Cauchy Integrals, Calderón Projectors, and Toeplitz Operators on Uniformly Rectifiable Domains *

Irina Mitrea   Marius Mitrea   Michael Taylor

Abstract

We develop properties of Cauchy integrals associated to a general class of first-order elliptic systems of differential operators D on a bounded, uniformly rectifiable (UR) domain Ω in a Riemannian manifold M. We show that associated to such Cauchy integrals are analogues of Hardy spaces of functions on Ω annihilated by D, and we produce projections, of Calderón type, onto subspaces of $L^p(\partial\Omega)$ consisting of boundary values of elements of such Hardy spaces. We consider Toeplitz operators associated to such projections and study their index properties. Of particular interest is a "cobordism argument," which often enables one to identify the index of a Toeplitz operator on a rough UR domain with that of one on a smoothly bounded domain.

Contents

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1 Introduction

Let M be a compact, connected, n-dimensional Riemannian manifold, and D a first-order elliptic differential operator on M, acting between sections of Hermitian vector bundles $\mathcal{F}_j \to M,\ j = 0, 1$, each of rank κ. We assume that there is a coordinate system on M for which the metric tensor is of class C^2 and, on such a local coordinate chart U, and with respect to trivializations of $\mathcal{F}_j$,

$$Du(x) = A_j(x)\partial_j u(x) + B(x)u(x) \tag{1.0.1}$$

(using the summation convention), with

$$A_j \in C^2(U, \operatorname{End}\mathbb{C}^\kappa), \quad B \in C^1(U, \operatorname{End}\mathbb{C}^\kappa). \tag{1.0.2}$$

Here $\operatorname{End}\mathbb{C}^\kappa$ is the space of $\kappa \times \kappa$ complex matrices. To say that D is elliptic is to say that the symbol $iA_j(x)\xi_j$ is invertible for each nonzero $\xi \in \mathbb{R}^n$. With $H^{s,p}$ denoting the L^p-Sobolev space of regularity s, we have

$$D : H^{s+1,p}(M, \mathcal{F}_0) \longrightarrow H^{s,p}(M, \mathcal{F}_1), \quad s \in [-2, 1],\ p \in (1, \infty). \tag{1.0.3}$$

Let $\Omega \subset M$ be an open subset. We will assume Ω is a uniformly rectifiable (UR) domain, a class we characterize as follows. First, we assume Ω has finite perimeter, which implies

$$\nabla\chi_\Omega = -\nu\,\sigma, \tag{1.0.4}$$

where χ_Ω is the characteristic function of Ω, ν is the outward pointing unit normal to $\partial\Omega$, and σ is "surface area" on $\partial\Omega$, carried by the measure-theoretic boundary $\partial_*\Omega \subset \partial\Omega$. We assume

$$\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0, \tag{1.0.5}$$

to avoid pathologies. Then $\sigma = \mathcal{H}^{n-1}\lfloor\partial\Omega$. (Here, $\mathcal{H}^{n-1}$ is $(n-1)$-dimensional Hausdorff measure.) Next, we assume $\partial\Omega$ is Ahlfors regular, i.e., there exist $C_0, C_1 \in (0, \infty)$ such that if $x_0 \in \partial\Omega,\ r \in (0, \operatorname{diam}\Omega)$,

$$C_0 r^{n-1} \le \mathcal{H}^{n-1}(\partial\Omega \cap B_r(x_0)) \le C_1 r^{n-1}. \tag{1.0.6}$$

Under these conditions, we say Ω is an Ahlfors regular domain. We say Ω is a UR domain if, in addition, $\partial\Omega$ is uniformly rectifiable, in the sense of G. David and S. Semmes. This means $\partial\Omega$ has, at all length scales $\le$ $\operatorname{diam}\Omega$, and in a uniformly controlled fashion, "large pieces" of Lipschitz surfaces. See §2.3 for a detailed definition. For such UR domains, fundamental work of David [9] yields bounds on $L^p(\partial\Omega)$ for singular integral operators of the form

$$Bf(x) = \mathrm{PV}\int_{\partial\Omega} E(x-y)f(y)\,d\sigma(y), \quad x \in \partial\Omega, \tag{1.0.7}$$

in case $\Omega \subset \mathbb{R}^n$, provided $E(z)$ is smooth on $\mathbb{R}^n \setminus 0$, odd in z, and homogeneous of degree $-(n-1)$ in z. Such estimates are established for certain variable coefficient versions

$$Bf(x) = \mathrm{PV} \int_{\partial\Omega} E(x,y) f(y)\, d\sigma(y), \quad x \in \partial\Omega, \tag{1.0.8}$$

and in the manifold context, $\overline{\Omega} \subset M$, in [11]. (We will say more about these operators later in this paper.) For $n = 2$, $\mathbb{R}^2 \approx \mathbb{C}$, and $E(z) = 1/z$, (1.0.7) is a standard version of a Cauchy integral.

We desire to study Cauchy integrals associated to the elliptic operator D mentioned above. If we replace M by $\mathbb{R}^n$ and take D to have constant coefficients,

$$Du(x) = A_j \partial_j u(x), \quad A_j \in \mathrm{End}(\mathbb{C}^\kappa), \tag{1.0.9}$$

then we can take $E \in C^\infty(\mathbb{R}^n \setminus 0)$ to be the fundamental solution to D, which is odd and homogeneous of degree $-(n-1)$, and produces an operator of the form (1.0.7). If we work in the manifold setting and D is invertible in (1.0.3), we can take $E(x,y)$ to be the integral kernel of D^{-1} and use (1.0.8). However, in many natural cases of great interest, D has nonzero index, and a different route is called for. We are motivated to consider

$$\mathcal{D} = \begin{pmatrix} iM_a & D^* \\ D & iM_a \end{pmatrix}, \tag{1.0.10}$$

acting on $H^{s+1,p}(M,\mathcal{E})$, where $\mathcal{E} = \mathcal{F}_0 \oplus \mathcal{F}_1$. Here D^* is the formal adjoint of D, defined via the Riemannian metric on M and C^2 Hermitian metrics on $\mathcal{F}_j$, and $M_a u(x) = a(x)u(x)$. Our hypotheses on the metric tensor (g_{jk}) and D, and on these Hermitian metrics, imply

$$\begin{aligned} D^* v(x) &= -A_j(x)^* \partial_j v(x) + \widetilde{B}(x) v(x), \\ \widetilde{B}(x) &= -g(x)^{-1/2} \partial_j (g(x)^{1/2} A_j(x)^*) + B(x)^*, \end{aligned} \tag{1.0.11}$$

so

$$A_j^* \in C^2(U, \mathrm{End}\,\mathbb{C}^\kappa), \quad \widetilde{B} \in C^1(U, \mathrm{End}\,\mathbb{C}^\kappa). \tag{1.0.12}$$

(Here, the "adjoints" A_j^*, etc., are computed using the Hermitian metrics on $\mathcal{F}_0$ and $\mathcal{F}_1$.) We also assume

$$a \in C^1(M), \quad a \geq 0, \tag{1.0.13}$$

and, if D in (1.0.3) is not invertible,

$$\mathcal{O} = \{x \in M : a(x) > 0\} \neq \emptyset, \quad \mathcal{O} \subset M \setminus \overline{\Omega}. \tag{1.0.14}$$

(If D is invertible in (1.0.3), we can just take $a \equiv 0$.) Then

$$\mathcal{D} : H^{s+1,p}(M,\mathcal{E}) \longrightarrow H^{s,p}(M,\mathcal{E}), \quad s \in [-2,1],\ p \in (1,\infty). \tag{1.0.15}$$

As shown in Appendix A.1, under these hypotheses, $\mathcal{D}$ in (1.0.15) is Fredholm, of index 0, and

$$\mathrm{Ker}\,\mathcal{D} = \big\{u \in \cap_{q<\infty} H^{2,q}(M,\mathcal{E}) : u\big|_{\mathcal{O}} = 0,\ Du_0 = 0,\ D^* u_1 = 0\big\}, \tag{1.0.16}$$

where $u = (u_0, u_1)^t$, $u_j \in H^{2,q}(M,\mathcal{F}_j)$. Thus $\mathcal{D}$ in (1.0.15) is invertible whenever the right side of (1.0.16) can be shown to be 0. Such a condition holds in particuler if

$$D \text{ and } D^* \text{ have the unique continuation property (UCP).} \tag{1.0.17}$$

See Appendix A.1 for the definition of the property UCP. This property holds if M has a real analytic metric tensor, $\mathcal{F}_j$ have real analytic Hermitian metrics, and the coefficients of D (and hence of D^*) are real analytic, by the Holmgren uniqueness theorem. Some classes of operators with limited regularity (1.0.2) and (1.0.12) that satisfy (1.0.17) are discussed in Appendix A.1. They include operators of Dirac type, and of "generalized Dirac type." From here on, we assume UCP.

Given that $\mathcal{D}$ in (1.0.15) is invertible, we denote the integral kernel of $\mathcal{D}^{-1}$ by $E(x,y)$:

$$\mathcal{D}^{-1}u(x) = \int_M E(x,y)u(y)\,dV(y), \tag{1.0.18}$$

and use this function $E(x,y)$ in (1.0.8). Results on $E(x,y)$ given in Appendix A.1, together with the variable coefficient extension of David's estimates given in [11], yield $L^p(\partial\Omega)$ boundedness of such an operator B, for each $p \in (1,\infty)$. See §2.3 for more details on this. Going further, we examine

$$\mathcal{B}f(x) = \int_{\partial\Omega} E(x,y)f(y)\,d\sigma(y), \quad x \in M \setminus \partial\Omega, \tag{1.0.19}$$

establish nontangential maximal function estimates, and show that, for $f \in L^p(\partial\Omega,\mathcal{E})$, $p \in (1,\infty)$, nontangential boundary values exist:

$$\lim_{z\to x, z\in\Gamma_x} \mathcal{B}f(z) = \frac{1}{2i}\sigma_E(x,\nu(x))f(x) + Bf(x), \quad \text{for } \sigma\text{-a.e. } x \in \partial\Omega, \tag{1.0.20}$$

where $\sigma_E(x,\nu) = \sigma_{\mathcal{D}}(x,\nu)^{-1}$ ($\sigma_{\mathcal{D}}$ standing for the principal symbol of $\mathcal{D}$), and $\Gamma_x \subset \Omega$ is a region of nontangential approach to $x \in \partial\Omega$. Thus, we are motivated to consider

$$\mathcal{C}_{\mathcal{D}}f(x) = i\int_{\partial\Omega} E(x,y)\sigma_{\mathcal{D}}(y,\nu(y))f(y)\,d\sigma(y), \quad x \in \Omega, \tag{1.0.21}$$

and the principal value integral

$$C_{\mathcal{D}}f(x) = \mathrm{PV}\ i\int_{\partial\Omega} E(x,y)\sigma_{\mathcal{D}}(y,\nu(y))f(y)\,d\sigma(y), \quad x \in \partial\Omega, \tag{1.0.22}$$

for which

$$\lim_{z\to x, z\in\Gamma_x} \mathcal{C}_{\mathcal{D}}f(x) = \frac{1}{2}f(x) + C_{\mathcal{D}}f(x), \quad \sigma\text{-a.e. } x \in \partial\Omega. \tag{1.0.23}$$

These results bear on the study of the following family of Hardy spaces:

$$\begin{aligned}\mathcal{H}^p(\Omega,\mathcal{D}) = \{u \in C(\Omega,\mathcal{E}) : \mathcal{D}u = 0,\ \mathcal{N}u \in L^p(\partial\Omega),\ &\text{and } u \text{ has a} \\ &\text{nontangential trace } u_b \in L^p(\partial\Omega,\mathcal{E})\}.\end{aligned} \tag{1.0.24}$$

(Here, $\mathcal{N}u$ denotes the nontangential maximal function associated with u.) In particular, for $p \in (1,\infty)$,

$$\mathcal{B}, \mathcal{C}_{\mathcal{D}} : L^p(\partial\Omega,\mathcal{E}) \longrightarrow \mathcal{H}^p(\Omega,\mathcal{D}). \tag{1.0.25}$$

One of the main results of §2 is the following *Cauchy-Pompieu reproducing formula*:

$$u \in \mathcal{H}^p(\Omega,\mathcal{D}) \Longrightarrow u = \mathcal{C}_{\mathcal{D}}(u_b). \tag{1.0.26}$$

It follows that

$$u = \mathcal{C}_{\mathcal{D}} f,\ \ f \in L^p(\partial\Omega, \mathcal{E}) \Longrightarrow u = \mathcal{C}_{\mathcal{D}}\big(u\big|_{\partial\Omega}\big), \tag{1.0.27}$$

where $u|_{\partial\Omega} = u_b$, and hence

$$\mathcal{P}_{\mathcal{D}} = \frac{1}{2} I + C_{\mathcal{D}} \Longrightarrow \mathcal{P}_{\mathcal{D}}^2 = \mathcal{P}_{\mathcal{D}}. \tag{1.0.28}$$

We furthermore show in §3 that, for $p \in (1,\infty)$, the range of $\mathcal{P}_{\mathcal{D}}$ on $L^p(\partial\Omega,\mathcal{E})$, which we denote by

$$\mathcal{H}^p(\partial\Omega, \mathcal{D}) = \mathcal{P}_{\mathcal{D}}\big(L^p(\partial\Omega, \mathcal{E})\big), \tag{1.0.29}$$

has the property that the trace map gives an isomorphism:

$$\tau : \mathcal{H}^p(\Omega, \mathcal{D}) \xrightarrow{\approx} \mathcal{H}^p(\partial\Omega, \mathcal{D}). \tag{1.0.30}$$

With respect to the splitting $\mathcal{E} = \mathcal{F}_0 \oplus \mathcal{F}_1$, $\mathcal{P}_{\mathcal{D}}$ has the diagonal form

$$\mathcal{P}_{\mathcal{D}} = \begin{pmatrix} \mathcal{P}_D & \mathcal{Q}_{01} \\ \mathcal{Q}_{10} & \mathcal{P}_{D^*} \end{pmatrix}. \tag{1.0.31}$$

General considerations readily yield that $\mathcal{Q}_{ab}$ are compact and $\mathcal{P}_D$ and $\mathcal{P}_{D^*}$ are projections modulo compacts. In fact, using (1.0.26), we show that $\mathcal{P}_D$ and $\mathcal{P}_{D^*}$ are projections. With obvious notation, for $p \in (1,\infty)$,

$$\mathcal{P}_D \ \text{ is a projection of } \ L^p(\partial\Omega, \mathcal{F}_0) \ \text{ onto } \ \mathcal{H}^p(\partial\Omega, D). \tag{1.0.32}$$

The projections $\mathcal{P}_{\mathcal{D}}, \mathcal{P}_D$, and $\mathcal{P}_{D^*}$ are of a sort considered by A. P. Calderón in his work on boundary problems for elliptic operators. They also play a role in the formulation of boundary problems of Atiyah-Patodi-Singer type for elliptic systems. Another related operator, also considered in §3, is

$$S_{\mathcal{D}} = \ \text{ orthogonal projection of } \ L^2(\partial\Omega, \mathcal{E}) \ \text{ onto } \ \mathcal{H}^2(\partial\Omega, \mathcal{D}). \tag{1.0.33}$$

This is analogous to the Szegö projection, onto spaces of boundary values of holomorphic functions. Extensions of $S_{\mathcal{D}}$ to $L^p(\partial\Omega,\mathcal{E})$ for a range of p, and comparisons with $\mathcal{P}_{\mathcal{D}}$ can be found in §3.

In §4 we study Toeplitz operators $\mathfrak{T}_\Phi$, initially for $\Phi \in C(\partial\Omega, \operatorname{End}\mathbb{C}^\ell)$, acting on $L^p(\partial\Omega, \mathcal{E} \otimes \mathbb{C}^\ell)$, given by

$$\mathfrak{T}_\Phi f = \mathcal{P}_{\mathcal{D}} \Phi \mathcal{P}_{\mathcal{D}} f + (I - \mathcal{P}_{\mathcal{D}}) f, \tag{1.0.34}$$

with $\mathcal{P}_{\mathcal{D}}$ acting on $L^p(\partial\Omega,\mathcal{E})$ and Φ acting on $\mathbb{C}^\ell$. We also define T_Φ on $L^p(\partial\Omega, \mathcal{F}_0 \otimes \mathbb{C}^\ell)$ by

$$T_\Phi f = \mathcal{P}_D \Phi \mathcal{P}_D f + (I - \mathcal{P}_D) f. \tag{1.0.35}$$

We show that $\mathfrak{T}_\Phi$ and T_Φ are Fredholm if $\Phi \in C(\partial\Omega, G\ell(\ell,\mathbb{C}))$, where $G\ell(\ell,\mathbb{C})$ denotes the group of invertible $\ell \times \ell$ matrices. We set

$$\iota(\Phi) = \operatorname{Index} T_\Phi \ \text{ on } \ L^p(\partial\Omega, \mathcal{F}_0 \otimes \mathbb{C}^\ell), \tag{1.0.36}$$

which we show is independent of $p \in (1,\infty)$, and depends only on the homotopy class of $\Phi : \partial\Omega \to G\ell(\ell,\mathbb{C})$. We extend this to

$$\Phi \in L^\infty \cap \operatorname{vmo}(\partial\Omega, \operatorname{End}\mathbb{C}^\ell), \tag{1.0.37}$$

again obtaining Fredholm operators if also $\Phi^{-1} \in L^\infty(\partial\Omega, \operatorname{End}\mathbb{C}^\ell)$. In this setting, we extend to the multi-dimensional setting of a UR domain Ω results on the index established by Brezis-Nirenberg [7] in the setting of $\Omega =$ the unit disk in $\mathbb{C}$ and $D = \partial/\partial\overline{z}$.

We also study T_Φ on L^p-Sobolev spaces $L^p_1(\partial\Omega, \mathcal{E})$, first for $\Phi \in C^1(\partial\Omega, \operatorname{End}\mathbb{C}^\ell)$, and then more generally for $\Phi \in L^q_1(\partial\Omega, \operatorname{End}\mathbb{C}^\ell)$, with $q > n-1,\ q \geq p$. Here we use the notation $L^p_1(\partial\Omega, \mathcal{E})$ to denote the space of L^p sections of $\mathcal{E}$ over $\partial\Omega$ whose gradients, suitably defined, belong to L^p. See Appendix A.2 for the precise definition.

In addition, we study a class of Toeplitz operators $\mathcal{T}_\Phi$, defined as in (1.0.34), but with $\mathcal{P}_\mathcal{D}$ replaced by the Calderón-Szegö projector $S_\mathcal{D}$.

Section 4.5 considers twisted Toeplitz operators, acting on sections of $\mathcal{F}_0 \otimes \mathcal{C}$, where $\mathcal{C}$ is an auxiliary vector bundle, with Φ a section of the bundle $G\ell(\mathcal{C})$. In §4.6, we study localization of Toeplitz operators. In §4.7, we establish an important cobordism invariance result for the index. With this, one can often show that the index of a Toeplitz operator on a rough UR domain is equal to one on a smoothly bounded domain. We make use of these results in §4.8 to obtain some explicit index formulas on rough UR domains.

This paper ends with some appendices, giving useful background material. Appendix A.1, already mentioned, discusses conditions under which $\mathcal{D}$ in (1.0.15) is invertible, and produces results on the integral kernel $E(x, y)$ of $\mathcal{D}^{-1}$ needed to establish the mapping properties (1.0.8) and (1.0.25). Appendix A.2 discusses L^p-Sobolev spaces on boundaries of Ahlfors regular domains, and Appendix A.3 gives some basic results on Cauchy integrals applied to elements of such spaces $L^p_1(\partial\Omega)$, when Ω is a UR domain. These results are useful for the study of Toeplitz operators on L^p-Sobolev spaces in §4. Appendix A.4 gives examples of UR domains with fairly wild boundaries.

1.1 Further directions

The results of this paper are applicable to the study of Riemann-Hilbert problems on uniformly rectifiable domains. This is being developed by the authors in [21].

2 Cauchy-Pompieu reproducing formulas

Here we prove the result stated in (1.0.26), namely, with $E(x, y)$ as in (1.0.18),

$$u(x) = i \int_{\partial\Omega} E(x, y)\sigma_\mathcal{D}(y, \nu(y))u(y)\, d\sigma(y), \quad \forall\, x \in \Omega, \tag{2.0.1}$$

provided

$$\mathcal{D}u = 0 \ \text{ on } \ \Omega, \tag{2.0.2}$$

and u satisfies certain regularity conditions up to $\partial\Omega$. More generally, we show that

$$\begin{aligned} u(x) = i \int_{\partial\Omega} & E(x, y)\sigma_\mathcal{D}(y, \nu(y))u(y)\, d\sigma(y) \\ & + \int_\Omega E(x, y)\mathcal{D}u(y)\, dV(y), \quad \forall\, x \in \Omega, \end{aligned} \tag{2.0.3}$$

for u having such regularity, a result that implies (2.1) if $\mathcal{D}u = 0$ on Ω. We take $\Omega \subset M$ and $\mathcal{D}$ as in §1.

We proceed in stages. In §2.1 we establish (2.0.3) when Ω is a finite perimeter domain and

$$u \in C(M, \mathcal{E}), \quad \mathcal{D}u \in L^1(M, \mathcal{E}). \tag{2.0.4}$$

In §2.2 we require Ω to be Ahlfors regular, and establish (2.0.3) when, for some $p > 1$,

$$u \in \mathfrak{L}^p \ \text{ and } \ \mathcal{D}u \in L^1(\Omega, \mathcal{E}), \tag{2.0.5}$$

where

$$\begin{aligned} \mathfrak{L}^p = \{u \in C(\Omega, \mathcal{E}) : \mathcal{N}u \in L^p(\partial\Omega), \text{ and} \\ \exists \text{ nontangential limit } u_b, \sigma\text{-a.e.}\}. \end{aligned} \tag{2.0.6}$$

In §2.3, we assume Ω is a UR domain and, as in (1.0.21), take

$$u(x) = \mathcal{C}_{\mathcal{D}} f(x) = i \int\limits_{\partial\Omega} E(x, y) \sigma_{\mathcal{D}}(y, \nu(y)) f(y) \, d\sigma(y), \tag{2.0.7}$$

with

$$f \in L^p(\partial\Omega, \mathcal{E}), \quad p \in (1, \infty). \tag{2.0.8}$$

In this situation, we show that the results of §2.2 apply, and draw conclusions about

$$\mathcal{P}_{\mathcal{D}} = \frac{1}{2} I + C_{\mathcal{D}} \tag{2.0.9}$$

(with $C_{\mathcal{D}}$ as in (1.0.22)–(1.0.23)), which will play a major role in §3.

2.1 Reproducing formulas on finite perimeter domains

Take M to be a compact, connected, Riemannian manifold with metric tensor of class C^2, as in §1. We work in the setting where the first-order, elliptic differential operator $\mathcal{D}$ is given by (1.0.10), with D as in (1.0.1). We assume

$$\mathcal{D} : H^{s+1,p}(M, \mathcal{E}) \longrightarrow H^{s,p}(M, \mathcal{E}) \tag{2.1.1}$$

is invertible, for

$$s \in [-2, 1], \quad p \in (1, \infty), \tag{2.1.2}$$

with inverse

$$E = \mathcal{D}^{-1} : H^{s,p}(M, \mathcal{E}) \longrightarrow H^{s+1,p}(M, \mathcal{E}). \tag{2.1.3}$$

We also let $E(x, y)$ denote the integral kernel of $\mathcal{D}^{-1}$:

$$Eu(x) = \int\limits_M E(x, y) u(y) \, dV(y), \tag{2.1.4}$$

and recall results on $E(x, y)$ established in Appendix A.1.

As a first step toward producing the reproducing formula (2.0.3), we start with the following "product formula." If $u \in H^{s+1,p}(M, \mathcal{E})$ and if $f \in C^1(M)$ is real (or complex) valued,

$$\mathcal{D}(fu) = f\mathcal{D}(u) + (\mathcal{D}_0 f)u, \tag{2.1.5}$$

where $\mathcal{D}_0$ is a first-order differential operator given by

$$\mathcal{D}_0 f(x) = \frac{1}{i} \sigma_{\mathcal{D}}(x, df(x)), \quad \mathcal{D}_0 : C^1(M) \longrightarrow C^0(M, \operatorname{End} \mathcal{E}). \tag{2.1.6}$$

In local coordinates (and this time eschewing the summation convention), if

$$\mathcal{D}u(x) = \sum A_j(x)\partial_j u + B(x)u, \quad A_j(x), B(x) \in \operatorname{End}\mathcal{E}_x, \tag{2.1.7}$$

then

$$(\mathcal{D}_0 f(x))u(x) = \sum \partial_j f(x)\, A_j(x)u(x). \tag{2.1.8}$$

Applying E to (2.1.5) yields

$$fu = E((\mathcal{D}_0 f)u) + E(f\mathcal{D}u). \tag{2.1.9}$$

We aim to extend the class of functions f to which (2.1.9) applies, first for u defined on M and having some moderate regularity, then, in subsequent sections, for more general u. As a first step, we consider the case when

$$u \in C(M,\mathcal{E}), \quad \mathcal{D}u \in L^1(M,\mathcal{E}). \tag{2.1.10}$$

We then assume

$$f \in L^\infty(M), \quad df \in \mathcal{M}(M), \tag{2.1.11}$$

where df is the exterior derivative of f, and $\mathcal{M}(M)$ denotes the space of finite (vector valued) measures on M. Then we can use convolutions in local coordinates to obtain

$$\begin{gathered} f_j \in C^1(M), \quad \|f_j\|_{L^\infty} \le C\|f\|_{L^\infty}, \quad f_j \to f \ \text{ pointwise a.e.,} \\ df_j \to df \ \text{ weak}^* \text{ in } \mathcal{M}. \end{gathered} \tag{2.1.12}$$

We have, for each j,

$$f_j u = E((\mathcal{D}_0 f_j)u) + E(f_j\mathcal{D}u). \tag{2.1.13}$$

As long as u satisfies (2.1.10),

$$f_j u \to fu, \quad \text{boundedly and a.e.,} \tag{2.1.14}$$

$$(\mathcal{D}_0 f_j)u \to (\mathcal{D}_0 f)u, \quad \text{weak}^* \text{ in } \mathcal{M}(M), \tag{2.1.15}$$

$$f_j\mathcal{D}u \to f\mathcal{D}u, \quad \text{in } L^1\text{-norm.} \tag{2.1.16}$$

Note that (2.1.14) implies norm convergence in L^p, for all $p < \infty$, and (2.1.15)–(2.1.16) imply norm convergence in $H^{-\varepsilon,p}$, for some $\varepsilon \in (0,1)$, $p \in (1,\infty)$. Hence

$$E((\mathcal{D}_0 f_j)u) \to E((\mathcal{D}_0 f)u), \quad E(f_j\mathcal{D}u) \to E(f\mathcal{D}u), \tag{2.1.17}$$

in $H^{1-\varepsilon,p}$, and we deduce from (2.1.13) that

$$fu = E((\mathcal{D}_0 f)u) + E(f\mathcal{D}u), \tag{2.1.18}$$

whenever u satisfies (2.1.10) and f satisfies (2.1.11).

We apply (2.1.18) to $f = \chi_\Omega$, when $\Omega \subset M$ is an open set satisfying

$$\Omega \text{ is a finite-perimeter domain,} \tag{2.1.19}$$

so

$$d\chi_\Omega = -\nu\sigma, \tag{2.1.20}$$

where σ is a positive (finite) Borel measure, supported on $\partial\Omega$, $\nu \in L^\infty(\partial\Omega, \sigma)$, and, for σ-a.e. x, $\nu(x) \in T_x^*M$ satisfies $|\nu(x)| = 1$. It follows from (2.1.6)–(2.1.8) that

$$\mathcal{D}_0\chi_\Omega = i\sigma_{\mathcal{D}}(x,\nu)\sigma. \tag{2.1.21}$$

Hence, if $E(x, y)$ denotes the integral kernel of E (taking values in $\mathrm{Hom}(\mathcal{E}_y, \mathcal{E}_x)$ for $x \neq y \in M$),

$$E((\mathcal{D}_0\chi_\Omega)u)(x) = i \int\limits_{\partial\Omega} E(x,y)\sigma_{\mathcal{D}}(y,\nu(y))u(y)\,d\sigma(y), \tag{2.1.22}$$

and (2.1.18) implies

$$\begin{aligned} u(x) = i \int\limits_{\partial\Omega} & E(x,y)\sigma_{\mathcal{D}}(y,\nu(y))u(y)\,d\sigma(y) \\ & + \int\limits_{\Omega} E(x,y)\mathcal{D}u(y)\,dV(y), \quad \forall\, x \in \Omega, \end{aligned} \tag{2.1.23}$$

given that u satisfies (2.1.10) and Ω satisfies (2.1.19). If, in addition,

$$\mathcal{D}u = 0 \ \text{ on } \ \Omega, \tag{2.1.24}$$

we obtain

$$u(x) = i \int\limits_{\partial\Omega} E(x,y)\sigma_{\mathcal{D}}(y,\nu(y))u(y)\,d\sigma(y), \quad \forall\, x \in \Omega. \tag{2.1.25}$$

We need to expand the class of functions u to which (2.1.25) applies, at least for a broad but more restricted class of domains Ω. We take this up in the following subsections.

2.2 Reproducing formulas on Ahlfors regular domains

If $\Omega \subset M$ is an open domain of finite perimeter, and σ is as in (2.1.20), it is known that σ is carried by the "measure theoretic boundary" $\partial_*\Omega$ and

$$\sigma = \mathcal{H}^{n-1}\lfloor\partial_*\Omega. \tag{2.2.1}$$

We say Ω is Ahlfors regular if the following two further conditions hold:

$$\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0, \tag{2.2.2}$$

and there exist $c_0, c_1 \in (0, \infty)$ such that for all $x \in \partial\Omega$, $r \in (0, 1]$,

$$c_0 r^{n-1} \le \sigma(B_r(x) \cap \partial\Omega) \le c_1 r^{n-1}. \tag{2.2.3}$$

We intend to extend the identity (2.1.23) to a broader family of functions u than were dealt with there (see (2.1.10)), when Ω is Ahlfors regular. To introduce our larger family, let us set

$$\begin{aligned} \mathfrak{L}^p = \{u \in C(\Omega, \mathcal{E}) : \mathcal{N}u \in L^p(\partial\Omega) \text{ and} & \\ \exists \text{ nontangential limit } u_b,\ \sigma\text{-a.e.}\}. & \end{aligned} \tag{2.2.4}$$

Here is the main result of this section.

Theorem 2.2.1 *Assume $\Omega \subset M$ is Ahlfors regular. If, for some $p > 1$,*

$$u \in \mathfrak{L}^p \quad and \quad \mathcal{D}u \in L^1(\Omega, \mathcal{E}), \tag{2.2.5}$$

then

$$u(x) = i \int\limits_{\partial\Omega} E(x,y)\sigma_{\mathcal{D}}(y,\nu(y))u(y)\, d\sigma(y) + \int\limits_{\Omega} E(x,y)\mathcal{D}u(y)\, dV(y), \tag{2.2.6}$$

for all $x \in \Omega$.

From §2.1, we have (2.2.6) for u satisfying (2.1.10), in particular for $u \in \mathrm{Lip}(\overline{\Omega})$, since such u has an extension to an element of $\mathrm{Lip}(M)$. More generally, results of §2.1 plus applications of a smooth cutoff give (2.2.6) whenever there exists an open $\widetilde{\Omega} \supset \overline{\Omega}$ such that $u \in C(\widetilde{\Omega})$ and $\mathcal{D}u \in L^1(\widetilde{\Omega}, \mathcal{E})$. Consequently, if u satisfies (2.2.5), then (2.2.6) holds with Ω replaced by a finite perimeter domain Ω_0 such that $\Omega_0 \subset \overline{\Omega}_0 \subset \Omega$, as long as $x \in \Omega_0$. We can take Ω_0 to be smoothly bounded. Replacing Ω by $\Omega \setminus \Omega_0$ (which does not contain x), we see that to prove Theorem 2.2.1 it suffices to prove the following.

Proposition 2.2.2 *Assume $\Omega \subset M$ is Ahlfors regular and u satisfies (2.2.5), for some $p > 1$. Then*

$$\int\limits_{\Omega} E(x,y)\mathcal{D}u(y)\, dV(y) = -i \int\limits_{\partial\Omega} E(x,y)\sigma_{\mathcal{D}}(y,\nu(y))u(y)\, d\sigma(y), \quad \forall\, x \notin \overline{\Omega}. \tag{2.2.7}$$

To prove Proposition 2.2.2, we use two lemmas that were established in §2.3 of [11].

