Chapter 16

Wiener Measure and Brownian Motion

Diffusion of particles is a product of their apparently random motion. The density $u(t,x)$ of diffusing particles satisfies the "diffusion equation"

$$\frac{\partial u}{\partial t} = \Delta u. \tag{16.1}$$

If the initial condition $u(0,x) = f(x)$ for $x \in \mathbb{R}^n$ is given, Fourier analysis, as described in (9.69)–(9.71), can be used to provide the solution

$$\begin{aligned} u(t,x) &= (2\pi)^{-n/2} \int \hat{f}(\xi) e^{-t|\xi|^2} e^{ix\cdot\xi}\, d\xi \\ &= \int p(t,x,y) f(y)\, dy, \end{aligned} \tag{16.2}$$

where $\hat{f}(\xi)$ is the Fourier transform of f and

$$p(t,x,y) = p(t,x-y) = (4\pi t)^{-n/2}\ e^{-|x-y|^2/4t}. \tag{16.3}$$

A suggestive notation for the solution operator provided by (16.2)–(16.3) is

$$u(t,x) = e^{t\Delta} f(x). \tag{16.4}$$

One property this "exponential" of the operator Δ has in common with the exponential of real numbers is the identity $e^{t\Delta}e^{s\Delta} = e^{(t+s)\Delta}$, which by (16.2)–(16.3) is equivalent to the identity

$$\int p(t,x-y)p(s,y)\, dy = p(t+s,x). \tag{16.5}$$

This identity can be verified directly, by manipulation of Gaussian integrals, as in (9.47)–(9.48), or via the identity $e^{-t|\xi|^2}e^{-s|\xi|^2} = e^{-(t+s)|\xi|^2}$, plus the sort of Fourier analysis behind (16.2).

Some other simple but important properties that can be deduced from (16.3) are

$$p(t,x,y) \geq 0 \tag{16.6}$$

and

$$\int p(t,x,y)\,dy = 1. \tag{16.7}$$

Consequently, for each $x \in \mathbb{R}^n$, $p(t,x,y)\,dy$ defines a probability distribution, which we can interpret as giving the probability that a particle starting at the point x at time 0 will be in a given region in $\mathbb{R}^n$ at time t.

We proceed to construct a probability measure, known as "Wiener measure," on the set of paths $\omega : [0,\infty) \to \mathbb{R}^n$, undergoing a random motion, called Brownian motion, described as follows. Given $t_1 < t_2$ and given that $\omega(t_1) = x_1$, the probability density for the location of $\omega(t_2)$ is

$$p(t, x - x_1) = (4\pi t)^{-n/2} e^{-|x-x_1|^2/4t}, \quad t = t_2 - t_1. \tag{16.8}$$

The motion of a random path for $t_1 \leq t \leq t_2$ is supposed to be independent of its past history. Thus, given $0 < t_1 < t_2 < \cdots < t_k$ and given Borel sets $E_j \subset \mathbb{R}^n$, the probability that a path, starting at $x = 0$ at $t = 0$, lies in E_j at time t_j for each $j \in [1,k]$ is

$$\int\limits_{E_1} \cdots \int\limits_{E_k} p(t_k - t_{k-1}, x_k - x_{k-1}) \cdots p(t_1, x_1)\; dx_k \cdots dx_1. \tag{16.9}$$

It is not obvious that there is a countably additive measure characterized by these properties, and Wiener's result was a great achievement. The construction we give here is a slight modification of one in Appendix A of [**Nel**].

Anticipating that Wiener measure is supported on the set of continuous paths, we will take a path to be characterized by its locations at all positive *rational* t. Thus, we consider the set of "paths"

$$\mathfrak{P} = \prod_{t \in \mathbb{Q}^+} \dot{\mathbb{R}}^n. \tag{16.10}$$

Here, $\dot{\mathbb{R}}^n$ is the one-point compactification of $\mathbb{R}^n$, i.e., $\dot{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$. Thus $\mathfrak{P}$ is a compact metrizable space. We construct Wiener measure W as a positive Borel measure on $\mathfrak{P}$.

