbb{R}^n),\ u \in H^{1,p}(\mathbb{R}^n)$. For example, the first identity in (10.4) is equivalent to

$$\begin{aligned} &-\iint \frac{\partial \psi}{\partial x_j}(x)\, \varphi(x-y) u(y)\, dy\, dx \\ &= \iint \psi(x)\, \frac{\partial \varphi}{\partial x_j}(x-y)\, u(y)\, dy\, dx, \quad \forall\ \psi \in C_0^\infty(\mathbb{R}^n), \end{aligned}$$

an identity that can be established by using Fubini's Theorem (to first do the x-integral) and integration by parts, via (7.67).

If $p < \infty,\ u \in H^{1,p}(\mathbb{R}^n)$, and (φ_j) is an approximate identity of the form (7.64), with $\varphi_j \in C_0^\infty(\mathbb{R}^n)$, then we can show that

$$\varphi_j * u \longrightarrow u \ \text{ in } \ H^{1,p}\text{-norm}, \tag{10.6}$$

using (10.4) and (7.65). Given $\varepsilon > 0$, we can take j such that

$$\|\varphi_j * u - u\|_{H^{1,p}} < \varepsilon.$$

Then we can pick $\psi \in C_0^\infty(\mathbb{R}^n)$ such that $\|\psi(\varphi_j * u) - \varphi_j * u\|_{H^{1,p}} < \varepsilon$. Of course, $\varphi_j * u$ is smooth, so $\psi(\varphi_j * u) \in C_0^\infty(\mathbb{R}^n)$. We have established

Proposition 10.1. *For $p \in [1, \infty)$, the space $C_0^\infty(\mathbb{R}^n)$ is dense in $H^{1,p}(\mathbb{R}^n)$.*

Sobolev spaces are very useful in analysis, particularly in the study of partial differential equations. We will establish just a few results here, some of which will be useful in Chapter 11. More material can be found in [**EG**], [**Fol**], [**T1**], and [**Yo**].

The following result is known as a Sobolev Imbedding Theorem.

Proposition 10.2. *If $p > n$ or if $p = n = 1$, then*

$$H^{1,p}(\mathbb{R}^n) \subset C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n). \tag{10.7}$$

For now we concentrate on the case $p \in (n, \infty)$. Since $C_0^\infty(\mathbb{R}^n)$ is then dense in $H^{1,p}(\mathbb{R}^n)$, it suffices to establish the estimate

$$\|u\|_{L^\infty} \le C\|u\|_{H^{1,p}}, \ \text{ for } u \in C_0^\infty(\mathbb{R}^n). \tag{10.8}$$

In turn, it suffices to establish

$$|u(0)| \le C\|u\|_{H^{1,p}}, \text{ for } u \in C_0^\infty(\mathbb{R}^n). \tag{10.9}$$

To get this, it suffices to show that, for a given $\varphi \in C_0^\infty(\overset{\circ}{B}_1)$ with $\varphi(0) = 1$,

$$|u(0)| \le C\|\nabla(\varphi u)\|_{L^p}, \tag{10.10}$$

where $\nabla v = (\partial_1 v, \dots, \partial_n v)$ or, equivalently, that

$$|u(0)| \le C\|\nabla u\|_{L^p}, \quad u \in C_0^\infty(\overset{\circ}{B}_1). \tag{10.11}$$

In turn, this will follow from an estimate of the form

$$|u(0) - u(\omega)| \le C\|\nabla u\|_{L^p(B_1)}, \quad u \in C^\infty(\mathbb{R}^n), \tag{10.12}$$

given $\omega \in \mathbb{R}^n$, $|\omega| = 1$. Thus we turn to a proof of (10.12).

