Traveling Wave Solutions to NLS and NLKG Equations in Non-Euclidean Settings *

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Abstract

We study traveling wave solutions to nonlinear Schrödinger (NLS) and nonlinear Klein-Gordon (NLKG) equations on a compact Riemannian manifold M, with a Killing field X, generating a group of isometries. The emphasis is on NLKG; then if X has length < 1 everywhere, one gets a semilinear elliptic PDE on M, to which standard variational techniques apply (for a natural class of nonlinearities), as reviewed in §1, though there remains the question of whether the associated waves are *really* (or just apparently) traveling, a point taken up in §2. In §§3–4 we consider sonic speed waves, in some situations that lead to subelliptic nonlinear PDE, and in §5 we consider some supersonic traveling waves.

1 Introduction and first results

Let M be a complete, n-dimensional Riemannian manifold, possibly with boundary. Assume $g(t)$ is a 1-parameter group of isometries of M, generated by the Killing vector field X (tangent to ∂M if the boundary is nonempty). We look for solutions to the nonlinear Schrödinger equation

$$i\partial_t v + \Delta v = -K|v|^{p-1}v, \tag{1.1}$$

or the nonlinear Klein-Gordon equation

$$\partial_t^2 v - \Delta v + m^2 v = K|v|^{p-1}v, \tag{1.2}$$

of the form

$$v(t,x) = e^{i\lambda t}u(g(t)x), \tag{1.3}$$

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with $\lambda \in \mathbb{R}$. Such solutions are called traveling waves.

In case $M = \mathbb{R}^n$ and $g(t)x = x + tv$, such traveling waves have been studied in such classical works as [9] and [1]. Here we look into curved Riemannian manifolds, and consider some phenomena that do not arise in the Euclidean setting. We mention recent interest in *standing wave* solutions to (1.1) and (1.2) (where $g(t)x \equiv x$), in non-euclidean settings, in [7], [2], and [3].

Note that if v is given by (1.3), then

$$i\partial_t v = e^{i\lambda t}\Big[-\lambda u(g(t)x) + iXu(g(t)x)\Big], \tag{1.4}$$

and

$$\partial_t^2 v = e^{i\lambda t}\Big[-\lambda^2 u(g(t)x) + 2i\lambda X u(g(t)x) + X^2 u(g(t)x)\Big]. \tag{1.5}$$

Thus (1.1) holds if and only if

$$-\Delta u + \lambda u - iXu = K|u|^{p-1}u, \tag{1.6}$$

and (1.2) holds if and only if

$$-\Delta u + (m^2 - \lambda^2)u + X^2 u + 2i\lambda X u = K|u|^{p-1}u. \tag{1.7}$$

If $\partial M \neq \emptyset$, we will for the sake of definiteness place the Dirichlet boundary condition on ∂M, though a similar analysis for the Neumann boundary condition could be done.

We assume

$$\langle X, X\rangle \leq b^2 < \infty. \tag{1.8}$$

Then iX is a relatively bounded perturbation of $-\Delta$, and $-\Delta + iX$ is self adjoint. Say

$$\mathrm{Spec}(-\Delta + iX) \subset [\alpha, \infty). \tag{1.9}$$

We will study (1.6) in case $\lambda > -\alpha$ and

$$1 < p < \frac{n+2}{n-2}. \tag{1.10}$$

We note that (1.8) is a significant restriction. There are no nonzero Killing fields on hyperbolic space $\mathcal{H}^n$ satisfying (1.8). In fact, here we will concentrate on the case of compact M (possibly with boundary).

To study (1.7), we strengthen (1.8) to

$$\langle X, X\rangle \leq b^2 < 1. \tag{1.11}$$

Then $-\Delta + X^2$ is a strongly elliptic, negative semidefinite self-adjoint operator, and $2i\lambda X$ a relatively bounded perturbation. Say

$$\operatorname{Spec}(-\Delta + X^2 + 2i\lambda X) \subset [\beta(\lambda), \infty). \tag{1.12}$$

Possibly $\beta(\lambda) < 0$.

Let us set

$$F_{\lambda,X}(u) = (-\Delta u - iXu + \lambda u, u), \tag{1.13}$$

and

$$\mathcal{F}_{m,\lambda,X}(u) = (-\Delta u + X^2 u + 2i\lambda X u + (m^2 - \lambda^2) u, u). \tag{1.14}$$

The following is readily established.