In order to construct this measure, we will construct a certain positive linear functional $E : C(\mathfrak{P}) \to \mathbb{R}$, on the space $C(\mathfrak{P})$ of real-valued continuous functions on $\mathfrak{P}$, satisfying $E(1) = 1$, and a condition motivated by (16.9), which we give in (16.12). We first define E on the subspace $\mathcal{C}^{\#}$ consisting of continuous functions that depend on only finitely many of the factors in (16.10), i.e., functions on $\mathfrak{P}$ of the form

$$\varphi(\omega) = F\big(\omega(t_1), \dots, \omega(t_k)\big), \quad t_1 < \cdots < t_k, \tag{16.11}$$

where F is continuous on $\prod_1^k \dot{\mathbb{R}}^n$ and $t_j \in \mathbb{Q}^+$. Motivated by (16.9), we take

$$E(\varphi) = \int \cdots \int p(t_1, x_1) p(t_2 - t_1, x_2 - x_1) \cdots p(t_k - t_{k-1}, x_k - x_{k-1}) \times F(x_1, \dots, x_k)\, dx_k \cdots dx_1. \tag{16.12}$$

If $\varphi(\omega)$ in (16.11) actually depends only on $\omega(t_\nu)$ for some proper subset $\{t_\nu\}$ of $\{t_1, \dots, t_k\}$, there arises a formula for $E(\varphi)$ with a different appearance from (16.12). The fact that these two expressions are equal follows from the identity (16.5). From this it follows that $E : \mathcal{C}^{\#} \to \mathbb{R}$ is well defined. It is also a positive linear functional, satisfying $E(1) = 1$.

Now, by the Stone-Weierstrass Theorem, $\mathcal{C}^{\#}$ is dense in $C(\mathfrak{P})$. Since $E : \mathcal{C}^{\#} \to \mathbb{R}$ is a positive linear functional and $E(1) = 1$, it follows that E has a unique continuous extension to $C(\mathfrak{P})$, possessing these properties. Theorem 13.5 associates to E the desired probability measure W. Therefore we have

Theorem 16.1. *There is a unique probability measure W on $\mathfrak{P}$ such that* (16.12) *is given by*

$$E(\varphi) = \int_{\mathfrak{P}} \varphi(\omega)\, dW(\omega), \tag{16.13}$$

for each $\varphi(\omega)$ of the form (16.11) *with F continuous on $\prod_1^k \dot{\mathbb{R}}^n$.*

This is the Wiener measure. We note that (16.12) then holds for any bounded Borel function F, and also for any positive Borel function F, on $\prod_1^k \dot{\mathbb{R}}^n$.

Remark. It is common to define Wiener measure slightly differently, taking $p(t, x)$ to be the integral kernel of $e^{t\Delta/2}$ rather than $e^{t\Delta}$. The path space $\{b\}$ so produced is related to the path space $\{\omega\}$ constructed here by $\omega(t) = b(2t)$.

Some basic examples of calculations of (16.13) include the following. Define functions X_t on $\mathfrak{P}$, taking values in $\mathbb{R}^n$, by

$$X_t(\omega) = \omega(t). \tag{16.14}$$

Then

$$E(|X_t|^2) = \int p(t,x)|x|^2\,dx = 2nt, \tag{16.15}$$

and, if $0 < s < t$,

$$\begin{aligned} E(|X_t - X_s|^2) &= \iint p(s,x_1)p(t-s,x_2-x_1)|x_2-x_1|^2\,dx_1\,dx_2 \\ &= \int p(t-s,y)|y|^2\,dy \\ &= 2n(t-s), \end{aligned} \tag{16.16}$$

a result that works for all $s, t \ge 0$, if $(t-s)$ is replaced by $|t-s|$. Another way to put (16.15)–(16.16) is

$$\|X_t\|_{L^2(\mathfrak{P})} = \sqrt{2nt}, \quad \|X_t - X_s\|_{L^2(\mathfrak{P})} = \sqrt{2n}\,|t-s|^{1/2}. \tag{16.17}$$