Lemma 2.2.3 *If $\Omega \subset M$ is Ahlfors regular, then there exists $C < \infty$ such that, for all $\delta \in (0,1]$,*

$$\frac{1}{\delta} \int\limits_{\mathcal{O}_\delta} |v|\, dV \le C\|\mathcal{N}v\|_{L^1(\partial\Omega)}, \quad \forall\, v \in \mathfrak{L}^1, \tag{2.2.8}$$

where $\mathcal{N}v$ is the nontangential maximal function and

$$\mathcal{O}_\delta = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \le \delta\}. \tag{2.2.9}$$

Lemma 2.2.4 *If $\Omega \subset M$ is Ahlfors regular and $p \in (1,\infty)$, then the following holds.*

$$\begin{aligned} &\text{If } u \in \mathfrak{L}^p,\ \exists w \in \mathfrak{L}^1 \text{ such that } w_b = u_b \text{ and } \exists w_k \in \mathrm{Lip}(\overline{\Omega}) \\ &\text{such that } \|\mathcal{N}(w - w_k)\|_{L^1(\partial\Omega)} \to 0. \end{aligned} \tag{2.2.10}$$

We begin the proof of Proposition 2.2.2. Let $\Omega_s = \{x \in \Omega : \mathrm{dist}(x,\partial\Omega) \ge s\}$. Let $\varphi_\delta(x) = \mathrm{dist}(x, \partial\Omega_{\delta/2})$, and set

$$\begin{aligned} \chi_\delta(x) = 1 \quad &\text{on} \quad \Omega_\delta, \\ 2\delta^{-1}\varphi_\delta(x) \quad &\text{on} \quad \Omega \setminus (\Omega_\delta \cup \mathcal{O}_{\delta/2}), \\ 0 \quad &\text{on} \quad \mathcal{O}_{\delta/2}. \end{aligned} \tag{2.2.11}$$

Note that each χ_δ is Lipschitz and

$$d\chi_\delta(y) = -\frac{2}{\delta}\chi_{\widetilde{\mathcal{O}}_\delta}(y)\tilde{\nu}(y), \quad \widetilde{\mathcal{O}}_\delta = \mathcal{O}_\delta \setminus \mathcal{O}_{\delta/2}, \tag{2.2.12}$$

where

$$\tilde{\nu}(y) = d\psi_\Omega(y), \quad \psi_\Omega(y) = \operatorname{dist}(y, \partial\Omega). \tag{2.2.13}$$

The function ψ_Ω has Lipschitz constant 1, so $|\tilde{\nu}(y)| \le 1$ a.e. The Lipschitz character of χ_δ suffices to yield

$$\mathcal{D}(\chi_\delta u) = (\mathcal{D}_0 \chi_\delta) u + \chi_\delta \mathcal{D} u, \tag{2.2.14}$$

if u satisfies (2.2.5). We have $\chi_\delta u$ continuous and compactly supported on Ω (we write $\chi_\delta u \in C_0^0(\Omega)$), and $\mathcal{D}(\chi_\delta u) \in L^1(\Omega, \mathcal{E})$. It is elementary that

$$\int_\Omega E(x,y) \mathcal{D}(\chi_\delta u(y))\, dV(y) = 0, \quad \forall\, x \notin \overline{\Omega}. \tag{2.2.15}$$

Noting that

$$\mathcal{D}_0 \chi_\delta(y) = \frac{2i}{\delta} \sigma_{\mathcal{D}}(y, \tilde{\nu}(y)) \chi_{\widetilde{\mathcal{O}}_\delta}(y), \quad \widetilde{\mathcal{O}}_\delta = \mathcal{O}_\delta \setminus \mathcal{O}_{\delta/2}, \tag{2.2.16}$$

we have, for $x \notin \overline{\Omega}$,

$$\begin{aligned} &\int_\Omega E(x,y) \chi_\delta(y) \mathcal{D} u(y)\, dV(y) \\ &\qquad = -\frac{2i}{\delta} \int_{\widetilde{\mathcal{O}}_\delta} E(x,y) \sigma_{\mathcal{D}}(y, \tilde{\nu}(y)) u(y)\, dV(y). \end{aligned} \tag{2.2.17}$$

Given $\mathcal{D} u \in L^1(\Omega)$, it is clear that the left side of (2.2.17) converges to the left side of (2.2.7) as $\delta \to 0$, provided $x \notin \overline{\Omega}$. Therefore, so does the right side of (2.2.17). What we need to show is that

$$\begin{aligned} &\frac{2}{\delta} \int_{\widetilde{\mathcal{O}}_\delta} E(x,y) \sigma_{\mathcal{D}}(y, \tilde{\nu}(y)) u(y)\, dV(y) \\ &\qquad \to \int_{\partial\Omega} E(x,y) \sigma_{\mathcal{D}}(y, \nu(y)) u(y)\, d\sigma(y), \end{aligned} \tag{2.2.18}$$

as $\delta \searrow 0$, provided u satisfies (2.2.5) and $x \notin \overline{\Omega}$. From what has just been said, we know that (2.2.18) holds whenever (2.2.7) holds. In particular, (2.2.18) holds whenever $u \in \operatorname{Lip}(\overline{\Omega}, \mathcal{E})$.

To complete the proof of Proposition 2.2.2, we take w and w_k as in Lemma 2.2.4. Given (2.2.8), and the estimates for E, we have, uniformly in $\delta \in (0,1]$,

$$\begin{aligned} &\Bigl| \frac{2}{\delta} \int_{\widetilde{\mathcal{O}}_\delta} E(x,y) \sigma_{\mathcal{D}}(y, \tilde{\nu}(y)) [w_k(y) - w(y)]\, dV(y) \Bigr| \\ &\qquad \le C \|\mathcal{N}(w - w_k)\|_{L^1(\partial\Omega)}, \end{aligned} \tag{2.2.19}$$

and we also have

$$\begin{aligned} &\Bigl| \int_{\partial\Omega} E(x,y) \sigma_{\mathcal{D}}(y, \nu(y)) [w_k(y) - w(y)]\, d\sigma(y) \Bigr| \\ &\qquad \le C \|\mathcal{N}(w_k - w)\|_{L^1(\partial\Omega)}, \end{aligned} \tag{2.2.20}$$

as long as $x \notin \overline{\Omega}$. Thus, since (2.2.18) holds for w_k, we have

$$\begin{aligned}
\frac{2}{\delta} \int\limits_{\widetilde{\mathcal{O}}_\delta} & E(x,y)\sigma_{\mathcal{D}}(y,\tilde{\nu}(y))w(y)\,dV(y) \\
&\to \int\limits_{\partial\Omega} E(x,y)\sigma_{\mathcal{D}}(y,\nu(y))w(y)\,d\sigma(y) \qquad (2.2.21)\\
&= \int\limits_{\partial\Omega} E(x,y)\sigma_{\mathcal{D}}(y,\nu(y))u(y)\,d\sigma(y),
\end{aligned}$$

as $\delta \searrow 0$. Thus to obtain (2.2.18) for u satisfying (2.2.5), it suffices to show that

$$v \in \mathfrak{L}^1,\ v_b = 0 \Longrightarrow \frac{2}{\delta} \int\limits_{\widetilde{\mathcal{O}}_\delta} |v|\,dV \to 0, \quad \text{as} \ \ \delta \searrow 0. \tag{2.2.22}$$

To show this, we note that the condition (2.2.8) is equivalent to the apparently stronger condition

$$\frac{1}{\delta} \int\limits_{\mathcal{O}_\delta} |v|\,dV \le C\|\mathcal{N}_\delta v\|_{L^1(\partial\Omega)}, \quad \forall\, v \in \mathfrak{L}^1,\ \delta \in (0,1], \tag{2.2.23}$$

where

$$\mathcal{N}_\delta v(x) = \sup\{|v(y)| : y \in \Gamma_x,\ \mathrm{dist}(x,y) \le 2\delta\}, \tag{2.2.24}$$

as a simple cutoff argument shows. Thus, to prove (2.2.22), it suffices to show that

$$v \in \mathfrak{L}^1,\ v_b = 0 \Longrightarrow \|\mathcal{N}_\delta v\|_{L^1(\partial\Omega)} \to 0, \ \ \text{as} \ \ \delta \searrow 0. \tag{2.2.25}$$

Indeed, the hypotheses of (2.2.25) yield $\mathcal{N}_\delta v(x) \to 0$, σ-a.e., and furthermore $\mathcal{N}_\delta v \le \mathcal{N} v$ for each δ, so (2.2.25) follows from the dominated convergence theorem.

Proposition 2.2.2 is proven. Hence Theorem 2.2.1 is proven.

2.3 Cauchy integrals and reproducing formulas on UR domains

Here, we take $\Omega \subset M$ to be a UR domain. As stated in the Introduction, this means Ω is an Ahlfors regular domain and that $\partial\Omega$ satisfies the uniform recifiability condition of David and Semmes. In more detail, this uniform rectifiability condition is defined as follows (given that $\partial\Omega$ is compact). There exist $\varepsilon, L \in (0,\infty)$ such that, for each $x \in \partial\Omega$, $R \in (0,1]$, there is a Lipschitz map $\varphi : B^{n-1}_R \to M$ (where B^{n-1}_R is a ball of radius R in $\mathbb{R}^{n-1}$) with Lipschitz constant $\le L$, such that

$$\mathcal{H}^{n-1}(\partial\Omega \cap B_R(x) \cap \varphi(B^{n-1}_R)) \ge \varepsilon R^{n-1}. \tag{2.3.1}$$

As shown in [9], if $\Omega \subset \mathbb{R}^n$ is bounded and UR and $k \in C^m(\mathbb{R}^n \setminus 0)$ is odd and homogeneous of degree $-(n-1)$, with m large enough, then, given

$$f \in L^p(\partial\Omega), \quad 1 < p < \infty, \tag{2.3.2}$$

$$\begin{aligned}
Kf(x) &= \mathrm{PV} \int\limits_{\partial\Omega} k(x-y)f(y)\,d\sigma(y) \\
&:= \lim_{\varepsilon \to 0} \int\limits_{\partial\Omega \setminus B_\varepsilon(x)} k(x-y)f(y)\,d\sigma(y)
\end{aligned} \tag{2.3.3}$$

exists for σ-a.e. $x \in \partial\Omega$, and

$$K : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \tag{2.3.4}$$

is bounded. In [11], this result is extended to "variable coefficient" kernels, such as

$$Kf(x) = \mathrm{PV} \int_{\partial\Omega} k(x-y, y) f(y)\, d\sigma(y), \tag{2.3.5}$$

where $k(z, y)$ is continuous on $(\mathbb{R}^n \setminus 0) \times \partial\Omega$, odd in z, and homogeneous of degree $-(n-1)$ in z, and satisfies

$$|\partial_z^\alpha k(z, x)| \le C_\alpha |z|^{-(n-1)-|\alpha|}, \quad z \in \mathbb{R}^n \setminus 0,\ x \in \partial\Omega,\ |\alpha| \le m, \tag{2.3.6}$$

for m large enough. Again one has (2.3.4). Going further, [11] established for

$$\mathcal{K}f(x) = \int_{\partial\Omega} k(x-y, y) f(y)\, d\sigma(y), \quad x \in \Omega, \tag{2.3.7}$$

nontangential maximal function estimates

$$\|\mathcal{N}\mathcal{K}f\|_{L^p(\partial\Omega)} \le C_p \|f\|_{L^p(\partial\Omega)}, \quad 1 < p < \infty, \tag{2.3.8}$$

and nontangential convergence

$$\lim_{z \to x, z \in \Gamma_x} \mathcal{K}f(z) = \frac{1}{2i} \hat{k}(\nu(x), x) f(x) + Kf(x), \quad \text{for } \sigma\text{-a.e. } x \in \partial\Omega, \tag{2.3.9}$$

with K as in (2.3.5) and $\hat{k}(\xi, x)$ the Fourier transform of $k(z, x)$ with respect to z. These results are established in §§3.2–3.4 of [11].

These results apply to

$$Bf(x) = \mathrm{PV} \int_{\partial\Omega} E(x, y) f(y)\, d\sigma(y), \quad x \in \partial\Omega, \tag{2.3.10}$$

and

$$\mathcal{B}f(x) = \int_{\partial\Omega} E(x, y) f(y)\, d\sigma(y), \quad x \in \Omega, \tag{2.3.11}$$

when $\Omega \subset M$ is a UR domain and $E(x, y)$ is the integral kernel of $\mathcal{D}^{-1}$. In fact, as shown in Appendix A.1, $E \in C^r(M \times M \setminus \mathrm{diag})$ for each $r < 2$, and, near the diagonal, $E(x, y)$ is given in local coordinates by

$$E(x, y) = e_0(x-y, y) + e_1(x, y), \tag{2.3.12}$$

with $e_0(x-y, y)$ as in (A.1.38)–(A.1.39). In particular, the results above on (2.3.5)–(2.3.7) apply to the first term on the right side of (2.3.12). As for the remainder $e_1(x, y)$, we have the estimate (A.1.51), i.e.,

$$|e_1(x, y)| \le C_\varepsilon |x-y|^{-(n-2+\varepsilon)}, \tag{2.3.13}$$

for each $\varepsilon > 0$. Thus $e_1(x, y)$ is weakly singular. For

$$K_1 f(x) = \int_{\partial\Omega} e_1(x, y) f(y)\, d\sigma(y), \quad x \in \partial\Omega, \tag{2.3.14}$$

we have compactness of $K_1 : L^p(\partial\Omega) \to L^p(\partial\Omega)$, for each $p \in (1,\infty)$, as well as nontangential maximal function estimates on

$$\mathcal{K}_1 f(x) = \int\limits_{\partial\Omega} e_1(x,y) f(y)\, d\sigma(y), \quad x \in \Omega, \tag{2.3.15}$$

and nontangential a.e. convergence $\mathcal{K}_1 f(z) \to K_1 f(x)$. These results, which are more straightforward than those on (2.3.5)–(2.3.9), are also demonstrated in [11]. Thus we have for (2.3.10)–(2.3.11) that

$$\|Bf\|_{L^p(\partial\Omega)},\ \|\mathcal{N}\mathcal{B}f\|_{L^p(\partial\Omega)} \le C_p \|f\|_{L^p(\partial\Omega)}, \quad 1 < p < \infty, \tag{2.3.16}$$

and, given $f \in L^p(\partial\Omega),\ 1 < p < \infty$,

$$\lim_{z\to x, z\in\Gamma_x} \mathcal{B}f(z) = \frac{1}{2i}\sigma_E(x,\nu(x))f(x) + Bf(x), \quad \text{for } \sigma\text{-a.e. } x \in \partial\Omega. \tag{2.3.17}$$

Here, $\sigma_E(x,\xi) = \sigma_{\mathcal{D}}(x,\xi)^{-1}$, and

$$\sigma_{\mathcal{D}}(x,\xi) = i\sum_j \mathcal{A}_j(x)\xi_j, \quad \mathcal{A}_j(x) = \begin{pmatrix} 0 & -A_j(x)^* \\ A_j(x) & 0 \end{pmatrix}. \tag{2.3.18}$$

In light of these results, we are motivated, even independently of the calculations of §§2.1–2.2, to consider

$$\mathcal{C}_{\mathcal{D}} f(x) = i \int\limits_{\partial\Omega} E(x,y)\sigma_{\mathcal{D}}(y,\nu(y)) f(y)\, d\sigma(y), \quad x \in \Omega, \tag{2.3.19}$$

and

$$C_{\mathcal{D}} f(x) = \mathrm{PV}\, i \int\limits_{\partial\Omega} E(x,y)\sigma_{\mathcal{D}}(y,\nu(y)) f(y)\, d\sigma(y), \quad x \in \partial\Omega. \tag{2.3.20}$$

We have

$$C_{\mathcal{D}} : L^p(\partial\Omega,\mathcal{E}) \longrightarrow L^p(\partial\Omega,\mathcal{E}), \quad 1 < p < \infty, \tag{2.3.21}$$

a bounded operator, nontangential maximal function estimates,

$$\|\mathcal{N}\mathcal{C}_{\mathcal{D}} f\|_{L^p(\partial\Omega)} \le C_p \|f\|_{L^p(\partial\Omega)}, \quad 1 < p < \infty, \tag{2.3.22}$$

and boundary behavior

$$\lim_{z\to x, z\in\Gamma_x} \mathcal{C}_{\mathcal{D}} f(x) = \frac{1}{2} f(x) + C_{\mathcal{D}} f(x), \quad \sigma\text{-a.e. } x \in \partial\Omega. \tag{2.3.23}$$

We see that if $f \in L^p(\partial\Omega,\mathcal{E}),\ 1 < p < \infty$, then $u = \mathcal{C}_{\mathcal{D}} f$ satisfies the hypotheses of Theorem 2.2.1, with $\mathcal{D}u = 0$ on Ω, so

$$u = \mathcal{C}_{\mathcal{D}} f,\ f \in L^p(\partial\Omega,\mathcal{E}) \Longrightarrow u = \mathcal{C}_{\mathcal{D}}\big(u\big|_{\partial\Omega}\big). \tag{2.3.24}$$

In concert with (2.3.23), this implies that

$$\mathcal{P}_{\mathcal{D}} = \frac{1}{2} I + C_{\mathcal{D}} \Longrightarrow \mathcal{P}_{\mathcal{D}}^2 = \mathcal{P}_{\mathcal{D}}, \tag{2.3.25}$$

so $\mathcal{P}_{\mathcal{D}}$ is a bounded projection operator on $L^p(\partial\Omega, \mathcal{E})$ for each $p \in (1, \infty)$.

With respect to the splitting $\mathcal{E} = \mathcal{F}_0 \oplus \mathcal{F}_1$, we can write

$$\mathcal{P}_{\mathcal{D}} = \begin{pmatrix} \mathcal{P}_D & \mathcal{Q}_{01} \\ \mathcal{Q}_{10} & \mathcal{P}_{D^*} \end{pmatrix}, \tag{2.3.26}$$

where $\mathcal{P}_0 = \mathcal{P}_D$ and $\mathcal{P}_1 = \mathcal{P}_{D^*}$ satisfy

$$\mathcal{P}_j : L^p(\partial\Omega, \mathcal{F}_j) \longrightarrow L^p(\partial\Omega, \mathcal{F}_j), \tag{2.3.27}$$

while

$$\mathcal{Q}_{ab} : L^p(\partial\Omega, \mathcal{F}_b) \longrightarrow L^p(\partial\Omega, \mathcal{F}_a), \tag{2.3.28}$$

for $a \neq b \in \{0, 1\}$. From (2.3.18), we see that the 2×2 matrix $\sigma_{\mathcal{D}}(y, \nu(y))$ in (2.3.20) is completely off-diagonal. As for $E(x, y)$, we have from (A.1.40) that its principal part is completely off-diagonal. We get

$$\mathcal{Q}_{01},\ \mathcal{Q}_{10}\ \text{ compact on }\ L^p(\partial\Omega),\ \text{ for }\ p \in (1, \infty). \tag{2.3.29}$$

It also follows that $(\mathcal{P}_j)^2 = \mathcal{P}_j$, modulo compacts. In fact, we have a stronger conclusion.

Proposition 2.3.1 *For each $j \in \{0, 1\},\ p \in (1, \infty)$, $\mathcal{P}_j$ is a projection on $L^p(\partial\Omega, \mathcal{F}_j)$.*

Proof. Consider $\mathcal{P}_0 = \mathcal{P}_D$, and take $f_0 \in L^p(\partial\Omega, \mathcal{F}_0)$. We have

$$u = \mathcal{C}_{\mathcal{D}} \begin{pmatrix} f_0 \\ 0 \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in C(\Omega, \mathcal{F}_0) \oplus C(\Omega, \mathcal{F}_1), \tag{2.3.30}$$

each factor u_j belonging to $\mathfrak{L}^p$, defined in (2.2.4), satisfying

$$Du_0 = 0, \quad D^* u_1 = 0\ \text{ on }\ \Omega, \tag{2.3.31}$$

since $\mathcal{D}u = 0$ on Ω and a is supported on the complement of Ω. By definition,

$$\mathcal{P}_D f_0 = u_0\big|_{\partial\Omega}. \tag{2.3.32}$$

Now

$$v = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = \begin{pmatrix} u_0 \\ 0 \end{pmatrix} \in \mathfrak{L}^p,\ \text{ and }\ \mathcal{D}v = 0\ \text{ on }\ \Omega, \tag{2.3.33}$$

so Theorem 2.2.1 implies

$$v = \mathcal{C}_{\mathcal{D}} \begin{pmatrix} u_0|_{\partial\Omega} \\ 0 \end{pmatrix} = \mathcal{C}_{\mathcal{D}} \begin{pmatrix} \mathcal{P}_D f_0 \\ 0 \end{pmatrix}, \tag{2.3.34}$$

and then, parallel to (2.3.32),

$$\mathcal{P}_D(\mathcal{P}_D f_0) = v_0\big|_{\partial\Omega} = u_0\big|_{\partial\Omega} = \mathcal{P}_D f_0, \tag{2.3.35}$$

proving that $\mathcal{P}_D$ is a projection. The argument for $\mathcal{P}_{D^*}$ is similar. □

3 Hardy spaces and Calderón projectors

Let M and $\mathcal{D}$ be as in §1 and let $\Omega \subset M$ be a UR domain. For $p \in [1,\infty)$, we define Hardy spaces of functions on Ω:

$$\mathcal{H}^p(\Omega,\mathcal{D}) = \{u \in C(\Omega,\mathcal{E}) : \mathcal{D}u = 0,\ \mathcal{N}u \in L^p(\partial\Omega),\ \text{and } u \text{ has a nontangential trace } u_b \in L^p(\partial\Omega,\mathcal{E})\}. \tag{3.0.1}$$

We also denote u_b by $u|_{\partial\Omega}$. This is a Banach space, with norm

$$\|u\|_{\mathcal{H}^p} = \|\mathcal{N}u\|_{L^p(\partial\Omega)}. \tag{3.0.2}$$

Results of §2.3 give

$$\mathcal{B}, \mathcal{C}_{\mathcal{D}} : L^p(\partial\Omega,\mathcal{E}) \longrightarrow \mathcal{H}^p(\Omega,\mathcal{D}), \quad 1 < p < \infty, \tag{3.0.3}$$

bounded operators. Also, given $f \in L^p(\partial\Omega,\mathcal{E}),\ p \in (1,\infty)$,

$$\mathcal{C}_{\mathcal{D}} f\big|_{\partial\Omega} = \frac{1}{2} f + C_{\mathcal{D}} f := \mathcal{P}_{\mathcal{D}} f. \tag{3.0.4}$$

It was further shown that $\mathcal{P}_{\mathcal{D}}$ is a projection on $L^p(\partial\Omega,\mathcal{E})$. This is a consequence of Theorem 2.2.1, which gives, for $p \in (1,\infty)$,

$$u \in \mathcal{H}^p(\Omega,\mathcal{D}) \Longrightarrow u = \mathcal{C}_{\mathcal{D}}\big(u\big|_{\partial\Omega}\big). \tag{3.0.5}$$

In addition, we have a projection

$$\mathcal{P}_D : L^p(\partial\Omega,\mathcal{F}_0) \longrightarrow L^p(\partial\Omega,\mathcal{F}_0), \tag{3.0.6}$$

given by

$$\mathcal{P}_D f_0 = \mathcal{C}_D f_0\big|_{\partial\Omega}, \tag{3.0.7}$$

where

$$\mathcal{C}_D : L^p(\partial\Omega,\mathcal{F}_0) \longrightarrow \mathcal{H}^p(\Omega, D) \tag{3.0.8}$$

is defined, via the decomposition $\mathcal{E} = \mathcal{F}_0 \oplus \mathcal{F}_1$, by

$$\mathcal{C}_{\mathcal{D}} \begin{pmatrix} f_0 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathcal{C}_D f_0 \\ u_1 \end{pmatrix}, \tag{3.0.9}$$

and the target space in (3.0.8) is

$$\mathcal{H}^p(\Omega, D) = \{u \in C(\Omega,\mathcal{F}_0) : Du = 0,\ \mathcal{N}u \in L^p(\partial\Omega),\ \text{and } u \text{ has a nontangential limit } u_b \in L^p(\partial\Omega,\mathcal{F}_0)\}. \tag{3.0.10}$$

The proofs that $\mathcal{P}_{\mathcal{D}}$ and $\mathcal{P}_D$ are projections made essential use of the Cauchy-Pompieu formula, Theorem 2.2.1.

In §3.1 we show that the trace map takes $\mathcal{H}^p(\Omega,\mathcal{D})$ isomorphically onto the range of $\mathcal{P}_{\mathcal{D}}$, i.e., $\mathcal{P}_{\mathcal{D}}(L^p(\partial\Omega,\mathcal{E}))$, which we denote $\mathcal{H}^p(\partial\Omega,\mathcal{D})$. Similarly, the trace map takes $\mathcal{H}^p(\Omega, D)$ isomorphically onto $\mathcal{P}_D(L^p(\partial\Omega,\mathcal{F}_0)) = \mathcal{H}^p(\partial\Omega, D)$. In §3.2 we define the Calderón-Szegö projector $S_{\mathcal{D}}$, as the orthogonal projection of $L^2(\partial\Omega,\mathcal{E})$ onto $\mathcal{H}^2(\partial\Omega,\mathcal{D})$, analyze the difference $S_{\mathcal{D}} - \mathcal{P}_{\mathcal{D}}$, and use this analysis to extend $S_{\mathcal{D}}$ to $L^p(\partial\Omega,\mathcal{E})$ for a range of exponents p.

We can restate the definition of the Hardy space (3.0.1) as

$$\mathcal{H}^p(\Omega,\mathcal{D}) = \widetilde{\mathcal{H}}^p(\Omega,\mathcal{D}) \cap \mathfrak{L}^p, \tag{3.0.11}$$

with $\mathfrak{L}^p$ as in (2.0.6) and

$$\widetilde{\mathcal{H}}^p(\Omega,\mathcal{D}) = \{u \in C(\Omega,\mathcal{E}) : \mathcal{D}u = 0,\ \mathcal{N}u \in L^p(\partial\Omega)\}. \tag{3.0.12}$$

A natural question is whether

$$\widetilde{\mathcal{H}}^p(\Omega,\mathcal{D}) = \mathcal{H}^p(\Omega,\mathcal{D}), \tag{3.0.13}$$

i.e., whether all elements of $\widetilde{\mathcal{H}}^p(\Omega,\mathcal{D})$ have nontangential traces, σ-a.e. on $\partial\Omega$. Such results are known as Fatou theorems. In [21], we show that (3.0.13) holds for certain classes of domains Ω.

In §3.3 we examine $\mathcal{P}_D$ as providing a nonlocal boundary condition on D, producing an operator $A_{\mathcal{P}} = D$ on

$$\operatorname{Dom}(A_{\mathcal{P}}) = \{u \in H^1(\Omega,\mathcal{F}_0) : \mathcal{P}_D u|_{\partial\Omega} = 0\}. \tag{3.0.14}$$

We compare this with A_Π, satisfying another nonlocal boundary condition, and study the index of A_Π. This is done in the framework of Lipschitz domains. We indicate possible contact with the Atiyah-Patodi-Singer theory of nonlocal boundary problems.

3.1 The trace isomorphism on $\mathcal{H}^p(\Omega,\mathcal{D})$

The map $u \mapsto u|_{\partial\Omega} = u_b$ acting on $\mathcal{H}^p(\Omega,\mathcal{D})$ gives a bounded trace map

$$\tau : \mathcal{H}^p(\Omega,\mathcal{D}) \longrightarrow L^p(\partial\Omega,\mathcal{E}), \tag{3.1.1}$$

for $p \in (1,\infty)$. In fact, in view of (3.0.4)–(3.0.5),

$$\tau : \mathcal{H}^p(\Omega,\mathcal{D}) \longrightarrow \mathcal{H}^p(\partial\Omega,\mathcal{D}), \tag{3.1.2}$$

for $p \in (1,\infty)$, where we set

$$\mathcal{H}^p(\partial\Omega,\mathcal{D}) = \mathcal{P}_{\mathcal{D}}(L^p(\partial\Omega,\mathcal{E})), \tag{3.1.3}$$

the range of $\mathcal{P}_{\mathcal{D}}$ on $L^p(\partial\Omega,\mathcal{E})$. We have the following.

Proposition 3.1.1 *The trace map τ in (3.1.2) is an isomorphism, for $p \in (1,\infty)$.*

Proof. That τ is injective follows from (3.0.5), which gives

$$u = \mathcal{C}_{\mathcal{D}}(\tau u), \quad \forall\, u \in \mathcal{H}^p(\Omega,\mathcal{D}). \tag{3.1.4}$$

The fact that τ is surjective follows from (3.0.6), which gives

$$\tau\mathcal{C}_{\mathcal{D}} f = \mathcal{P}_{\mathcal{D}} f, \quad \forall\, f \in L^p(\partial\Omega,\mathcal{E}). \tag{3.1.5}$$

□

Similarly, we have

$$\tau : \mathcal{H}^p(\Omega, D) \longrightarrow \mathcal{H}^p(\partial\Omega, D), \tag{3.1.6}$$

with $\mathcal{H}^p(\Omega, D)$ as in (3.0.10) and

$$\mathcal{H}^p(\partial\Omega, D) = \mathcal{P}_D(L^p(\partial\Omega, \mathcal{F}_0)), \tag{3.1.7}$$

the range of $\mathcal{P}_D$ on $L^p(\partial\Omega, \mathcal{F}_0)$. We see that

$$u = \mathcal{C}_D(\tau u), \quad \forall\, u \in \mathcal{H}^p(\Omega, D), \tag{3.1.8}$$

and

$$\tau\mathcal{C}_D f_0 = \mathcal{P}_D f_0, \quad \forall\, f_0 \in L^p(\partial\Omega, \mathcal{F}_0), \tag{3.1.9}$$

so, parallel to Proposition 3.1.1, we have

$$\tau : \mathcal{H}^p(\Omega, D) \xrightarrow{\approx} \mathcal{H}^p(\partial\Omega, D). \tag{3.1.10}$$

3.2 Calderón projectors and the Calderón-Szegö projector

The projections $\mathcal{P}_{\mathcal{D}}$ and $\mathcal{P}_D$, defined respectively on $L^p(\partial\Omega, \mathcal{E})$ and on $L^p(\partial\Omega, \mathcal{F}_0)$, are singular integral operators of the type often called Calderón projectors. They play a role in Calderón's approach to the theory of elliptic boundary problems. They also arise in the Atiyah-Patodi-Singer index theory. Here, we introduce a variant, a Calderón type projector which, in the setting of holomorphic function theory is also called a Szegö projector.

We define the Calderón-Szegö projector

$$S_{\mathcal{D}} : L^2(\partial\Omega, \mathcal{E}) \longrightarrow \mathcal{H}^2(\partial\Omega, \mathcal{D}) \tag{3.2.1}$$

to be the orthogonal projection of $L^2(\partial\Omega, \mathcal{E})$ onto the closed linear subspace

$$\mathcal{H}^2(\partial\Omega, \mathcal{D}) = \mathcal{P}_{\mathcal{D}}(L^2(\partial\Omega, \mathcal{E})) = \operatorname{Ker}(I - \mathcal{P}_{\mathcal{D}}). \tag{3.2.2}$$

We aim to extend $S_{\mathcal{D}}$ to act on $L^p(\partial\Omega, \mathcal{E})$, at least for p close to 2. To this end, note from (3.2.2) that

$$S_{\mathcal{D}}\mathcal{P}_{\mathcal{D}} = \mathcal{P}_{\mathcal{D}}, \quad (I - \mathcal{P}_{\mathcal{D}})S_{\mathcal{D}} = 0, \quad \text{hence} \;\; S_{\mathcal{D}}(I - \mathcal{P}_{\mathcal{D}}^*) = 0. \tag{3.2.3}$$

Now we can set

$$A = \mathcal{P}_{\mathcal{D}} - \mathcal{P}_{\mathcal{D}}^* = C_{\mathcal{D}} - C_{\mathcal{D}}^* : L^p(\partial\Omega) \to L^p(\partial\Omega), \quad 1 < p < \infty, \tag{3.2.4}$$

and then (3.2.3) implies

$$S_{\mathcal{D}}(I + A) = \mathcal{P}_{\mathcal{D}} \;\text{ on }\; L^2(\partial\Omega, \mathcal{E}). \tag{3.2.5}$$

Since $A^* = -A$, $I+A$ is clearly invertible on $L^2(\partial\Omega, \mathcal{E})$. An extrapolation result of Sneiberg implies that there exist $p_0 < 2$ and $p_1 > 2$ such that

$$I + A : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \;\text{ is invertible for }\; p \in (p_0, p_1). \tag{3.2.6}$$

We have proved the following.

Proposition 3.2.1 *If $\Omega \subset M$ is a UR domain, then there exist $p_0 < 2$ and $p_1 > 2$ such that*

$$S_{\mathcal{D}} = \mathcal{P}_{\mathcal{D}}(I + A)^{-1} : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega), \quad p \in (p_0, p_1). \tag{3.2.7}$$

The following result shows how a compactness condition allows one to extend the range of p.