Proposition 1.1 *Assume M is compact. If (1.9) holds and*

$$\lambda > -\alpha, \tag{1.15}$$

then

$$F_{\lambda,X}(u) \approx \|u\|_{H^1}^2, \quad \forall\, u \in H_0^1(M). \tag{1.16}$$

If (1.11)–(1.12) hold and

$$m^2 > \lambda^2 - \beta(\lambda), \tag{1.17}$$

then

$$\mathcal{F}_{m,\lambda,X}(u) \approx \|u\|_{H^1}^2, \quad \forall\, u \in H_0^1(M). \tag{1.18}$$

We can produce a solution to (1.6) (under hypothesis (1.15)) by minimizing $F_{\lambda,X}(u)$, over $u \in H_0^1(M)$, subject to the constraint

$$I_p(u) = \int_M |u|^{p+1}\, dV = A, \tag{1.19}$$

with $A \in (0, \infty)$ fixed. The role of (1.10) in this minimization procedure is that (if M has bounded geometry)

$$H_0^1(M) \subset L^q(M), \quad \forall\, q \in \Big[2, \frac{2n}{n-2}\Big], \tag{1.20}$$

if $n \geq 3$, $\forall\, q \in [2, \infty)$ if $n = 2$. Furthermore, if M is compact,

$$H_0^1(M) \hookrightarrow L^q(M) \text{ is compact, } \forall\, q \in \Big[2, \frac{2n}{n-2}\Big). \tag{1.21}$$

Such a result holds for $q = p + 1$ provided p satisfies (1.10). Note that if $u, v \in H_0^1(M)$,

$$\frac{d}{d\tau} F_{\lambda,X}(u + \tau v)\Big|_{\tau=0} = 2 \operatorname{Re} (-\Delta u - iXu + \lambda u, v), \tag{1.22}$$

and

$$\frac{d}{d\tau} I_p(u + \tau v)\Big|_{\tau=0} = (p + 1) \operatorname{Re} \int |u|^{p-1} u\overline{v}\, dV. \tag{1.23}$$

Hence, if u is such a minimizer, we have the implication

$$v \in H_0^1(M),\ \operatorname{Re} (|u|^{p-1}u, v) = 0 \Longrightarrow \operatorname{Re} (-\Delta u - iXu + \lambda u, v) = 0. \tag{1.24}$$

Now Re (,) is a nondegenerate, $\mathbb{R}$-bilinear dual pairing of (complex) $H_0^1(M)$ and $H^{-1}(M)$, so (1.24) implies

$$-\Delta u + \lambda u - iXu = K_0|u|^{p-1}u, \tag{1.25}$$

for some $K_0 \in \mathbb{R}$. Pairing both sides of (1.25) with u, using $H_0^1 - H^{-1}$ duality, we have

$$K_0 I_p(u) = F_{\lambda,X}(u) \geq (\lambda + \alpha)\|u\|_{L^2}^2, \tag{1.26}$$

and in particular $K_0 > 0$. If u solves (1.25), then $u_a = au$ solves

$$-\Delta u_a + \lambda u_a - iXu_a = |a|^{-(p-1)} K_0|u_a|^{p-1}u_a, \tag{1.27}$$

so we have a solution to (1.6), given any $K > 0$.

We produce a solution to (1.7) (under hypothesis (1.17)) by minimizing $\mathcal{F}_{m,\lambda,X}(u)$ over $H_0^1(M)$, subject to the constraint (1.19). In this case, for $u, v \in H_0^1(M)$,

$$\frac{d}{d\tau}\mathcal{F}_{m,\lambda,X}(u+\tau v)\Big|_{\tau=0} = 2 \operatorname{Re} (-\Delta u+X^2u+2i\lambda Xu+(m^2-\lambda^2)u, v). \tag{1.28}$$

Comparing (1.23), we see that if u is such a minimizer, then

$$\begin{aligned} &v \in H_0^1(M),\ \operatorname{Re} (|u|^{p-1}u, v) = 0 \\ &\Longrightarrow \operatorname{Re} (-\Delta u + X^2u + 2i\lambda Xu + (m^2 - \lambda^2)u, v) = 0. \end{aligned} \tag{1.29}$$

Hence

$$-\Delta u + X^2u + 2i\lambda Xu + (m^2 - \lambda^2)u = K_0|u|^{p-1}u, \tag{1.30}$$

for some $K_0 \in \mathbb{R}$. Parallel to (1.26), we have

$$K_0 I_p(u) = \mathcal{F}_{m,\lambda,X}(u) \geq (m^2 - \lambda^2 + \beta(\lambda))\|u\|_{L^2}^2, \tag{1.31}$$

so $K_0 > 0$. If u solves (1.30), then $u_a = au$ solves

$$-\Delta u_a + X^2 u_a + 2i\lambda X u_a + (m^2 - \lambda^2) u_a = |a|^{-(p-1)} K_0 |u_a|^{p-1} u_a, \quad (1.32)$$

so we have a solution to (1.7), give $K > 0$.