Note that the latter result implies $t \mapsto X_t$ is uniformly continuous from $\mathbb{Q}^+$ to $L^2(\mathfrak{P}, W)$ and hence has a unique continuous extension to $\mathbb{R}^+ = [0,\infty)$:

$$\mathfrak{X} : \mathbb{R}^+ \longrightarrow L^2(\mathfrak{P}, W), \tag{16.18}$$

such that $\mathfrak{X}(t) = X_t$, given by (16.14) for $t \in \mathbb{Q}^+$, and then (16.15)–(16.16) are valid for all real $s, t \ge 0$. This is evidence in favor of the assertion made above that W-almost every $\omega \in \mathfrak{P}$ extends continuously from $t \in \mathbb{Q}^+$ to $t \in \mathbb{R}^+$, though it does not prove it. Before we tackle that proof, we make some more observations.

Let us take $t > s > 0$ and calculate

$$\begin{aligned} (X_s, X_t)_{L^2(\mathfrak{P})} &= E(X_s \cdot X_t) \\ &= \int p(s,x_1)p(t-s,x_2-x_1)\,x_1 \cdot x_2\,dx_1\,dx_2 \\ &= \int p(s,x_1)p(t-s,y)\,x_1 \cdot (y+x_1)\,dx_1\,dy. \end{aligned} \tag{16.19}$$

Now $x_1 \cdot (y + x_1) = x_1 \cdot y + |x_1|^2$. The latter contribution is evaluated as in (16.15), and the former contribution is the dot product $A(s) \cdot A(t-s)$, where

$$A(s) = \int p(s,x_1)x_1\,dx_1 = 0. \tag{16.20}$$

So (16.19) is equal to $2ns$ if $t > s > 0$. Hence, by symmetry,

$$(X_s, X_t)_{L^2(\mathfrak{P})} = 2n \min(s,t). \tag{16.21}$$

One can also obtain this by noting $|X_t - X_s|^2 = |X_t|^2 + |X_s|^2 - 2X_s \cdot X_t$ and comparing (16.15) and (16.16). Furthermore, comparing (16.21) and (16.15), we see that

$$t > s \ge 0 \Longrightarrow (X_t - X_s, X_s)_{L^2(\mathfrak{P})} = 0. \tag{16.22}$$

This result is a special case of the following, whose content can be phrased as the statement that if $t > s \ge 0$, then $X_t - X_s$ is independent of X_σ for $\sigma \le s$, and also that $X_t - X_s$ has the same statistical behavior as X_{t-s}. For more on this independence, see the exercises at the end of this chapter and Chapter 17.

Proposition 16.2. *Assume $0 < s_1 < \cdots < s_k < s < t$ $(\in \mathbb{Q}^+)$, and consider functions on $\mathfrak{P}$ of the form*

$$\varphi(\omega) = F(\omega(s_1), \dots, \omega(s_k)), \quad \psi(\omega) = G(\omega(t) - \omega(s)). \tag{16.23}$$

Then

$$E(\varphi\psi) = E(\varphi)E(\psi), \tag{16.24}$$

and

$$E(\psi) = E(\tilde{\psi}), \quad \tilde{\psi}(\omega) = G(\omega(t-s)). \tag{16.25}$$

Proof. By (16.12), we have

$$\begin{aligned} E(\psi) &= \int p(s,y_1)p(t-s,y_2-y_2)G(y_2-y_1)\,dy_1\,dy_2 \\ &= \int p(s,y_1)p(t-s,z)G(z)\,dy_1\,dz \\ &= \int p(t-s,z)G(z)\,dz, \end{aligned} \tag{16.26}$$

which establishes (16.25). Next, we have

(16.27)
$$\begin{aligned} E(\varphi\psi) = \int p(s_1,x_1)p(s_2-s_1,x_2-x_1)\cdots p(s_k-s_{k-1},x_k-x_{k-1}) \\ p(s-s_k,y_1-x_k)p(t-s,y_2-y_1)F(x_1,\dots,x_k) \\ \times G(y_2-y_1)\,dx_1\cdots dx_k\,dy_1\,dy_2. \end{aligned}$$

If we change variables to $x_1, \dots, x_k, y_1, z = y_2 - y_1$, then comparison with (16.26) shows that $E(\psi)$ factors out of (16.27). Then use of $\int p(s-s_k, y_1 - x_k)\,dy_1 = 1$ shows that the other factor is equal to $E(\varphi)$, so we have (16.24).