Proposition 3.2.2 *Suppose A, given by (3.2.4), is compact on $L^p(\partial\Omega)$ for some $p \in (1,\infty)$. Then (3.2.6) and (3.2.7) hold with $p_0 = 1$ and $p_1 = \infty$.*

Proof. First, an interpolation result of Krasnoselski (see [4], p. 203) implies that A is compact on $L^p(\partial\Omega)$ for all $p \in (1,\infty)$ if it is compact for one such p, and bounded for all such p. From this, we have

$$I - A, I + A : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \ \text{ Fredholm, of index 0, } \forall\, p \in (1,\infty). \tag{3.2.8}$$

To get invertibility, it suffices to get injectivity. We clearly have injectivity on $L^2(\partial\Omega)$, hence on $L^p(\partial\Omega)$ for $2 \le p < \infty$. This gives invertibility in (3.2.8) for $p \in [2,\infty)$. Taking adjoints gives invertibility for $p \in (1,2]$. $\square$

We want to give explicit conditions on Ω and D that guarantee that the operator A is compact. As a preliminary, we write A as a singular integral operator. We have

$$C^*_{\mathcal{D}} f(x) = \mathrm{PV}(-i) \int_{\partial\Omega} \bigl[E(y,x)\sigma_{\mathcal{D}}(x,\nu(x))\bigr]^* f(y)\, d\sigma(y). \tag{3.2.9}$$

Since $\mathcal{D}$ is a zero-order perturbation of a first order elliptic, self adjoint operator, it follows from the analysis of §A.1 that

$$E(y,x)^* = E(x,y) + R_0(x,y), \tag{3.2.10}$$

where R_0 has a weak singularity on $x = y \in \partial\Omega$, hence is the integral kernel of a compact operator on $L^p(\partial\Omega)$, for $p \in (1,\infty)$. Hence

$$Af(x) = \mathrm{PV}\, i \int_{\partial\Omega} \bigl\{E(x,y)\sigma_{\mathcal{D}}(y,\nu(y)) + \sigma_{\mathcal{D}}(x,\nu(x))E(x,y)\bigr\} f(y)\, d\sigma(y) + \mathcal{R}_1 f(x), \tag{3.2.11}$$

where $\mathcal{R}_1$ is compact on $L^p(\partial\Omega)$ for $p \in (1,\infty)$.

We now formulate our condition for the next compactness result. First, we assume D is of Dirac type, i.e.,

$$\sigma_D(x,\xi)^*\sigma_D(x,\xi) = \langle \xi,\xi\rangle_x, \ \text{ hence } \ \sigma_{\mathcal{D}}(x,\xi)^2 = \langle \xi,\xi\rangle_x, \tag{3.2.12}$$

where $\langle\ ,\ \rangle_x$ is the inner product on $T^*_x M$ associatd to the Riemannian metric. Next, we assume Ω is a regular SKT (Semmes-Kenig-Toro) domain. This class of domains was introduced and studied in [26] and [13]–[15], where they were called cord-arc domains with vanishing constant. The label "regular SKT domain" was proposed in [11], where it was shown that this class of domains can be characterized as follows:

$$\begin{aligned} &\Omega \text{ is an Ahlfors regular domain,} \\ &\Omega \text{ satisfies a two-sided local John condition, and} \\ &\text{the unit normal } \nu \text{ belongs to } \mathrm{vmo}(\partial\Omega). \end{aligned} \tag{3.2.13}$$

In such a case, Ω is a UR domain and also an NTA domain. See the references cited above for more details. Here is our result.

Proposition 3.2.3 *Assume D is of Dirac type and $\Omega \subset M$ is a regular SKT domain. Then $A = C_{\mathcal{D}} - C_{\mathcal{D}}^*$ is compact on $L^p(\partial\Omega)$, for all $p \in (1, \infty)$.*

In order to expose the main lines of the argument, we first treat the case where $\Omega \subset \mathbb{R}^n$ is a bounded, regular SKT domain, D has constant coefficients, and $\mathcal{D}$ is given by (1.0.10) with $a \equiv 0$, so

$$\mathcal{D}u(x) = \sum_j A_j \partial_j u(x), \quad A_j^* = -A_j. \tag{3.2.14}$$

In such a case, we have $R_0 = 0$ and $\mathcal{R}_1 = 0$ in (3.2.10)–(3.2.11), and

$$Af(x) = \mathrm{PV}\, i \int_{\partial\Omega} \bigl\{E(x-y)\sigma_{\mathcal{D}}(y, \nu(y)) + \sigma_{\mathcal{D}}(x, \nu(x))E(x-y)\bigr\} f(y)\, d\sigma(y), \tag{3.2.15}$$

with

$$E(z) = \mathcal{D}G(z), \quad G(z) = C_n |z|^{-(n-2)}, \tag{3.2.16}$$

for $n \geq 3$, $G(z) = C_2 \log|z|$ if $n = 2$, since $\mathcal{D}^2 = -\Delta$. Hence

$$Af(x) = \mathrm{PV}\, i \int_{\partial\Omega} \Gamma(x, y) f(y)\, d\sigma(y), \tag{3.2.17}$$

with

$$\Gamma(x, y) = \sum_{j,k} \bigl\{A_j A_k \nu_k(y) + A_k A_j \nu_k(x)\bigr\} \partial_j G(x-y). \tag{3.2.18}$$

Now the Dirac type condition (3.2.12) translates to the anticommutator condition

$$A_j A_k + A_k A_j = -2\delta_{jk}. \tag{3.2.19}$$

Hence

$$\begin{aligned} \Gamma(x, y) = &\sum_{j \neq k} A_j A_k [\nu_k(y) - \nu_k(x)] \partial_j G(x-y) \\ &- \sum_k [\nu_k(y) + \nu_k(x)] \partial_k G(x-y) \\ = &\ \Gamma_1(x, y) + \Gamma_2(x, y). \end{aligned} \tag{3.2.20}$$

It follows that A_1, given by

$$A_1 f(x) = \mathrm{PV}\, i \int_{\partial\Omega} \Gamma_1(x, y) f(y)\, d\sigma(y), \tag{3.2.21}$$

is a sum of commutators of bounded singular integral integral operators with multiplication by ν_k. As long as $\nu \in \mathrm{vmo}(\partial\Omega)$, these commutators are all compact on $L^p(\partial\Omega)$, for $p \in (1, \infty)$; cf. Theorem 2.19 of [11]. Meanwhile, A_2, given by

$$A_2 f(x) = \mathrm{PV}\, i \int_{\partial\Omega} \Gamma_2(x, y) f(y)\, d\sigma(y), \tag{3.2.22}$$

is a sum of two terms, namely

$$\begin{aligned} A_{21}f(x) &= C_n' \,\mathrm{PV} \int\limits_{\partial\Omega} \langle x-y, \nu(y)\rangle |x-y|^{-n} f(y)\, d\sigma(y), \\ A_{22}f(x) &= C_n' \,\mathrm{PV} \int\limits_{\partial\Omega} \langle x-y, \nu(x)\rangle |x-y|^{-n} f(y)\, d\sigma(y). \end{aligned} \tag{3.2.23}$$

Compactness of A_{21} on $L^p(\partial\Omega)$ for all $p \in (1,\infty)$, provided Ω is a regular SKT domain, is a special case of the main result in §4.5 of [11], and compactness of A_{22} on $L^p(\partial\Omega)$ for all such p follows by duality. This takes care of the Euclidean case of Proposition 3.2.3.

The extension of Proposition 3.2.3 to the manifold case is mainly technical, making use of results on $E(x,y)$ given in Appendix A.1.

3.3 Nonlocal boundary conditions on D

Let $\Omega \subset M$ be open, $D : H^{1,p}(M,\mathcal{F}_0) \to L^p(M,\mathcal{F}_1)$ a first order elliptic differential operator, as presented in the introduction, in particular, satisfying UCP. Construct the Calderon projector $\mathcal{P}_D$ on $L^p(\partial\Omega,\mathcal{F}_0)$ as done previously.

In this subsection, we will assume that Ω is a *Lipschitz domain.* Extending this analysis to a more general class of domains is an intriguing problem.

We set $\mathcal{P} = \mathcal{P}_D$ and let Π be another projection on $L^2(\partial\Omega,\mathcal{F}_0)$. We assume boundedness on the L^2-Sobolev space $H^{1/2}$:

$$\mathcal{P}, \Pi : H^{1/2}(\partial\Omega,\mathcal{F}_0) \longrightarrow H^{1/2}(\partial\Omega,\mathcal{F}_0). \tag{3.3.1}$$

We know this holds for $\mathcal{P} = \mathcal{P}_D$, in case Ω is a UR domain, since then $\mathcal{P} : H^1(\partial\Omega) \to H^1(\partial\Omega)$, and we can interpolate.

Let us define unbounded operators $A_{\mathcal{P}}$ and A_Π on $L^2(\partial\Omega,\mathcal{F}_0)$ by

$$\begin{aligned} \mathrm{Dom}(A_{\mathcal{P}}) &= \{u \in H^1(\Omega,\mathcal{F}_0) : \mathcal{P}u|_{\partial\Omega} = 0\}, \\ \mathrm{Dom}(A_\Pi) &= \{u \in H^1(\Omega,\mathcal{F}_0) : \Pi u|_{\partial\Omega} = 0\}. \end{aligned} \tag{3.3.2}$$

We set $A_{\mathcal{P}}u = Du$ and $A_\Pi u = Du$, on their respective domains.

Proposition 3.3.1 *If Ω is a Lipschitz domain,*

$$A_{\mathcal{P}} : \mathrm{Dom}(A_{\mathcal{P}}) \longrightarrow L^2(\Omega,\mathcal{F}_1) \quad \textit{is bijective.} \tag{3.3.3}$$

Proof. Given $u \in \mathrm{Ker}\, A_{\mathcal{P}}$, we have

$$Du = 0 \ \text{ on } \ \Omega, \quad \text{and} \quad u\big|_{\partial\Omega} = 0, \tag{3.3.4}$$

the latter result because $Du = 0$ on $\Omega \Rightarrow \mathcal{P}u|_{\partial\Omega} = u|_{\partial\Omega}$. Then UCP $\Rightarrow u = 0$, so $A_{\mathcal{P}}$ is injective.

Next, given $f \in L^2(\Omega,\mathcal{F}_1)$, we can find $u_0 \in H^1(\Omega,\mathcal{F}_0)$ such that $Du_0 = f$ on Ω (via the construction of $\mathcal{D}^{-1}$). Then (for Ω Lipschitz) one has

$$u_0\big|_{\partial\Omega} = \psi \in H^{1/2}(\partial\Omega,\mathcal{F}_0). \tag{3.3.5}$$

We need to find $u_1 \in H^1(\Omega, \mathcal{F}_0)$ such that

$$Du_1 = 0 \text{ on } \Omega, \quad \mathcal{P}u_1\big|_{\partial\Omega} = \mathcal{P}\psi, \tag{3.3.6}$$

so $u_0 - u_1 \in \mathcal{D}(A_{\mathcal{P}})$ and $A_{\mathcal{P}}(u_0 - u_1) = f$. In fact, we can take

$$u_1 = \mathcal{C}_D\psi, \tag{3.3.7}$$

and satisfy (3.3.6). We claim that, if Ω is a Lipschitz domain,

$$\mathcal{C}_D : H^{1/2}(\partial\Omega, \mathcal{F}_0) \longrightarrow H^1(\Omega, \mathcal{F}_0). \tag{3.3.8}$$

See Proposition 3.3.6 below. Given this, we have Proposition 3.3.1. □

Proposition 3.3.2 *If Ω is a Lipschitz domain,*

$$\operatorname{Ker} A_\Pi \approx \operatorname{Ker} \Pi\big|_{\mathcal{R}(\mathcal{P})}. \tag{3.3.9}$$

with $\mathcal{R}(\mathcal{P})$ denoting the range of $\mathcal{P}$ as a projection on $H^{1/2}(\partial\Omega, \mathcal{F}_0)$.

Proof. Given $u \in \operatorname{Ker} A_\Pi$, we have $u \in H^1(\Omega, \mathcal{F}_0)$, and the defining condition is

$$Du = 0 \text{ on } \Omega, \quad \Pi\big(u\big|_{\partial\Omega}\big) = 0. \tag{3.3.10}$$

Now the first condition in (3.3.10) implies $u|_{\partial\Omega} \in \mathcal{R}(\mathcal{P})$, and in fact $u = \mathcal{C}_D f$, for a uniquely defined $f \in \mathcal{R}(\mathcal{P})$, so

$$\operatorname{Ker} A_\Pi \approx \{f \in \mathcal{R}(\mathcal{P}) \subset H^{1/2}(\partial\Omega, \mathcal{F}_0) : \Pi f = 0\}, \tag{3.3.11}$$

giving (3.3.9). □

Proposition 3.3.3 *If Ω is a Lipschitz domain,*

$$L^2(\Omega, \mathcal{F}_1)/\mathcal{R}(A_\Pi) \approx \mathcal{R}(\Pi)/\mathcal{R}(\Pi\mathcal{P}), \tag{3.3.12}$$

where $\mathcal{R}(\Pi)$ denotes the range of Π as a projection on $H^{1/2}(\partial\Omega, \mathcal{F}_0)$, and $\mathcal{R}(\Pi\mathcal{P})$ denotes the image of $H^{1/2}(\partial\Omega, \mathcal{F}_0)$ under $\Pi\mathcal{P}$.

Proof. As in the proof of Proposition 3.3.1, given $f \in L^2(\Omega, \mathcal{F}_1)$, we can take $u_0 \in H^1(\Omega, \mathcal{F}_0)$ such that $Du_0 = f$ on Ω. Such u_0 is determined uniquely mod $\{u_1 \in H^1(\Omega, \mathcal{F}_0) : Du_1 = 0 \text{ on } \Omega\}$. We then form $u_0|_{\partial\Omega} = \psi \in H^{1/2}(\partial\Omega, \mathcal{F}_0)$, as in (3.3.5), and ψ is determined uniquely, mod $\mathcal{R}(\mathcal{P})$. Note that $u_0 \in H^1(\Omega, \mathcal{F}_0)$ can be arbitrary, hence $\psi \in H^{1/2}(\partial\Omega, \mathcal{F}_0)$ can be arbitrary, due to the surjectivity of the trace map. We hence have a linear isomorphism

$$L^2(\Omega, \mathcal{F}_1) \xrightarrow{\approx} H^{1/2}(\partial\Omega, \mathcal{F}_0)/\mathcal{R}(\mathcal{P}), \quad f \mapsto \psi \ (\text{mod } \mathcal{R}(\mathcal{P})). \tag{3.3.13}$$

Following this with Π then yields

$$f \mapsto \Pi\psi \ (\text{mod } \mathcal{R}(\Pi\mathcal{P})), \tag{3.3.14}$$

and hence a surjective map

$$\vartheta : L^2(\Omega, \mathcal{F}_1) \longrightarrow \mathcal{R}(\Pi)/\mathcal{R}(\Pi\mathcal{P}). \tag{3.3.15}$$

Now, to see if $f \in L^2(\Omega, \mathcal{F}_1)$ belongs to $\mathcal{R}(A_\Pi)$, having u_0 as above, we seek $u_1 \in H^1(\Omega, \mathcal{F}_0)$ such that

$$Du_1 = 0 \ \text{ on } \ \Omega, \quad \Pi u_1\big|_{\partial\Omega} = \Pi\psi, \tag{3.3.16}$$

so $u_0 - u_1 \in \mathcal{D}(A_\Pi)$. This is equivalent to seeking

$$u_1 = \mathcal{C}_D\varphi, \quad \varphi \in \mathcal{R}(\mathcal{P}), \quad \Pi\varphi = \Pi\psi. \tag{3.3.17}$$

It follows that, in (3.3.15), $\operatorname{Ker}\vartheta = \mathcal{R}(A_\Pi)$, so we have the isomorphism (3.3.12). □

From here, we have the following generalization of Proposition 2 of [33].

Corollary 3.3.4 *Assume Ω is a Lipschitz domain, and consider the maps*

$$A_\Pi : \operatorname{Dom}(A_\Pi) \longrightarrow L^2(\Omega, \mathcal{F}_1) \tag{3.3.18}$$

and

$$\Pi\big|_{\mathcal{R}(\mathcal{P})} : \mathcal{R}(\mathcal{P}) \longrightarrow \mathcal{R}(\Pi). \tag{3.3.19}$$

If either (3.3.18) *or* (3.3.19) *is semi-Fredholm, so is the other, and the two maps have the same index.*

Regarding the applicability of Corollary 3.3.4, note the following.

Proposition 3.3.5 *If $\mathcal{P}$ and Π are bounded projections on $H^{1/2}(\partial\Omega, \mathcal{F}_0)$ and*

$$\mathcal{P} - \Pi \ \text{ is compact on } \ H^{1/2}(\partial\Omega, \mathcal{F}_0), \tag{3.3.20}$$

then the map (3.3.19) *is Fredholm. Hence, in the setting of Corollary 3.3.4,*

$$\operatorname{Index} A_\Pi = \operatorname{Index} \Pi\big|_{\mathcal{R}(\mathcal{P})} : \mathcal{R}(\mathcal{P}) \longrightarrow \mathcal{R}(\Pi). \tag{3.3.21}$$

It remains to establish (3.3.8) when Ω is a Lipschitz domain. This result (which is well known for smoothly bounded Ω) is the case $s = 1/2$ of the following.

Proposition 3.3.6 *Let Ω be a Lipschitz domain. Then*

$$\mathcal{C}_D : H^s(\partial\Omega, \mathcal{F}_0) \longrightarrow H^{s+1/2}(\Omega, \mathcal{F}_0), \quad \forall\, s \in [0, 1]. \tag{3.3.22}$$

Proof. It suffices to get (3.3.22) for $s = 0$ and $s = 1$. The rest follows by interpolation. The facts that

$$\begin{aligned} \mathcal{C}_D &: L^2(\partial\Omega) \longrightarrow H^{1/2}(\Omega), \\ \mathcal{C}_D &: H^1(\partial\Omega) \longrightarrow H^{3/2}(\Omega), \end{aligned} \tag{3.3.23}$$

follow from Theorem 3.1 of [20], applied to $u = \mathcal{C}_D f$, which solves $Lu = 0$, with $L = D^*D$, a second order, strongly elliptic, formally self adjoint system. In fact, the first part of (3.3.23) follows from Theorem 1.1 of [20], plus some elementary auxiliary estimates (carried out in the proof of Theorem 3.1). □

Remark. In [33], the emphasis was on Π arising from the Atiyah-Patodi-Singer boundary condition.

4 Toeplitz operators on UR domains

We continue to take M and $\mathcal{D}$ as in §1, and let $\Omega \subset M$ be a UR domain. Recall the projection $\mathcal{P}_{\mathcal{D}}$ on $L^p(\partial\Omega, \mathcal{E})$, for $p \in (1, \infty)$, given by

$$\mathcal{P}_{\mathcal{D}} = \frac{1}{2} I + C_{\mathcal{D}}, \tag{4.0.1}$$

with

$$C_{\mathcal{D}} f(x) = \mathrm{PV}\, i \int_{\partial\Omega} E(x, y) \sigma_{\mathcal{D}}(y, \nu(y)) f(y)\, d\sigma(y), \quad x \in \partial\Omega, \tag{4.0.2}$$

where, for $x \neq y \in \partial\Omega$, $E(x, y) \in \mathrm{Hom}(\mathcal{E}_y, \mathcal{E}_x)$. Given $\ell \in \mathbb{N}$, we extend $\mathcal{P}_{\mathcal{D}}$ to a projection

$$\mathcal{P}_{\mathcal{D}} : L^p(\partial\Omega, \mathcal{E} \otimes \mathbb{C}^\ell) \longrightarrow L^p(\partial\Omega, \mathcal{E} \otimes \mathbb{C}^\ell), \tag{4.0.3}$$

in the standard fashion, i.e., componentwise. Then, if

$$\Phi \in L^\infty(\partial\Omega, \mathrm{End}\, \mathbb{C}^\ell), \tag{4.0.4}$$

we define the "Toeplitz operator"

$$\mathfrak{T}_\Phi : L^p(\partial\Omega, \mathcal{E} \otimes \mathbb{C}^\ell) \longrightarrow L^p(\partial\Omega, \mathcal{E} \otimes \mathbb{C}^\ell) \tag{4.0.5}$$

by

$$\mathfrak{T}_\Phi f = \mathcal{P}_{\mathcal{D}} \Phi \mathcal{P}_{\mathcal{D}} f + (I - \mathcal{P}_{\mathcal{D}}) f. \tag{4.0.6}$$

The structure described above implies the commutativity

$$\Phi(x) E(x, y) = E(x, y) \Phi(x), \quad \Phi(x) \sigma_{\mathcal{D}}(y, \nu(y)) = \sigma_{\mathcal{D}}(y, \nu(y)) \Phi(x), \tag{4.0.7}$$

which will be useful in the analysis.

We also define

$$T_\Phi : L^p(\partial\Omega, \mathcal{F}_0 \otimes \mathbb{C}^\ell) \longrightarrow L^p(\partial\Omega, \mathcal{F}_0 \otimes \mathbb{C}^\ell) \tag{4.0.8}$$

by

$$T_\Phi f = \mathcal{P}_D \Phi \mathcal{P}_D f + (I - \mathcal{P}_D) f, \tag{4.0.9}$$

where $\mathcal{P}_D$ arises in

$$\mathcal{P}_{\mathcal{D}} = \begin{pmatrix} \mathcal{P}_D & \mathcal{Q}_{01} \\ \mathcal{Q}_{10} & \mathcal{P}_{D^*} \end{pmatrix}, \tag{4.0.10}$$

and we have also seen that $\mathcal{P}_D$ is a projection. As noted before, the operators $\mathcal{Q}_{01}$ and $\mathcal{Q}_{10}$ are compact. There is a similar definition of T^1_Φ, and, for Φ as in (4.0.4),

$$\mathfrak{T}_\Phi - \begin{pmatrix} T_\Phi & \\ & T^1_\Phi \end{pmatrix} \text{ is compact on } L^p(\partial\Omega, \mathcal{E} \otimes \mathbb{C}^\ell), \tag{4.0.11}$$

for each $p \in (1, \infty)$, thanks to the compactness of $\mathcal{Q}_{01}$ and $\mathcal{Q}_{10}$.

Our main goal here is to investigate Fredholm properties of $\mathfrak{T}_\Phi$ and T_Φ, for subclasses of functions Φ which, together with Φ^{-1}, satisfy (4.0.4). In §4.1, we take

$$\Phi \in C(\partial\Omega, G\ell(\ell, \mathbb{C})), \tag{4.0.12}$$

and show that these operators are Fredholm on $L^p(\partial\Omega)$. In §4.2, we take

$$\Phi, \Phi^{-1} \in L^\infty \cap \mathrm{vmo}(\partial\Omega, \mathrm{End}\, \mathbb{C}^\ell), \tag{4.0.13}$$

and demonstrate such Fredholm properties. We study the index $\iota(\Phi)$ of T_Φ in cases (4.0.12) and (4.0.13). We show it is independent of $p \in (1, \infty)$. In case (4.0.12), $\iota(\Phi)$ depends only on the homotopy class of $\Phi : \partial\Omega \to G\ell(\ell, \mathbb{C})$. The case (4.0.13), which involves discontinuous Φ, requires a more delicate analysis. For this, we extend results of Brezis-Nirenberg [7], involving a generalized notion of degree.

In §4.3 we examine Toeplitz operators on the L^p-Sobolev spaces $L^p_1(\partial\Omega, \mathcal{E} \otimes \mathbb{C}^\ell)$, first for $\Phi \in C^1(\partial\Omega, G\ell(\ell, \mathbb{C}))$, then for $\Phi \in L^q_1(\partial\Omega, G\ell(\ell, \mathbb{C}))$, provided $q > n - 1$ and $q \geq p$.

In §4.4 we consider Toeplitz operators of the form

$$\mathcal{T}_\Phi f = S_\mathcal{D} \Phi S_\mathcal{D} f + (I - S_\mathcal{D}) f, \tag{4.0.14}$$

i.e., like (4.0.6), but with $\mathcal{P}_\mathcal{D}$ replaced by the Szegö projector $S_\mathcal{D}$.

Section 4.5 considers twisted Toeplitz operators, replacing $\mathcal{E} \otimes \mathbb{C}^\ell$ by $\mathcal{E} \otimes \mathcal{C}$, where $\mathcal{C}$ is a vector bundle over M. Section 4.6 investigates localizations of Toeplitz operators. Results of these sections are applied in §4.7 to establish an important cobordism invariance result, which can be used to show that a Toeplitz operator on a rough UR domain has the same index as one on a smoothly bounded domain.

Section 4.8 applies these results to the computation of the index of some examples of Toeplitz operators on rough UR domains.

Section 4.9 considers "Toeplitz operators" associated to the orthogonal projection of $L^2(\Omega, \mathcal{E})$ onto $\mathfrak{H}^2(\Omega, D)$, the subspace of elements annihilated by D.

For simplicity, we will use the notation $L^p(\partial\Omega)$, in place of $L^p(\partial\Omega, \mathcal{E} \otimes \mathbb{C}^\ell)$, etc., when the context is clear.

4.1 Toeplitz operators with continuous coefficients

In this subsection, we study $\mathfrak{T}_\Phi$ and T_Φ for

$$\Phi \in C(\partial\Omega, \mathrm{End}\, \mathbb{C}^\ell). \tag{4.1.1}$$

We want to show that $\mathfrak{T}_\Phi$ is Fredholm on $L^p(\partial\Omega)$ provided $\Phi(x)$ is invertible for each $x \in \partial\Omega$. A key to this is to demonstrate compactness of the commutator

$$[\mathcal{P}_\mathcal{D}, \Phi] = \mathcal{P}_\mathcal{D}\Phi - \Phi\mathcal{P}_\mathcal{D} = [C_\mathcal{D}, \Phi], \tag{4.1.2}$$

with $C_\mathcal{D}$ given by (4.0.2). Now (4.0.7) implies

$$[\mathcal{P}_\mathcal{D}, \Phi] f(x) = \mathrm{PV} \int_{\partial\Omega} E(x, y) \{\Phi(y) - \Phi(x)\} \sigma_\mathcal{D}(y, \nu(y)) f(y)\, d\sigma(y) = \mathcal{K} g(x), \tag{4.1.3}$$

with

$$\begin{aligned} g(y) &= \sigma_\mathcal{D}(y, \nu(y)) f(y), \\ \mathcal{K} g(x) &= \mathrm{PV} \int_{\partial\Omega} E(x, y) \{\Phi(y) - \Phi(x)\} g(y)\, d\sigma(y). \end{aligned} \tag{4.1.4}$$

Lemma 4.1.1 *If* $\Phi \in C(\partial\Omega, \mathrm{End}\,\mathbb{C}^\ell)$*, then*

$$[\mathcal{P}_{\mathcal{D}}, \Phi] : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \quad \textit{is compact}, \quad \forall\, p \in (1,\infty). \tag{4.1.5}$$

Proof. By a limiting argument, it suffices to prove (4.1.5) when Φ is Hölder continuous, of exponent $r \in (0,1)$, i.e., $\Phi \in C^r(\partial\Omega, \mathrm{End}\,\mathbb{C}^\ell)$. Then we have (4.1.3) with

$$\mathcal{K}g(x) = \int_{\partial\Omega} k(x,y)g(y)\,d\sigma(y), \quad |k(x,y)| \le C d(x,y)^{-(n-1)+r}. \tag{4.1.6}$$

Since Ω is Ahlfors regular, the compactness of $\mathcal{K}$ on $L^p(\partial\Omega)$ for $p \in (1,\infty)$ follows from Lemma 2.20 (or Proposition 5.1) of [11]. ☐

To proceed, assume

$$\Phi, \Psi \in C(\partial\Omega, \mathrm{End}\,\mathbb{C}^\ell). \tag{4.1.7}$$

Then

$$\begin{aligned} \mathfrak{T}_\Psi \mathfrak{T}_\Phi &= \big(\mathcal{P}_{\mathcal{D}}\Psi\mathcal{P}_{\mathcal{D}} + (I - \mathcal{P}_{\mathcal{D}})\big)\big(\mathcal{P}_{\mathcal{D}}\Phi\mathcal{P}_{\mathcal{D}} + (I - \mathcal{P}_{\mathcal{D}})\big) \\ &= \mathcal{P}_{\mathcal{D}}\Psi\mathcal{P}_{\mathcal{D}}\Phi\mathcal{P}_{\mathcal{D}} + (I - \mathcal{P}_{\mathcal{D}}) \\ &= \mathfrak{T}_{\Psi\Phi} + \mathcal{P}_{\mathcal{D}}\Psi[\mathcal{P}_{\mathcal{D}}, \Phi]\mathcal{P}_{\mathcal{D}}. \end{aligned} \tag{4.1.8}$$

(We could also write the last term as $-\mathcal{P}_{\mathcal{D}}[\mathcal{P}_{\mathcal{D}}, \Psi]\Phi\mathcal{P}_{\mathcal{D}}$.) Consequently,

$$\mathfrak{T}_\Psi \mathfrak{T}_\Phi - \mathfrak{T}_{\Psi\Phi} \ \text{ is compact on } \ L^p(\partial\Omega), \quad \forall\, p \in (1,\infty). \tag{4.1.9}$$

Similarly, we have compactness of $\mathfrak{T}_\Phi \mathfrak{T}_\Psi - \mathfrak{T}_{\Phi\Psi}$. This yields the following.

Proposition 4.1.2 *Let* $\Omega \subset M$ *be a UR domain, and suppose*

$$\Phi : \partial\Omega \longrightarrow G\ell(\ell, \mathbb{C}) \tag{4.1.10}$$

is continuous. Then

$$\mathfrak{T}_{\Phi^{-1}}\mathfrak{T}_\Phi - I \ \textit{ and } \ \mathfrak{T}_\Phi\mathfrak{T}_{\Phi^{-1}} - I \ \textit{ are compact on } \ L^p(\partial\Omega), \tag{4.1.11}$$

for all $p \in (1,\infty)$*, so*

$$\mathfrak{T}_\Phi : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \ \textit{ is Fredholm}, \quad \forall\, p \in (1,\infty). \tag{4.1.12}$$

Similarly we have

$$T_\Phi T_\Psi - T_{\Phi\Psi} \ \text{ compact on } \ L^p(\partial\Omega), \tag{4.1.13}$$

which yields the following.

Corollary 4.1.3 *In the setting of Proposition 4.1.2,*

$$T_{\Phi^{-1}}T_\Phi - I \ \textit{ and } \ T_\Phi T_{\Phi^{-1}} - I \ \textit{ are compact on } \ L^p(\partial\Omega), \tag{4.1.14}$$

for all $p \in (1,\infty)$*, so*

$$T_\Phi : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \ \textit{ is Fredholm}, \ \forall\, p \in (1,\infty). \tag{4.1.15}$$

We set

$$\iota_p(\Phi) = \operatorname{Index} T_\Phi \quad \text{on} \quad L^p(\partial\Omega). \tag{4.1.16}$$

From (4.1.9) we have

$$\Phi, \Psi \in C(\partial\Omega, G\ell(\ell, \mathbb{C})) \Longrightarrow \iota_p(\Phi\Psi) = \iota_p(\Phi) + \iota_p(\Psi). \tag{4.1.17}$$

Also

$$\Phi_0, \Phi_1 \in C(\partial\Omega, G\ell(\ell, \mathbb{C})) \ \text{ homotopic } \ \Longrightarrow \iota_p(\Phi_0) = \iota_p(\Phi_1), \tag{4.1.18}$$

since then T_{Φ_0} and T_{Φ_1} are connected by an operator-norm continuous family of Fredholm Toeplitz operators.

In fact, $\iota_p(\Phi)$ is independent of p. We record a simple proof of this, which simultaneously establishes a regularity result. Given $p \in (1, \infty)$, let us set

$$\begin{aligned} K_{\Phi,p} &= \operatorname{Ker} T_\Phi \quad \text{on} \quad L^p(\partial\Omega), \\ K^*_{\Phi,p} &= \operatorname{Ker} (T_\Phi)^* \quad \text{on} \quad L^{p'}(\partial\Omega), \end{aligned} \tag{4.1.19}$$

where p' is the dual index to p.

Proposition 4.1.4 *If $\Phi \in C(\partial\Omega, G\ell(\ell, \mathbb{C}))$, then, given $1 < p < q < \infty$,*

$$K_{\Phi,p} = K_{\Phi,q} \quad \text{and} \quad K^*_{\Phi,p} = K^*_{\Phi,q}, \tag{4.1.20}$$

and

$$\iota_p(\Phi) = \iota_q(\Phi). \tag{4.1.21}$$

Proof. Clearly

$$\begin{aligned} p < q \Rightarrow K_{\Phi,q} \subset K_{\Phi,p} \ \text{ and } \ K^*_{\Phi,p} \subset K^*_{\Phi,q} \\ \Rightarrow \iota_q(\Phi) \le \iota_p(\Phi). \end{aligned} \tag{4.1.22}$$

Similarly,

$$p < q \Longrightarrow \iota_q(\Phi^{-1}) \le \iota_p(\Phi^{-1}). \tag{4.1.23}$$

But since $\iota_p(\Phi) + \iota_p(\Phi^{-1}) = 0 = \iota_q(\Phi) + \iota_q(\Phi^{-1})$, this yields the asserted identity (4.1.21), and thus forces equality in (4.1.20). □

Let us note the following further regularity result.