Given (1.16), (1.18), and (1.21), a straightforward variant of the proof of Theorem 1 in [8] yields the following existence result.

Proposition 1.2 *Assume M is compact and p satisfies (1.10).*

If (1.8)–(1.9) and (1.15) hold, then, given $A \in (0, \infty)$, there exists $u \in H_0^1(M)$ minimizing $F_{\lambda,X}(u)$, subject to the constraint (1.19). Such u is a solution to (1.6), and (1.3) then gives a traveling wave solution to the NLS equation (1.1).

If (1.11)–(1.12) and (1.17) hold, then, given $A \in (0, \infty)$, there exists $u \in H_0^1(M)$ minimizing $\mathcal{F}_{m,\lambda,X}(u)$, subject to the constraint (1.19). Such u is a solution to (1.7), and then (1.3) gives a traveling wave solution to the NLKG equation (1.2).

Proof. The details for the first part go as follows. Take $u_\nu \in H_0^1(M)$ such that $I_p(u_\nu) = A$ and $F_{\lambda,X}(u_\nu)$ tends to the infimum, say B. Then, by (1.16), $\{u_\nu\}$ is bounded in $H_0^1(M)$, so there is a weakly convergent subsequence $u_\nu \to u \in H_0^1(M)$, and $F_{\lambda,B}(u) \le B$. By (1.21), $u_\nu \to u$ in L^{p+1}-norm, so $I_p(u) = A$. Hence u is a minimizer (and $F_{\lambda,X}(u) = B$, and $u_\nu \to u$ in H^1-norm). The details for the second part are similar. $\square$

Now that we have minimizers, which satisfy (1.25) and (1.30), we need to verify that passing to $v(t,x)$ in (1.3) yields waves that are actually traveling. Here is a potential source of a problem. Suppose $\partial M = \emptyset$. Then the constant $u = [(m^2 - \lambda^2)/K_0]^{1/p}$ solves (1.25) and (1.30), and one might wonder if this is a minimizer. Even if $\partial M \neq \emptyset$, there are solutions to (1.25) and (1.30) that are constant on each orbit of X, obtained by minimizing $F_{\lambda,X}(u)$ and $\mathcal{F}_{m,\lambda,X}(u)$ over

$$\{u \in H_0^1(M) : Xu = 0\}, \quad (1.33)$$

subject to the constraint $I_p(u) = A$. Can one be sure this is not also a minimizer over all $u \in H_0^1(M)$, subject to this constraint? We tackle this question in §2. There we show that, when $M = S^n$, the standard unit sphere in $\mathbb{R}^{n+1}$, minimizers for $\mathcal{F}_{m,0,X}(u)$, subject to $I_p(u) = A$, give rise to genuinely traveling waves, at least for a nonempty set of parameters. Part of the argument will involve transfering minimizers of related functionals attached to $\mathbb{R}^n$ to S_R^n, the sphere of radius R, taken large, and then scaling.

In sections 3–5 we pursue further results for traveling waves, for NLKG, first relaxing and then erasing the "subsonic speed" condition (1.11). In §3 we take $M = S^2$ and let X generate the 2π-periodic rotation of S^2 about the x-axis. Then $\langle X, X\rangle < 1$ except on the great circle $\gamma = \{(x, y, z) \in S^2 : x = 0\}$, where $\langle X, X\rangle = 1$. Thus an associated traveling wave will travel at the "speed of sound" on γ. Our analysis of the needed analogue of Proposition 1.2 makes use of subelliptic estimates for $-\Delta + X^2 + i\alpha X$, valid for $|\alpha| < 1$. See Proposition 3.3 for an existence result.

In §4, we construct mach 1 traveling waves on S^3. The construction is somewhat parallel to that in §3, in particular making essential use of subelliptic estimates. One difference is that the waves we produce on S^3 travel at the speed of sound everywhere, not just along an "equator." See Proposition 4.4 for the existence result.