Here is the promised result on path continuity.

Proposition 16.3. *The set $\mathfrak{P}_0$ of paths from $\mathbb{Q}^+$ to $\mathbb{R}^n$ that are uniformly continuous on bounded subsets of $\mathbb{Q}^+$ (and that hence extend uniquely to continuous paths from $[0,\infty)$ to $\mathbb{R}^n$) is a Borel subset of $\mathfrak{P}$ with Wiener measure* 1.

For a set S, let $\operatorname{osc}_S(\omega)$ denote $\sup_{s,t\in S} |\omega(s)-\omega(t)|$. Set

$$E(a,b,\varepsilon) = \big\{\omega \in \mathfrak{P} : \operatorname{osc}_{[a,b]}(\omega) > 2\varepsilon\big\}; \tag{16.28}$$

here $[a,b]$ denotes $\{s \in \mathbb{Q}^+ : a \le s \le b\}$. The complement is

$$E^c(a,b,\varepsilon) = \bigcap_{t,s\in[a,b]} \big\{\omega \in \mathfrak{P} : |\omega(s)-\omega(t)| \le 2\varepsilon\big\}, \tag{16.29}$$

which is closed in $\mathfrak{P}$. Below we will demonstrate the following estimate on the Wiener measure of $E(a,b,\varepsilon)$:

$$W\big(E(a,b,\varepsilon)\big) \le 2\rho\Big(\frac{\varepsilon}{2}, |b-a|\Big), \tag{16.30}$$

where

$$\begin{aligned}\rho(\varepsilon,\delta) &= \sup_{t\le\delta} \int\limits_{|x|>\varepsilon} p(t,x)\,dx \\ &= \sup_{t\le\delta} \int\limits_{|y|>\varepsilon/\sqrt{t}} p(1,y)\,dy,\end{aligned} \tag{16.31}$$

with $p(t,x)$ as in (16.3). Clearly the sup is assumed at $t=\delta$, so

$$\rho(\varepsilon,\delta) = \int\limits_{|y|>\varepsilon/\sqrt{\delta}} p(1,y)\,dy = \psi_n\Big(\frac{\varepsilon}{\sqrt{\delta}}\Big), \tag{16.32}$$

where

$$\psi_n(r) = (4\pi)^{-n/2} \int\limits_{|y|>r} e^{-|y|^2/4}\,dy \le \alpha_n r^n e^{-r^2/4}, \tag{16.33}$$

as $r\to\infty$.

The relevance of the analysis of $E(a,b,\varepsilon)$ is that, if we set

$$F(k,\varepsilon,\delta) = \Big\{\omega \in \mathfrak{P} : \operatorname{osc}_J(\omega) > 4\varepsilon,\ \text{for some } J \subset [0,k]\cap\mathbb{Q}^+, \ell(J) \le \frac{\delta}{2}\Big\}, \tag{16.34}$$

where $\ell(J)$ is the length of the interval J, then

$$F(k,\varepsilon,\delta)=\bigcup\Bigl\{E(a,b,2\varepsilon):[a,b]\subset[0,k],|b-a|\le\frac{\delta}{2}\Bigr\} \tag{16.35}$$

is an open set, and, via (16.30), we have

$$W\bigl(F(k,\varepsilon,\delta)\bigr)\le 2k\,\frac{\rho(\varepsilon,\delta)}{\delta}. \tag{16.36}$$

Furthermore, with $F^c(k,\varepsilon,\delta)=\mathfrak{P}\setminus F(k,\varepsilon,\delta)$,

$$\begin{aligned}\mathfrak{P}_0&=\bigl\{\omega:\forall\,k<\infty,\forall\,\varepsilon>0,\exists\,\delta>0\text{ such that }\omega\in F^c(k,\varepsilon,\delta)\bigr\}\\&=\bigcap_k\bigcap_{\varepsilon=1/\nu}\bigcup_{\delta=1/\mu}F^c(k,\varepsilon,\delta)\end{aligned} \tag{16.37}$$

is a Borel set (in fact, an $\mathcal{F}_{\sigma\delta}$ set, i.e., a countable intersection of $\mathcal{F}_\sigma$ sets), and we can conclude that $W(\mathfrak{P}_0)=1$ from (16.36), given the observation that, for any $\varepsilon>0$,

$$\frac{\rho(\varepsilon,\delta)}{\delta}\longrightarrow 0,\ \text{ as }\ \delta\to 0, \tag{16.38}$$

which follows immediately from (16.32)–(16.33). Thus, to complete the proof of Proposition 16.3, it remains to establish the estimate (16.30). The next lemma goes most of the way towards that goal.