Proposition 4.1.5 *In the setting of Proposition 4.1.4, with $1 < p < q < \infty$,*

$$f \in L^p(\partial\Omega), \ T_\Phi f \in L^q(\partial\Omega) \Longrightarrow f \in L^q(\partial\Omega). \tag{4.1.24}$$

Proof. Set $g = T_\Phi f$, so g is in the range of T_Φ, acting on $L^p(\partial\Omega)$. Hence $\langle \varphi, g \rangle = 0$ for all $\varphi \in K^*_{\Phi,p}$. By (4.1.20), $\langle \varphi, g \rangle = 0$ for all $\varphi \in K^*_{\Phi,q}$, so g is in the range of T_Φ, acting on $L^q(\partial\Omega)$. We have

$$T_\Phi f = T_\Phi \tilde{f}, \quad \text{for some} \quad \tilde{f} \in L^q(\partial\Omega) \subset L^p(\partial\Omega). \tag{4.1.25}$$

Hence $f - \tilde{f} \in K_{\Phi,p}$. Again by (4.1.20), $f - \tilde{f} \in K_{\Phi,q}$, and we have the conclusion in (4.1.24). □

Remark. We mention previous works that deduce regularity results from Fredholmness.

See [28] and [24]. See also [12] for a general index stability result on a complex interpolation scale, from which (4.1.21) also follows.

Having the identity (4.1.21), we set

$$\iota(\Phi) = \iota_p(\Phi), \quad 1 < p < \infty. \tag{4.1.26}$$

By (4.1.17)–(4.1.18), this induces

$$\iota : [\partial\Omega; G\ell(\ell, \mathbb{C})] \longrightarrow \mathbb{Z}, \quad \text{homomorphism}, \tag{4.1.27}$$

where $[\partial\Omega; G\ell(\ell, \mathbb{C})]$ is the set of homotopy classes of continuous maps $\partial\Omega \to G\ell(\ell, \mathbb{C})$, with group structure given by pointwise multiplication.

Returning to (4.1.16), we will often find it useful to record the dependence on D, and use the notation

$$\iota(\Phi; D) = \operatorname{Index} T_\Phi \ \text{ on } \ L^p(\partial\Omega). \tag{4.1.28}$$

Note that, in the decomposition (4.0.10), we have projections

$$\begin{aligned} \mathcal{P}_{\mathcal{D}} &: L^p(\partial\Omega, \mathcal{E}) \longrightarrow \mathcal{H}^p(\partial\Omega, \mathcal{D}), \\ \mathcal{P}_D &: L^p(\partial\Omega, \mathcal{F}_0) \longrightarrow \mathcal{H}^p(\partial\Omega, D), \\ \mathcal{P}_{D^*} &: L^p(\partial\Omega, \mathcal{F}_1) \longrightarrow \mathcal{H}^p(\partial\Omega, D^*), \end{aligned} \tag{4.1.29}$$

each tensored with $\mathbb{C}^\ell$. We also have

$$\iota(\Phi; D) = \operatorname{Index} \mathcal{P}_D \Phi \ \text{ on } \ \mathcal{H}^p(\partial\Omega, D). \tag{4.1.30}$$

Note that switching D and D^* effectively switches $\mathcal{P}_D$ and $\mathcal{P}_{D^*}$, so

$$\iota(\Phi; D^*) = \operatorname{Index} T^1_\Phi \ \text{ on } \ L^p(\partial\Omega). \tag{4.1.31}$$

Also,

$$\begin{aligned} \iota(\Phi; \mathcal{D}) &= \operatorname{Index} \mathfrak{T}_\Phi \ \text{ on } \ L^p(\partial\Omega) \\ &= \iota(\Phi; D) + \iota(\Phi; D^*). \end{aligned} \tag{4.1.32}$$

In cases where the use of D is understood, we will use the notation $\iota(\Phi)$ for $\iota(\Phi; D)$, but we use the notation (4.1.28) when additional precision is desired.

To illustrate material developed above, let us take

$$\Omega \subset \mathbb{R}^2 \approx \mathbb{C}, \quad \text{bounded, connected UR domain}, \tag{4.1.33}$$

and

$$k = \ell = 1, \quad D = \overline{\partial} = \frac{\partial}{\partial \overline{z}}. \tag{4.1.34}$$

Here, $\Phi : \partial\Omega \to \mathbb{C} \setminus 0$. If Ω is the unit disk, it is classical that $\iota(\Phi) = -w(\Phi)$, where $w(\Phi)$ is the winding number of the curve $\Phi(\partial\Omega)$ about 0. We can extend this, as follows. Assume $\mathbb{C} \setminus \overline{\Omega}$ has $\mu + 1$ connected components and

$$\partial\Omega = \bigcup_{j=0}^{\mu} \gamma_j, \tag{4.1.35}$$

where γ_0 is the outer boundary and γ_j for $j \geq 1$ enclose bounded components of $\mathbb{C} \setminus \overline{\Omega}$. We assume each γ_j is homeomorphic to the circle S^1, and gets the orientation induced as a boundary component of Ω (counterclockwise for γ_0, clockwise for other γ_j). Let $w_j(\Phi)$ denote the winding number of $\Phi|_{\gamma_j}$ about 0.

Proposition 4.1.6 *In the setting of (4.1.33)–(4.1.34), with $\Phi \in C(\partial\Omega, \mathbb{C} \setminus 0)$,*

$$\iota(\Phi; \overline{\partial}) = -\sum_{j=0}^{\mu} w_j(\Phi). \tag{4.1.36}$$

Proof. It follows from (4.1.27) that there exist $c_j \in \mathbb{Z}$ such that

$$\iota(\Phi; \overline{\partial}) = \sum_{j=0}^{\mu} c_j w_j(\Phi), \quad \forall\, \Phi \in C(\partial\Omega, \mathbb{C} \setminus 0). \tag{4.1.37}$$

We find c_j by picking certain special cases of Φ. In fact, pick

$$a_0 \in \Omega, \quad a_k \in \mathcal{O}_k,\ 1 \le k \le \mu, \tag{4.1.38}$$

where $\mathcal{O}_k$ are the bounded components of $\mathbb{C} \setminus \overline{\Omega}$, with boundary $-\gamma_k$. Then set

$$\Phi_k(z) = z - a_k, \quad 0 \le k \le \mu. \tag{4.1.39}$$

It is clear that $T_{\Phi_k} : L^p(\partial\Omega) \to L^p(\partial\Omega)$ is injective for each $k \in \{0, \dots, \mu\}$. In fact it is bijective for $k \ge 1$, with inverse T_{Ψ_k}, $\Psi_k(z) = (z - a_k)^{-1}$. On the other hand, $f \in L^p(\partial\Omega)$ belongs to the range of T_{Φ_0} if and only if $\mathcal{P}_D f$, extended to be holomorphic on Ω, vanishes at a_0. Hence

$$\iota(\Phi_0) = -1, \quad \iota(\Phi_k) = 0,\ k \ge 1, \tag{4.1.40}$$

while

$$w_0(\Phi_0) = 1, \quad w_j(\Phi_0) = 0 \ \text{ for } \ j \ge 1, \tag{4.1.41}$$

and, for $k \ge 1$,

$$\begin{aligned} w_j(\Phi_k) = \ & 1, \quad j = 0, \\ & -1, \quad j = k, \\ & 0, \quad \text{other } j. \end{aligned} \tag{4.1.42}$$

These identities force $c_j = -1$ for all j in (4.1.37). □

Note that in the setting of (4.1.33)–(4.1.34), we have

$$D^* = -\partial = -\frac{\partial}{\partial z}. \tag{4.1.43}$$

Since complex conjugation takes $\mathcal{H}^p(\Omega, \overline{\partial})$ to $\mathcal{H}^p(\Omega, \partial)$ (and hence $\mathcal{H}^p(\partial\Omega, \overline{\partial})$ to $\mathcal{H}^p(\partial\Omega, \partial)$) and vice-versa, we obtain

$$\iota(\Phi; \partial) = \iota(\overline{\Phi}; \overline{\partial}) = -\iota(\Phi; \overline{\partial}), \tag{4.1.44}$$

the latter identity holding because, with $\ell = 1$, $\overline{\Phi}$ is homotopic to Φ^{-1}. Generally, relations can be more elaborate. In particular, it is not always the case that $\iota(\overline{\Phi}; D) = -\iota(\Phi; D)$. We return to this issue later on.

4.2 Toeplitz operators with coefficients in $L^\infty \cap \mathrm{vmo}$

Here, we extend the setting of Φ in (4.0.6) and (4.0.8) from $\Phi \in C(\partial\Omega, \mathrm{End}\,\mathbb{C}^\ell)$ to

$$\Phi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, \mathrm{End}\,\mathbb{C}^\ell). \tag{4.2.1}$$

We note the following useful result, for scalar valued functions.

Lemma 4.2.1 *$L^\infty \cap \mathrm{vmo}(\partial\Omega)$ is a closed linear subspace of $L^\infty(\partial\Omega)$, closed under products, hence a closed $*$-subalgebra of the C^*-algebra $L^\infty(\partial\Omega)$.*

A proof can be found in [30], p. 81. This extends to $L^\infty \cap \mathrm{vmo}(\partial\Omega, \mathrm{End}\,\mathbb{C}^\ell)$. Generally, if $\mathcal{A}$ is a C^*-algebra with unit 1 and $\mathcal{B}$ a C^*-subalgebra, containing 1, then an element $\varphi \in \mathcal{B}$ is invertible in $\mathcal{B}$ if and only if it is invertible in $\mathcal{A}$. This has the following consequence:

$$\begin{aligned} &\Phi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, \mathrm{End}\,\mathbb{C}^\ell), \quad \Phi^{-1} \in L^\infty(\partial\Omega, \mathrm{End}\,\mathbb{C}^\ell) \\ &\quad \Longrightarrow \Phi^{-1} \in L^\infty \cap \mathrm{vmo}(\partial\Omega, \mathrm{End}\,\mathbb{C}^\ell). \end{aligned} \tag{4.2.2}$$

When Φ satisfies (4.2.2), we say

$$\Phi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, G\ell(\ell, \mathbb{C})). \tag{4.2.3}$$

Our goal here is to treat Toeplitz operators $\mathfrak{T}_\Phi$ and T_Φ for Φ as in (4.2.1), with special attention to Fredholm properties for Φ as in (4.2.3). We start with the following extension of Lemma 4.1.1.

Lemma 4.2.2 *If $\Omega \subset M$ is a UR domain and $\Phi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, \mathrm{End}\,\mathbb{C}^\ell)$, then*

$$[\mathcal{P}_\mathcal{D}, \Phi] : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \ \ \textit{is compact,} \ \ \forall\, p \in (1, \infty). \tag{4.2.4}$$

Proof. The computation (4.1.3) continues to hold, so it suffices to obtain compactness on $L^p(\partial\Omega)$ of $\mathcal{K}$, given by

$$\mathcal{K}g(x) = \mathrm{PV} \int_{\partial\Omega} E(x,y)\{\Phi(y) - \Phi(x)\} g(y)\, d\sigma(y). \tag{4.2.5}$$

Given the results on $E(x,y)$ in Appendix A.1, such compactness follows from Theorem 2.19 of [11]. □

For later use, we remark that Theorem 2.19 of [11] also gives

$$\|[\mathcal{P}_\mathcal{D}, \Phi]\|_{\mathcal{L}(L^p(\partial\Omega))} \le C_p \|\Phi\|_{\mathrm{BMO}}, \tag{4.2.6}$$

where the BMO-seminorm is given by

$$\|\Phi\|_{\mathrm{BMO}} = \sup_B \frac{1}{\sigma(B)} \|\Phi - \Phi_B\|_{L^1(B)}, \tag{4.2.7}$$

where B runs over all balls in $\partial\Omega$ and

$$\Phi_B = \frac{1}{\sigma(B)} \int_B \Phi(y)\, d\sigma(y). \tag{4.2.8}$$

This is only a seminorm, since Φ constant $\Rightarrow \|\Phi\|_{\mathrm{BMO}} = 0$. We have the norm

$$\|\Phi\|_{\mathrm{bmo}} = \|\Phi\|_{\mathrm{BMO}} + \|\Phi\|_{L^1(\partial\Omega)}. \tag{4.2.9}$$

To proceed, assume

$$\Phi, \Psi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, \mathrm{End}\,\mathbb{C}^\ell). \tag{4.2.10}$$

Then, as in (4.1.8),

$$\mathfrak{T}_\Psi \mathfrak{T}_\Phi = \mathfrak{T}_{\Psi\Phi} + \mathcal{P}_{\mathcal{D}} \Psi [\mathcal{P}_{\mathcal{D}}, \Phi] \mathcal{P}_{\mathcal{D}}. \tag{4.2.11}$$

Consequently,

$$\mathfrak{T}_\Psi \mathfrak{T}_\Phi - \mathfrak{T}_{\Psi\Phi} \ \text{ is compact on } \ L^p(\partial\Omega), \quad \forall\, p \in (1,\infty). \tag{4.2.12}$$

Similarly, we have compactness of $\mathfrak{T}_\Phi \mathfrak{T}_\Psi - \mathfrak{T}_{\Phi\Psi}$. This yields the following extension of Proposition 4.1.2.

Proposition 4.2.3 *Let $\Omega \subset M$ be a UR domain, and suppose*

$$\Phi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, G\ell(\ell,\mathbb{C})). \tag{4.2.13}$$

Then

$$\mathfrak{T}_{\Phi^{-1}} \mathfrak{T}_\Phi - I \ \textit{ and } \ \mathfrak{T}_\Phi \mathfrak{T}_{\Phi^{-1}} - I \ \textit{ are compact on } \ L^p(\partial\Omega), \tag{4.2.14}$$

for all $p \in (1,\infty)$, so

$$\mathfrak{T}_\Phi : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \ \textit{ is Fredholm}, \ \forall\, p \in (1,\infty). \tag{4.2.15}$$

We have analogous results for T_Φ.

We set

$$\iota_p(\Phi) = \mathrm{Index}\, T_\Phi \ \text{ on } \ L^p(\partial\Omega). \tag{4.2.16}$$

From (4.2.12) we have

$$\Phi, \Psi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, G\ell(\ell,\mathbb{C})) \Longrightarrow \iota_p(\Phi\Psi) = \iota_p(\Phi) + \iota_p(\Psi), \tag{4.2.17}$$

extending (5.25).

Propositions 4.1.4–4.1.5 extend immediately to the current setting. In particular, $\iota_p(\Phi)$ is independent of $p \in (0,\infty)$, so we simply set

$$\iota(\Phi) = \iota_p(\Phi). \tag{4.2.18}$$

The appropriate extension of the homotopy invariance (4.1.18) to the current setting is less straightforward. As a first step, given $\Phi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, G\ell(\ell,\mathbb{C}))$, we write

$$A = (\Phi\Phi^*)^{1/2}. \tag{4.2.19}$$

Using the Riesz functional calculus

$$A = \frac{1}{2\pi i} \int_\gamma \zeta^{1/2} (\zeta I - \Phi\Phi^*)^{-1}\, d\zeta, \tag{4.2.20}$$

for an appropriate contour γ, and using Lemma 4.2.1, we have

$$A \in L^\infty \cap \mathrm{vmo}(\partial\Omega, G\ell(\ell,\mathbb{C})), \tag{4.2.21}$$

and similarly for A^{-1}. Hence

$$\Phi = AU, \quad U = A^{-1}\Phi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, G\ell(\ell,\mathbb{C})), \tag{4.2.22}$$

and in fact

$$U \in L^\infty \cap \mathrm{vmo}(\partial\Omega, U(\ell)), \tag{4.2.23}$$

where $U(\ell)$ is the compact group of unitary operators on $\mathbb{C}^\ell$. We see that T_A and T_U are Fredholm on $L^p(\partial\Omega)$ for each $p \in (1,\infty)$, and

$$\iota(\Phi) = \iota(U) + \iota(A). \tag{4.2.24}$$

On the other hand,

$$\tau_t = T_{(1-t)A+tI} = (1-t)T_A + tT_I, \tag{4.2.25}$$

is a norm continuous family of bounded operators on $L^p(\partial\Omega)$, and

$$(1-t)A + tI \in L^\infty \cap \mathrm{vmo}(\partial\Omega, G\ell(\ell,\mathbb{C})), \quad \forall\, t \in [0,1], \tag{4.2.26}$$

so (4.2.25) is a norm continuous family of Fredholm operators on $L^p(\partial\Omega)$, for $t \in [0,1]$, and we have

$$\iota(A) = \iota(I) = 0, \tag{4.2.27}$$

hence

$$\iota(\Phi) = \iota(U), \tag{4.2.28}$$

when Φ and U are related by (4.2.19)–(4.2.22).

We are left with the task of understanding $\iota(\Phi)$ for $\Phi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, U(\ell))$. The following is a key homotopy result.

Proposition 4.2.4 *Assume $\Phi_t \in L^\infty \cap \mathrm{vmo}(\partial\Omega, U(\ell))$ for each $t \in [0,1]$, and*

$$t \mapsto \Phi_t \text{ is continuous from } [0,1] \text{ to } \mathrm{bmo}(\partial\Omega, \mathrm{End}\,\mathbb{C}^\ell). \tag{4.2.29}$$

Then $\iota(\Phi_t)$ is independent of $t \in [0,1]$.

To prove Proposition 4.2.4, we use an argument adapted from a treatment of Toeplitz operators on the disk in [7], Appendix 2. It suffices to show that, under the hypotheses of Proposition 4.2.4, $\iota(\Phi_t) = \iota(\Phi_0)$ for t close enough to 0. Now

$$\iota(\Phi_t) - \iota(\Phi_0) = \iota(\Phi_t\Phi_0^*), \tag{4.2.30}$$

so it suffices to show that $\iota(\Phi_t\Phi_0^*) = 0$ for t close enough to 0. We bring in a couple of lemmas.

Lemma 4.2.5 *Let $\Psi_t \in L^\infty \cap \mathrm{vmo}(\partial\Omega, U(\ell))$ satisfy*

$$\|\Psi_t\|_{\mathrm{BMO}} \to 0, \quad \text{as} \quad t \to 0. \tag{4.2.31}$$

Then, for each $p \in (1,\infty)$, T_{Ψ_t} is invertible on $L^p(\partial\Omega)$ for t small enough.

Proof. As in (4.2.11) (cf. also (4.1.8)), we have

$$T_{\Psi_t} T_{\Psi_t^*} = I + R_t, \quad T_{\Psi_t^*} T_{\Psi_t} = I + \widetilde{R}_t, \tag{4.2.32}$$

with

$$R_t = \mathcal{P}_D \Psi_t [\mathcal{P}_D, \Psi_t^*] \mathcal{P}_D, \quad \widetilde{R}_t = \mathcal{P}_D \Psi_t^* [\mathcal{P}_D, \Psi_t] \mathcal{P}_D. \tag{4.2.33}$$

Using (4.2.6), we have from (4.2.31) (which also implies $\|\Psi_t^*\|_{\mathrm{BMO}} \to 0$) that $\|R_t\|_{\mathcal{L}(L^p)}$ and $\|\widetilde{R}_t\|_{\mathcal{L}(L^p)} \to 0$ as $t \to 0$, for each $p \in (1, \infty)$. Hence (4.2.32) yields the asserted invertibility. □

With Φ_t as in Proposition 4.2.4, we have

$$\|\Phi_t \Phi_0^*\|_{\mathrm{BMO}} = \|(\Phi_t - \Phi_0)\Phi_0^*\|_{\mathrm{BMO}}, \tag{4.2.34}$$

since $\Phi_0 \Phi_0^* = I$ is constant. The following lemma will be useful.

Lemma 4.2.6 *Fix $g \in L^\infty \cap \mathrm{vmo}(\partial\Omega)$. For $\delta > 0$, there exists $C(\delta, g) < \infty$ such that*

$$\|fg\|_{\mathrm{BMO}} \leq \delta \|f\|_{L^\infty} + C(\delta, g)\|f\|_{\mathrm{bmo}}, \quad \forall\, f \in L^\infty(\partial\Omega). \tag{4.2.35}$$

Proof. This is Lemma A.2.5 of [7], when $\partial\Omega$ is a smooth compact manifold, but the argument works when $\partial\Omega$ is the boundary of a bounded, Ahlfors regular domain. □

Proof of Proposition 4.2.4. As seen from (4.2.30), we need to show that $\iota_p(\Phi_t \Phi_0^*) = 0$ for t close to 0. We apply Lemma 4.2.5, with $\Psi_t = \Phi_t \Phi_0^*$, and use (4.2.34). By Lemma 4.2.6,

$$\|\Psi_t\|_{\mathrm{BMO}} \leq \delta \|\Phi_t - \Phi_0\|_{L^\infty} + C(\delta, \Phi_0^*)\|\Phi_t - \Phi_0\|_{\mathrm{bmo}}. \tag{4.2.36}$$

By the hypotheses of Proposition 4.2.4, this implies $\limsup_{t \to 0} \|\Psi_t\|_{\mathrm{BMO}} \leq C\delta$ for each $\delta > 0$, and hence $\|\Psi_t\|_{\mathrm{BMO}} \to 0$ as $t \to 0$, which by Lemma 4.2.5 gives $\iota(\Psi_t) = 0$ for t small, as desired. □

To tie in Proposition 4.2.4 with material from §4.1, we bring in some constructions, which in the case when $\partial\Omega$ is a compact smooth manifold were made in [6]. We then make modifications to deal with Ahlfors regular domains. To set things up, let $u \in BMO(\partial\Omega)$, and, for $\varepsilon > 0$, set

$$\overline{u}_\varepsilon(x) = \frac{1}{\sigma(B_\varepsilon(x))} \int\limits_{B_\varepsilon(x)} u(y)\, d\sigma(y). \tag{4.2.37}$$

With

$$M_a(u) = \sup_{\varepsilon < a} \sup_{x \in \partial\Omega} \frac{1}{\sigma(B_\varepsilon(x))} \int\limits_{B_\varepsilon(x)} |u(y) - \overline{u}_\varepsilon(x)|\, d\sigma(y), \tag{4.2.38}$$

we have $M_a(u) \leq \|u\|_{\mathrm{BMO}}$, and

$$u \in \mathrm{VMO}(\partial\Omega) \Longrightarrow \lim_{a \to 0} M_a(u) = 0. \tag{4.2.39}$$

One has the following result of D. Sarason.

Lemma 4.2.7 *Assume $\partial\Omega$ is smooth. There exists A, depending on $\partial\Omega$, such that*

$$\|u - \overline{u}_\varepsilon\|_{\mathrm{BMO}} \leq AM_\varepsilon(u), \quad \forall\, u \in \mathrm{BMO}(\partial\Omega). \tag{4.2.40}$$

Hence, for $u \in \mathrm{VMO}(\partial\Omega)$,

$$\|u - \overline{u}_\varepsilon\|_{\mathrm{bmo}} \longrightarrow 0, \quad \text{as } \varepsilon \to 0. \tag{4.2.41}$$

The following is key to results of [6]. It arose as an observation of L. Boutet de Monvel and O. Gabber. Let Y be a smooth compact submanifold of some Euclidean space $\mathbb{R}^N$. For us, $Y = U(\ell) \subset \mathrm{End}\, \mathbb{C}^\ell$.

Lemma 4.2.8 *Assume Ω is Ahlfors regular. Given $u \in \mathrm{vmo}(\partial\Omega, Y)$, $x \in \partial\Omega$, $\varepsilon > 0$,*

$$\mathrm{dist}(\overline{u}_\varepsilon(x), Y) \leq \frac{1}{\sigma(B_\varepsilon(x))} \int\limits_{B_\varepsilon(x)} |u(y) - \overline{u}_\varepsilon(x)|\, d\sigma(y) \leq M_\varepsilon(u). \tag{4.2.42}$$

Proof. The first inequality in (4.2.42) holds because each $u(y) \in Y$, and the second follows from the definition (4.2.38). ☐

In the setting of [6], where $\partial\Omega$ is smooth, $\overline{u}_\varepsilon$ is continuous, at least for small $\varepsilon > 0$, but such continuity is not guaranteed for rougher $\partial\Omega$ (even Lipschitz). Hence we produce the following modification of $\overline{u}_\varepsilon$. It is convenient to embed M in $\mathbb{R}^m$, for some $m > n$. Fix $h \in C(\mathbb{R}^m)$ such that

$$h \geq 0, \quad h(x) = 1 \text{ for } |x| \leq \frac{1}{2}, \quad 0 \text{ for } |x| \geq 1, \tag{4.2.43}$$

and introduce

$$h_\varepsilon(x) = h(\varepsilon^{-1}x). \tag{4.2.44}$$

Then set

$$\begin{aligned} \tilde{u}_\varepsilon(x) &= \frac{1}{A_\varepsilon(x)} \int\limits_{\partial\Omega} u(y) h_\varepsilon(x-y)\, d\sigma(y), \\ A_\varepsilon(x) &= \int\limits_{\partial\Omega} h_\varepsilon(x-y)\, d\sigma(y). \end{aligned} \tag{4.2.45}$$

Note that

$$\tilde{u}_\varepsilon(x) = \frac{u\sigma * h_\varepsilon(x)}{\sigma * h_\varepsilon(x)} \tag{4.2.46}$$

is a quotient of two continuous functions on $\mathbb{R}^m$, and Ahlfors regularity of $\partial\Omega$ implies

$$A_\varepsilon(x) \geq C\varepsilon^{n-1}, \quad \forall\, x \in \partial\Omega,\ \varepsilon \in (0,1], \tag{4.2.47}$$

so

$$\tilde{u}_\varepsilon \text{ is continuous on } \partial\Omega, \quad \forall\, \varepsilon \in (0,1]. \tag{4.2.48}$$

We can write (4.2.45)–(4.2.46) as

$$\tilde{u}_\varepsilon(x) = \int\limits_{\partial\Omega} p_\varepsilon(x,y) u(y)\, d\sigma(y), \tag{4.2.49}$$

where

$$p_\varepsilon(x,y)=\frac{h_\varepsilon(x-y)}{A_\varepsilon(x)}. \tag{4.2.50}$$

We have the following estimates:

$$\begin{aligned}
&p_\varepsilon(x,u)=0, \quad \text{if } \ |x-y|\geq \varepsilon,\\
&|p_\varepsilon(x,y)|\leq C\varepsilon^{-(n-1)}, \quad \forall\, x,y\in\partial\Omega,\\
&|p_\varepsilon(x,y)-p_\varepsilon(x',y)|\leq C\varepsilon^{-(n-1)}\cdot\frac{|x-x'|}{\varepsilon}, \quad \text{for } \ x,x',y\in\partial\Omega,\\
&\int\limits_{\partial\Omega} p_\varepsilon(x,y)\,d\sigma(y)=1, \quad \forall\, x\in\partial\Omega.
\end{aligned} \tag{4.2.51}$$

Of these four properties, the first and last are obvious, and the second follows from (4.2.47). It suffices to establish the third estimate for $|x-x'|\leq\varepsilon/4$, since otherwise it follows from the second. This in turn follows from the estimate

$$|\nabla_x p_\varepsilon(x,y)|\leq C\varepsilon^{-n}, \quad \text{for } \ y\in\partial\Omega, \ \operatorname{dist}(x,\partial\Omega)\leq\frac{\varepsilon}{4}, \tag{4.2.52}$$

which in turn follows directly from the definition (4.2.50) and from (4.2.47), also valid when $\operatorname{dist}(x,\partial\Omega)\leq\varepsilon/4$.

Remark. It is convenient to have (4.2.52) on a tubular neighborhood of $\partial\Omega$, since we do not want to assume that each $x,y\in\partial\Omega$ can be joined by a path in $\partial\Omega$ of length $\leq C|x-y|$.

Given the estimate (4.2.51), we have from the proof of Proposition 2.22 in [11] that, for $u\in\operatorname{bmo}(\partial\Omega)$,

$$\|u-\tilde u_\varepsilon\|_{\mathrm{bmo}}\leq C_1 M_{c_2\varepsilon}(u), \tag{4.2.53}$$

which leads to the following.

Lemma 4.2.9 *If Ω is Ahlfors regular,*

$$u\in\operatorname{vmo}(\partial\Omega)\Longrightarrow\|u-\tilde u_\varepsilon\|_{\mathrm{bmo}}\to 0 \quad \text{as} \ \ \varepsilon\to 0. \tag{4.2.54}$$

Next, parallel to (4.2.42), we have, for $u\in\operatorname{vmo}(\partial\Omega,Y), \ x\in\partial\Omega$,

$$\begin{aligned}
\operatorname{dist}(\tilde u_\varepsilon(x),Y)&\leq\frac{1}{\sigma(B_\varepsilon(x))}\int\limits_{B_\varepsilon(x)}|u(y)-\tilde u_\varepsilon(x)|\,d\sigma(y)\\
&\leq\widetilde M_\varepsilon(u),
\end{aligned} \tag{4.2.55}$$

where

$$\widetilde M_a(u)=\sup_{\varepsilon<a}\sup_{x\in\partial\Omega}\frac{1}{\sigma(B_\varepsilon(x))}\int\limits_{B_\varepsilon(x)}|u(y)-\tilde u_\varepsilon(x)|\,d\sigma(y). \tag{4.2.56}$$

This leads us to the following.

Proposition 4.2.10 *If Ω is Ahlfors regular, and $u\in\operatorname{vmo}(\partial\Omega,Y)$, then*

$$\sup_{x\in\partial\Omega}\operatorname{dist}(\tilde u_\varepsilon(x),Y)\longrightarrow 0, \quad \text{as} \ \ \varepsilon\to 0. \tag{4.2.57}$$

Proof. By (4.2.55), this follows from

$$u \in \mathrm{vmo}(\partial\Omega) \Longrightarrow \widetilde{M}_\varepsilon(u) \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{4.2.58}$$

This in turn is a consequence of the estimate

$$\widetilde{M}_1(u) \le C\|u\|_{\mathrm{bmo}}, \tag{4.2.59}$$

since it is clear that $\widetilde{M}_\varepsilon(u) \to 0$ for $u \in C(\partial\Omega)$, and $\widetilde{M}_\varepsilon(u_1 + u_2) \le \widetilde{M}_\varepsilon(u_1) + \widetilde{M}_\varepsilon(u_2)$.

It remains to prove (4.2.59), i.e.,

$$\frac{1}{\sigma(B_\varepsilon(x))} \int\limits_{B_\varepsilon(x)} |u - \tilde{u}_\varepsilon(x)|\, d\sigma \le C\|u\|_{\mathrm{bmo}}, \quad \forall\, x \in \partial\Omega,\ \varepsilon \in (0,1]. \tag{4.2.60}$$

Indeed, given $\overline{u}_\varepsilon(x)$ as in (4.2.37), we have, for each $x \in \partial\Omega,\ \varepsilon > 0$,

$$\begin{aligned} |\tilde{u}_\varepsilon(x) - \overline{u}_\varepsilon(x)| &\le \int\limits_{\partial\Omega} p_\varepsilon(x,y)|u(y) - \overline{u}_\varepsilon(x)|\, d\sigma(y) \\ &\le \frac{C}{\sigma(B_\varepsilon(x))} \int\limits_{B_\varepsilon(x)} |u - \overline{u}_\varepsilon(x)|\, d\sigma \\ &\le C\|u\|_{\mathrm{bmo}}, \end{aligned} \tag{4.2.61}$$

the second inequality in (4.2.61) by the second estimate in (4.2.51), coupled with Ahlfors regularity of $\partial\Omega$. In other words,

$$\|\tilde{u} - \overline{u}\|_{L^\infty} \le C\|u\|_{\mathrm{bmo}}. \tag{4.2.62}$$

Hence (4.2.60) follows from (4.2.7) and (4.2.62). □

Having Proposition 4.2.10, we proceed as follows. Given $u \in \mathrm{vmo}(\partial\Omega, Y)$, there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$, $\tilde{u}_\varepsilon$ has range in a small tubular neighborhood $\mathcal{O}$ of Y. We have $N : \mathcal{O} \to Y$, mapping $z \in \mathcal{O}$ to the nearest point in Y, and

$$u_\varepsilon(x) = N\tilde{u}_\varepsilon(x) \in Y \quad \text{is well defined.} \tag{4.2.63}$$

Since $\|\tilde{u}_\varepsilon - u_\varepsilon\|_{L^\infty} \to 0$ as $\varepsilon \to 0$, we have from (4.2.54) that

$$\|u_\varepsilon - u\|_{\mathrm{bmo}} \longrightarrow 0, \quad \text{as} \quad \varepsilon \to 0. \tag{4.2.64}$$

We apply these results to $\Phi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, U(\ell))$, obtaining $\varepsilon_0 > 0$ and

$$\Phi_t \in C(\partial\Omega, U(\ell)), \quad \forall\, t \in (0, \varepsilon_0], \tag{4.2.65}$$

such that

$$\|\Phi_t - \Phi\|_{\mathrm{bmo}} \longrightarrow 0, \quad \text{as} \quad t \to 0. \tag{4.2.66}$$

The same argument used to establish Proposition 4.2.4 now yields the following.