Results of §§3–4 have no parallel in the setting of flat Euclidean space. A certain degree of curvature is necessary for the production of operators satisfying subelliptic estimates. Here, we have chosen to illustrate these subelliptic phenomena on two and three dimensional spheres, but one could extend the analysis to much larger classes of manifolds. For example, we could take compact surfaces of rotation in $\mathbb{R}^3$ which "bulge out" at their equator. We could also take higher dimensional manifolds, including spheres, compact semisimple Lie groups, and other cases.

In §5, we allow $\langle X, X\rangle > 1$. Associated traveling waves are "supersonic." In such a case, we replace Proposition 1.2 by the existence of constrained minima on spaces

$$V_\mu = \{u \in H_0^1(M) : Xu = i\mu u\},$$

when such a space is $\neq 0$.

2 Nontriviality of solutions

In this section we take $M = S^n$, the standard n-sphere, and let X be a nonzero Killing field on S^n (satisfying (1.11)), so X generates a 1-parameter subgroup of $SO(n+1)$. We take $\lambda = 0$ and $m > 0$ and assume $u \in H^1(S^n)$ minimizes $\mathcal{F}_{m,0,X}(u)$, subject to the constraint (1.19), so u solves

$$-\Delta u + X^2 u + m^2 u = K_0 |u|^{p-1} u, \tag{2.1}$$

for some $K_0 > 0$. Our first observation is that if u were constant on each orbit of X, it would have to be trivial.

Lemma 2.1 *If $u \in H^1(S^n)$ minimizes $\mathcal{F}_{m,0,X}(u)$, subject to (1.19), and $Xu = 0$, then u must minimize $\mathcal{F}_{m,0,0}(u)$, subject to (1.19), and furthermore, u must be constant.*

Proof. Comparing

$$\begin{aligned} \mathcal{F}_{m,0,X}(u) &= ((-\Delta + X^2 + m^2)u, u) \\ &= \|\nabla u\|_{L^2}^2 - \|Xu\|_{L^2}^2 + m^2\|u\|_{L^2}^2 \end{aligned} \tag{2.2}$$

with

$$\begin{aligned} \mathcal{F}_{m,0,0}(u) &= ((-\Delta + m^2)u, u) \\ &= \|\nabla u\|_{L^2}^2 + m^2\|u\|_{L^2}^2, \end{aligned} \tag{2.3}$$

we see that

$$\mathcal{F}_{m,0,X}(u) \le \mathcal{F}_{m,0,0}(u), \quad \forall\, u \in H^1(S^n). \tag{2.4}$$

Now if u satisfies the hypotheses of Lemma 2.1, then $\mathcal{F}_{m,0,X}(u) = \mathcal{F}_{m,0,0}(u)$, so, by (2.4), u must also minimize $\mathcal{F}_{m,0,0}(u)$, subject to (1.19). To proceed, set

$$u_g(x) = u(gx), \quad g \in SO(n+1). \tag{2.5}$$

Then $\mathcal{F}_{m,0,0}(u_g) = \mathcal{F}_{m,0,0}(u)$ for all $g \in SO(n+1)$. On the other hand, $\mathcal{F}_{m,0,X}(u_g) \ge \mathcal{F}_{m,0,X}(u)$. Hence, by (2.4) with u replaced by u_g, we must have

$$\mathcal{F}_{m,0,X}(u_g) = \mathcal{F}_{m,0,0}(u_g). \tag{2.6}$$

Then (2.2)–(2.3), with u replaced by u_g, yields

$$Xu_g = 0, \quad \forall\, g \in SO(n+1), \tag{2.7}$$

which implies u is constant. $\square$

We now scale the metric up. Let S^n_R denote S^n with the distance magnified by a factor of R. Thus S^n_R has circumference $2\pi R$, and $\mathrm{Vol}(S^n_R) = A_n R^n$, where $A_n = \mathrm{Vol}(S^n)$. We set $\varepsilon = 1/R$ and let Δ_ε denote the Laplace-Beltrami operator on S^n_R. Also, given a vector field X on S^n satisfying (1.11), let X_ε denote the constant multiple of X whose rescaled norm is pointwise equal to the original norm of X. If we pick a "north pole" $o \in S^n_R$ and use exponential coordinates centered at o, then, as $R \to \infty$, S^n_R approaches flat Euclidean space $\mathbb{R}^n$, Δ_ε approaches the flat Laplacian on $\mathbb{R}^n$, and we can arrange that X_ε approaches $b\partial_1$, with $b \in (0,1)$.

