Toeplitz Operators on Uniformly Rectifiable Domains
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Joint work with Irina Mitrea and Marius Mitrea,
building on work with Steve Hofmann and M. Mitrea
Notes available on my website, under Downloadable Lecture Notes

7. Seminar talks and AMS talks

See also
2. Singular integral operators
15. Index theory

Reproducing Formula

First order elliptic differential operator $D : C^\infty(M, \mathcal{E}_0) \to C^\infty(M, \mathcal{E}_1)$.
$\Omega \subset M$, UR domain (defined below).
E fundamental solution of D over neighborhood $\mathcal{O}$ of $\overline{\Omega}$
Leibniz formula:

$$D(fu) = fDu + (D_0 f)u, \quad \text{supp } f \subset \mathcal{O}. \tag{1}$$

$Du = \sum A_j \partial_j u + Bu, \quad (D_0 f)u = \sum A_j(\partial_j f)u = i^{-1}\sigma_D(x, df)u.$
Apply E to (1).

$$fu = \int_M E(x,y) D_0 f(y)\, u(y)\, dV(y) + \int_M E(x,y) f(y) Du(y)\, dV(y). \tag{2}$$

Assume Ω has finite perimeter, so $d\chi_\Omega$ is finite measure.
Let $f = f_\nu \to \chi_\Omega$, boundedly, $df_\nu \to \mu = d\chi_\Omega$, weak* in measure.
Assume $u \in C(M),\ Du \in L^1(M)$. Get **reproducing formula:**

$$u(x) = i \int_{\partial\Omega} E(x,y)\sigma_D(y, \nu(y)) u(y)\, d\sigma(y) + \int_\Omega E(x,y) Du(y)\, dV(y), \tag{3}$$

for $x \in \Omega$. Last term vanishes if $Du = 0$ on Ω.
Assume now that Ω is Ahlfors regular, so, with $n = \dim \Omega$,
$\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$ and $\sigma(\partial\Omega \cap B_r(q)) \approx Cr^{n-1},\ q \in \partial\Omega$.
Hofmann-Mitrea-Taylor: (3) holds provided $Du \in L^1(\Omega)$ and

$$u \in C(\Omega), \quad \mathcal{N}u \in L^p(\partial\Omega), \quad \exists \text{nontangential limit } u\big|_{\partial\Omega}, \text{ a.e.} \tag{4}$$

Here, $\mathcal{N}u$ denotes the nontangential maximal function, $p > 1$.

Uniformly Rectifiable Domains

An Ahlfors regular domain Ω is a UR domain provided
$\partial\Omega$ contains big pieces of Lipschitz surfaces, at all length scales, satisfying uniform Lipschitz bounds.

That is, $\exists\,\varepsilon, L \in (0,\infty)$ such that for each $x \in \partial\Omega,\ R \in (0,1]$,
$\exists$ Lipschitz map $\varphi : B_R^{n-1} \to M$, with Lipschitz constant $\le L$, such that

$$\mathcal{H}^{n-1}(\partial\Omega \cap B_R(x) \cap \varphi(B_R^{n-1})) \ge \varepsilon R^{n-1}. \tag{5}$$

Here B_R^{n-1} is a ball of radius R in $\mathbb{R}^{n-1}$, $n = \dim \Omega$.

Layer Potentials

$$\mathcal{B}f(x) = \int_{\partial\Omega} E(x,y) f(y)\, d\sigma(y), \quad x \in \Omega. \tag{6}$$

$$Bf(x) = \mathrm{PV}\int_{\partial\Omega} E(x,y) f(y)\, d\sigma(y), \quad x \in \partial\Omega. \tag{7}$$

G. David: If Ω is a UR domain,

$$B : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega), \quad 1 < p < \infty. \tag{8}$$

Hofmann-Mitrea-Taylor: If Ω is a UR domain,

$$\|\mathcal{N}\mathcal{B}f\|_{L^p(\partial\Omega)} \le C_p \|f\|_{L^p(\partial\Omega)}, \quad 1 < p < \infty, \tag{9}$$

and then there exists a nontangential limit a.e. on $\partial\Omega$,

$$\mathcal{B}f\big|_{\partial\Omega}(x) = \frac{1}{2i}\sigma_E(x,\nu(x)) f(x) + Bf(x). \tag{10}$$

Cauchy Transform

Given $f \in L^p(\partial\Omega)$, set

$$\mathcal{C}f(x) = i\mathcal{B}(\vartheta f)(x), \quad Cf(x) = iB(\vartheta f)(x), \quad \vartheta(x) = \sigma_D(x,\nu(x)). \tag{11}$$

In particular,

$$\mathcal{C}f(x) = i\int_{\partial\Omega} E(x,y)\sigma_D(y,\nu(y)) f(y)\, d\sigma(y), \quad x \in \Omega. \tag{12}$$