Proposition 4.2.11 *Let Ω be a bounded UR domain. Given $\Phi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, U(\ell))$, there exists $\varepsilon_1 > 0$ such that*

$$\iota(\Phi) = \iota(\Phi_t), \quad \forall\, t \in (0, \varepsilon_1]. \tag{4.2.67}$$

Here, Φ_t is as in the paragraph above.

4.3 Toeplitz operators on L^p-Sobolev spaces

In this subsection, we assume $\Omega \subset \mathbb{R}^n$ is a bounded UR domain and

$$\Phi \in \operatorname{Lip}(\partial\Omega, \operatorname{End}\mathbb{C}^\ell), \tag{4.3.1}$$

though later we impose other conditions. We assume $\mathcal{D}$ has the form

$$\mathcal{D} = \begin{pmatrix} & D^* \\ D & \end{pmatrix}, \quad Du = A_j \partial_j u, \quad A_j \in \operatorname{End}\mathbb{C}^\kappa. \tag{4.3.2}$$

In this case $\mathcal{E} \to \mathbb{R}^n$ is the trivial vector bundle, with fiber $\mathbb{C}^\kappa \oplus \mathbb{C}^\kappa$. We define $\mathfrak{T}_\Phi$ as in (4.0.6), and make use of results of §§A.2–A.3 to get

$$\mathfrak{T}_\Phi f = \mathcal{P}_\mathcal{D} \Phi \mathcal{P}_\mathcal{D} f + (I - \mathcal{P}_\mathcal{D}) f, \quad \mathfrak{T}_\Phi : L^p_1(\partial\Omega) \longrightarrow L^p_1(\partial\Omega), \quad 1 < p < \infty. \tag{4.3.3}$$

We will seek conditions under which we can show that $\mathfrak{T}_\Psi \mathfrak{T}_\Phi - \mathfrak{T}_{\Psi\Phi}$ is compact on $L^p_1(\partial\Omega)$, given also $\Psi \in \operatorname{Lip}(\partial\Omega, \operatorname{End}\mathbb{C}^\ell)$. In fact, we will start by seeking conditions on Φ and Ψ that imply

$$\mathfrak{T}_\Psi \mathfrak{T}_\Phi - \mathfrak{T}_{\Psi\Phi} : L^p(\partial\Omega) \longrightarrow L^p_1(\partial\Omega). \tag{4.3.4}$$

Such a property will imply compactness on $L^p_1(\partial\Omega)$, whenever $\partial\Omega$ has the property that the natural injection

$$L^p_1(\partial\Omega) \hookrightarrow L^p(\partial\Omega) \ \text{ is compact.} \tag{4.3.5}$$

We recall the following sufficient condition for (4.3.5) to hold, established in Corollary 4.31 of [11].

Lemma 4.3.1 *If Ω is a bounded UR domain satisfying a two-sided John condition, then (4.3.5) holds, for each $p \in (1, \infty)$.*

To continue, we recall from (4.1.8) that

$$\mathfrak{T}_\Psi \mathfrak{T}_\Phi - \mathfrak{T}_{\Psi\Phi} = \mathcal{P}_\mathcal{D} \Psi [\mathcal{P}_\mathcal{D}, \Phi] \mathcal{P}_\mathcal{D}, \tag{4.3.6}$$

so (4.3.4) will hold provided

$$[\mathcal{P}_\mathcal{D}, \Phi] = [C_\mathcal{D}, \Phi] : L^p(\partial\Omega) \longrightarrow L^p_1(\partial\Omega). \tag{4.3.7}$$

Since $C_\mathcal{D} f = iB(\sigma_\mathcal{D}(x, \nu) f)$, this holds provided

$$[B, \Phi] : L^p(\partial\Omega) \longrightarrow L^p_1(\partial\Omega). \tag{4.3.8}$$

Note that

$$[B, \Phi] f(x) = \mathrm{PV} \int_{\partial\Omega} E(x - y) \{\Phi(y) - \Phi(x)\} f(y)\, d\sigma(y). \tag{4.3.9}$$

Here is our first result.

Proposition 4.3.2 *Assume*

$$\Phi \in C^{1,\omega}(\mathbb{R}^n, \operatorname{End}\mathbb{C}^\ell), \tag{4.3.10}$$

so $\partial_j \Phi$ has modulus of continuity ω for each j, i.e.,

$$|\partial_j \Phi(x) - \partial_j \Phi(y)| \le C\omega(|x - y|). \tag{4.3.11}$$

Assume ω satisfies the Dini condition

$$\int_0^1 \frac{\omega(t)}{t}\,dt < \infty. \tag{4.3.12}$$

Then (4.3.7)–(4.3.8) hold, and hence so does (4.3.4) for all $\Psi \in \mathrm{Lip}(\partial\Omega, \mathrm{End}\,\mathbb{C}^\ell)$.

Proof. Our strategy is to apply Proposition A.2.3 to

$$u(x) = \int\limits_{\partial\Omega} E(x-y)\{\Phi(y) - \Phi(x)\} f(y)\,d\sigma(y), \quad x \in \Omega. \tag{4.3.13}$$

Note that $E(x-y)$ and $\Phi(x)$ are defined for $x \in \Omega$. We know that

$$\mathcal{N}(u) \in L^p(\partial\Omega), \quad u\big|_{\partial\Omega} = [B, \Phi]f, \tag{4.3.14}$$

given $f \in L^p(\partial\Omega)$. We need to show that

$$\mathcal{N}(\nabla u) \in L^p(\partial\Omega), \tag{4.3.15}$$

and that $\partial_j u$ has a nontangential limit on $\partial\Omega$, for each j. To proceed, for $x \in \Omega$ we have

$$\begin{aligned} \partial_j u(x) = &-\Phi_j(x) \int\limits_{\partial\Omega} E(x-y) f(y)\,d\sigma(y) \\ &+ \int\limits_{\partial\Omega} E_j(x-y)\{\Phi(y) - \Phi(x)\} f(y)\,d\sigma(y) \\ =\ & v_j(x) + w_j(x), \end{aligned} \tag{4.3.16}$$

with

$$\Phi_j = \partial_j \Phi, \quad E_j = \partial_j E. \tag{4.3.17}$$

Since $v_j = -\Phi_j \mathcal{B}(f)$, results of §2.3 apply to yield $\mathcal{N}(v_j) \in L^p(\partial\Omega)$ and $v_j|_{\partial\Omega} = -\Phi_j \mathcal{B}(f)|_{\partial\Omega}$. To analyze w_j, write

$$\begin{aligned} \Phi(x) - \Phi(y) &= \nabla\Phi(x)(x-y) + R(x,y)(x-y), \\ R(x,y) &= \int_0^1 \{\nabla\Phi(sx + (1-s)y) - D\Phi(x)\}\,ds, \end{aligned} \tag{4.3.18}$$

so R is continuous on $\mathbb{R}^n \times \mathbb{R}^n$, and

$$|R(x,y)| \le C\omega(|x-y|). \tag{4.3.19}$$

We have

$$\begin{aligned} w_j(x) = &-\nabla\Phi(x) \int\limits_{\partial\Omega} E_j(x-y) \otimes (x-y)\, f(y)\,d\sigma(y) \\ &- \int\limits_{\partial\Omega} R(x,y) E_j(x-y) \otimes (x-y)\, f(y)\,d\sigma(y) \\ =\ & w_{j1}(x) + w_{j2}(x). \end{aligned} \tag{4.3.20}$$

Now $E_j(x) \otimes x$ shares with $E(x)$ the properties of being smooth on $\mathbb{R}^n \setminus 0$, odd, and homogeneous of degree $-(n-1)$ in x. Hence the results in §3.4 of [11] that yield (2.3.8)–(2.3.9) also give $\mathcal{N}(w_{j1}) \in L^p(\partial\Omega)$ and nontangential limits of w_{j1} on $\partial\Omega$.

It remains to consider w_{j2}. We have

$$\sup_{z\in\Gamma_x} |R(z,y)E_j(z-y) \otimes (z-y)| \le C\frac{\omega(|x-y|)}{|x-y|^{n-1}}, \quad \forall\, x,y \in \partial\Omega, \tag{4.3.21}$$

hence

$$\mathcal{N}w_{j2}(x) \le C\int\limits_{\partial\Omega} \frac{\omega(|x-y|)}{|x-y|^{n-1}}|f(y)|\,d\sigma(y), \quad x \in \partial\Omega. \tag{4.3.22}$$

As long as the Dini condition (4.3.12) holds, Proposition 5.1 of [11] yields

$$\|\mathcal{N}w_{j2}\|_{L^p(\partial\Omega)} \le C_p\|f\|_{L^p(\partial\Omega)}. \tag{4.3.23}$$

To get nontangential a.e. convergence of w_{j2} for all $f \in L^p(\partial\Omega)$, we can use (4.3.23) to deduce this from such convergence for all f on a dense linear subspace, e.g., for $f \in C(\partial\Omega)$. However, for $f \in C(\partial\Omega)$, such convergence at each $x \in \partial\Omega$ follows by the Lebesgue dominated convergence theorem. This finishes the proof of Proposition 4.3.2. □

This leads to the following compactness result.

Proposition 4.3.3 *Assume Ω is a bounded UR domain and that the natural injection $L^p_1(\partial\Omega) \hookrightarrow L^p(\partial\Omega)$ is compact. Then*

$$\mathfrak{T}_\Psi\mathfrak{T}_\Phi - \mathfrak{T}_{\Psi\Phi} : L^p_1(\partial\Omega) \longrightarrow L^p_1(\partial\Omega) \ \text{ is compact,} \tag{4.3.24}$$

for all $\Psi \in \mathrm{Lip}(\partial\Omega, \mathrm{End}\,\mathbb{C}^\ell)$, whenever

$$\Phi \in C^1(\partial\Omega, \mathrm{End}\,\mathbb{C}^\ell). \tag{4.3.25}$$

Proof. Compactness in (4.3.24) follows from (4.3.5) and (4.3.4) if $\Phi \in C^{1,\omega}(\partial\Omega, \mathrm{End}\,\mathbb{C}^\ell)$, given ω as in (4.3.12). It then follows for all $\Phi \in C^1(\partial\Omega, \mathrm{End}\,\mathbb{C}^\ell)$, by a standard approximation argument. □

Corollary 4.3.4 *Assume Ω is a bounded UR domain satisfying* (4.3.5)*, and*

$$\Phi \in C^1(\partial\Omega, G\ell(\ell,\mathbb{C})). \tag{4.3.26}$$

Then

$$\mathfrak{T}_\Phi : L^p_1(\partial\Omega) \longrightarrow L^p_1(\partial\Omega) \ \text{ is Fredholm,} \tag{4.3.27}$$

with Fredholm inverse $\mathfrak{T}_{\Phi^{-1}}$.

We want to expand the scope of Proposition 4.3.3 and Corollary 4.3.4, to a broader class of Toeplitz operators, going beyond even $\Phi, \Psi \in \mathrm{Lip}(\partial\Omega, \mathrm{End}\,\mathbb{C}^\ell)$. For this, we will require on Ω the assumptions used in Lemma 4.3.1, namely that Ω be a bounded UR domain satisfying a two-sided John condition (equivalently, Ω is an Ahlfors regular domain and satisfies a two-sided John condition, since these hypotheses imply Ω is a UR domain). In such a case, we have from Corollary 4.31 of [11] that

$$q \in (n-1, \infty] \Longrightarrow L^q_1(\partial\Omega) \subset C^r(\partial\Omega), \quad r = 1 - \frac{n-1}{q}, \tag{4.3.28}$$

and, from Proposition 4.29 of [11], that

$$C^\infty(\partial\Omega) \text{ is dense in } L^q_1(\partial\Omega), \quad \forall\, q \in (1,\infty), \tag{4.3.29}$$

where $C^\infty(\partial\Omega)$ is the space of restrictions to $\partial\Omega$ of elements of $C^\infty(\mathbb{R}^n)$. To complement these results, we have the following, proven in Appendix A.2.

$$\begin{aligned} &p \in (1,\infty), \quad q \in (n-1,\infty), \quad q \ge p \\ &\Longrightarrow L^p_1(\partial\Omega) \text{ is a module over } L^q_1(\partial\Omega), \end{aligned} \tag{4.3.30}$$

under our current hypotheses on Ω. In such a circumstance,

$$\Phi \in L^q_1(\partial\Omega) \Longrightarrow \mathfrak{T}_\Phi : L^p_1(\partial\Omega) \to L^p_1(\partial\Omega), \tag{4.3.31}$$

and

$$\Phi_k \in C^\infty(\partial\Omega), \ \|\Phi_k - \Phi\|_{L^q_1} \to 0 \Longrightarrow \|\mathfrak{T}_{\Phi_k} - \mathfrak{T}_\Phi\|_{\mathcal{L}(L^p_1)} \to 0. \tag{4.3.32}$$

We also have (thanks to the validity of the Leibniz rule)

$$\Phi \in L^q_1(\partial\Omega, G\ell(\ell,\mathbb{C})), \ q > n-1 \Longrightarrow \Phi^{-1} \in L^q_1(\partial\Omega, G\ell(\ell,\mathbb{C})). \tag{4.3.33}$$

Then standard limiting arguments applied to Corollary 4.3.4 yield the following.

Proposition 4.3.5 *Assume Ω is a bounded Ahlfors regular domain, satisfying a two-sided John condition. Take $p \in (1,\infty)$ and assume*

$$\Phi \in L^q_1(\partial\Omega, G\ell(\ell,\mathbb{C})), \quad q \ge p, \ q \in (n-1,\infty). \tag{4.3.34}$$

Then

$$\mathfrak{T}_\Phi : L^p_1(\partial\Omega) \longrightarrow L^p_1(\partial\Omega) \ \text{ is Fredholm,} \tag{4.3.35}$$

with Fredholm inverse $\mathfrak{T}_{\Phi^{-1}}$.

Under the hypotheses of Proposition 4.3.5, we also have

$$T_\Phi : L^p_1(\partial\Omega) \longrightarrow L^p_1(\partial\Omega), \text{ Fredholm}, \quad \forall\, p \in (1,q]. \tag{4.3.36}$$

Parallel to (4.1.16), we set

$$\iota_{p,1}(\Phi) = \operatorname{Index} T_\Phi \quad \text{on} \quad L^p_1(\partial\Omega), \tag{4.3.37}$$

and we get

$$\Phi, \Psi \in L^q_1(\partial\Omega, G\ell(\ell,\mathbb{C})) \Longrightarrow \iota_{p,1}(\Phi\Psi) = \iota_{p,1}(\Phi) + \iota_{p,1}(\Psi). \tag{4.3.38}$$

We have a regularity result and index identity parallel to that of Proposition 4.1.4, given as follows. Define $K_{\Phi,p}$ and $K^*_{\Phi,p}$ as in (4.1.19), and set

$$\begin{aligned} L_{\Phi,p} &= \operatorname{Ker} T_\Phi \quad \text{on} \quad L^p_1(\partial\Omega), \\ L^*_{\Phi,p} &= \operatorname{Ker}(T_\Phi)^* \quad \text{on} \quad L^p_1(\partial\Omega)^*, \end{aligned} \tag{4.3.39}$$

so

$$\begin{aligned} \iota(\Phi) = \iota_p(\Phi) &= \dim K_{\Phi,p} - \dim K^*_{\Phi,p}, \\ \iota_{p,1}(\Phi) &= \dim L_{\Phi,p} - \dim L^*_{\Phi,p}. \end{aligned} \tag{4.3.40}$$

We have the following.

Proposition 4.3.6 *In the setting of Proposition 4.3.5, in particular, with the assumption that* $\Phi \in L^q_1(\partial\Omega, G\ell(\ell,\mathbb{C}))$ *for* $q > n-1,\ q \ge p$,

$$K_{\Phi,p} = L_{\Phi,p} \quad \textit{and} \quad K^*_{\Phi,p} = L^*_{\Phi,p}, \tag{4.3.41}$$

and

$$\iota_{p,1}(\Phi) = \iota(\Phi). \tag{4.3.42}$$

Proof. The proof is similar to that of Proposition 4.1.4. We clearly have $L_{\Phi,p} \subset K_{\Phi,p}$ and $K^*_{\Phi,p} \subset L^*_{\Phi,p}$, so

$$\iota_{p,1}(\Phi) \le \iota_p(\Phi). \tag{4.3.43}$$

Similarly, $\iota_{p,1}(\Phi^{-1}) \le \iota_p(\Phi^{-1})$, but since $\iota_p(\Phi) + \iota_p(\Phi^{-1}) = 0 = \iota_{p,1}(\Phi) + \iota_{p,1}(\Phi^{-1})$, this yields (4.3.42), and then this forces (4.3.41). □

Remark. Comparison with (4.1.20) yields other regularity results. Going further, one can extend Proposition 4.1.5 to cases where $f \in L^p(\partial\Omega), T_\Phi f \in L^r_1(\partial\Omega)$. We leave this to the interested reader.

4.4 Toeplitz operators associated to Calderón-Szegö projectors

We return to the setting of §§4.1–4.2, but replace $\mathcal{P}_{\mathcal{D}}$ by the Calderón-Szegö projector $S_{\mathcal{D}}$. Thus, given $\Phi \in L^\infty(\partial\Omega, \operatorname{End}\mathbb{C}^\ell)$, we set

$$\mathcal{T}_\Phi f = S_{\mathcal{D}}\Phi S_{\mathcal{D}} f + (I - S_{\mathcal{D}})f, \tag{4.4.1}$$

so $\mathcal{T}_\Phi$ is bounded on $L^p(\partial\Omega)$ whenever $S_{\mathcal{D}}$ is bounded on $L^p(\partial\Omega)$ (cf. Proposition 3.2.1). Note that the splitting $\mathcal{E} = \mathcal{F}_0 \oplus \mathcal{F}_1$, as an orthogonal direct sum, yields

$$S_{\mathcal{D}} = \begin{pmatrix} S_D & \\ & S_{D^*} \end{pmatrix}, \tag{4.4.2}$$

and hence

$$\mathcal{T}_\Phi = \begin{pmatrix} \mathcal{T}^0_\Phi & \\ & \mathcal{T}^1_\Phi \end{pmatrix}. \tag{4.4.3}$$

It is clear (via Proposition 3.2.2) that

$$\begin{aligned} &A = C_{\mathcal{D}} - C^*_{\mathcal{D}} \ \text{ compact on } \ L^p(\partial\Omega) \\ &\Rightarrow \mathfrak{T}_\Phi - \mathcal{T}_\Phi \ \text{ compact on } \ L^p(\partial\Omega), \end{aligned} \tag{4.4.4}$$

for all $\Phi \in L^\infty(\partial\Omega, \operatorname{End}\mathbb{C}^\ell)$. Hence, when D is of Dirac type and Ω is a regular SKT domain, Proposition 3.2.3 implies that $\mathfrak{T}_\Phi - \mathcal{T}_\Phi$ is compact for all such Φ, for all $p \in (1,\infty)$. Hence all the Fredholm results of §§4.1–4.2 apply to such a situation.

Here we look at the behavior of $\mathcal{T}_\Phi$ when Ω is not necessarily a regular SKT domain (and D is perhaps not of Dirac type), based on the identity

$$S_{\mathcal{D}}(I + A) = \mathcal{P}_{\mathcal{D}} \quad \text{on} \ \ L^2(\partial\Omega), \tag{4.4.5}$$

and the fact that

$$I + A : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \ \text{ is invertible for } \ p \in (p_0, p_1), \tag{4.4.6}$$

for some $p_0 < 2, p_1 > 2$. Recall that

$$A = C_{\mathcal{D}} - C_{\mathcal{D}}^* : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega), \quad p \in (1,\infty), \tag{4.4.7}$$

with

$$C_{\mathcal{D}} f(x) = \mathrm{PV}\, i \int_{\partial\Omega} E(x,y)\sigma_{\mathcal{D}}(y,\nu(y)) f(y)\, d\sigma(y). \tag{4.4.8}$$

It follows from (4.4.5) that, with $M_\Phi f = \Phi f$,

$$[S_{\mathcal{D}}, M_\Phi](I + A) = [\mathcal{P}_{\mathcal{D}}, M_\Phi] - S_{\mathcal{D}}[A, M_\Phi]. \tag{4.4.9}$$

As seen in §§4.1–4.2, if $\Phi \in C(\partial\Omega, \mathrm{End}\,\mathbb{C}^\ell)$, or more generally

$$\Phi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, \mathrm{End}\,\mathbb{C}^\ell), \tag{4.4.10}$$

then

$$[\mathcal{P}_{\mathcal{D}}, M_\Phi] \ \text{ is compact on } \ L^p(\partial\Omega), \quad \forall\, p \in (1,\infty). \tag{4.4.11}$$

Similarly, $[C_{\mathcal{D}}, M_\Phi]$ enjoys such compactness, and, by duality, so does $[C_{\mathcal{D}}^*, M_\Phi]$, and consequently $[A, M_\Phi]$. This gives the following analogue of Proposition 4.1.2.

Proposition 4.4.1 *Given $p_0 < 2$, $p_1 > 2$ such that (4.4.6) holds,*

$$[S_{\mathcal{D}}, M_\Phi] \ \textit{ is compact on } \ L^p(\partial\Omega), \quad \forall\, p \in (p_0, p_1), \tag{4.4.12}$$

whenever $\Phi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, \mathrm{End}\,\mathbb{C}^\ell)$.

From here, using (parallel to (4.1.8)) the identity

$$\mathcal{T}_\Psi \mathcal{T}_\Phi = \mathcal{T}_{\Psi\Phi} + S_{\mathcal{D}}\Psi[S_{\mathcal{D}}, \Phi]S_{\mathcal{D}}, \tag{4.4.13}$$

we deduce that

$$\mathcal{T}_\Psi \mathcal{T}_\Phi - \mathcal{T}_{\Psi\Phi} \ \text{ is compact on } L^p(\partial\Omega), \quad \forall\, p \in (p_0, p_1), \tag{4.4.14}$$

provided

$$\Phi, \Psi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, \mathrm{End}\,\mathbb{C}^\ell). \tag{4.4.15}$$

This leads to the following.

Proposition 4.4.2 *If $\Omega \subset M$ is a UR domain and*

$$\Phi, \ \Phi^{-1} \in L^\infty \cap \mathrm{vmo}(\partial\Omega, \mathrm{End}\,\mathbb{C}^\ell), \tag{4.4.16}$$

then

$$\mathcal{T}_{\Phi^{-1}}\mathcal{T}_\Phi - I \ \textit{ and } \ \mathcal{T}_\Phi \mathcal{T}_{\Phi^{-1}} - I \textit{ are compact on } \ L^p(\partial\Omega), \quad \forall\, p \in (p_0, p_1), \tag{4.4.17}$$

so

$$\mathcal{T}_\Phi : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \ \textit{ is Fredholm}, \ \forall\, p \in (p_0, p_1). \tag{4.4.18}$$

We have analogous results for $\mathcal{T}_\Phi^0$.

Parallel to (4.1.16), we set

$$\tilde{\iota}_p(\Phi) = \text{ Index } \mathcal{T}^0_\Phi \text{ on } L^p(\partial\Omega), \tag{4.4.19}$$

for $p \in (p_0, p_1)$. Recall from §§4.1–4.2 the quantity

$$\iota(\Phi) = \text{Index}\, T_\Phi \text{ on } L^p(\partial\Omega), \tag{4.4.20}$$

which was shown to be independent of $p \in (1, \infty)$. We have the following key result.

Proposition 4.4.3 *In the setting of Proposition 4.4.2,*

$$\tilde{\iota}_p(\Phi) = \iota(\Phi), \quad \forall\, p \in (p_0, p_1). \tag{4.4.21}$$

Proof. We produce a norm-continuous path $\mathcal{T}_{\Phi,t},\ 0 \le t \le 1$, consisting of Fredholm operators, such that $\mathcal{T}_{\Phi,0} = \mathfrak{T}_\Phi$ and $\mathcal{T}_{\Phi,1} = \mathcal{T}_\Phi$. To get this, take

$$\mathcal{P}_t = tS_{\mathcal{D}} + (1-t)\mathcal{P}_{\mathcal{D}}, \quad 0 \le t \le 1. \tag{4.4.22}$$

This is a norm-continuous path of operators on $L^p(\partial\Omega)$, for $p \in (p_0, p_1)$. A computation, using $S_{\mathcal{D}}\mathcal{P}_{\mathcal{D}} = \mathcal{P}_{\mathcal{D}},\ \mathcal{P}_{\mathcal{D}}S_{\mathcal{D}} = S_{\mathcal{D}}$, gives

$$\mathcal{P}_t^2 = \mathcal{P}_t. \tag{4.4.23}$$

Furthermore, it is clear from (4.4.12) and results of §4.2 that

$$[\mathcal{P}_t, M_\Phi] \text{ is compact on } L^p(\partial\Omega), \ \ \forall\, p \in (p_0, p_1), \tag{4.4.24}$$

from which we deduce that, if

$$\mathcal{T}_{\Phi,t} f = \mathcal{P}_t \Phi \mathcal{P}_t f + (I - \mathcal{P}_t) f, \tag{4.4.25}$$

then $\mathcal{T}_{\Phi,t}$ is Fredholm for each $t \in [0,1]$, and hence is the desired path from $\mathfrak{T}_\Phi$ to $\mathcal{T}_\Phi$. The index identity (4.4.21) is an immediate consequence. □

4.5 Twisted Toeplitz operators

In previous sections, we have extended the action of $\mathcal{D}$ from sections of $\mathcal{E} \to M$ to sections of $\mathcal{E} \otimes \mathbb{C}^\ell$, in a canonical fashion. Then we have taken $\Phi \in C(\partial\Omega, \text{End}\,\mathbb{C}^\ell)$, or a variant, and defined $\mathfrak{T}_\Phi$ on $L^p(\partial\Omega, \mathcal{E} \otimes \mathbb{C}^\ell)$ and T_Φ on $L^p(\partial\Omega, \mathcal{F}_0 \otimes \mathbb{C}^\ell)$. Here, we replace $\mathbb{C}^\ell$ by a complex vector bundle $\mathcal{C} \to M$, with a Hermitian metric, and define "twisted" Toeplitz operators

$$\mathfrak{T}_\Phi \text{ on } L^p(\partial\Omega, \mathcal{E} \otimes \mathcal{C}) \text{ and } T_\Phi \text{ on } L^p(\partial\Omega, \mathcal{F}_0 \otimes \mathcal{C}). \tag{4.5.1}$$

The first order of business is to define twisted versions of D and $\mathcal{D}$,

$$\begin{aligned} D_{\mathcal{C}} &: H^{s+1,p}(M, \mathcal{F}_0 \otimes \mathcal{C}) \longrightarrow H^{s,p}(M, \mathcal{F}_1 \otimes \mathcal{C}), \\ \mathcal{D}_{\mathcal{C}} &: H^{s+1,p}(M, \mathcal{E} \otimes \mathcal{C}) \longrightarrow H^{s,p}(M, \mathcal{E} \otimes \mathcal{C}), \end{aligned} \tag{4.5.2}$$

such that

$$\sigma_{D_{\mathcal{C}}}(x,\xi) = \sigma_D(x,\xi) \otimes I, \quad \sigma_{\mathcal{D}_{\mathcal{C}}}(x,\xi) = \sigma_{\mathcal{D}}(x,\xi) \otimes I. \tag{4.5.3}$$

For the trivial bundle $\mathbb{C}^\ell \times M$, this is done componentwise. However, if $\mathcal{C}$ is a nontrivial bundle, more work is required. We will construct $D_\mathcal{C}$ and $\mathcal{D}_\mathcal{C}$ when $\mathcal{C}$ has a connection, i.e., a covariant derivative. To motivate the construction, recall from Chapter 2 that, if f is scalar, and u is a section of $\mathcal{F}_0$, then

$$D(fu) = fDu + (D_0 f)u, \quad D_0 f(x) = \frac{1}{i}\sigma_D(x, df(x)). \tag{4.5.4}$$

We aim to define $D_\mathcal{C}$ via

$$D_\mathcal{C}(u \otimes v) = Du \otimes v + (D_0 v)u, \tag{4.5.5}$$

when u is a section of $\mathcal{F}_0$ and v is a section of $\mathcal{C}$. It remains to define $D_0 v$ when v is a section of the bundle $\mathcal{C}$, which is equipped with a connection ∇. To get this, we note that $\sigma_D(x,\xi)$ is linear in ξ, so

$$\sigma_D(x,\xi) = \sigma_D(x)\xi, \quad \sigma_D(x) : T^*_x \longrightarrow \operatorname{Hom}(\mathcal{F}_{0x}, \mathcal{F}_{1x}), \tag{4.5.6}$$

or equivalently

$$\sigma_D(x) : \mathcal{F}_{0x} \otimes T^*_x \longrightarrow \mathcal{F}_{1x}. \tag{4.5.7}$$

We can tensor with the identity on $\mathcal{C}_x$ to extend $\sigma_D(x)$ to

$$\sigma_D(x) : \mathcal{F}_{0x} \otimes T^*_x \otimes \mathcal{C}_x \longrightarrow \mathcal{F}_{1x} \otimes \mathcal{C}_x. \tag{4.5.8}$$

Now, if v is a section of $\mathcal{C}$, then its covariant derivative ∇v is a section of $T^* \otimes \mathcal{C}$. We complete the definition of $D_\mathcal{C}$ in (4.5.5) by setting

$$(D_0 v(x))u(x) = \frac{1}{i}\sigma_D(x)\big(u(x) \otimes \nabla v(x)\big). \tag{4.5.9}$$

Having $D_\mathcal{C}$, we can define $\mathcal{D}_\mathcal{C}$ as in the Introduction. This operator is invertible provided $D_\mathcal{C}$ and $D^*_\mathcal{C}$ possess UCP. In particular, if D is of Dirac type, so is $D_\mathcal{C}$, and this leads to invertibility. From here on, we will assume $\mathcal{D}_\mathcal{C}$ is invertible.

Having constructed the twisted operators $D_\mathcal{C}$ and $\mathcal{D}_\mathcal{C}$, we lighten our notational load, and simply denote these twisted operators by D and $\mathcal{D}$, respectively. With $E(x,y)$ denoting the integral kernel of $\mathcal{D}^{-1}$, acting on $H^{s,p}(M, \mathcal{E} \otimes \mathcal{C})$, we define

$$C_\mathcal{D} : L^p(\partial\Omega, \mathcal{E} \otimes \mathcal{C}) \longrightarrow L^p(\partial\Omega, \mathcal{E} \otimes \mathcal{C}) \tag{4.5.10}$$

as before, i.e., by

$$C_\mathcal{D} f(x) = \mathrm{PV}\, i \int_{\partial\Omega} E(x,y)\sigma_\mathcal{D}(y, \nu(y)) f(y)\, d\sigma(y), \tag{4.5.11}$$

and similarly define $C_D, \mathcal{P}_\mathcal{D}$, and $\mathcal{P}_D$.

We now define the operators $\mathfrak{T}_\Phi$ and T_Φ by the same formulas as before, i.e.,

$$\begin{aligned} \mathfrak{T}_\Phi f &= \mathcal{P}_\mathcal{D} \Phi \mathcal{P}_\mathcal{D} f + (I - \mathcal{P}_\mathcal{D}) f, \\ T_\Phi f &= \mathcal{P}_D \Phi \mathcal{P}_D f + (I - \mathcal{P}_D) f, \end{aligned} \tag{4.5.12}$$

for $f \in L^p(\partial\Omega, \mathcal{E} \otimes \mathcal{C})$ or $L^p(\partial\Omega, \mathcal{F}_0 \otimes \mathcal{C})$, respectively, where, in the current setting,

$$\Phi \in C(\partial\Omega, \operatorname{End} \mathcal{C}), \tag{4.5.13}$$

or more generally

$$\Phi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, \mathrm{End}\,\mathcal{C}). \tag{4.5.14}$$

From (4.5.3), which follows from our twisting construction, we have the crucial identity

$$\sigma_{\mathcal{D}}(x,\xi)\Phi(x)f(x) = \Phi(x)\sigma_{\mathcal{D}}(x,\xi)f(x), \tag{4.5.15}$$

for $x \in \partial\Omega,\ \xi \in T_x^*M$. The key to the extension of the previous results to the current setting is the following.

Proposition 4.5.1 *If $\Omega \subset M$ is a UR domain and Φ satisfies* (4.5.14)*, then*

$$[\mathcal{P}_{\mathcal{D}}, \Phi] : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \ \ \textit{is compact}, \ \ \forall\, p \in (1,\infty). \tag{4.5.16}$$

Proof. As before, we have $[\mathcal{P}_{\mathcal{D}}, \Phi] = [C_{\mathcal{D}}, \Phi]$, hence

$$[\mathcal{P}_{\mathcal{D}}, \Phi]f(x) = \mathrm{PV}\, i \int_{\partial\Omega} \big\{E(x,y)\Phi(y) - \Phi(x)E(x,y)\big\}\sigma_{\mathcal{D}}(y,\nu(y))f(y)\,d\sigma(y), \tag{4.5.17}$$

where we have used (4.5.15), with $x = y,\ \xi = \nu(y)$. As opposed to arguments in previous sections, we do not necessarily have $E(x,y)\Phi(y) = \Phi(y)E(x,y)$ here, but we can come close enough, as follows.