Fix $p \in (1, (n+2)/(n-2))$, $m > 0$, and $A \in (0, \infty)$, and, for $\varepsilon > 0$, let $u^\varepsilon \in H^1(S^n_R)$ denote a minimizer of

$$\begin{aligned} \mathcal{F}^\varepsilon_{m,0,X}(u) &= ((-\Delta_\varepsilon + X^2_\varepsilon + m^2)u, u) \\ &= \|\nabla_\varepsilon u\|^2_{L^2(S^n_R)} - \|X_\varepsilon u\|^2_{L^2(S^n_R)} + m^2\|u\|^2_{L^2(S^n_R)}, \end{aligned} \tag{2.8}$$

subject to the constraint

$$I^\varepsilon_p(u) = \int\limits_{S^n_R} |u|^{p+1}\, dV = A. \tag{2.9}$$

The analogue of Lemma 2.1 applies. For each $\varepsilon > 0$, if $X_\varepsilon u^\varepsilon = 0$, then u^ε must be constant on S^n_R. From (2.9), we get

$$X_\varepsilon u^\varepsilon = 0 \Longrightarrow u^\varepsilon \equiv \Big(\frac{A}{A_n R^n}\Big)^{1/(p+1)}, \tag{2.10}$$

hence

$$\begin{aligned} \mathcal{F}^\varepsilon_{m,0,X}(u^\varepsilon) &= m^2|u^\varepsilon|^2 A_n R^n \\ &= m^2 A^{1/(p+1)}(A_n R^n)^{p/(p+1)}. \end{aligned} \tag{2.11}$$

Since $X_\varepsilon u^\varepsilon = 0$, this must also be the infimum of $\mathcal{F}^\varepsilon_{m,0,0}(u)$, subject to the constraint (2.9). We will show that this is not the case if R is sufficiently large.

We do this via the following construction. With p and A as in (2.8)–(2.9), set

$$\begin{aligned} F_m(u) = \mathcal{F}^0_{m,0,0}(u) &= ((-\Delta_0 + m^2)u, u) \\ &= \|\nabla u\|^2_{L^2(\mathbb{R}^n)} + m^2\|u\|^2_{L^2(\mathbb{R}^n)}, \end{aligned} \tag{2.12}$$

and

$$I^0_p(u) = \int\limits_{\mathbb{R}^n} |u|^{p+1}\, dx. \tag{2.13}$$

The following holds.

Lemma 2.2 *Given*

$$n \geq 2, \quad p \in \Big(1, \frac{n+2}{n-2}\Big), \quad A \in (0, \infty), \tag{2.14}$$

there is a minimizer $u^0 \in H^1(\mathbb{R}^n)$ to $F_m(u) = \mathcal{F}^0_{m,0,0}(u)$, subject to the constraint $I^0_p(u^0) = A$.

This is essentially a special case of work of [1]. Since that paper treated a slightly different constrained minimization problem, we will sketch the argument. Parallel to (1.20), we have

$$H^1(\mathbb{R}^n) \subset L^q(\mathbb{R}^n), \quad \forall\, q \in \Big[2, \frac{2n}{n-2}\Big], \tag{2.15}$$

if $n \geq 3$, $\forall\, q \in [2, \infty)$ if $n = 2$, but (1.21) must be weakened to the result that restriction to the ball $B_K = \{x \in \mathbb{R}^n : |x| \leq K\}$ gives

$$H^1(\mathbb{R}^n) \longrightarrow L^q(B_K) \ \text{ compact}, \ \ \forall\, q \in \Big[2, \frac{2n}{n-2}\Big), \quad K < \infty. \tag{2.16}$$

An extra argument is required to show that a minimizing sequence w_ν for F_m, satisfying $I_p^0(w_\nu) \equiv A$, has a subsequence that converges to the desired minimizer.

To get this, let w_ν be such a minimizing sequence; say $F_m(w_\nu) \to B$. By (2.15), $B > 0$. The following is the key to success. Let u_ν be the radially symmetric, decreasing rearrangement of w_ν. Then clearly

$$I_p^0(u_\nu) = I_p^0(w_\nu) \ \text{ and } \ \|u_\nu\|_{L^2} = \|w_\nu\|_{L^2}. \tag{2.17}$$

Furthermore, as shown in [6],

$$\int\limits_{\mathbb{R}^n} |\nabla u_\nu|^2\, dx \leq \int\limits_{\mathbb{R}^n} |\nabla w_\nu|^2\, dx. \tag{2.18}$$