Lemma 16.4. *Given $\varepsilon,\delta>0$, take ν numbers $t_j\in\mathbb{Q}^+$, $0\le t_1<\cdots<t_\nu$, such that $t_\nu-t_1\le\delta$. Let*

$$A=\bigl\{\omega\in\mathfrak{P}:|\omega(t_1)-\omega(t_j)|>\varepsilon,\ \textit{for some } j=1,\dots,\nu\bigr\}. \tag{16.39}$$

Then

$$W(A)\le 2\rho\Bigl(\frac{\varepsilon}{2},\delta\Bigr). \tag{16.40}$$

Proof. Let

$$\begin{aligned}B&=\{\omega:|\omega(t_1)-\omega(t_\nu)|>\varepsilon/2\},\\C_j&=\{\omega:|\omega(t_j)-\omega(t_\nu)|>\varepsilon/2\},\\D_j&=\{\omega:|\omega(t_1)-\omega(t_j)|>\varepsilon\text{ and}\\&\qquad|\omega(t_1)-\omega(t_k)|\le\varepsilon,\ \text{for all }k\le j-1\}.\end{aligned} \tag{16.41}$$

Then $A\subset B\cup\bigcup_{j=1}^{\nu}\bigl(C_j\cap D_j\bigr)$, so

$$W(A)\le W(B)+\sum_{j=1}^{\nu}W\bigl(C_j\cap D_j\bigr). \tag{16.42}$$

Clearly $W(B) \le \rho(\varepsilon/2, \delta)$. Furthermore, we have

$$W(C_j \cap D_j) = W(C_j)W(D_j) \le \rho\Big(\frac{\varepsilon}{2}, \delta\Big) W(D_j), \tag{16.43}$$

the first identity by Proposition 16.2 (i.e., the independence of C_j and D_j) and the subsequent inequality by the easy estimate $W(C_j) \le \rho(\varepsilon/2, \delta)$. Hence

$$\sum_j W\big(C_j \cap D_j\big) \le \rho\Big(\frac{\varepsilon}{2}, \delta\Big), \tag{16.44}$$

since the D_j are mutually disjoint. This proves (16.40). Note that this estimate is independent of ν.

We now finish the demonstration of (16.30). Given such t_j as in the statement of Lemma 16.4, if we set

$$E = \big\{\omega : |\omega(t_j) - \omega(t_k)| > 2\varepsilon, \text{ for some } j, k \in [1, \nu]\big\}, \tag{16.45}$$

it follows that

$$W(E) \le 2\rho\Big(\frac{\varepsilon}{2}, \delta\Big), \tag{16.46}$$

since E is a subset of A, given by (16.39). Now, $E(a, b, \varepsilon)$, given by (16.28), is a countable increasing union of sets of the form (16.45), obtained, e.g., by letting $\{t_1, \dots, t_\nu\}$ consist of all $t \in [a, b]$ that are rational with denominator $\le K$ and taking $K \nearrow +\infty$. Thus we have (16.30), and the proof of Proposition 16.3 is complete.

We make the natural identification of paths $\omega \in \mathfrak{P}_0$ with continuous paths $\omega : [0, \infty) \to \mathbb{R}^n$. Note that a function φ on $\mathfrak{P}_0$ of the form (16.11), with $t_j \in \mathbb{R}^+$, not necessarily rational, is a pointwise limit on $\mathfrak{P}_0$ of functions in $\mathcal{C}^\#$, as long as F is continuous on $\prod_1^k \dot{\mathbb{R}}^n$, and consequently such φ is measurable. Furthermore, (16.12) continues to hold, by the Dominated Convergence Theorem.