By the regularity of $E(x,y)$ off the diagonal, it suffices to get compactness when f is supported on a coordinate chart and x in (4.5.17) is restricted to that chart. Then we have

$$E(x,y)\sqrt{g(y)} = e_0(x-y,y) + e_1(x,y), \tag{4.5.18}$$

where $e_1(x,y)$ has a weak singularity and

$$\begin{aligned} e_0(x-y,y) &= (2\pi)^{-n}\int E_0(\xi,y)e^{i(x-y)\cdot\xi}\,d\xi, \\ E_0(\xi,y) &= \sigma_{\mathcal{D}}(y,\xi)^{-1}. \end{aligned} \tag{4.5.19}$$

Thus, modulo a compact opeator, the right side of (4.5.17) becomes

$$\begin{aligned} &\mathrm{PV}\, i \int_{\partial\Omega} \big\{e_0(x-y,y)\Phi(y) - \Phi(x)e_0(x-y,y)\big\}g(y)^{-1/2}\sigma_{\mathcal{D}}(y,\nu(y))f(y)\,d\sigma(y) \\ &= \mathrm{PV}\, i \int_{\partial\Omega} \big\{\Phi(y) - \Phi(x)\big\}e_0(x-y,y)g(y)^{-1/2}\sigma_{\mathcal{D}}(y,\nu(y))f(y)\,d\sigma(y), \end{aligned} \tag{4.5.20}$$

the latter identity by (4.5.15) and (4.5.19). At this point, we can again deduce compactness from Theorem 2.19 of [11]. □

Corollary 4.5.2 *In the seting of Proposition 4.5.1, if*

$$\Phi \in C(\partial\Omega, G\ell(\mathcal{C})), \tag{4.5.21}$$

or more generally

$$\Phi, \Phi^{-1} \in L^\infty \cap \mathrm{vmo}(\partial\Omega, \mathrm{End}\,\mathcal{C}), \tag{4.5.22}$$

then

$$\mathfrak{T}_\Phi : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \ \ \textit{is Fredholm}, \ \ \forall\, p \in (1,\infty), \tag{4.5.23}$$

with a similar result for T_Φ.

Thus we can set

$$\iota(\Phi) = \operatorname{Index} T_\Phi \quad \text{on} \quad L^p(\partial\Omega, \mathcal{F}_0 \otimes \mathcal{C}), \quad p \in (1,\infty). \tag{4.5.24}$$

As before, this index is independent of $p \in (1,\infty)$. If also Ψ satisfies (4.5.21) or (4.5.22), we have

$$\iota(\Phi\Psi) = \iota(\Phi) + \iota(\Psi). \tag{4.5.25}$$

Furthermore, if Φ satisfies (4.5.21), $\iota(\Phi)$ depends only on the homotopy class of Φ (within the class of continuous sections of $G\ell(\mathcal{C})$). Also, results on the stability of the index in the setting of (4.5.22) extend. We leave the details to the reader.

So far in this subsection, we have defined and studied

$$T_\Phi : L^p(\partial\Omega, \mathcal{F}_0 \otimes \mathcal{C}) \longrightarrow L^p(\partial\Omega, \mathcal{F}_0 \otimes \mathcal{C}), \tag{4.5.26}$$

when $\mathcal{C} \to M$ is a vector bundle, equipped with a connection, and seen that it is Fredholm when $\Phi \in C(\partial\Omega, G\ell(\mathcal{C}))$, or more generally when $\Phi, \Phi^{-1} \in L^\infty \cap \operatorname{vmo}(\partial\Omega, \operatorname{End}\mathcal{C})$. Let us also use the notation

$$\widehat{T}_\Phi = T_\Phi\big|_{\mathcal{R}(\mathcal{P}_D)}, \quad \widehat{T}_\Phi : \mathcal{R}(\mathcal{P}_D) \longrightarrow \mathcal{R}(\mathcal{P}_D), \tag{4.5.27}$$

where

$$\mathcal{R}(\mathcal{P}_D) = \mathcal{P}_D(L^p(\partial\Omega, \mathcal{F}_0 \otimes \mathcal{C})). \tag{4.5.28}$$

We suppress the p-dependence; of course, we assume $p \in (1,\infty)$. Note that

$$\widehat{T}_\Phi f = \mathcal{P}_D \Phi f, \quad f \in \mathcal{R}(\mathcal{P}_D). \tag{4.5.29}$$

Also, if $\Phi, \Psi \in L^\infty \cap \operatorname{vmo}(\partial\Omega, \operatorname{End}\mathcal{C})$, then

$$\widehat{T}_{\Phi\Psi} = \widehat{T}_\Phi \widehat{T}_\Psi \ \text{ is compact on } \ \mathcal{R}(\mathcal{P}_D). \tag{4.5.30}$$

If Φ satisfies (4.5.22), then $\widehat{T}_\Phi$ is Fredholm, and

$$\operatorname{Index} \widehat{T}_\Phi = \operatorname{Index} T_\Phi. \tag{4.5.31}$$

Since the bundle $\mathcal{C}$ was equipped with a connection in order to define $D_\mathcal{C}$, and hence T_Φ and $\widehat{T}_\Phi$, it is useful to record the following.

Proposition 4.5.3 *The index of T_Φ (hence of $\widehat{T}_\Phi$) is independent of the choice of connection on $\mathcal{C}$.*

Proof. Two connections on $\mathcal{C}$ give two operators $D_\mathcal{C}$ that differ by an operator of order zero. Hence the integral kernels $E(x,y)$ differ by a weakly singular term, and so the two versions of T_Φ differ by a compact operator. □

We now extend our notion of twisted Toeplitz operators. For simplicity, we assume D is of Dirac type. Let $\mathcal{C}_0$ and $\mathcal{C}_1$ be two vector bundles (of the same rank) over M, equipped with connections. Then, associated to $D : H^{s+1,p}(M, \mathcal{F}_0) \to H^{s,p}(M, \mathcal{F}_1)$, we have twisted operators

$$D_j : H^{s+1,p}(M, \mathcal{F}_0 \otimes \mathcal{C}_j) \longrightarrow H^{s,p}(M, \mathcal{F}_1 \otimes \mathcal{C}_j), \quad j = 0, 1, \tag{4.5.32}$$

and then associated Calderón-type projectors

$$\mathcal{P}_j : L^p(\partial\Omega, \mathcal{F}_0 \otimes \mathcal{C}_j) \longrightarrow L^p(\partial\Omega, \mathcal{F}_0 \otimes \mathcal{C}_j), \quad j = 0, 1. \tag{4.5.33}$$

Let us take

$$\Phi_0 \in C(\partial\Omega, \mathrm{Hom}(\mathcal{C}_0, \mathcal{C}_1)), \quad \Phi_1 \in C(\partial\Omega, \mathrm{Hom}(\mathcal{C}_1, \mathcal{C}_0)). \tag{4.5.34}$$

(We could replace C by $L^\infty \cap \mathrm{vmo}$.) Then we form $\mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_1$,

$$D_{\mathcal{C}} = \begin{pmatrix} D_0 & \\ & D_1 \end{pmatrix} : H^{s+1,p}(M, \mathcal{F}_0 \otimes \mathcal{C}) \longrightarrow H^{s,p}(M, \mathcal{F}_1 \otimes \mathcal{C}), \tag{4.5.35}$$

and

$$\Phi = \begin{pmatrix} & \Phi_1 \\ \Phi_0 & \end{pmatrix} \in C(\partial\Omega, \mathrm{End}\, \mathcal{C}). \tag{4.5.36}$$

Construction of $\mathcal{P} = \mathcal{P}_{D_{\mathcal{C}}}$ gives

$$\mathcal{P} = \begin{pmatrix} \mathcal{P}_0 & \\ & \mathcal{P}_1 \end{pmatrix} : L^p(\partial\Omega, \mathcal{F}_0 \otimes \mathcal{C}) \longrightarrow L^p(\partial\Omega, \mathcal{F}_0 \otimes \mathcal{C}). \tag{4.5.37}$$

As before, we form

$$T_\Phi = \mathcal{P}\Phi\mathcal{P} + (I - \mathcal{P}). \tag{4.5.38}$$

A calculation gives

$$T_\Phi = \begin{pmatrix} \mathcal{P}_0^c & \mathcal{P}_0\Phi_1\mathcal{P}_1 \\ \mathcal{P}_1\Phi_0\mathcal{P}_0 & \mathcal{P}_1^c \end{pmatrix}, \quad \mathcal{P}_j^c = I - \mathcal{P}_j. \tag{4.5.39}$$

If also

$$\Psi = \begin{pmatrix} & \Psi_1 \\ \Psi_0 & \end{pmatrix} \in C(\partial\Omega, \mathrm{End}\, \mathcal{C}), \tag{4.5.40}$$

(or more generally, in $L^\infty \cap \mathrm{vmo}(\partial\Omega, \mathrm{End}\, \mathcal{C})$), then

$$T_\Phi T_\Psi = \begin{pmatrix} \mathcal{P}_0\Phi_1\mathcal{P}_1\Psi_0\mathcal{P}_0 + \mathcal{P}_0^c & 0 \\ 0 & \mathcal{P}_1\Phi_0\mathcal{P}_0\Psi_1\mathcal{P}_1 + \mathcal{P}_1^c \end{pmatrix}. \tag{4.5.41}$$

By contrast, we have

$$\Phi\Psi = \begin{pmatrix} \Phi_1\Psi_0 & \\ & \Phi_0\Psi_1 \end{pmatrix}, \tag{4.5.42}$$

and

$$T_{\Phi\Psi} = \begin{pmatrix} \mathcal{P}_0\Phi_1\Psi_0\mathcal{P}_0 + \mathcal{P}_0^c & 0 \\ 0 & \mathcal{P}_1\Phi_0\Psi_1\mathcal{P}_1 + \mathcal{P}_1^c \end{pmatrix}. \tag{4.5.43}$$

Now we know

$$T_{\Phi\Psi} - T_\Phi T_\Psi \ \text{ is compact on } \ L^p(\partial\Omega, \mathcal{F}_0 \otimes \mathcal{C}). \tag{4.5.44}$$

It follows that

$$(\mathcal{P}_0\Phi_1\mathcal{P}_1)(\mathcal{P}_1\Psi_0\mathcal{P}_0) - \mathcal{P}_0\Phi_1\Psi_0\mathcal{P}_0 \ \text{ is compact on } \ L^p(\partial\Omega, \mathcal{F}_0 \otimes \mathcal{C}_0). \tag{4.5.45}$$

We are in a position to define a further class of Toeplitz operators. First, we simplify notation (altering the convention used in (4.5.36)), and consider

$$\Psi \in C(\partial\Omega, \mathrm{Hom}(\mathcal{C}_0, \mathcal{C}_1)), \quad \Phi \in C(\partial\Omega, \mathrm{Hom}(\mathcal{C}_1, \mathcal{C}_0)). \tag{4.5.46}$$

(As before, we can replace C by $L^\infty \cap \mathrm{vmo}$.) Then we define

$$\widehat{T}_\Psi : \mathcal{R}(\mathcal{P}_0) \longrightarrow \mathcal{R}(\mathcal{P}_1), \quad \widehat{T}_\Phi : \mathcal{R}(\mathcal{P}_1) \longrightarrow \mathcal{R}(\mathcal{P}_0), \tag{4.5.47}$$

by

$$\widehat{T}_\Psi = \mathcal{P}_1 \Psi \ \text{ on } \ \mathcal{R}(\mathcal{P}_0), \quad \widehat{T}_\Phi = \mathcal{P}_0 \Phi \ \text{ on } \ \mathcal{R}(\mathcal{P}_1). \tag{4.5.48}$$

As in (4.5.28), we set

$$\mathcal{R}(\mathcal{P}_j) = \mathcal{P}_j(L^p(\partial\Omega, \mathcal{F}_0 \otimes \mathcal{C}_j)), \quad j = 0, 1, \tag{4.5.49}$$

and suppress dependence on $p \in (1, \infty)$. It follows from (4.5.45) that

$$\widehat{T}_\Phi \widehat{T}_\Psi - \widehat{T}_{\Phi\Psi} \ \text{ is compact on } \ \mathcal{R}(\mathcal{P}_0), \tag{4.5.50}$$

and

$$\widehat{T}_\Psi \widehat{T}_\Phi - \widehat{T}_{\Psi\Phi} \ \text{ is compact on } \ \mathcal{R}(\mathcal{P}_1). \tag{4.5.51}$$

Note that the Toeplitz operators $\widehat{T}_{\Phi\Psi}$ and $\widehat{T}_{\Psi\Phi}$ are of the sort treated in (4.5.27)–(4.5.29). These results imply the following.

Proposition 4.5.4 *Assume $\Psi \in C(\partial\Omega, \mathrm{Hom}(\mathcal{C}_0, \mathcal{C}_1))$ is invertible at each point of $\partial\Omega$, or more generally*

$$\Psi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, \mathrm{Hom}(\mathcal{C}_0, \mathcal{C}_1)), \quad \Psi^{-1} \in L^\infty \cap \mathrm{vmo}(\partial\Omega, \mathrm{Hom}(\mathcal{C}_1, \mathcal{C}_0)). \tag{4.5.52}$$

Then $\widehat{T}_\Psi : \mathcal{R}(\mathcal{P}_0) \to \mathcal{R}(\mathcal{P}_1)$ is Fredholm, with index independent of $p \in (1, \infty)$.

The next proposition is a useful precursor of an important "cobordism invariance" result, which will be treated in §4.7.

Proposition 4.5.5 *In the setting of Proposition 4.5.4, assume Ψ extends to M, satisfying*

$$\Psi \in C(M, \mathrm{Hom}(\mathcal{C}_0, \mathcal{C}_1)), \quad \Psi^{-1} \in C(M, \mathrm{Hom}(\mathcal{C}_1, \mathcal{C}_0)). \tag{4.5.53}$$

Then

$$\mathrm{Index}\, \widehat{T}_\Psi = 0. \tag{4.5.54}$$

Proof. Using a homotopy, we can assume Ψ is smooth, of class C^2. Then, using Ψ, we can pull back $\mathcal{C}_1$ to the vector bundle $\mathcal{C}_0$, and hence conjugate D_1, acting on sections of $\mathcal{F}_0 \otimes \mathcal{C}_1$, to an operator $\widetilde{D}_1$, acting on sections of $\mathcal{F}_0 \otimes \mathcal{C}_0$. Then $\mathcal{P}_1$ is conjugated to a projection acting on sections of $\mathcal{F}_0 \otimes \mathcal{C}_0$. The difference between D_0 and $\widetilde{D}_1$ is that they are associated to different connections on $\mathcal{C}_0$, the given one and the one pulled back via Ψ. Now these two connections can be joined by a path, producing a continuous family D_s of elliptic operators, acting on sections of $\mathcal{F}_0 \otimes \mathcal{C}_0$, all with the same principal symbol. They give rise to a norm-continuous family of projections $\mathcal{P}_s$, differing by compact operators from $\mathcal{P}_0$, and $\widehat{T}_\Psi$ is conjugated to T_1, where

$$T_s = \mathcal{P}_s\big|_{\mathcal{R}(\mathcal{P}_0)}, \quad T_s : \mathcal{R}(\mathcal{P}_0) \to \mathcal{R}(\mathcal{P}_s). \tag{4.5.55}$$

The proof of Proposition 4.5.5 is completed by the following result.

Lemma 4.5.6 *Let $\{P_s : 0 \le s \le 1\}$ be a norm continuous family of projections on a Banach space V. Assume $P_s - P_0$ is compact for each s. Set*

$$T_s = P_s\big|_{\mathcal{R}(P_0)}, \quad T_s : \mathcal{R}(P_0) \longrightarrow \mathcal{R}(P_s). \tag{4.5.56}$$

Then, for each $s \in [0,1]$, T_s is Fredholm, of index zero.

Proof. Write $P_s = P_0 + K_s$, and consider

$$\widetilde{T}_s = P_s P_0 + (I - P_s)(I - P_0) = I + K_s(2P_0 - I). \tag{4.5.57}$$

For s small, $\widetilde{T}_s$ is invertible on V, and hence $T_s : \mathcal{R}(P_0) \to \mathcal{R}(P_s)$ is an isomorphism. Similarly, we can partition $[0,1]$ into intervals $[s_j, s_{j+1}]$, $0 = s_0 < s_1 < \cdots < s_N = 1$, such that

$$P_s\big|_{\mathcal{R}(P_{s_j})} : \mathcal{R}(P_{s_j}) \longrightarrow \mathcal{R}(P_s) \tag{4.5.58}$$

is an isomorphism, for each j, and each $s \in [s_j, s_{j+1}]$. In fact, this isomorphism is the restriction to $\mathcal{R}(P_{s_j})$ of

$$\widetilde{T}_{s,j} = P_s P_{s_j} + (I - P_s)(I - P_{s_j}), \quad s \in [s_j, s_{j+1}], \tag{4.5.59}$$

which is invertible on V, and differs from I by a compact operator. Composing these, we get a norm-continuous family Q_s of invertible operators on V, each differing from I by a compact operator, such that, for each $s \in [0,1]$,

$$Q_s : \mathcal{R}(P_0) \longrightarrow \mathcal{R}(P_s) \ \text{ is an isomorphism.} \tag{4.5.60}$$

Then

$$Q_s^{-1} T_s : \mathcal{R}(P_0) \longrightarrow \mathcal{R}(P_0) \tag{4.5.61}$$

is a norm continuous family of operators on $\mathcal{R}(P_0)$, each differing from the identity by a compact operator, hence each Fredholm of index zero. Since Q_s is invertible, this implies each T_s is Fredholm, of index zero. □

4.6 Localization of Toeplitz operators

We take D, acting on sections of $\mathcal{F}_0 \otimes \mathcal{C}$, of Dirac type, $\Omega \subset M$ a UR domain, $\Phi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, \mathrm{End}\,\mathcal{C})$, and

$$T_\Phi = \mathcal{P}_D \Phi \mathcal{P}_D + (I - \mathcal{P}_D) : L^p(\partial\Omega, \mathcal{F}_0 \otimes \mathcal{C}) \to L^p(\partial\Omega, \mathcal{F}_0 \otimes \mathcal{C}). \tag{4.6.1}$$

Recall that $\mathcal{P}_D = (1/2)I + C_D$ and C_D is the upper left block of $C_{\mathcal{D}}$, defined by

$$C_{\mathcal{D}} f(x) = i\,\mathrm{PV} \int_{\partial\Omega} E(x,y) \sigma_{\mathcal{D}}(y, \nu(y)) f(y)\, d\sigma(y), \tag{4.6.2}$$

where $E(x,y)$ is the integral kernel of $\mathcal{D}^{-1}$. Hence with $E = \begin{pmatrix} E_{00} & E_{01} \\ E_{10} & E_{11} \end{pmatrix}$,

$$C_D f(x) = i\,\mathrm{PV} \int_{\partial\Omega} E_{01}(x,y) \sigma_D(y, \nu(y)) f(y)\, d\sigma(y). \tag{4.6.3}$$

Here, we consider some localization phenomena for Toeplitz operators.

To begin, we consider localizing T_Φ, when $\partial\Omega$ is not connected. Suppose

$$\partial\Omega = \bigcup_{j=1}^{J} \Gamma_j, \quad \text{disjoint closed subsets.} \tag{4.6.4}$$

We do not assume the sets Γ_j are connected. Let us set

$$C_j f(x) = i\, \mathrm{PV} \int_{\Gamma_j} E_{01}(x,y)\sigma_D(y,\nu(y)) f(y)\, d\sigma(y), \tag{4.6.5}$$

for

$$x \in \Gamma_j, \quad f \in L^p(\Gamma_j, \mathcal{F}_0 \otimes \mathcal{C}). \tag{4.6.6}$$

We have

$$C_j : L^p(\Gamma_j) \longrightarrow L^p(\Gamma_j), \tag{4.6.7}$$

for $1 \le j \le J,\ p \in (1,\infty)$, hence $\oplus_1^J C_j : L^p(\partial\Omega) \to L^p(\partial\Omega)$. It is clear from the behavior of $E(x,y)$ that

$$C - \bigoplus_{j=1}^{J} C_j \ \text{ is compact on } L^p(\partial\Omega), \quad \forall\, p \in (1,\infty). \tag{4.6.8}$$

Consequently, with

$$\mathcal{P}_j = \frac{1}{2} I + C_j : L^p(\Gamma_j) \to L^p(\Gamma_j), \tag{4.6.9}$$

we have

$$\mathcal{P}_D - \bigoplus_{j=1}^{J} \mathcal{P}_j \text{ compact on } L^p(\partial\Omega), \quad \mathcal{P}_j^2 - \mathcal{P}_j \text{ compact on } L^p(\Gamma_j). \tag{4.6.10}$$

Then, with

$$T_{\Gamma_j,\Omega,\Phi} f = \mathcal{P}_j \Phi \mathcal{P}_j f + (I - \mathcal{P}_j) f, \quad f \in L^p(\Gamma_j), \tag{4.6.11}$$

we have

$$T_\Phi - \bigoplus_{j=1}^{J} T_{\Gamma_j,\Omega,\Phi} \text{ compact on } L^p(\partial\Omega). \tag{4.6.12}$$

Clearly $T_{\Gamma_j,\Omega,\Phi}$ depends only on $\Phi|_{\Gamma_j}$. Note that, if

$$\Phi, \Phi^{-1} \in L^\infty \cap \mathrm{vmo}(\partial\Omega, \mathrm{End}\, \mathcal{C}), \tag{4.6.13}$$

then each operator $T_{\Gamma_j,\Omega,\Phi}$ is Fredholm on $L^p(\Gamma_j, \mathcal{F}_0 \otimes \mathcal{C})$, if $p \in (1,\infty)$, by (4.6.12). Furthermore,

$$\mathrm{Index}\, T_\Phi = \sum_{j=1}^{J} \mathrm{Index}\, T_{\Gamma_j,\Omega,\Phi}. \tag{4.6.14}$$

We move on to another localization phenomenon. Namely, with $\Omega \subset M$ as above, assume there exists another Riemannian manifold $\widetilde{M}$, a neighborhood $\mathcal{O}$ of $\overline{\Omega}$ in M, and an open $\widetilde{\mathcal{O}} \subset \widetilde{M}$, isometric to $\mathcal{O}$. (From here on, we identify $\mathcal{O}$ and $\widetilde{\mathcal{O}}$.) Assume that there exists a first order elliptic differential operator $\widetilde{D}$ on $\widetilde{M}$, acting on sections of a vector bundle over $\widetilde{M}$ which agrees with $\mathcal{F}_0 \otimes \mathcal{C}$ on $\widetilde{\mathcal{O}} = \mathcal{O}$, such that the coefficients of $\widetilde{D}$ on $\widetilde{\mathcal{O}}$ agree with those of D on $\mathcal{O}$. We can then form

$$\widetilde{\mathcal{D}} = \begin{pmatrix} ib & \widetilde{D}^* \\ \widetilde{D} & ib \end{pmatrix}, \tag{4.6.15}$$

with $b \geq 0$ on $\widetilde{M}$, $b = 0$ on a neighborhood of $\overline{\Omega}$, $b > 0$ on $\widetilde{M} \setminus \widetilde{\mathcal{O}}$, so $\widetilde{\mathcal{D}}$ is invertible, and then we have the associated Toeplitz operator

$$T_{\widetilde{M},\Phi} : L^p(\partial\Omega, \mathcal{F}_0 \otimes \mathcal{C}) \longrightarrow L^p(\partial\Omega, \mathcal{F}_0 \otimes \mathcal{C}). \tag{4.6.16}$$

Here, $T_{\widetilde{M},\Phi} = \mathcal{P}_{\widetilde{D}} \Phi \mathcal{P}_{\widetilde{D}} + (I - \mathcal{P}_{\widetilde{D}})$, $\mathcal{P}_{\widetilde{D}} = (1/2)I + C_{\widetilde{D}}$, and $C_{\widetilde{D}}$ is given as in (4.6.3), with $E_{01}(x,y)$ replaced by $\widetilde{E}_{01}(x,y)$, and clearly the difference $E_{01}(x,y) - \widetilde{E}_{01}(x,y)$ is weakly singular. Hence, for $\Phi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, \mathrm{End}\,\mathcal{C})$,

$$T_\Phi - T_{\widetilde{M},\Phi} \ \text{ is compact on } \ L^p(\partial\Omega, \mathcal{F}_0 \otimes \mathcal{C}), \tag{4.6.17}$$

for all $p \in (1,\infty)$. Thus, if also $\Phi^{-1} \in L^\infty \cap \mathrm{vmo}(\partial\Omega, \mathrm{End}\,\mathcal{C})$,

$$\mathrm{Index}\, T_\Phi = \mathrm{Index}\, T_{\widetilde{M},\Phi}. \tag{4.6.18}$$

4.7 Cobordism invariance of the index

As in §4.6, we take D, of Dirac type, acting on sections of $\mathcal{F}_0 \otimes \mathcal{C}$, and assume $\Omega \subset M$ is a UR domain. We begin with the following significant sharpening of Proposition 4.5.5.

Proposition 4.7.1 *If $\Phi \in C(\overline{\Omega}, G\ell(\mathcal{C}))$, then*

$$\mathrm{Index}\, T_\Phi = 0. \tag{4.7.1}$$

Proof. Extend Φ to M as a continuous section of $\mathrm{End}\,\mathcal{C}$. There is a neighborhood $\mathcal{O} \supset \overline{\Omega}$ on which Φ is invertible. Then one can take a smoothly bounded M_0 such that $\overline{\Omega} \subset M_0 \subset \overline{M}_0 \subset \mathcal{O}$. Let $\widetilde{M}$ denote the double of $\overline{M}_0$. Using a gluing construction, it is shown in [33], p. 111, that there are vector bundles $\widetilde{\mathcal{F}}_j \to \widetilde{M}$, extending $\mathcal{F}_j|_{\overline{M}_0}$, and that there is a first order elliptic differential operator $\widetilde{D}$ on $\widetilde{M}$, extending $D|_{\overline{M}_0}$. In addition, one can use reflection to extend $\mathcal{C}|_{\overline{M}_0}$ to $\widetilde{\mathcal{C}} \to \widetilde{M}$ and to extend $\Phi|_{\overline{M}_0}$ to $\widetilde{\Phi} \in C(\widetilde{M}, G\ell(\widetilde{\mathcal{C}}))$. One can also give $\widetilde{\mathcal{C}}$ an extended connection.

Now Proposition 4.5.5 applies, to give

$$\mathrm{Index}\, T_{\widetilde{M},\Phi} = 0. \tag{4.7.2}$$

On the other hand, (4.6.18) implies

$$\mathrm{Index}\, T_\Phi = \mathrm{Index}\, T_{\widetilde{M},\Phi}, \tag{4.7.3}$$

so we have (4.7.1). □

Proposition 4.7.1 applies in the following setting. Let us take an open set $\mathcal{O} \subset \Omega$, with the properties that $\mathcal{O}$ is a UR domain and

$$\partial\mathcal{O} = \partial\Omega \cup \Gamma, \quad \text{disjoint closed subsets.} \tag{4.7.4}$$

Let

$$\Phi \in C(\overline{\mathcal{O}}, G\ell(\mathcal{C})). \tag{4.7.5}$$

Then, using notation as in (4.6.11), we have

$$\begin{aligned} T_\Phi &\text{ Fredholm on } L^p(\partial\Omega, \mathcal{F}_0 \otimes \mathcal{C}), \\ T_{\partial\Omega,\mathcal{O},\Phi} &\text{ Fredholm on } L^p(\partial\Omega, \mathcal{F}_0 \otimes \mathcal{C}), \\ T_{\Gamma,\mathcal{O},\Phi} &\text{ Fredholm on } L^p(\Gamma, \mathcal{F}_0 \otimes \mathcal{C}). \end{aligned} \tag{4.7.6}$$

Furthermore,

$$T_\Phi - T_{\partial\Omega,\mathcal{O},\Phi} \ \text{ is compact on } \ L^p(\partial\Omega), \tag{4.7.7}$$

so

$$\operatorname{Index} T_\Phi = \operatorname{Index} T_{\partial\Omega,\mathcal{O},\Phi}. \tag{4.7.8}$$

We also have the analogue of T_Φ, which we will denote $T_{\mathcal{O},\Phi}$, defined by replacing Ω by $\mathcal{O}$ in our basic construction, and (4.6.14) gives, in this setting,

$$\operatorname{Index} T_{\mathcal{O},\Phi} = \operatorname{Index} T_{\partial\Omega,\mathcal{O},\Phi} + \operatorname{Index} T_{\Gamma,\mathcal{O},\Phi}. \tag{4.7.9}$$

Given (4.7.5), we can apply Proposition 4.7.1 (with $\mathcal{O}$ in place of Ω) and deduce that

$$\operatorname{Index} T_{\mathcal{O},\Phi} = 0. \tag{4.7.10}$$

Putting this together with (4.7.8)–(4.7.9) then yields the identity

$$\operatorname{Index} T_\Phi = -\operatorname{Index} T_{\Gamma,\mathcal{O},\Phi}. \tag{4.7.11}$$

Furthermore, with $\widetilde{\Omega} = \Omega \setminus \overline{\mathcal{O}}$, we have $\partial\widetilde{\Omega} = \Gamma$ and

$$\operatorname{Index} T_{\Gamma,\mathcal{O},\Phi} = -\operatorname{Index} T_{\widetilde{\Omega},\Phi}. \tag{4.7.12}$$

Hence

$$\operatorname{Index} T_\Phi = \operatorname{Index} T_{\widetilde{\Omega},\Phi}. \tag{4.7.13}$$

This chain of reasoning can be used in cases where $\partial\Omega$ is rough but $\partial\widetilde{\Omega}$ is smooth. There are tools available for calculating the right side of (4.7.13) (including the Atiyah-Singer index theorem) when $\partial\widetilde{\Omega}$ is smooth, so the identity (4.7.13) provides a path for the calculation of the index of T_Φ, in many cases where $\partial\Omega$ is rough.

4.8 Further results on index computations

Here we use results from previous sections to draw conclusions about computing

$$\iota(\Phi) = \iota(\Phi; D) = \operatorname{Index} T_\Phi. \tag{4.8.1}$$

We assume for simplicity that

$$\Phi \in C(\partial\Omega, G\ell(\ell)). \tag{4.8.2}$$

Results for $\Phi \in L^\infty \cap \operatorname{vmo}(\partial\Omega, G\ell(\ell))$ follow via results of §4.2.

We begin with general conclusions that can be drawn from the fact that (4.8.1) yields a group homomorphism

$$\iota : [\partial\Omega; G\ell(\ell)] \longrightarrow \mathbb{Z}, \tag{4.8.3}$$

where $[\partial\Omega; G\ell(\ell)]$ is the group of homotopy classes of continuous maps of $\partial\Omega$ to $G\ell(\ell)$. Writing

$$\Phi(x) = A(x)\Psi(x), \quad A(x) = \big(\Phi(x)\Phi(x)^*\big)^{1/2}, \quad \Psi \in C(\partial\Omega, U(\ell)), \tag{4.8.4}$$

we have $\iota(\Phi) = \iota(A) + \iota(\Psi) = \iota(\Psi)$, since a homotopy argument gives $\iota(A) = 0$, thanks to the contractibility of the set of positive-definite $\ell \times \ell$ matrices. Hence we can focus attention on the case

$$\Phi \in C(\partial\Omega, U(\ell)). \tag{4.8.5}$$

Furthermore, we can write

$$\Phi(x) = \Phi_0(x)\Phi_1(x), \tag{4.8.6}$$

with

$$\Phi_0(x) = \begin{pmatrix} \varphi(x) & \\ & I \end{pmatrix}, \quad \varphi(x) = \det \Phi(x), \quad \Phi_1 \in C(\partial\Omega, SU(\ell)), \tag{4.8.7}$$

and

$$\begin{aligned} \iota(\Phi) &= \iota(\Phi_0) + \iota(\Phi_1) \\ &= \iota(\varphi) + \iota(\Phi_1), \end{aligned} \tag{4.8.8}$$

with $\varphi \in C(\partial\Omega, S^1)$, $S^1 \subset \mathbb{C}$. Then

$$[\partial\Omega; S^1] = 0 \Longrightarrow \iota(\Phi) = \iota(\Phi_1), \tag{4.8.9}$$

and

$$[\partial\Omega; SU(\ell)] = 0 \Longrightarrow \iota(\Phi) = \iota(\varphi). \tag{4.8.10}$$

As for the applicability of (4.8.9)–(4.8.10), we note that

$$\partial\Omega \ \text{ simply connected } \Longrightarrow [\partial\Omega; S^1] = 0, \tag{4.8.11}$$

and

$$\dim \Omega \le 3 \Longrightarrow [\partial\Omega; SU(2)] = 0. \tag{4.8.12}$$

On the other hand, $[\mathbb{T}^3; S^1] \neq 0$ and $[\mathbb{T}^3; SU(2)] \neq 0$.