Without loss of generality, we can take $\omega = e_n = (0, \dots, 0, 1)$. We will work with the set $\Sigma = \{z \in \mathbb{R}^{n-1} : |z| \le \sqrt{3}/2\}$. For $z \in \Sigma$, let γ_z be the path from 0 to e_n consisting of a line segment from 0 to $(z, 1/2)$, followed by a line segment from $(z, 1/2)$ to e_n, as illustrated in Figure 10.1. Then (with $A = \text{Area } \Sigma$)

$$u(e_n) - u(0) = \int_\Sigma \Bigl(\int_{\gamma_z} du\Bigr) \frac{dz}{A} = \int_{B_1} \nabla u(x) \cdot \psi(x)\, dx, \tag{10.13}$$

where the last identity applies the change of variable formula. The behavior of the Jacobian determinant of the map $(t, z) \mapsto \gamma_z(t)$ yields

$$|\psi(x)| \le C|x|^{-(n-1)} + C|x - e_n|^{-(n-1)}. \tag{10.14}$$

Thus

$$\int_{B_{1/2}} |\psi(x)|^q\, dx \le C \int_0^{1/2} r^{-nq+q} r^{n-1}\, dr. \tag{10.15}$$

It follows that

$$\psi \in L^q(B_1), \quad \forall\, q < \frac{n}{n-1}. \tag{10.16}$$

Thus

$$|u(e_n) - u(0)| \le \|\nabla u\|_{L^p(B_1)} \|\psi\|_{L^{p'}(B_1)} \le C\|\nabla u\|_{L^p(B_1)}, \tag{10.17}$$

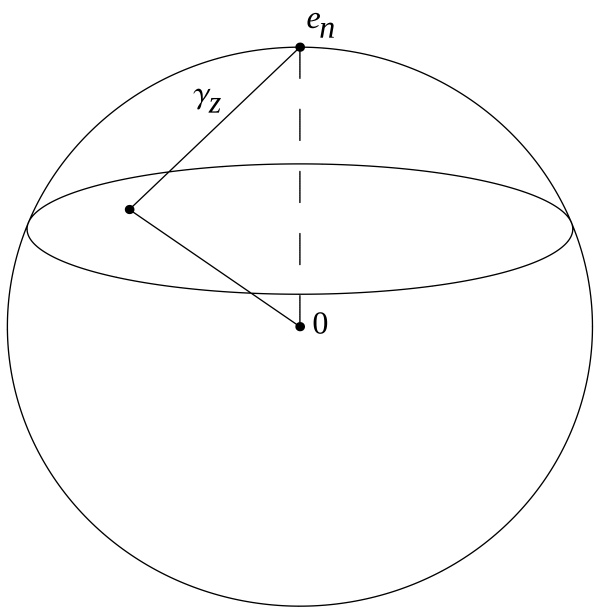

Figure 10.1

as long as $p' < n/(n-1)$, which is the same as $p > n$.

This proves (10.12) and hence Proposition 10.2, for $p \in (n, \infty)$. For $n = 1$, (10.13) simplifies to $u(1) - u(0) = \int_0^1 u'(x)\, dx$, which immediately gives the estimate (10.12) for $p = n = 1$.

We can refine Proposition 10.2 to the following.

Proposition 10.3. *If $p \in (n, \infty)$, then every $u \in H^{1,p}(\mathbb{R}^n)$ satisfies a Hölder condition:*

$$H^{1,p}(\mathbb{R}^n) \subset C^s(\mathbb{R}^n), \quad s = 1 - \frac{n}{p}. \tag{10.18}$$

Proof. Applying (10.12) to $v(x) = u(rx)$, we have, for $|\omega| = 1$,

$$|u(r\omega) - u(0)|^p \le Cr^p \int_{B_1} |\nabla u(rx)|^p\, dx = Cr^{p-n} \int_{B_r} |\nabla u(x)|^p\, dx. \tag{10.19}$$

This implies

$$|u(x) - u(y)| \le C'|x-y|^{1-n/p} \Bigl(\int_{B_r(x)} |\nabla u(z)|^p\, dz \Bigr)^{1/p}, \quad r = |x-y|, \tag{10.20}$$

which gives (10.18).