An alternative approach to the construction of W would be to replace (16.10) by $\widetilde{\mathfrak{P}} = \prod\{\dot{\mathbb{R}}^n : t \in \mathbb{R}^+\}$. With the product topology, this is compact but not metrizable. The set of continuous paths is a Borel subset of $\widetilde{\mathfrak{P}}$, but not a Baire set, so some extra measure-theoretic considerations arise if one takes this route, which was taken in [**Nel**].

Looking more closely at the estimate (16.36) of the measure of the set $F(k, \varepsilon, \delta)$, defined by (16.34), we note that we can take $\varepsilon = K\sqrt{\delta \log(1/\delta)}$, in which case

$$\rho\Big(\frac{\varepsilon}{2}, \delta\Big) = \psi_n\Big(\frac{K}{2}\sqrt{\log 1/\delta}\Big) \le C_{n,K}\Big(\log\frac{1}{\delta}\Big)^{n/2} \delta^{K^2/16}. \tag{16.47}$$

Then we obtain the following refinement of Proposition 16.3.

Proposition 16.5. *For almost all $\omega \in \mathfrak{P}_0$, we have, for each $T < \infty$,*

$$\limsup_{|s-t|=\delta\to 0} \left(\bigl|\omega(s) - \omega(t)\bigr| - 8\sqrt{\delta \log \frac{1}{\delta}} \right) \le 0, \quad s,t \in [0,T]. \tag{16.48}$$

Consequently, given $T < \infty$,

$$|\omega(s) - \omega(t)| \le C(\omega, T)\sqrt{\delta \log \frac{1}{\delta}}, \quad s,t \in [0,T], \tag{16.49}$$

with $C(\omega, T) < \infty$ for almost all $\omega \in \mathfrak{P}_0$.

In fact, (16.47) gives $W(S_k) = 1$ where S_k is the set of paths satisfying (16.48), with 8 replaced by $8 + 1/k$, since then (16.47) applies with $K > 4$, so (16.38) holds. Then $\bigcap_k S_k$ is precisely the set of paths satisfying (16.48).

The estimate (16.48) is not quite sharp; P. Lévy showed that for almost all $\omega \in \mathfrak{P}$, with $\mu(\delta) = 2\sqrt{\delta \log 1/\delta}$,

$$\limsup_{|s-t|\to 0} \frac{|\omega(s) - \omega(t)|}{\mu(|s-t|)} = 1. \tag{16.50}$$

See [**McK**] for a proof.

Wiener proved that almost all Brownian paths are nowhere differentiable. We refer to [**McK**] for a proof of this. The following result specifies another respect in which Brownian paths are highly irregular.

Proposition 16.6. *Assume $n \ge 2$, and pick $T \in (0, \infty)$. Then, for almost all $\omega \in \mathfrak{P}_0$,*

$$\omega([0,T]) = \{\omega(t) : 0 \le t \le T\} \subset \mathbb{R}^n \ \ \textit{has Hausdorff dimension} \ \ 2. \tag{16.51}$$

Proof. The fact that $\operatorname{Hdim} \omega([0,T]) \le 2$ for W-a.e. ω follows from the modulus of continuity estimate (16.48), which implies that for each $\delta > 0, \omega$ is Hölder continuous of order $1/(2+\delta)$. This implies by Exercise 9 of Chapter 12 that $\mathcal{H}^r(\omega([0,T])) < \infty$ for $r = 2 + \delta$. (Of course this upper bound is trivial in the case $n = 2$.)