We now specialize to the case where $\partial\Omega$ is homeomorphic to a sphere:

$$\partial\Omega \approx S^m, \quad m = n - 1 \quad (n = \dim \Omega). \tag{4.8.13}$$

In such a case, $[\partial\Omega; U(\ell)] \approx \pi_m(U(\ell))$. That is to say, we are in the setting of $\pi_m(Y)$, the group of homotopy classes of maps from the sphere S^m to a space Y (with $Y = U(\ell)$). Classical results of Bott (cf. [18]) imply

$$m = 2\mu - 1 \Longrightarrow \pi_m(U(\ell)) \approx \mathbb{Z}, \ \text{ if } \ \ell \ge \mu. \tag{4.8.14}$$

By contrast,

$$m \notin \{1, 3, \dots, 2\ell - 1\} \Longrightarrow \pi_m(U(\ell)) \ \text{ is finite.} \tag{4.8.15}$$

When (4.8.14) holds, let

$$\vartheta : [\partial\Omega; U(\ell)] \xrightarrow{\approx} \mathbb{Z} \tag{4.8.16}$$

denote the induced isomorphism (uniquely defined up to sign). We have the following.

Proposition 4.8.1 *Assume $\Omega \subset M$ is a UR domain and* (4.8.13) *holds. If $m = 2\mu - 1$ and $\ell \geq \mu$, then there exists $\alpha = \alpha(\Omega, D) \in \mathbb{Z}$ such that*

$$\iota(\Phi, D) = \alpha\vartheta([\Phi]), \quad \forall\, \Phi \in C(\partial\Omega, U(\ell)). \tag{4.8.17}$$

If $m \notin \{1, 3, \dots, 2\ell - 1\}$, then

$$\iota(\Phi; D) = 0, \quad \forall\, \Phi \in C(\partial\Omega, U(\ell)). \tag{4.8.18}$$

Actually, the argument given so far yields (4.8.17) with $\alpha = \alpha_\ell$, possibly depending on ℓ (satisfying $\ell \geq \mu,\ m = 2\mu - 1$). We now establish that α is independent of such ℓ (up to sign). This uses the fact that the natural inclusion $U(\ell) \hookrightarrow U(\ell + 1)$ induces an isomorphism $\pi_m(U(\ell)) \approx \pi_m(U(\ell + 1))$ for $m = 2\mu - 1,\ \ell \geq \mu$ (cf. [18]). In more detail, let $\Phi_\ell \in C(\partial\Omega, U(\ell))$ have the property that its homotopy class generates $\pi_m(U(\ell))$. Then $\iota(\Phi_\ell, D) = \alpha_\ell$ (up to sign). Now $\Phi_{\ell+1} \in C(\partial\Omega, U(\ell + 1))$, given by

$$\Phi_{\ell+1} = \begin{pmatrix} \Phi_\ell & \\ & 1 \end{pmatrix}$$

gives a generator of $\pi_m(U(\ell + 1))$, by the isomorphism mentioned above, and clearly $\iota(\Phi_{\ell+1}, D) = \iota(\Phi_\ell, D)$. Hence $\alpha_{\ell+1} = \alpha_\ell$ (up to sign), as asserted.

This argument also yields the following.

Corollary 4.8.2 *In the setting of Proposition 4.8.1, if $m = 2\mu - 1$ and $\ell_1 \geq \mu$, and if there exists $\Phi_1 \in C(\partial\Omega, U(\ell_1))$ such that*

$$\operatorname{Index} T_{\Phi_1} = 1, \tag{4.8.19}$$

then (4.8.17) *holds with $\alpha = \pm 1$, for all $\ell \geq \mu$.*

In fact, we see that α must be a nonzero integer of magnitude ≤ 1.

Our next goal is to produce some cases where Corollary 4.8.2 applies. We begin with an apparent digression. Namely, let $B \subset \mathbb{C}^\mu$ be the unit ball. Assume $\mu \geq 2$. Let $S_h : L^2(B) \to L^2(B)$ be the Szegö projector onto the space of boundary values of functions holomorphic on B. Since holomorphic functions satisfy an overdetermined elliptic system, this is a different sort of projector from what we have been considering. For example,

$$S_h \in OPS^0_{1/2,1/2}(\partial B). \tag{4.8.20}$$

This is sufficient to imply that operators $\tau_\Phi = S_h \Phi S_h + (I - S_h)$ are Fredholm if $\Phi \in C(\partial B, U(\ell))$, and one has an analogue of (4.8.17):

$$\operatorname{Index} \tau_\Phi = \alpha_h \vartheta([\Phi]). \tag{4.8.21}$$

In [32], it is shown that (4.8.21) holds with $\alpha_h = \pm 1$. An alternative treatment of such an index formula, in a more general setting, was done by Boutet de Monvel in [5]. His formula, valid when $B \subset \mathbb{C}^\mu$ is a smoothly bounded, strongly pseudoconvex domain, can be described as follows. Consider

$$D = \overline{\partial} + \overline{\partial}^* : \Lambda^{0,\text{even}}(\mathbb{C}^\mu) \longrightarrow \Lambda^{0,\text{odd}}(\mathbb{C}^\mu). \tag{4.8.22}$$

This is an operator of Dirac type. Then

$$\operatorname{Index} \tau_\Phi = \iota(\Phi; D). \tag{4.8.23}$$

See also [3], for a proof of (4.8.23) using K-homology. We have the following consequence.

Proposition 4.8.3 *When $\Omega = B$ is the unit ball in $\mathbb{C}^\mu$ and D is given by* (4.8.22), *then* (4.8.17) *holds with $\alpha = \pm 1$, provided $\ell \geq \mu$.*

From here, we obtain the following.

Proposition 4.8.4 *Let $\Omega \subset \mathbb{C}^\mu$ be a bounded UR domain and let D be given by* (4.8.22). *Let $\ell \geq \mu$. Then*

$$\text{there exists } \Phi_1 \in C(\partial\Omega, U(\ell)) \text{ such that } \operatorname{Index} T_{\Phi_1} = 1. \tag{4.8.24}$$

Proof. We can assume $0 \in B \subset \overline{B} \subset \Omega$. Take $\Phi_1 \in C(\partial B, U(\ell))$ such that T_{B,Φ_1} has index 1, using Proposition 4.8.3. Then extend Φ_1 to an element of $C(\mathbb{C}^\mu \setminus 0, U(\ell))$, homogeneous of degree 0, and restrict to $\partial\Omega$. The cobordism argument of §4.7 implies

$$\operatorname{Index} T_{\Omega,\Phi_1} = \operatorname{Index} T_{B,\Phi_1}, \tag{4.8.25}$$

so we have (4.8.24). □

Corollary 4.8.5 *Let $\Omega \subset \mathbb{C}^\mu$ be a bounded UR domain and let D be given by* (4.8.22). *If $\partial\Omega$ is homeomorphic to $S^{2\mu-1}$, then* (4.8.17) *holds, with $\alpha = \pm 1$.*

4.9 Another class of Toeplitz operators

Let D be a first order elliptic differential operator on a compact manifold M, as in §1. Let $\Omega \subset M$ be open, possibly with nasty boundary. We define

$$\mathcal{D} = \begin{pmatrix} & D^* \\ D & \end{pmatrix} \tag{4.9.1}$$

as a closed, unbounded operator on $H_0 \oplus H_1$ (with $H_j = L^2(\Omega, \mathcal{F}_j)$), using the maximal extension of D, so

$$\operatorname{Dom} D = \{u \in H_0 : Du \in H_1\}, \quad \operatorname{Dom} D^* = H_0^1(\Omega, \mathcal{F}_1). \tag{4.9.2}$$

Here $H_0^1(\Omega, \mathcal{F}_1)$ denotes the closure in $H^1(\Omega, \mathcal{F}_1)$ of the space of smooth sections with compact support in Ω. Then $\mathcal{D}^2$ has compact resolvent on H_1, though not on H_0. (In fact, D has infinite dimensional null space on H_0.) We recall some results from [3]. From Proposition 1.1 of [3] we have

$$[M_\varphi, \mathcal{D}(\mathcal{D}^2 + 1)^{-1/2}] \text{ compact on } H_0 \oplus H_1, \quad \forall\, \varphi \in C(\overline{\Omega}), \tag{4.9.3}$$

where M_φ acts on H_j by scalar multiplication. Furthermore, $\mathcal{D}$ has closed range. As shown in Proposition 3.1 of [3], (4.9.3) implies that the pair $(M, \mathcal{D})$ defines a relative cycle

$$[D] \in K_0(\overline{\Omega}, \partial\Omega). \tag{4.9.4}$$

Here, $K_0(\overline{\Omega}, \partial\Omega)$ denotes a relative K-homology group. We refer the reader to [3] for the definition and basic properties of this group, and also to the groups $K_1(\partial\Omega)$ and $K^1(\partial\Omega)$ mentioned below.

If P_0 denotes the orthogonal projection of H_0 onto

$$\mathfrak{H}^2(\Omega, D) = \{u \in L^2(\Omega, \mathcal{F}_0) : Du = 0\}, \tag{4.9.5}$$

and we set

$$\tilde{\tau}_\varphi : \mathfrak{H}^2(\Omega, D) \longrightarrow \mathfrak{H}^2(\Omega, D), \quad \tilde{\tau}_\varphi u = P_0 M_\varphi u, \tag{4.9.6}$$

for $u \in \mathfrak{H}^2(\Omega, D),\ \varphi \in C(\overline{\Omega})$, then with

$$C_*(\Omega) = \{\varphi \in C(\overline{\Omega}) : \varphi|_{\partial\Omega} = 0\}, \tag{4.9.7}$$

we have

$$\varphi \in C_*(\Omega) \Longrightarrow \tilde{\tau}_\varphi \ \text{ compact on } \ \mathfrak{H}^2(\Omega, D), \tag{4.9.8}$$

so we get a linear map

$$\tau : C(\partial\Omega) \longrightarrow \mathcal{Q}(\mathfrak{H}^2(\Omega, D)), \quad \tau_f = \tilde{\tau}_\varphi,\ \varphi|_{\partial\Omega} = f, \tag{4.9.9}$$

where if H is a Hilbert space, $\mathcal{Q}(H) = \mathcal{L}(H)/\mathcal{K}(H)$ is the Calkin algebra. Furthermore, closely related to (4.9.3), we have

$$\varphi \in C(\overline{\Omega}) \Longrightarrow [M_\varphi, P_0] \ \text{ compact}, \tag{4.9.10}$$

so

$$\tau_{fg} = \tau_f \tau_g, \quad f, g \in C(\partial\Omega). \tag{4.9.11}$$

The map τ defines a K-homology cycle,

$$[\tau] \in K_1(\partial\Omega), \tag{4.9.12}$$

and (cf. [3], Proposition 4.1) we have

$$[\tau] = \partial[D], \quad \partial : K_0(\overline{\Omega}, \partial\Omega) \to K_1(\partial\Omega). \tag{4.9.13}$$

We move from scalar multipliers to matrix multipliers. If $\Phi \in C(\partial\Omega, \operatorname{End}\mathbb{C}^\ell)$, we have

$$\tau_\Phi \in \mathcal{Q}(\mathfrak{H}^2(\Omega, D) \otimes \mathbb{C}^\ell), \tag{4.9.14}$$

and if also $\Psi \in C(\partial\Omega, \operatorname{End}\mathbb{C}^\ell)$,

$$\tau_\Phi \tau_\Psi = \tau_{\Phi\Psi}. \tag{4.9.15}$$

Hence

$$\Phi \in C(\partial\Omega, G\ell(\ell, \mathbb{C})) \Longrightarrow \tau_\Phi \ \text{ invertible in } \ \mathcal{Q}(\mathfrak{H}^2(\Omega, D) \otimes \mathbb{C}^\ell), \tag{4.9.16}$$

and we have an index map

$$j(\Phi) = \ \text{index of } \ \tau_\Phi, \tag{4.9.17}$$

giving

$$j : [\partial\Omega; G\ell(\ell, \mathbb{C})] \longrightarrow \mathbb{Z}, \quad \text{homomorphism}. \tag{4.9.18}$$

As in (4.1.30), $[\partial\Omega; G\ell(\ell, \mathbb{C})]$ denotes the group of homotopy classes of continuous maps $\partial\Omega \to G\ell(\ell, \mathbb{C})$.

This also has a K-theoretic interpretation. The homotopy class of $\Phi \in C(\partial\Omega, G\ell(\ell, \mathbb{C}))$ defines

$$[\Phi] \in K^1(\partial\Omega), \tag{4.9.19}$$

and, with $[\tau]$ as in (4.9.12)–(4.9.13),

$$j(\Phi) = \langle [\tau], [\Phi] \rangle \tag{4.9.20}$$

is given by the intersection product

$$K_1(\partial\Omega) \times K^1(\partial\Omega) \longrightarrow \mathbb{Z}. \tag{4.9.21}$$

To compare $j(\Phi)$ with $\iota(\Phi)$ from §4.1, we note that Proposition 4.3 of [3] yields the following.

Proposition 4.9.1 *If Ω has smooth boundary, then*

$$j(\Phi) = \iota(\Phi), \quad \forall\, \Phi \in C(\partial\Omega, G\ell(\ell,\mathbb{C})). \tag{4.9.22}$$

Generally, when Ω is a UR domain, the association $\varphi \mapsto T_\varphi$ constructed in §4.1 yields an element

$$[T] \in K_1(\partial\Omega). \tag{4.9.23}$$

Proposition 4.3 of [3] implies

$$[T] = [\tau] \ \text{ in } \ K_1(\partial\Omega), \tag{4.9.24}$$

when Ω has smooth boundary. One might conjecture that (4.9.24) holds for general UR domains. This would imply that (4.9.22) holds for general UR domains, since both sides of (4.9.22) are given by the intersection product (4.9.21).

A Auxiliary results

A.1 Invertibility of $\mathcal{D}$ and behavior of $\mathcal{D}^{-1}$

As in §1, we have a first order elliptic differential operator D, mapping from sections of $\mathcal{F}_0$ to sections of $\mathcal{F}_1$ (each of rank κ), on a compact, connected Riemannian manifold M, equipped with a C^2 metric tensor. Such an operator is given in a local coordinate chart U (and with respect to local trivializations of $\mathcal{F}_j$) by

$$Du(x) = A_j(x)\partial_j u(x) + B(x)u(x), \tag{A.1.1}$$

(using the summation convention), and we assume

$$A_j \in C^2(U, \operatorname{End}\mathbb{C}^\kappa), \quad B \in C^1(U, \operatorname{End}\mathbb{C}^\kappa). \tag{A.1.2}$$

We take $a \in C^1(M),\ a \geq 0$, and set

$$\mathcal{D} = \begin{pmatrix} iM_a & D^* \\ D & iM_a \end{pmatrix}, \tag{A.1.3}$$

where $M_a u = au$. Here, in such local coordinates,

$$\begin{aligned} D^*v(x) &= -A_j(x)^*\partial_j v(x) + \widetilde{B}(x)v(x), \\ \widetilde{B}(x) &= -g(x)^{-1/2}\partial_j(g(x)^{1/2}A_j(x)^*) + B(x)^*, \end{aligned} \tag{A.1.4}$$

so $A_j^* \in C^2(U, \operatorname{End}\mathbb{C}^\kappa)$ and $\widetilde{B} \in C^1(U, \operatorname{End}\mathbb{C}^\kappa)$. In this situation, we have

$$\mathcal{D} : H^{s+1,p}(M,\mathcal{E}) \longrightarrow H^{s,p}(M,\mathcal{E}), \quad s \in [-2,1], \ \ p \in (1,\infty), \tag{A.1.5}$$

where $\mathcal{E} = \mathcal{F}_0 \oplus \mathcal{F}_1$. We begin our investigation of conditions under which $\mathcal{D}$ is invertible in (A.1.5) with the following result.

Proposition A.1.1 *Under the hypotheses given above, $\mathcal{D}$ in (A.1.5) is Fredholm, of index zero, and in each such case,*

$$\operatorname{Ker}\mathcal{D} \subset H^{2,q}(M,\mathcal{E}), \quad \forall\, q < \infty. \tag{A.1.6}$$

Proof. The symbol smoothing technique described in Chapter 2 of [29] (see also [31], Chapter 13, §9) gives, for $\delta \in (0,1)$,

$$\begin{aligned} \mathcal{D} = \mathcal{D}^{\#} + \mathcal{D}^b + M_{\widehat{B}}, \quad \mathcal{D}^{\#} \in OPS^1_{1,\delta}, \text{ elliptic}, \\ \mathcal{D}^b \in OPC^2 S^{1-2\delta}_{1,\delta}, \quad \widehat{B} \in C^1. \end{aligned} \tag{A.1.7}$$

Then $\mathcal{D}^{\#}$ has a two-sided parametrix

$$E^{\#} \in OPS^{-1}_{1,\delta}, \tag{A.1.8}$$

with the mapping property

$$E^{\#} : H^{s,p}(M,\mathcal{E}) \longrightarrow H^{s+1,p}(M,\mathcal{E}) \tag{A.1.9}$$

(valid for all $s \in \mathbb{R},\ p \in (1,\infty)$), and a standard iterative argument applied to

$$\mathcal{D}u = f \Rightarrow u = E^{\#} f - E^{\#}\mathcal{D}^b u - E^{\#}(\widehat{B}u), \quad \text{mod} \quad C^\infty, \tag{A.1.10}$$

making use of the mapping property (A.1.9) and

$$\mathcal{D}^b : H^{\sigma+1-2\delta,p} \longrightarrow H^{\sigma,p}, \quad \forall\, \sigma \in (-2(1-\delta),2),\ p \in (1,\infty), \tag{A.1.11}$$

gives

$$u \in H^{-1,p}(M,\mathcal{E}),\ \mathcal{D}u \in H^{s,q}(M,\mathcal{E}) \Longrightarrow u \in H^{s+1,q}(M,\mathcal{E}), \tag{A.1.12}$$

for each $s \in (-2,1],\ p,q \in (1,\infty)$. This implies (A.1.6). Furthermore, $E^{\#}$ is a two-sided Fredholm inverse of $\mathcal{D}$ in (A.1.5). As for the index calculation, note that $\mathcal{D}$ differs by a compact operator from what one gets by taking $a \equiv 0$, so

$$\operatorname{Index} \mathcal{D} = \operatorname{Index} D + \operatorname{Index} D^* = 0, \tag{A.1.13}$$

where the last calculation takes into account the regularity result (A.1.6). □

Proposition A.1.2 *In the setting of Proposition A.1.1, for* $u = (u_0,u_1)^t \in \cap_q H^{2,q}(M,\mathcal{E})$,

$$\mathcal{D}u = 0 \Longleftrightarrow u\big|_{\mathcal{O}} = 0, \quad Du_0 = 0, \quad D^* u_1 = 0, \tag{A.1.14}$$

where

$$\mathcal{O} = \{x \in M : a(x) > 0\}. \tag{A.1.15}$$

Proof. The implication $\Leftarrow$ is obvious. For the implication $\Rightarrow$, note that if $u \in \operatorname{Ker} \mathcal{D}$, then

$$\operatorname{Im}(\mathcal{D}u,u)_{L^2} = \int_M a\langle u,u\rangle\, dV = 0. \tag{A.1.16}$$

Hence $u = (u_0,u_1)^t \in \operatorname{Ker}\mathcal{D}$ satisfies $au = 0$, so $u = 0$ on $\mathcal{O}$, and hence $Du_0 = 0$ and $D^* u_1 = 0$ on M. □

Putting together Propositions A.1.1–A.1.2 gives the following.

Corollary A.1.3 *In the setting of Propositions A.1.1–A.1.2, $\mathcal{D}$ in (A.1.5) is invertible provided that, given $u = (u_0, u_1)^t \in \cap_q H^{2,q}(M, \mathcal{E})$,*

$$u_0\big|_{\mathcal{O}} = 0, \quad Du_0 = 0 \quad on \quad M \Longrightarrow u_0 = 0 \quad on \quad M, \tag{A.1.17}$$

and

$$u_1\big|_{\mathcal{O}} = 0, \quad D^* u_1 = 0 \quad on \quad M \Longrightarrow u_1 = 0 \quad on \quad M. \tag{A.1.18}$$

The operator D is said to have the unique continuation property (UCP) provided the implication (A.1.17) holds for arbitrary nonempty $\mathcal{O} \subset M$ (and without the requirement that M be compact). There is a similar notion for D^* to have UCP. We thus have the following.

Corollary A.1.4 *In the setting of Proposition A.1.1, if $\mathcal{O}$ in (A.1.15) is nonempty, then $\mathcal{D}$ in (A.1.5) is invertible, provided*

$$D \text{ and } D^* \text{ have UCP.} \tag{A.1.19}$$

One well known case where Corollary A.1.4 applies is when M has a real analytic metric tensor and the coefficients of D (and hence of D^*) are real analytic. Then Holmgren's uniqueness theorem implies (A.1.19). Another is when $\sigma_D(x,\xi)^*\sigma_D(x,\xi)$ is a scalar multiple of the identity. Here, $\sigma_D(x,\xi)$ denotes the principal symbol of D. In local coordinates (with the summation convention)

$$\sigma_D(x,\xi) = iA_j(x)\xi_j. \tag{A.1.20}$$

In such a case, one sometimes says D is of Dirac type. Here is a more general class of operators to which we will show (A.1.19) applies.

Definition. We say D is of *generalized Dirac type* provided there exists a first order elliptic differential operator $\widetilde{D} : H^{s+1,p}(M, \mathcal{F}_1) \to H^{s,p}(M, \mathcal{F}_0)$, given in local coordinates by

$$\widetilde{D}v(x) = \widetilde{A}_j(x)\partial_j v(x) + \widetilde{B}(x)v(x), \quad \widetilde{A}_j \in C^2, \ \widetilde{B} \in C^1, \tag{A.1.21}$$

such that

$$\sigma_{\widetilde{D}}(x,\xi)\sigma_D(x,\xi) = \gamma(x,\xi)I, \quad \gamma(x,\xi) \in (0,\infty) \ \text{ for } \ \xi \neq 0. \tag{A.1.22}$$

Note that

$$\gamma(x,\xi) = \gamma^{jk}(x)\xi_j\xi_k, \quad \gamma^{jk} \in C^2(U). \tag{A.1.23}$$

Proposition A.1.5 *If D is of generalized Dirac type, with D and $\widetilde{D}$ having the regularity of (A.1.2) and (A.1.21), then D, $\widetilde{D}$, and D^* all have UCP.*

Proof. If $u_0 \in \cap_q H^{2,q}(M, \mathcal{F}_0)$ satisfies $Du_0 = 0$, then

$$Lu_0 = 0 \ \text{ on } \ M, \quad L = \widetilde{D}D, \tag{A.1.24}$$

and, in local coordinates

$$Lu_0(x) = -\gamma^{jk}(x)\partial_j\partial_k u_0(x) + X_j(x)\partial_j u_0(x) + Y(x)u_0(x), \tag{A.1.25}$$

where

$$\begin{aligned} X_j &= \widetilde{A}_k(\partial_k A_j) + \widetilde{A}_j B + \widetilde{B}A_j \in C^1(U), \\ Y &= \widetilde{A}_j(\partial_j B) + \widetilde{B}B \in C^0(U). \end{aligned} \tag{A.1.26}$$

Thus L is a strongly elliptic, second order operator, with real principal symbol. Classic work of [1] and [8] yields

$$L \text{ has UCP.} \tag{A.1.27}$$

Hence D has UCP, if D is of generalized Dirac type. Note that (A.1.22) implies

$$\sigma_D(x,\xi)\sigma_{\widetilde{D}}(x,\xi) = \gamma(x,\xi)I, \tag{A.1.28}$$

so also $\widetilde{D}$ is of generalized Dirac type. Applying adjoints to (A.1.28) gives

$$\sigma_{\widetilde{D}^*}(x,\xi)\sigma_{D^*}(x,\xi) = \gamma(x,\xi)I, \tag{A.1.29}$$

so also D^* is of generalized Dirac type. This finishes the proof of Proposition A.1.5. □

For the rest of this subsection, we assume that $\mathcal{D}$ in (A.1.5) is invertible, with inverse

$$\mathcal{D}^{-1} : H^{s,p}(M,\mathcal{E}) \longrightarrow H^{s+1,p}(M,\mathcal{E}), \quad s \in [-2,1],\ p \in (1,\infty). \tag{A.1.30}$$

We investigate properties of its integral kernel $E(x,y)$, given by

$$\mathcal{D}^{-1}u(x) = \int_M E(x,y)u(y)\,dV(y). \tag{A.1.31}$$

Note that $\delta_y \in H^{-\varepsilon,p}$ for each $\varepsilon > 0$, for some $p = p(\varepsilon) > 1$, and $E(x,y) = \mathcal{D}^{-1}\delta_y(x)$. We have $E(\cdot,y) \in H^{1-\varepsilon,p(\varepsilon)}(M)$. Furthermore, the arguments yielding the regularity result (A.1.12) are of a local nature, and we have

$$E(\cdot,y) \in H^{2,q}_{\text{loc}}(M \setminus \{y\}), \quad \forall\, q < \infty. \tag{A.1.32}$$

Similar arguments apply to $E^*(x,y) = E(y,x)^*$, the integral kernel of $(\mathcal{D}^*)^{-1}$, yielding

$$E(y,\cdot) \in H^{2,q}_{\text{loc}}(M \setminus \{y\}), \quad \forall\, q < \infty. \tag{A.1.33}$$

It follows that $(\Delta_x + \Delta_y)E \in L^q_{\text{loc}}(M \times M \setminus \text{diag})$, and hence

$$E \in H^{2,q}_{\text{loc}}(M \times M \setminus \text{diag}), \quad \forall\, q < \infty. \tag{A.1.34}$$

In particular,

$$E \in C^r_{\text{loc}}(M \times M \setminus \text{diag}), \quad \forall\, r < 2. \tag{A.1.35}$$

It remains to investigate E on a small neighborhood of the diagonal. Hence, given $y_0 \in M$, we want to investigate E on $\mathcal{O} \times \mathcal{O}$, where $\mathcal{O}$ is a coordinate neighborhood of y_0. Our subsequent calculations will be done in such a coordinate chart.

Recall that the class of classical symbols S^m_{cl} is defined by requiring that (the matrix-valued) function $q(x,\xi)$ has an asymptotic expansion of the form

$$q(x,\xi) \sim q_m(x,\xi) + q_{m-1}(x,\xi) + \cdots, \tag{A.1.36}$$

with q_j smooth in x and ξ and homogeneous of degree j in ξ (for $|\xi| \geq 1$). Here we find it convenient to work with classes of symbols $C^rS^m_{1,0}$ which are only C^r in the spatial variable, while still C^∞ in the Fourier variable. The family of classical pseudodifferential operators associated with such symbols whose symbols can be expanded as in (A.1.36), where $q_j(X,\xi) \in C^r S^{m-j}_{1,0}$ is homogeneous of degree j in ξ for $|\xi| \ge 1$, $j = m, m-1, \dots$, will be denoted $\mathrm{OP}C^r S^m_{\mathrm{cl}}$. Finally, we set $\text{Ø}\mathrm{P}C^r S^m_{\mathrm{cl}}$ for the space of all formal adjoints of operators in $\mathrm{OP}C^r S^m_{\mathrm{cl}}$.

Let $E_0(D,x)$ denote the operator in $\text{Ø}\mathrm{P}C^2 S^{-1}_{\mathrm{cl}}$, given by

$$E_0(D,x)u(x) = (2\pi)^{-n} \int E_0(\xi,y) e^{i(x-y)\cdot\xi} u(y)\, dy\, d\xi, \tag{A.1.37}$$

with

$$\begin{aligned} E_0(\xi,y) &= \mathcal{A}(y,\xi)^{-1}, \quad \mathcal{A}(y,\xi) = \sum_j i\mathcal{A}_j(y)\xi_j, \\ \mathcal{A}_j(y) &= \begin{pmatrix} 0 & -A_j(y)^* \\ A_j(y) & 0 \end{pmatrix}. \end{aligned} \tag{A.1.38}$$

The Schwartz kernel of $E_0(D,x)$ has the form

$$e_0(x-y,y) = (2\pi)^{-n} \int E_0(\xi,y) e^{i(x-y)\cdot\xi}\, d\xi. \tag{A.1.39}$$

It follows that $e_0(z,y)$ is odd in z, smooth in $z \in \mathbb{R}^n \setminus 0$, and homogeneous of degree $-(n-1)$ in z, with C^2 y-dependence. Let us also note that $e_0(x-y,y)$ has a strictly off-diagonal form:

$$e_0(x-y,y) = \begin{pmatrix} 0 & e_{01}(x-y,y) \\ e_{10}(x-y,y) & 0 \end{pmatrix}. \tag{A.1.40}$$

As a first step toward comparing $e_0(x-y,y)$ and $E(x,y)$, we apply $\mathcal{D}$ to (A.1.37), obtaining

$$\mathcal{D}E_0(D,x)u(x) = (2\pi)^{-n} \int \big[\mathcal{A}(x,\xi) + B^\star(x)\big] E_0(\xi,y) e^{i(x-y)\cdot\xi} u(y)\, dy\, d\xi, \tag{A.1.41}$$

with $\mathcal{A}(x,\xi)$ as in (A.1.38) and $B^\star \in C^1(\mathcal{O})$. Note that $\mathcal{A}(y,\xi)E_0(\xi,y) = I$. Hence

$$\begin{aligned} \mathcal{A}(x,\xi)E_0(y,\xi) &= I + [\mathcal{A}(x,\xi) - \mathcal{A}(y,\xi)]E_0(\xi,y) \\ &= I + \sum H_{\ell j}(x,y)(x_\ell - y_\ell)\xi_j E_0(\xi,y), \end{aligned} \tag{A.1.42}$$

where

$$i\mathcal{A}_j(x) - i\mathcal{A}_j(y) = \sum_\ell H_{\ell j}(x,y)(x_\ell - y_\ell), \quad H_{\ell j} \in C^1(\mathcal{O} \times \mathcal{O}). \tag{A.1.43}$$

Then an integration by parts gives

$$\mathcal{D}E_0(D,x)u(x) = u(x) + \int R(x,y)u(y)\, dy, \tag{A.1.44}$$

where

$$\begin{aligned} R(x,y) = {} & (2\pi)^{-n} \int B^\star E_0(\xi,y) e^{i(x-y)\cdot\xi}\, d\xi \\ & - i(2\pi)^{-n} \int \frac{\partial}{\partial \xi_\ell} \big\{ H_{\ell j}(x,y)\xi_j E_0(\xi,y) \big\} e^{i(x-y)\cdot\xi}\, d\xi. \end{aligned} \tag{A.1.45}$$

The amplitudes in the integrands in (A.1.45) are homogeneous in ξ of degree -1. Hence

$$|R(x,y)| \le C|x-y|^{-(n-1)}. \tag{A.1.46}$$

In terms of (A.1.39), we have

$$\mathcal{D}_x e_0(x-y,y) = \delta_y(x) + R(x,y). \tag{A.1.47}$$

In local coordinates,

$$\mathcal{D}^{-1}u(x) = \int E(x,y)u(y)\sqrt{g(y)}\,dy, \tag{A.1.48}$$

since $dV(y) = \sqrt{g(y)}\,dy$. We desire to estimate the difference

$$e_1(x,y) = E(x,y)\sqrt{g(y)} - e_0(x-y,y). \tag{A.1.49}$$

Note that, by (A.1.48),

$$\mathcal{D}_x e_1(x,y) = -R(x,y). \tag{A.1.50}$$

Given (A.1.46), a dilation argument parallel to that used on pp. 200–201 of [23] gives, for each $\varepsilon > 0$,

$$|e_1(x,y)| \le C_\varepsilon |x-y|^{-(n-2+\varepsilon)}, \tag{A.1.51}$$

and

$$|\nabla_x e_1(x,y)| \le C_\varepsilon |x-y|^{-(n-1+\varepsilon)}. \tag{A.1.52}$$

From the results on $e_0(x-y,y)$ above plus (A.1.51)–(A.1.52), we deduce that

$$|E(x,y)| \le C\,\mathrm{dist}(x,y)^{-(n-1)}, \quad |\nabla_x E(x,y)| \le C\,\mathrm{dist}(x,y)^{-n}. \tag{A.1.53}$$

Since the integral kernel of $(\mathcal{D}^*)^{-1}$ is $E^*(x,y) = E(y,x)^*$, we deduce that also

$$|\nabla_y E(x,y)| \le C\,\mathrm{dist}(x,y)^{-n}. \tag{A.1.54}$$

Remark. If the metric tensor of M and coefficients of $\mathcal{D}$ are C^∞, the analysis of $E(x,y)$ can be done much more briefly. In that case,

$$\mathcal{D}^{-1},\ E_0(D,x) \in OPS^{-1}_{\mathrm{cl}}(M), \tag{A.1.55}$$

and

$$\mathcal{D}^{-1} - E_0(D,x) \in OPS^{-2}_{\mathrm{cl}}(M), \tag{A.1.56}$$

so $e_1(x,y)$ is the integral kernel of an operator in $OPS^{-2}_{\mathrm{cl}}(M)$. These results imply (A.1.53)–(A.1.54) and also (A.1.51)–(A.1.52), with $\varepsilon = 0$, except that, when $n = 2$, (A.1.51) becomes

$$|e_1(x,y)| \le C \log \frac{1}{|x-y|}, \tag{A.1.57}$$

for $|x-y|$ small.