If $u \in H^{1,\infty}(\mathbb{R}^n)$ and if $\varphi \in C_0^\infty(\mathbb{R}^n)$, then $\varphi u \in H^{1,p}(\mathbb{R}^n)$ for all $p \in [1,\infty)$, so Proposition 10.3 applies. We next show that in fact $H^{1,\infty}(\mathbb{R}^n)$ coincides with the space of Lipschitz functions:

$$\text{Lip}(\mathbb{R}^n) = \{u \in L^\infty(\mathbb{R}^n) : |u(x) - u(y)| \le K|x-y|\}. \tag{10.21}$$

Proposition 10.4. *We have the identity*

$$H^{1,\infty}(\mathbb{R}^n) = \text{Lip}(\mathbb{R}^n). \tag{10.22}$$

Proof. First, suppose $u \in \text{Lip}(\mathbb{R}^n)$. Thus

$$h^{-1}\big[u(x+he_j) - u(x)\big] \text{ is bounded in } L^\infty(\mathbb{R}^n). \tag{10.23}$$

Hence, by Proposition 9.4, there is a sequence $h_\nu \to 0$ and $f_j \in L^\infty(\mathbb{R}^n)$ such that

$$h_\nu^{-1}\big[u(x+h_\nu e_j) - u(x)\big] \to f_j \text{ weak}^* \text{ in } L^\infty(\mathbb{R}^n). \tag{10.24}$$

In particular, for all $\varphi \in C_0^\infty(\mathbb{R}^n)$,

$$h_\nu^{-1}\int \varphi(x)\big[u(x+h_\nu e_j) - u(x)\big]\,dx \longrightarrow \int \varphi(x) f_j(x)\,dx. \tag{10.25}$$

But the left side of (10.25) is equal to

$$h_\nu^{-1}\int \big[\varphi(x-h_\nu e_j) - \varphi(x)\big]u(x)\,dx \longrightarrow -\int \frac{\partial\varphi}{\partial x_j}u(x)\,dx. \tag{10.26}$$

This shows that $\partial_j u = f_j$. Hence $\text{Lip}(\mathbb{R}^n) \subset H^{1,\infty}(\mathbb{R}^n)$.

Next, suppose $u \in H^{1,\infty}(\mathbb{R}^n)$. Let $\varphi_j(x) = j^n\varphi(jx)$ be an approximate identity as in (10.6), with $\varphi \in C_0^\infty(\mathbb{R}^n)$. We do not get $\varphi_j * u \to u$ in $H^{1,\infty}$-norm, but we do have $u_j = \varphi_j * u$ bounded in $H^{1,\infty}(\mathbb{R}^n)$; in fact, each u_j is C^∞, and we have

$$\|u_j\|_{L^\infty} \le K_1, \quad \|\nabla u_j\|_{L^\infty} \le K_2. \tag{10.27}$$

Also $u_j \to u$ locally uniformly. The second estimate in (10.27) implies

$$|u_j(x) - u_j(y)| \le K_2|x-y|, \tag{10.28}$$

since $u_j(x) - u_j(y) = \int_0^1 (x-y)\cdot\nabla u(tx + (1-t)y)\,dt$. Thus in the limit $j \to \infty$, we get also $|u(x) - u(y)| \le K_2|x-y|$. This completes the proof.

We next show that, when $p \in [1,n)$, $H^{1,p}(\mathbb{R}^n)$ is contained in $L^q(\mathbb{R}^n)$ for some $q > p$. One technical tool which is useful for our estimates is the following generalized Hölder inequality.

Lemma 10.5. *If $p_j \in [1,\infty],\ \sum p_j^{-1} = 1,$ then*

$$\int_M |u_1 \cdots u_m|\ dx \le \|u_1\|_{L^{p_1}(M)} \cdots \|u_m\|_{L^{p_m}(M)}. \tag{10.29}$$

The proof follows by induction from the case $m = 2$, which is the usual Hölder inequality.

Proposition 10.6. *For $p \in [1, n)$,*

$$H^{1,p}(\mathbb{R}^n) \subset L^{np/(n-p)}(\mathbb{R}^n). \tag{10.30}$$

In fact, there is an estimate

$$\|u\|_{L^{np/(n-p)}} \le C\|\nabla u\|_{L^p} \tag{10.31}$$

for $u \in H^{1,p}(\mathbb{R}^n)$, with $C = C(p, n)$.