We will obtain the estimate $\operatorname{Hdim} \omega([0,T]) \ge 2$ for a.e. ω as an application of Proposition 12.19. To get this, we start with the following generalization of (16.16): for $0 < s < t$,

$$\begin{aligned} E(\varphi(X_t - X_s)) &= \iint p(s, x_1) p(t-s, x_2 - x_1) \varphi(x_2 - x_1)\, dx_1\, dx_2 \\ &= \int p(t-s, y)\varphi(y)\, dy \\ &= \bigl(4\pi(t-s)\bigr)^{-n/2} \int e^{-|y|^2/4|t-s|} \varphi(y)\, dy. \end{aligned} \tag{16.52}$$

We now assume φ is radial. We switch to spherical polar coordinates. We also allow $t < s$ and obtain

$$E(\varphi(X_t - X_s)) = \frac{A_{n-1}}{(4\pi|t-s|)^{n/2}} \int_0^\infty e^{-r^2/4|t-s|}\, \varphi(r) r^{n-1}\, dr, \tag{16.53}$$

where $A_{n-1} = \text{Area}(S^{n-1})$. We apply this to $\varphi(r) = r^{-a}$ to get

$$\begin{aligned} E(|X_t - X_s|^{-a}) &= C_n |t-s|^{-n/2} \int_0^\infty e^{-r^2/4|t-s|}\, r^{n-1-a}\, dr \\ &= C_{n,a} |t-s|^{-a/2}, \end{aligned} \tag{16.54}$$

where $C_{n,a} < \infty$ provided $a < n$. We deduce that

$$\begin{aligned} &\int_{\mathfrak{P}} \int_0^T \int_0^T \frac{ds\, dt}{|\omega(t) - \omega(s)|^a}\, dW(\omega) \\ &= C_{n,a} \int_0^T \int_0^T \frac{ds\, dt}{|t-s|^{a/2}} = C'_{n,a} < \infty, \quad \text{if} \ \ a < 2. \end{aligned} \tag{16.55}$$

Consequently, as long as $a < 2 \le n$,

$$\int_0^T \int_0^T \frac{ds\, dt}{|\omega(t) - \omega(s)|^a} < \infty, \quad \text{for } W\text{-a.e. } \omega. \tag{16.56}$$

We can rewrite this as

$$\int\limits_{\omega([0,T])} \int\limits_{\omega([0,T])} \frac{d\mu^\omega(x)\, d\mu^\omega(y)}{|x-y|^a} < \infty, \quad \text{for } W\text{-a.e. } \omega, \tag{16.57}$$

where μ^ω is the measure on $\omega([0,T])$ given by

$$\mu^\omega(S) = m(\{t \in [0,T] : \omega(t) \in S\}), \tag{16.58}$$

m denoting Lebesgue measure on $\mathbb{R}$. The existence of a nonzero positive Borel measure on $\omega([0,T])$ satisfying (16.57) implies $\text{Hdim}\, \omega([0,T]) \ge 2$, by Proposition 12.19, so Proposition 16.6 is proven.

So far we have considered Brownian paths starting at the origin in $\mathbb{R}^n$. Via a simple translation of coordinates, we have a similar construction for the set of Brownian paths ω starting at a general point $x \in \mathbb{R}^n$, yielding the positive functional $E_x : C(\mathfrak{P}) \to \mathbb{R}$, and Wiener measure W_x, such that

$$E_x(\varphi) = \int_{\mathfrak{P}} \varphi(\omega)\, dW_x(\omega). \tag{16.59}$$

When $\varphi(\omega)$ is given by (16.11), $E_x(\varphi)$ has the form (16.12), with the function $p(t_1, x_1)$ replaced by $p(t_1, x_1 - x)$. To put it another way, $E_x(\varphi)$ has the form (16.12) with $F(x_1, \dots, x_k)$ replaced by $F(x_1 + x, \dots, x_k + x)$. One often uses such notation as $E_x\big(f(\omega(t))\big)$ instead of $\int_{\mathfrak{P}} f\big(X_t(\omega)\big)\, dW_x(\omega)$ or $E_x\big(f(X_t(\omega))\big)$.

The following simple observation is useful.

Proposition 16.7. *If $\varphi \in C(\mathfrak{P})$, then $E_x(\varphi)$ is continuous in x.*

Proof. Continuity for $\varphi \in \mathcal{C}^{\#}$, the set of functions of the form (16.11), is clear from (16.12), and its extension to $x \neq 0$ discussed above. Since $\mathcal{C}^{\#}$ is dense in $C(\mathfrak{P})$, the result follows easily.

In Chapter 17 we discuss further results on Brownian motion, arising from the study of martingales. For other reading on the topic, we mention the books [**Dur**], [**McK**], and [**Si**] and also Chapter 11 of [**T1**].