A.2 L^p-Sobolev spaces on boundaries of Ahlfors regular domains

We recall some results from §3.6 of [11], but with some slightly different arguments. Let $\Omega \subset \mathbb{R}^n$ be bounded and Ahlfors regular. Given $\varphi \in C_0^1(\mathbb{R}^n)$, we set

$$\partial_{\tau_{jk}} \varphi = \nu_k \partial_j \varphi - \nu_j \partial_k \varphi \big|_{\partial\Omega}. \tag{A.2.1}$$

Note that if also $\psi \in C_0^1(\mathbb{R}^n)$,

$$\begin{aligned} \int_{\partial\Omega} (\partial_{\tau_{jk}} \varphi)\psi \, d\sigma &= \int_{\partial\Omega} \bigl\{ \nu_k (\partial_j \varphi)\psi - \nu_j (\partial_k \varphi)\psi \bigr\} \, d\sigma \\ &= \int_{\Omega} \bigl\{ (\partial_j \varphi)(\partial_k \psi) - (\partial_k \varphi)(\partial_j \psi) \bigr\} \, dV \\ &= \int_{\partial\Omega} \bigl\{ \nu_j \varphi(\partial_k \psi) - \nu_k \varphi(\partial_j \psi) \bigr\} \, d\sigma \\ &= -\int_{\partial\Omega} \varphi(\partial_{\tau_{jk}} \psi) \, d\sigma, \end{aligned} \tag{A.2.2}$$

the second and third identities by the Gauss-Green formula ("easy" version),

$$\int_{\partial\Omega} \nu_k F_j \, d\sigma = \int_{\Omega} \partial_k F_j \, dV, \tag{A.2.3}$$

applied first to $F_j = (\partial_j \varphi)\psi$, and its counterpart with j and k switched, so the resulting integral is

$$\int_{\Omega} \bigl\{ (\partial_j \varphi)(\partial_k \psi) + (\partial_k \partial_j \varphi)\psi - (\partial_k \varphi)(\partial_j \psi) - (\partial_j \partial_k \varphi)\psi \bigr\} \, dV, \tag{A.2.4}$$

and the resulting cancellation yields the second line in (A.2.2), provided $\varphi, \psi \in C_0^2(\mathbb{R}^n)$. This gives (A.2.2) for such φ, ψ, and a limiting argument gives (A.2.2) for $\varphi, \psi \in C_0^1(\mathbb{R}^n)$.

For later use, we recast this argument. We set up the vector fields

$$\begin{aligned} G_{jk} &= (\partial_j \varphi)\psi e_k - (\partial_k \varphi)\psi e_j, \\ H_{jk} &= \varphi(\partial_k \psi) e_j - \varphi(\partial_j \psi) e_k, \end{aligned} \tag{A.2.5}$$

where $\{e_1, \dots, e_n\}$ is the standard orthonormal basis of $\mathbb{R}^n$. Then

$$\begin{aligned} \operatorname{div} G_{jk} = \operatorname{div} H_{jk} &= (\partial_j \varphi)(\partial_k \psi) - (\partial_k \varphi)(\partial_j \psi), \\ \nu \cdot G_{jk} = (\partial_{\tau_{jk}} \varphi)\psi, &\quad \nu \cdot H_{jk} = -\varphi(\partial_{\tau_{jk}} \psi), \end{aligned} \tag{A.2.6}$$

Then (A.2.2) can be rewritten

$$\begin{aligned} \int_{\partial\Omega} (\partial_{\tau_{jk}} \varphi)\psi \, d\sigma &= \int_{\partial\Omega} \nu \cdot G_{jk} \, d\sigma \\ &= \int_{\Omega} \operatorname{div} G_{jk} \, dV \\ &= \int_{\partial\Omega} \nu \cdot H_{jk} \, d\sigma \\ &= -\int_{\partial\Omega} \varphi(\partial_{\tau_{jk}} \psi) \, d\sigma, \end{aligned} \tag{A.2.7}$$

a sequence of identities that applies directly to all $\varphi, \psi \in C^1_0(\mathbb{R}^n)$, using a slightly more sophisticated version of the Gauss-Green theorem, given in §2.2 of [11].

To proceed, given $f \in L^p(\partial\Omega)$, $p \in [1,\infty]$, we say $f \in L^p_1(\partial\Omega)$ provided that for each j,k, there exists $f_{jk} \in L^p(\partial\Omega)$ such that

$$\int_{\partial\Omega} (\partial_{\tau_{jk}}\varphi) f \, d\sigma = -\int_{\partial\Omega} \varphi f_{jk} \, d\sigma, \quad \forall\, \varphi \in C^1_0(\mathbb{R}^n). \tag{A.2.8}$$

In such a case, we say

$$\partial_{\tau_{jk}} f = f_{jk}. \tag{A.2.9}$$

By (A.2.2), or (A.2.7), if $f = \psi|_{\partial\Omega}$, with $\psi \in C^1_0(\mathbb{R}^n)$, then $f \in L^p_1(\partial\Omega)$ and $f_{jk} = \nu_k \partial_j \psi - \nu_j \partial_k \psi|_{\partial\Omega}$.

Proposition A.2.1 *For each $p \in [1,\infty]$, $L^p_1(\partial\Omega)$ is a Banach space, with norm*

$$\|f\|_{L^p_1} = \|f\|_{L^p} + \sum_{j,k} \|\partial_{\tau_{jk}} f\|_{L^p}. \tag{A.2.10}$$

Proof. The right side of (A.2.10) makes $L^p_1(\partial\Omega)$ a normed linear space. To check completeness, suppose $(f^\mu)_\mu$ is Cauchy in $L^p_1(\partial\Omega)$, in such a norm. Then we have $f,\ f_{jk} \in L^p(\partial\Omega)$ such that

$$f^\mu \to f, \quad \partial_{\tau_{jk}} f^\mu \to f_{jk} \ \text{ in } \ L^p(\partial\Omega). \tag{A.2.11}$$

It suffices to show that

$$\partial_{\tau_{jk}} f = f_{jk}. \tag{A.2.12}$$

This follows from

$$\int_{\partial\Omega} (\partial_{\tau_{jk}}\varphi) f^\mu \, d\sigma = -\int_{\partial\Omega} \varphi\, \partial_{\tau_{jk}} f^\mu \, d\sigma, \tag{A.2.13}$$

since taking $\mu \to \infty$ yields (A.2.8). □

The following is useful information on $\mathrm{Lip}(\partial\Omega)$.

Proposition A.2.2 *We have*

$$\mathrm{Lip}(\partial\Omega) \subset L^\infty_1(\partial\Omega). \tag{A.2.14}$$

Proof. Suppose $f \in \mathrm{Lip}(\partial\Omega)$, so $f = \psi|_{\partial\Omega}$, with $\psi \in \mathrm{Lip}_c(\mathbb{R}^n)$. We can use a mollifier to construct $\psi^\mu \in C^\infty_0(\mathbb{R}^n)$ such that $\psi^\mu \to \psi$ uniformly and $\|\nabla\psi^\mu\|_{L^\infty} \le \|\psi\|_{\mathrm{Lip}}$; set $f^\mu = \psi^\mu|_{\partial\Omega}$. Then, for all $\varphi \in C^1_0(\mathbb{R}^n)$,

$$\int_{\partial\Omega} (\partial_{\tau_{jk}}\varphi) f \, d\sigma = \lim_{\mu\to\infty} \int_{\partial\Omega} (\partial_{\tau_{jk}}\varphi) f^\mu \, d\sigma. \tag{A.2.15}$$

Meanwhile, for all ν,

$$\int_{\partial\Omega} (\partial_{\tau_{jk}}\varphi) f^\mu \, d\sigma = -\int_{\partial\Omega} \varphi f^\mu_{jk} \, d\sigma, \tag{A.2.16}$$

with $f^\mu_{jk} = \nu_k \partial_j \psi^\mu - \nu_j \partial_k \psi^\mu|_{\partial\Omega}$. We have

$$\int_{\partial\Omega} (\partial_{\tau_{jk}}\varphi) f \, d\sigma = -\lim_{\mu\to\infty} \int_{\partial\Omega} \varphi f^\mu_{jk} \, d\sigma, \quad \sup_{\partial\Omega} |f^\mu_{jk}| \le \|f\|_{\mathrm{Lip}}. \tag{A.2.17}$$

Hence

$$\Big|\int\limits_{\partial\Omega}(\partial_{\tau_{jk}}\varphi)f\,d\sigma\Big| \le \|f\|_{\mathrm{Lip}(\partial\Omega)}\|\varphi\|_{L^1(\partial\Omega)}, \quad \forall\,\varphi\in C_0^1(\mathbb{R}^n), \tag{A.2.18}$$

so there exist

$$f_{jk}\in L^\infty(\partial\Omega,\sigma), \quad \|f_{jk}\|_{L^\infty(\partial\Omega)}\le\|f\|_{\mathrm{Lip}(\partial\Omega)}, \tag{A.2.19}$$

such that

$$\int\limits_{\partial\Omega}(\partial_{\tau_{jk}}\varphi)f\,d\sigma = -\int\limits_{\partial\Omega}\varphi f_{jk}\,d\sigma, \quad \forall\,\varphi\in C_0^1(\mathbb{R}^n). \tag{A.2.20}$$

This completes the proof of the proposition. □

Remark. From (A.2.17), we have

$$\int\limits_{\partial\Omega}\varphi f_{jk}\,d\sigma = \lim_{\mu\to\infty}\int\limits_{\partial\Omega}\varphi f^\mu_{jk}\,d\sigma, \tag{A.2.21}$$

for all $\varphi\in C_0^1(\mathbb{R}^n)$, hence, passing to the limit, for all $\varphi\in L^1(\partial\Omega,\sigma)$.

The following result is useful in Appendix A.3.

Proposition A.2.3 *Take $p\in(1,\infty)$. Assume $u\in C^1(\Omega)$, $\mathcal{N}(u),\mathcal{N}(\nabla u)\in L^p(\partial\Omega)$, and that there are nontangential a.e. limits in $L^p(\partial\Omega)$,*

$$u\to f, \quad \partial_j u\to f_j. \tag{A.2.22}$$

Then $f\in L^p_1(\partial\Omega)$ and

$$\partial_{\tau_{jk}}f=\nu_k f_j-\nu_j f_k. \tag{A.2.23}$$

Proof. Take $\varphi\in C_0^1(\mathbb{R}^n)$ and set

$$\begin{aligned} G_{jk}&=(\partial_j\varphi)ue_k-(\partial_k\varphi)ue_j,\\ H_{jk}&=\varphi(\partial_k u)e_j-\varphi(\partial_j u)e_k. \end{aligned} \tag{A.2.24}$$

We have $G_{jk},H_{jk}\in\mathfrak{L}^p$, defined by (2.4). As a consequence of [11], Proposition 3.20 (§3.2),

$$\mathcal{N}(\nabla u)\in L^p(\partial\Omega)\Longrightarrow\nabla u\in L^r(\Omega), \quad r=\frac{n}{n-1}p. \tag{A.2.25}$$

We hence have

$$\begin{aligned} \operatorname{div}G_{jk}=\operatorname{div}H_{jk}&=(\partial_j\varphi)(\partial_k u)-(\partial_k\varphi)(\partial_j u)\in L^1(\Omega),\\ \nu\cdot G_{jk}=(\partial_{\tau_{jk}}\varphi)f,&\quad \nu\cdot H_{jk}=-\varphi(\nu_j f_k-\nu_k f_j). \end{aligned} \tag{A.2.26}$$

Hence, parallel to (A.2.7),

$$\begin{aligned}\int_{\partial\Omega} (\partial_{\tau_{jk}}\varphi)u\,d\sigma &= \int_{\partial\Omega} \nu\cdot G_{jk}\,d\sigma \\ &= \int_{\Omega} \operatorname{div} G_{jk}\,dV \\ &= \int_{\partial\Omega} \nu\cdot H_{jk}\,d\sigma \\ &= -\int_{\partial\Omega} \varphi(\nu_j f_k - \nu_k f_j)\,d\sigma.\end{aligned} \tag{A.2.27}$$

In this case, the second and third identities hold by the "hard" Gauss-Green theorem, from §2.3 of [11]. The last identity establishes (A.2.23). □

The next result extends the scope of (A.2.8)–(A.2.9).

Proposition A.2.4 *Given* $f \in \mathrm{Lip}(\partial\Omega),\ g \in L^p_1(\partial\Omega),$

$$\int_{\partial\Omega} (\partial_{\tau_{jk}} f)g\,d\sigma = -\int_{\partial\Omega} f(\partial_{\tau_{jk}} g)\,d\sigma. \tag{A.2.28}$$

Proof. Take ψ, ψ^μ, f^μ as in the argument involving (A.11)–(A.16). Since each $\psi^\mu \in C_0^\infty(\mathbb{R}^n)$, we know that

$$\int_{\partial\Omega} (\partial_{\tau_{jk}} f^\mu)g\,d\sigma = -\int_{\partial\Omega} f^\mu(\partial_{\tau_{jk}} g)\,d\sigma. \tag{A.2.29}$$

As $\mu \to \infty$, the left side of (A.2.29) approaches the left side of (A.2.28), by (A.2.21), with $\varphi = g$, extended from $\varphi \in C_0^1(\mathbb{R}^n)$ to $\varphi \in L^1(\partial\Omega, \sigma)$, as indicated there. Meanwhile, the right side of (A.2.29) tends to the right side of (A.2.28), so (A.2.28) is established. □

We now show that each space $L^p_1(\partial\Omega)$ is a module over $\mathrm{Lip}(\partial\Omega)$. We start with the following.

Lemma A.2.5 *Given* $f \in \mathrm{Lip}(\partial\Omega),\ \varphi \in C_0^1(\partial\Omega),$

$$\partial_{\tau_{jk}}(f\varphi) = (\partial_{\tau_{jk}} f)\varphi + f\partial_{\tau_{jk}}\varphi. \tag{A.2.30}$$

Proof. We know that $f\varphi \in \mathrm{Lip}(\partial\Omega)$, hence $\partial_{\tau_{jk}}(f\varphi) \in L^\infty(\partial\Omega, \sigma)$, so, given $h \in C_0^1(\mathbb{R}^n)$, we have

$$\int_{\partial\Omega} (\partial_{\tau_{jk}} h)f\varphi\,d\sigma = -\int_{\partial\Omega} h\,\partial_{\tau_{jk}}(f\varphi)\,d\sigma. \tag{A.2.31}$$

Taking ψ^μ as in the proof of Proposition A.4, we see that the left side of (A.2.31) equals the limit as $\mu \to \infty$ of

$$\int_{\partial\Omega} (\partial_{\tau_{jk}} h)\psi^\mu\varphi\,d\sigma = -\int_{\partial\Omega} h\{(\partial_{\tau_{jk}}\psi^\mu)\varphi + (\partial_{\tau_{jk}}\varphi)\psi^\mu\}\,d\sigma. \tag{A.2.32}$$

Arguments mentioned above give $\partial_{\tau_{jk}}\psi^\mu \to \partial_{\tau_{jk}} f$, weak* in $L^\infty(\partial\Omega, \sigma)$, and $\psi^\mu \to f$ uniformly on $\partial\Omega$, so, as $\mu \to \infty$, (A.2.32) tends to

$$-\int_{\partial\Omega} h\{(\partial_{\tau_{jk}} f)\varphi + (\partial_{\tau_{jk}}\varphi) f\}\, d\sigma, \tag{A.2.33}$$

which is hence equal to (A.2.31). This proves (A.2.30). □

Here is the promised module result.

Proposition A.2.6 *Given $f \in \mathrm{Lip}(\partial\Omega)$, $g \in L^p_1(\partial\Omega)$, we have*

$$fg \in L^p_1(\partial\Omega), \quad \text{and} \;\; \partial_{\tau_{jk}}(fg) = (\partial_{\tau_{jk}} f)g + f(\partial_{\tau_{jk}} g). \tag{A.2.34}$$

Proof. Take $\varphi \in C^1_0(\mathbb{R}^n)$. We have

$$\begin{aligned}
\int_{\partial\Omega} (\partial_{\tau_{jk}}\varphi) fg\, d\sigma &= \int_{\partial\Omega} \{\partial_{\tau_{jk}}(f\varphi) - \varphi(\partial_{\tau_{jk}} f)\} g\, d\sigma \\
&= -\int_{\partial\Omega} f\varphi\, \partial_{\tau_{jk}} g\, d\sigma - \int_{\partial\Omega} \varphi(\partial_{\tau_{jk}} f) g\, d\sigma \\
&= -\int_{\partial\Omega} \varphi\{f\partial_{\tau_{jk}} g + (\partial_{\tau_{jk}} f) g\}\, d\sigma,
\end{aligned} \tag{A.2.35}$$

the first identity by (A.2.30) and the second by (A.2.28). The last identity proves $fg \in L^p_1(\partial\Omega)$ and establishes (A.2.34). □

We next aim to extend the scope of Proposition A.2.6, from $f \in \mathrm{Lip}(\partial\Omega)$ to $f \in L^q_1(\partial\Omega)$, for sufficiently large q. For this, we restrict the class of domains Ω under consideration; we assume Ω is a bounded, Ahlfors regular domain, and that

$$\Omega \text{ satisfies a two-sided John condition.} \tag{A.2.36}$$

(These hypotheses imply Ω is a UR domain.) In such a case, we have from Theorem 4.27 of [11] that

$$\begin{aligned}
L^p_1(\partial\Omega) \subset L^{p^*}(\partial\Omega) \;\; &\text{for} \;\; p^* = \frac{(n-1)p}{n-1-p}, \quad \text{if} \;\; p \in (1, n-1), \\
L^q(\partial\Omega) \;\; &\text{for all} \;\; q \in (1,\infty), \quad \text{if} \;\; p = n-1, \\
C^r(\partial\Omega) \;\; &\text{for} \;\; r = 1 - \frac{n-1}{p}, \quad \text{if} \;\; p \in (n-1, \infty).
\end{aligned} \tag{A.2.37}$$

Furthermore, by Proposition 4.29 of [11],

$$C^\infty(\partial\Omega) \text{ is dense in } L^q_1(\partial\Omega), \;\; \forall\, q \in (1,\infty), \tag{A.2.38}$$

where $C^\infty(\partial\Omega)$ is the space of restrictions to $\partial\Omega$ of elements of $C^\infty(\mathbb{R}^n)$. Using these results, we prove the following.

Proposition A.2.7 *Assume Ω is a bounded, Ahlfors regular domain, of dimension n, satisfying a two-sided John condition. Assume*

$$p \in (1, \infty), \quad q \in (n-1, \infty), \quad q \geq p. \tag{A.2.39}$$

Then

$$L^p_1(\partial\Omega) \text{ is a module over } L^q_1(\partial\Omega), \tag{A.2.40}$$

i.e.,

$$f \in L^q_1(\partial\Omega), \; g \in L^p_1(\partial\Omega) \Longrightarrow fg \in L^p_1(\partial\Omega). \tag{A.2.41}$$

Furthermore, the Leibniz formula (A.2.35) holds.

Proof. Given f, g as in (A.2.41), pick

$$f_\nu, g_\nu \in C^\infty(\partial\Omega), \quad \|f - f_\nu\|_{L^q_1} \to 0, \;\; \|g - g_\nu\|_{L^p_1} \to 0. \tag{A.2.42}$$

Then $f_\nu g_\nu \in C^\infty(\partial\Omega)$, and, by (A.2.37),

$$f_\nu \longrightarrow f \;\; \text{uniformly on } \;\partial\Omega, \tag{A.2.43}$$

so

$$f_\nu g_\nu \longrightarrow fg \;\; \text{in} \;\; L^p(\partial\Omega). \tag{A.2.44}$$

Also

$$\partial_{\tau_{jk}}(f_\nu g_\nu) = (\partial_{\tau_{jk}} f_\nu) g_\nu + f_\nu(\partial_{\tau_{jk}} g_\nu), \tag{A.2.45}$$

and $\partial_{\tau_{jk}} g_\nu \to \partial_{\tau_{jk}} g$ in $L^p(\partial\Omega)$, so

$$f_\nu(\partial_{\tau_{jk}} g_\nu) \longrightarrow f\partial_{\tau_{jk}} g \;\; \text{in} \;\; L^p(\partial\Omega). \tag{A.2.46}$$

Furthermore, $\partial_{\tau_{jk}} f_\nu \to \partial_{\tau_{jk}} f$ in $L^q(\partial\Omega)$, and, by (A.2.37),

$$\begin{aligned} g_\nu \to g \;\; \text{in} \;\; & L^{p^*}(\partial\Omega), \;\; \text{for } p^* = \frac{(n-1)p}{n-n-p}, \; \text{if } p \in (1, n-1), \\ & L^r(\partial\Omega), \;\; \text{for all } \; r \in (1,\infty), \; \text{if } p = n-1, \\ & C(\partial\Omega), \;\; \text{if } \; p > n-1. \end{aligned} \tag{A.2.47}$$

Under the hypothesis (A.2.39), we hence have

$$(\partial_{\tau_{jk}} f_\nu) g_\nu \longrightarrow (\partial_{\tau_{jk}} f) g \;\; \text{in} \;\; L^p(\partial\Omega), \tag{A.2.48}$$

hence

$$\partial_{\tau_{jk}}(f_\nu g_\nu) \longrightarrow (\partial_{\tau_{jk}} f) g + f(\partial_{\tau_{jk}} g) \;\; \text{in} \;\; L^p(\partial\Omega). \tag{A.2.49}$$

It follows that $(f_\nu g_\nu)$ is Cauchy in $L^p_1(\partial\Omega)$, so, by Proposition A.2.1, it has a limit in $L^p_1(\partial\Omega)$, and by (A.2.44) that limit is fg. The proof of Proposition A.2.1 then also yields the Leibniz formula (A.2.34). ☐

A.3 Gradient estimates

Here, we take $M = \mathbb{R}^n$ and assume (with a slight change in notation) that $\mathcal{D}$ has the form

$$\mathcal{D}u(x) = \sum A_j \partial_j u, \quad A_j \in \mathrm{End}(\mathbb{R}^k), \quad A_j^* = -A_j. \tag{A.3.1}$$

We continue to assume $\mathcal{D}$ is elliptic, so it has a fundamental solution $E \in \mathcal{S}'(\mathbb{R}^n)$, smooth on $\mathbb{R}^n \setminus 0$, homogeneous of degree $-(n-1)$, and satisfying

$$E(x) = E(-x)^* = -E(-x). \tag{A.3.2}$$

We take $\Omega \subset \mathbb{R}^n$ to be a bounded UR domain. In place of (1.0.19), we take

$$\mathcal{B}f(x) = \int_{\partial\Omega} E(x-y) f(y)\, d\sigma(y), \quad x \in \Omega, \tag{A.3.3}$$

and in place of (1.0.21), we take

$$\mathcal{C}f(x) = i \int_{\partial\Omega} E(x-y)\sigma_{\mathcal{D}}(y, \nu(y)) f(y)\, d\sigma(y), \quad x \in \Omega. \tag{A.3.4}$$

In this case,

$$i\sigma_{\mathcal{D}}(y, \nu(y)) = \sum A_j \nu_j(y). \tag{A.3.5}$$

Parallel to (1.0.20), we have

$$\mathcal{B}f\Big|_{\partial\Omega}(x) = \frac{1}{2i}\sigma_E(x, \nu(x)) f(x) + Bf(x), \tag{A.3.6}$$

and parallel to (1.0.23), we have

$$\mathcal{C}f\Big|_{\partial\Omega}(x) = \frac{1}{2} f(x) + Cf(x). \tag{A.3.7}$$

We also have estimates parallel to (2.3.16), in particular

$$\|\mathcal{N}\mathcal{B}f\|_{L^p(\partial\Omega)} \le C_p \|f\|_{L^p(\partial\Omega)}, \quad 1 < p < \infty. \tag{A.3.8}$$

Our goal here is to estimate $\nabla \mathcal{C}f$ on Ω when $f \in L^p_1(\partial\Omega)$. The following is the first key result. From here on, we sum over repeated indices.

Proposition A.3.1 *If $p \in (1, \infty)$ and $f \in L^p_1(\partial\Omega)$, then, for $x \in \Omega$,*

$$\partial_k \mathcal{C}f(x) = -\mathcal{B}(A_j \partial_{\tau_{jk}} f)(x). \tag{A.3.9}$$

Proof. We have

$$\begin{aligned} \partial_k \mathcal{C}f(x) &= \int_{\partial\Omega} \partial_{x_k} E(x-y) A_j \nu_j(y) f(y)\, d\sigma(y) \\ &= -\int_{\partial\Omega} \partial_{y_k} E(x-y) A_j \nu_j(y) f(y)\, d\sigma(y), \end{aligned} \tag{A.3.10}$$

for $x \in \Omega$. Note that

$$A_j \partial_{x_j} E(x-y) = \mathcal{D} E(x-y) = \delta(x-y) I, \tag{A.3.11}$$

hence, by (A.3.2),

$$\partial_{y_j} E(x-y) A_j = \delta(x-y) I. \tag{A.3.12}$$

In particular, $x \neq y \Rightarrow \partial_{y_j} E(x-y) A_j = 0$, so we can go from (A.3.10) to

$$\begin{aligned}
\partial_k \mathcal{C} f(x) &= \int\limits_{\partial\Omega} \big\{ -\partial_{y_k} E(x-y) A_j \nu_j(y) + \partial_{y_j} E(x-y) A_j \nu_k(y) \big\} f(y)\, d\sigma(y) \\
&= \int\limits_{\partial\Omega} \partial_{\tau_{jk}} E(x-y) A_j f(y)\, d\sigma(y) \\
&= - \int\limits_{\partial\Omega} E(x-y) A_j \partial_{\tau_{jk}} f(y)\, d\sigma(y) \\
&= -\mathcal{B}(A_j \partial_{\tau_{jk}} f)(x),
\end{aligned} \tag{A.3.13}$$

for $x \in \Omega$, as asserted in (A.3.9). □

Corollary A.3.2 *If $p \in (1, \infty)$ and $f \in L^p_1(\partial\Omega)$, then*

$$\| \mathcal{N} \nabla \mathcal{C} f \|_{L^p(\partial\Omega)} \leq C_p \| f \|_{L^p_1(\partial\Omega)}, \tag{A.3.14}$$

and there exists a σ-a.e. nontangential limit

$$\partial_k \mathcal{C} f \Big|_{\partial\Omega}(x) = -\frac{1}{2i} \sigma_E(x, \nu(x)) A_j \partial_{\tau_{jk}} f(x) - B(A_j \partial_{\tau_{jk}} f)(x), \tag{A.3.15}$$

for $x \in \partial\Omega$.

From (A.3.15), (A.3.7), and Proposition A.2.3, we deduce the following.

Corollary A.3.3 *For $p \in (1, \infty)$,*

$$C : L^p_1(\partial\Omega) \longrightarrow L^p_1(\partial\Omega). \tag{A.3.16}$$

Hence, for $\mathcal{P}_{\mathcal{D}}$, given by (3.0.4), we have

$$\mathcal{P}_{\mathcal{D}} : L^p_1(\partial\Omega) \longrightarrow L^p_1(\partial\Omega), \quad 1 < p < \infty. \tag{A.3.17}$$

It would be interesting to know when the Calderń-Szegö projector S_D, defined in §3.2, satisfies

$$S_D : L^p_1(\partial\Omega) \longrightarrow L^p_1(\partial\Omega). \tag{A.3.18}$$

Returning to (A.3.15), we have, for $f \in L^p_1(\partial\Omega)$,

$$\begin{aligned}
\partial_{\tau_{k\ell}} \mathcal{C} f \big|_{\partial\Omega} &= \nu_\ell \partial_k \mathcal{C} f \big|_{\partial\Omega} - \nu_k \partial_\ell \mathcal{C} f \big|_{\partial\Omega} \\
&= -\frac{1}{2i} \sigma_E(x, \nu(x)) \Big\{ \nu_\ell A_j \partial_{\tau_{jk}} f - \nu_k A_j \partial_{\tau_{j\ell}} f \Big\} \\
&\quad - \nu_\ell B(A_j \partial_{\tau_{jk}} f) + \nu_k B(A_j \partial_{\tau_{j\ell}} f).
\end{aligned} \tag{A.3.19}$$

Furthermore (cf. [11], Lemma 3.36), for $f \in L^p_1(\partial\Omega)$, Ω a UR domain,

$$\nu_k \partial_{\tau_{j\ell}} f - \nu_\ell \partial_{\tau_{jk}} f = \nu_j \partial_{\tau_{k\ell}} f, \tag{A.3.20}$$

so, since $i\sigma_E(x, \nu(x)) A_j \nu_j(x) = I$,

$$\partial_{\tau_{k\ell}} \mathcal{C} f\big|_{\partial\Omega} = \frac{1}{2} \partial_{\tau_{k\ell}} f + \nu_k B(A_j \partial_{\tau_{j\ell}} f) - \nu_\ell B(A_j \partial_{\tau_{jk}} f). \tag{A.3.21}$$

Hence, for $f \in L^p_1(\partial\Omega)$,

$$\partial_{\tau_{k\ell}} C f = \nu_k B(A_j \partial_{\tau_{j\ell}} f) - \nu_\ell B(A_j \partial_{\tau_{jk}} f). \tag{A.3.22}$$

A.4 UR domains with infinite topology

Recall that a compact surface $\Sigma \subset \mathbb{R}^n$ is an Ahlfors regular surface provided there exist $c_j \in (0, \infty)$ such that $c_0 r^{n-1} \le \mathcal{H}^{n-1}(B_r(p) \cap \Sigma) \le c_1 r^{n-1}$ for each $p \in \Sigma,\ r \in (0, 1]$; that a bounded open set $\Omega \subset \mathbb{R}^n$ is an Ahlfors regular domain provided $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$ and $\partial\Omega$ is an Ahlfors regular surface; and that such Ω is a UR domain provided, in addition, $\partial\Omega$ contains large pieces of Lipschitz surfaces. We aim to describe examples of UR domains of infinite topological type.

We begin with an Ahlfors regular surface $\overline{\mathcal{O}}$ that is a bounded subset of $\mathbb{R}^{n-1} \subset \mathbb{R}^n$. For example, we might have

$$\mathcal{O} = D_1(0) \setminus \bigcup_{k \ge 1} \overline{D_{2^{-k-2}}(2^{-k} v_k)}, \tag{A.4.1}$$

where $D_\rho(p) = \{x' \in \mathbb{R}^{n-1} : |x' - p| < \rho\}$, and v_k are unit vectors in $\mathbb{R}^{n-1}$. The following is easily established.

Lemma A.4.1 *If $f : \mathbb{R}^{n-1} \to \mathbb{R}$ is Lipschitz, then the set*

$$\Sigma = \{(x', f(x')) : x' \in \overline{\mathcal{O}}\} \tag{A.4.2}$$

is an Ahlfors regular surface.

Proof. Given $p = (q, f(q)) \in \Sigma,\ r \in (0, 1]$, the desired upper bound on $\mathcal{H}^{n-1}(B_r(p) \cap \Sigma)$ is straightforward. It remains to establish a lower bound. For this, assume the Lipschitz constant of f is $\le L$, and set $\beta = (1 + L^2)^{-1/2}$. Then

$$x' \in D_{\beta r}(q) \cap \mathcal{O} \Longrightarrow (x', f(x')) \in B_r(p) \cap \Sigma,$$

so

$$\mathcal{H}^{n-1}(B_r(p) \cap \Sigma) \ge \mathcal{H}^{n-1}(D_{\beta r}(q) \cap \mathcal{O}),$$

yielding the desired lower bound. □

This lemma leads to the following.

Proposition A.4.2 *If $f, g : \mathbb{R}^{n-1} \to \mathbb{R}$ are Lipschitz,*

$$f = g \quad \text{on} \quad \partial\mathcal{O}, \quad \text{and} \quad f > g \quad \text{on} \quad \mathcal{O}, \tag{A.4.3}$$

then

$$\Omega = \{(x', x_n) : x' \in \mathcal{O},\ g(x') < x_n < f(x')\} \tag{A.4.4}$$

is a UR domain.

Proof. That Ω is an Ahlfors regular domain follows from Lemma A.4.1. The UR property then follows directly from the definition. ☐

REMARK. For such $\mathcal{O}$ as in (A.4.1), one could take $f(x') = \mathrm{dist}(x', \mathbb{R}^{n-1} \setminus \mathcal{O})$, and $g \equiv 0$, or perhaps $g = -f$.

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