Proof. It suffices to establish (10.31) for $u \in C_0^\infty(\mathbb{R}^n)$. Clearly

$$|u(x)| \le \int_{-\infty}^{\infty} |\partial_j u|\ dy_j, \tag{10.32}$$

where the integrand, written more fully, is $|\partial_j u(x_1, \dots, y_j, \dots, x_n)|$. (Note that the right side of (10.32) is independent of x_j.) Hence

$$|u(x)|^{n/(n-1)} \le \prod_{j=1}^{n} \Bigl(\int_{-\infty}^{\infty} |\partial_j u|\, dy_j\Bigr)^{1/(n-1)}. \tag{10.33}$$

We can integrate (10.33) successively over each variable $x_j,\ j = 1, \dots, n$, and apply the generalized Hölder inequality (10.29) with $m = p_1 = \cdots = p_m = n - 1$ after each integration. We get

$$\|u\|_{L^{n/(n-1)}} \le \Bigl\{\prod_{j=1}^{n} \int_{\mathbb{R}^n} |\partial_j u|\, dx\Bigr\}^{1/n} \le C\|\nabla u\|_{L^1}. \tag{10.34}$$

This establishes (10.31) in the case $p = 1$. We can apply this to $v = |u|^\gamma,\ \gamma > 1$, obtaining

$$\bigl\||u|^\gamma\bigr\|_{L^{n/(n-1)}} \le C\bigl\||u|^{\gamma-1}|\nabla u|\bigr\|_{L^1} \le C\bigl\||u|^{\gamma-1}\bigr\|_{L^{p'}}\bigl\|\nabla u\bigr\|_{L^p}. \tag{10.35}$$

For $p < n$, pick $\gamma = (n-1)p/(n-p)$. Then (10.35) gives (10.31) and the proposition is proved.

There are also Sobolev spaces $H^{k,p}(\mathbb{R}^n)$, for each $k \in \mathbb{Z}^+$. By definition $u \in H^{k,p}(\mathbb{R}^n)$ provided

$$\partial^\alpha u = f_\alpha \in L^p(\mathbb{R}^n), \quad \forall\, |\alpha| \le k, \tag{10.36}$$

where $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and, as in (10.2), (10.36) means

$$(-1)^{|\alpha|} \int \frac{\partial^\alpha \varphi}{\partial x^\alpha} u\, dx = \int \varphi f_\alpha\, dx, \quad \forall\, \varphi \in C_0^\infty(\mathbb{R}^n). \tag{10.37}$$

Given $u \in H^{k,p}(\mathbb{R}^n)$, we can apply Proposition 10.6 to estimate the $L^{np/(n-p)}$-norm of $\partial^{k-1} u$ in terms of $\|\partial^k u\|_{L^p}$, where we use the notation

$$\partial^k u = \{\partial^\alpha u : |\alpha| = k\}, \quad \|\partial^k u\|_{L^p} = \sum_{|\alpha|=k} \|\partial^\alpha u\|_{L^p}, \tag{10.38}$$

and proceed inductively, obtaining the following corollary.

Proposition 10.7. *For $kp < n$,*

$$H^{k,p}(\mathbb{R}^n) \subset L^{np/(n-kp)}(\mathbb{R}^n). \tag{10.39}$$

The next result provides a generalization of Proposition 10.2.

Proposition 10.8. *We have*

$$H^{k,p}(\mathbb{R}^n) \subset C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \quad \text{for} \quad kp > n. \tag{10.40}$$

Proof. If $p > n$, we can apply Proposition 10.2. If $p = n$ and $k \ge 2$, since it suffices to obtain an L^∞ bound for $u \in H^{k,p}(\mathbb{R}^n)$ with support in the unit ball, just use $u \in H^{2,n-\varepsilon}(\mathbb{R}^n)$ and proceed to the next step of the argument.

If $p \in [1, n)$, it follows from Proposition 10.6 that

$$H^{k,p}(\mathbb{R}^n) \subset H^{k-1,p_1}(\mathbb{R}^n), \quad p_1 = \frac{np}{n-p}. \tag{10.41}$$

Thus the hypothesis $kp > n$ implies $(k-1)p_1 > kp > n$. Iterating this argument, we obtain $H^{k,p}(\mathbb{R}^n) \subset H^{\ell,q}(\mathbb{R}^n)$, for some $\ell \ge 1$ and $q > n$, and again we can apply Proposition 10.2.