The family of random variables X_t on the probability space $(\mathfrak{P}, W)$ is a special case of a *stochastic process*, often called the Wiener process. More general stochastic processes include Lévy processes. They have a characterization similar to (16.12), with $p(t,x)\,dx$ replaced by probability measures ν_t discussed in Exercises 17–18 of Chapter 15. Theorem 16.1 extends to the construction of such Lévy processes. However, the paths are not a.e. continuous in the non-Gaussian case, but rather there are jumps. For material on Lévy processes, see [**Sat**] and references therein.

Exercises

1. With $X_t(\omega) = \omega(t)$ as in (16.14), show that for all $\xi \in \mathbb{R}^n$

$$E(e^{i\xi\cdot X_t}) = e^{-t|\xi|^2}. \tag{16.60}$$

   Hence each component of X_t is a Gaussian random variable on $(\mathfrak{P}_0, W)$, of mean 0 and variance $2t$, by (15.41).

2. More generally, if $0 < t_1 < \cdots < t_k$ and $\xi_j \in \mathbb{R}^n$, show that

$$\begin{aligned} &E\big(e^{i\xi_1\cdot X_{t_1} + i\xi_2\cdot(X_{t_2}-X_{t_1}) + \cdots + i\xi_k\cdot(X_{t_k}-X_{t_{k-1}})}\big) \\ &\quad = e^{-t_1|\xi_1|^2 - (t_2-t_1)|\xi_2|^2 - \cdots - (t_k-t_{k-1})|\xi_k|^2}. \end{aligned} \tag{16.61}$$

   Deduce that each component of $\zeta_1 \cdot X_{t_1} + \cdots + \zeta_k \cdot X_{t_k}$ is a Gaussian random variable on $(\mathfrak{P}_0, W)$, for each $\zeta_1, \dots, \zeta_k \in \mathbb{R}^n$.

3. Show that if $0 < t_1 < \cdots < t_k$, then $X_{t_1}, X_{t_2} - X_{t_1}, \dots, X_{t_k} - X_{t_{k-1}}$ are independent ($\mathbb{R}^n$-valued) random variables on $(\mathfrak{P}_0, W)$.
   Hint. Use (16.61) and Exercise 3 of Chapter 15. Alternatively (but less directly), use the orthogonality (16.22), the Gaussian behavior given in

Exercise 2, and the independence result of Exercise 4 in Chapter 15.

4. Compute $E(e^{\lambda|X_t|^2})$. Show this is finite if and only if $\lambda < 1/4t$.

5. Show that

$$E\big([X_t - X_s]^{2k}\big) = E\big(X^{2k}_{|t-s|}\big) = \frac{(2k)!}{k!}\,|t-s|^k.$$

6. Show that $L^p(\mathfrak{P}_0, W)$ is *separable*, for $1 \le p < \infty$.
   Hint. $\mathfrak{P}$ is a compact *metric* space.

7. Given $a > 0$, define a transformation $D_a : \mathfrak{P}_0 \to \mathfrak{P}_0$ by

$$(D_a\omega)(t) = a\omega(t/a^2).$$

   Show that D_a preserves the Wiener measure W. This transformation is called Brownian scaling.

8. Let

$$\widetilde{\mathfrak{P}}_0 = \big\{\omega \in \mathfrak{P}_0 : \lim_{s\to\infty} s^{-1}\omega(s) = 0\big\}.$$

   Show that $W(\widetilde{\mathfrak{P}}_0) = 1$. Define a transformation $\rho : \widetilde{\mathfrak{P}}_0 \to \mathfrak{P}_0$ by

$$(\rho\omega)(t) = t\omega(1/t),$$

   for $t > 0$. Show that ρ preserves the Wiener measure W.

9. Given $a > 0$, define a transformation $R_a : \mathfrak{P}_0 \to \mathfrak{P}_0$ by

$$\begin{aligned}(R_a\omega)(t) &= \omega(t) && \text{for } \ 0 \le t \le a,\\ &\quad 2\omega(a) - \omega(t) && \text{for } \ t \ge a.\end{aligned}$$

   Show that R_a preserves the Wiener measure W.