By (10), for a.e. $x \in \partial\Omega$,

$$\mathcal{C}f\big|_{\partial\Omega}(x) = \frac{1}{2}f(x) + Cf(x). \tag{13}$$

Note that $D\mathcal{C}f = 0$ on Ω, so $\mathcal{C} : L^p(\partial\Omega) \longrightarrow \mathcal{H}^p(\Omega, D)$, for $1 < p < \infty$, where

$$\mathcal{H}^p(\Omega, D) = \{u \in C^1(\Omega) : Du = 0,\ \mathcal{N}u \in L^p(\partial\Omega),\ u|_{\partial\Omega} \text{ exists}\}. \tag{14}$$

Calderon Projections

Comparing (12)–(13) and the reproducing formula (3) gives that

$$\mathcal{C}f\big|_{\partial\Omega} = \mathcal{P}f, \tag{15}$$

where

$$\mathcal{P}f = \Big(\frac{1}{2}I + C\Big)f, \quad \mathcal{P} : L^p(\partial\Omega) \to L^p(\partial\Omega), \tag{16}$$

satisfies

$$\mathcal{P}^2 = \mathcal{P}. \tag{17}$$

This is a Calderon-type projection.

Toeplitz Operators

Given a UR domain Ω, $\Phi \in L^\infty(\partial\Omega, M(\ell, \mathbb{C}))$, $f \in L^p(\partial\Omega, \mathcal{E}_0 \otimes \mathbb{C}^\ell)$,

$$T_\Phi f = \mathcal{P}\Phi\mathcal{P}f + (I - \mathcal{P})f. \tag{18}$$

$$T_\Phi : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega), \quad 1 < p < \infty. \tag{19}$$

Theorem. ([MMT]) If $\Phi, \Psi \in C(\partial\Omega)$,

$$T_{\Psi\Phi} - T_\Psi T_\Phi : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega), \text{ is compact}, \tag{20}$$

for $1 < p < \infty$. More generally, such compactness holds for

$$\Phi,\ \Psi \in L^\infty \cap \text{vmo}(\partial\Omega). \tag{21}$$

Fredholm Properties

If

$$\Phi \in C(\partial\Omega, G\ell(\ell,\mathbb{C}), \tag{22}$$

or more generally if $\Phi,\ \Phi^{-1} \in L^\infty \cap \mathrm{vmo}(\partial\Omega)$,
then $T_{\Phi^{-1}}$ inverts T_Φ, mod compacts, so

$$T_\Phi : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \ \text{ Is Fredholm, for } p \in (1,\infty). \tag{23}$$

We set

$$\iota(\Phi) = \text{ Index } T_\Phi. \tag{24}$$

The index $\iota(\Phi)$ is independent of $p \in (1,\infty)$. ([MMT])
This implies some global regularity results, such as, for $1 < p < q < \infty$,

$$f \in L^p(\partial\Omega),\ T_\Phi f \in L^q(\partial\Omega) \Longrightarrow f \in L^q(\partial\Omega). \tag{25}$$

Homotopy Properties of Index

If $\Phi_t \in C(\partial\Omega, G\ell(\ell,\mathbb{C}))$ varies continuously with t, then $\iota(\Phi_t)$ is constant. So we get a group homomorphism (on the group of homotopy classes)

$$\iota : [\partial\Omega; G\ell(\ell,\mathbb{C})] \longrightarrow \mathbb{Z}. \tag{26}$$

Polar decomposition: $\Phi = AU, \quad A = (\Phi\Phi^*)^{1/2}, \quad U$ unitary. $\iota(\Phi) = \iota(U)$.
More delicate result:
Theorem. ([MMT]) Assume $\Phi_t \in L^\infty \cap \mathrm{vmo}(\partial\Omega, U(\ell))$ for $t \in [0,1]$, and

$$t \mapsto \Phi_t \text{ continuous from } [0,1] \text{ to } \mathrm{bmo}(\partial\Omega, M(\ell,\mathbb{C})). \tag{27}$$

Then $\iota(\Phi_t)$ is independent of t.

Proof involves an extension of the bmo-homotopy theory of maps of Brezis-Nirenberg.

Cobordism Invariance

Theorem. ([MMT]) If Ω is a UR domain and $\Phi \in C(\overline{\Omega}, G\ell(\ell,\mathbb{C}))$, then

$$\text{Index } T_\Phi = 0. \tag{28}$$

Key application. $\widetilde{\Omega} \subset\subset \Omega$, $\mathcal{O} = \Omega \setminus \widetilde{\Omega}$, $\partial\mathcal{O} = \partial\Omega \cup \partial\widetilde{\Omega}$.

$$\Phi \in C(\overline{\mathcal{O}}, G\ell(\ell,\mathbb{C})) \Longrightarrow \text{ Index } T_\Phi = \text{ Index } T_{\widetilde{\Omega},\Phi}. \tag{29}$$

Sometimes $\partial\widetilde{\Omega}$ is smooth, and the Atiyah-Singer theorem is available.