Exercises

1. Write down the details for the proof of the identities in (10.4).

2. Verify the estimates in (10.14).
Hint. Write the first integral in (10.13) as $1/A$ times
$$\int\limits_{\Sigma}\int_0^1 v_+(z)\cdot\nabla u(tz,\tfrac{1}{2}t)\,dt\,dz+\int\limits_{\Sigma}\int_0^1 v_-(z)\cdot\nabla u(tz,1-\tfrac{1}{2}t)\,dt\,dz,$$
where $v_\pm(z)=(\pm z,1/2)$. Then calculate an appropriate Jacobian determinant to obtain the second integral in (10.13).

3. Suppose $1<p<\infty$. If $\tau_y f(x)=f(x-y)$, show that f belongs to $H^{1,p}(\mathbb{R}^n)$ if and only if $\tau_y f$ is a Lipschitz function of y with values in $L^p(\mathbb{R}^n)$, i.e.,
$$\|\tau_y f-\tau_z f\|_{L^p}\le C|y-z|. \tag{10.42}$$
Hint. Consider the proof of Proposition 10.4.
What happens in the case $p=1$?

4. Show that $H^{n,1}(\mathbb{R}^n)\subset C(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$.
Hint. $u(x)=\int_{-\infty}^0\cdots\int_{-\infty}^0 \partial_1\cdots\partial_n u(x+y)\;dy_1\cdots dy_n$.

5. If $p_j\in[1,\infty]$ and $u_j\in L^{p_j}$, show that $u_1u_2\in L^r$ provided $1/r=1/p_1+1/p_2$ and
$$\|u_1u_2\|_{L^r}\le\|u_1\|_{L^{p_1}}\|u_2\|_{L^{p_2}}. \tag{10.43}$$
Show that this implies (10.29).

6. Given $u\in L^2(\mathbb{R}^n)$, show that
$$u\in H^{k,2}(\mathbb{R}^n)\Longleftrightarrow\big(1+|\xi|\big)^k\hat{u}\in L^2(\mathbb{R}^n). \tag{10.44}$$

7. Let $f\in L^1(\mathbb{R})$, and set $g(x)=\int_{-\infty}^x f(y)\;dy$. Continuity of g follows from the Dominated Convergence Theorem. Show that
$$\partial_1 g=f. \tag{10.45}$$

Hint. Given $\varphi \in C_0^\infty(\mathbb{R})$, start with

$$\int \frac{d\varphi}{dx} g(x)\,dx = \int \int_{-\infty}^{x} \varphi'(x) f(y)\,dy\,dx, \tag{10.46}$$

and use Fubini's Theorem. Then use $\int_y^\infty \varphi'(x)\,dx = -\varphi(y)$.
Alternative. Write the left side of (10.46) as

$$\lim_{h\to 0} \frac{1}{h} \int \bigl[\varphi(x+h) - \varphi(x)\bigr] g(x)\,dx = -\lim_{h\to 0} \int \int_{x}^{x+h} f(y)\varphi(x)\,dy\,dx,$$

and use (4.64).

8. If $u \in H^{1,p}(\mathbb{R}^n)$ for some $p \in [1,\infty)$ and $\partial_j u = 0$ on a connected open set $U \subset \mathbb{R}^n$, for $1 \le j \le n$, show that u is (equal a.e. to a) constant on U.
   Hint. Approximate u by (10.6), i.e., by $u_\nu = \varphi_\nu * u$, where $\varphi_\nu \in C_0^\infty(\mathbb{R}^n)$ has support in $\{|x| < 1/\nu\}$, $\int \varphi_\nu\,dx = 1$. Show that $\partial_j(\varphi_\nu * u) = 0$ on $U_\nu \subset\subset U$, where $U_\nu \nearrow U$ as $\nu \to \infty$.

   More generally, if $\partial_j u = f_j \in C(U),\ 1 \le j \le n$, show that u is equal a.e. to a function in $C^1(U)$.

9. In case $n = 1$, deduce from Exercises 7 and 8 that, if $u \in L^1_{\mathrm{loc}}(\mathbb{R})$,

$$\partial_1 u = f \in L^1(\mathbb{R}) \Longrightarrow u(x) = c + \int_{-\infty}^{x} f(y)\,dy, \quad \text{a.e. } x \in \mathbb{R}, \tag{10.47}$$

   for some constant c.

10. Let $g \in H^{2,1}(\mathbb{R}),\ I = [a,b]$, and $f = g\big|_I$. Show that the estimate (9.75) concerning the trapezoidal rule holds in this setting.