Application of Bott's Theorem

Bott's Theorem.

$$m = 2\mu - 1 \Longrightarrow \pi_m(U(\ell)) \approx \mathbb{Z} \text{ if } \ell \geq \mu. \tag{30}$$

$$m \notin \{1, 3, \dots, 2\ell - 1\} \Longrightarrow \pi_m(U(\ell)) \text{ finite.} \tag{31}$$

Assume Ω is a UR domain and $\partial\Omega$ is homeomorphic to a sphere:

$$\partial\Omega \approx S^m, \quad m = n - 1 \quad (n = \text{ dim } \Omega). \tag{32}$$

If (30) holds, consider the isomorphism

$$\vartheta : [\partial\Omega; U(\ell)] \approx \mathbb{Z}, \tag{33}$$

unique up to sign.

Proposition. ([MMT]) Assume Ω is a UR domain and $\partial\Omega \approx S^m$. If $m = 2\mu - 1$ and $\ell \geq \mu$, there exists $\alpha = \alpha(\Omega, D) \in \mathbb{Z}$ such that

$$\iota(\Phi; D) = \alpha\vartheta([\Phi]), \quad \forall\, \Phi \in C(\partial\Omega, U(\ell)). \tag{34}$$

If $m \notin \{1, 3, \dots, 2\ell - 1\}$, then

$$\iota(\Phi; D) = 0, \quad \forall\, \Phi \in C(\partial\Omega, U(\ell)). \tag{35}$$

NOTE. α in (34) is independent of ℓ, up to sign, when $\ell \geq \mu,\ m = 2\mu - 1$.

Corollary. If $m = 2\mu - 1$ and $\ell_1 \geq \mu$, and if $\exists\, \Phi_1 \in C(\partial\Omega, U(\ell_1))$ such that

$$\text{Index } T_{\Phi_1} = 1, \tag{36}$$

then (34) holds with $\alpha = \pm 1$, for all $\ell \geq \mu$.

"Digression"

$B \subset \mathbb{C}^\mu$ unit ball, $\mu \geq 2$. Szego projector

$$S_h : L^2(\partial B) \longrightarrow L^2(\partial B), \quad S_h \in OPS^0_{1/2,1/2}(\partial B). \tag{37}$$

Associated Toeplitz operator

$$\tau_\Phi = S_h \Phi S_h + (I - S_h), \tag{38}$$

Fredholm if $\Phi \in C(\partial B, U(\ell))$. As in (34),

$$\text{Index } \tau_\Phi = \alpha_h \vartheta([\Phi]). \tag{39}$$

Venugopalkrishna:

$$\alpha_h = \pm 1. \tag{40}$$

Boutet de Monvel Index Theorem

$$D = \overline{\partial} + \overline{\partial}^* : \Lambda^{0,\mathrm{even}}(\mathbb{C}^\mu) \longrightarrow \Lambda^{0,\mathrm{odd}}(\mathbb{C}^\mu). \tag{41}$$

Theorem. (Boutet de Monvel) For $B \subset \mathbb{C}^\mu$ strongly pseudoconvex, $\Phi \in C(\partial B, U(\ell))$,

$$\text{Index } \tau_\Phi = \iota(\Phi; D). \tag{42}$$

K-homology proof (Baum-Douglas-Taylor, 1989)
$[\tau]$ and $[T]$ define the same element of $K_1(\partial B)$. Both are equal to $\partial[D]$, $[D] \in K_0(B, \partial B)$. Then (42) follows from the intersection pairing

$$K_1(\partial B) \times K^1(\partial B) \longrightarrow \mathbb{Z}. \tag{43}$$

Application of (39)–(42) and Cobordism Invariance

Proposition. ([MMT]) When $\Omega = B$ is the unit ball in $\mathbb{C}^\mu$ and D is given by (41), then

$$\iota(\Phi; D) = \pm\vartheta([\Phi]), \quad \forall\, \Phi \in C(\partial\Omega, U(\ell)), \tag{44}$$

provided $\ell \geq \mu$.
Via cobordism invariance:
Proposition. Let $\Omega \subset \mathbb{C}^\mu$ be a bounded UR domain and let D be given by (41). Let $\ell \geq \mu$. Then

$$\exists\, \Phi_1 \in C(\partial\Omega, U(\ell)) \text{ such that Index } T_{\Phi_1} = 1. \tag{45}$$

Corollary. Let $\Omega \subset \mathbb{C}^\mu$ be a UR domain and let D be given by (41). If $\partial\Omega$ is homeomorphic to $S^{2\mu-1}$, then (44) holds.

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