Then $I_p^0(u_\nu) \equiv A$ and $F_m(u_\nu) \leq F_m(w_\nu)$, so we have a minimizing sequence, consisting of positive, radial, decreasing functions. We have convergence (of a subsequence) $u_\nu \to u$, weakly in $H^1(\mathbb{R}^n)$, and clearly $F_m(u) \leq B$. We can conclude that u is the desired minimizer if we can establish

$$u_\nu \longrightarrow u \ \text{ in norm, in } \ L^{p+1}(\mathbb{R}^n). \tag{2.19}$$

Now (2.16) gives such L^{p+1}-norm convergence on any ball B_K of finite radius. The proof of (2.19) is completed via the following result (Radial Lemma A.II of [1]). If $n \geq 2$,

$$v \in H^1(\mathbb{R}^n) \text{ radial} \Longrightarrow |v(x)| \leq C_n |x|^{-(n-1)/2} \|v\|_{H^1}, \ \ \forall\, |x| \geq 1. \tag{2.20}$$

We can apply this to $v_\nu = u - u_\nu$, to get

$$\begin{aligned} \int\limits_{|x|>K} |v_\nu|^{p+1}\, dx &\leq C \sup_{|x|>K} |v_\nu|^{p-1} \int\limits_{|x|>K} |v_\nu|^2\, dx \\ &\leq C K^{-(p-1)(n-1)/2}\, \|v_\nu\|_{H^1}^{p+1}, \end{aligned} \tag{2.21}$$

and finish off the proof of (2.19). This yields Lemma 2.2.

REMARK. As shown in [1], the resulting minimizer $u^0 \in H^1(\mathbb{R}^n)$ satisfies

$$|u^0(x)|,\ |\nabla u^0(x)| \le Ce^{-\delta|x|}, \tag{2.22}$$

for some $\delta > 0$.

We can use the minimizer u^0 of Lemma 2.2 to construct functions v^ε on S^n_R $(R = 1/\varepsilon)$, as follows. As we have noted, u^0 is radial, i.e.,

$$u^0(x) = u^\#(|x|), \quad x \in \mathbb{R}^n. \tag{2.23}$$

We fix a pole $o \in S^n_R$, and set

$$v^\varepsilon(x) = u^\#(\operatorname{dist}(o,x)), \quad x \in S^n_R, \tag{2.24}$$

where dist denotes geodesic distance in S^n_R. We have

$$I^\varepsilon_p(v^\varepsilon) \longrightarrow A, \tag{2.25}$$

and

$$\mathcal{F}^\varepsilon_{m,0,0}(v^\varepsilon) \longrightarrow \mathcal{F}^0_{m,0,0}(u^0) < \infty, \tag{2.26}$$

as $\varepsilon \to 0$ $(R \to \infty)$. Comparison with (2.10)–(2.11) contradicts the possibility that (2.10) gives a minimizer, for small $\varepsilon > 0$. Hence, for $\varepsilon > 0$ sufficiently small, a minimizer u^ε of (2.8), subject to the constraint (2.9), has the property $X_\varepsilon u^\varepsilon \ne 0$. Such a function u^ε solves

$$-\Delta_\varepsilon u^\varepsilon + X^2_\varepsilon u^\varepsilon + m^2 u^\varepsilon = K|u^\varepsilon|^{p-1}u^\varepsilon, \tag{2.27}$$

on S^n_R, with $K = K_\varepsilon > 0$, and as before the substitution $u^\varepsilon \mapsto au^\varepsilon$ yields a solution to (2.27) for arbitrary $K > 0$.

Using radial projection, we can identify S^n_R with the unit sphere S^n, and hence identify functions on S^n_R with functions on S^n. By slight abuse of notation, we continue to denote the associated function on S^n by u^ε. Seeing how Δ_ε and X_ε scale, we have

$$-\Delta u^\varepsilon + X^2 u^\varepsilon + R^2 m^2 u^\varepsilon = R^2 K|u^\varepsilon|^{p-1}u^\varepsilon, \quad \text{on } \ S^n. \tag{2.28}$$

We have established the following result.

Proposition 2.3 *Given $n \ge 2$, $p \in (1, (n+2)/(n-2))$, $m > 0$, $K > 0$, and a Killing field X on S^n such that $\langle X, X\rangle \le b^2 < 1$, there exists $\varepsilon_0 > 0$ such that, for $\varepsilon \in (0, \varepsilon_0]$, the minimization procedure described above produces a solution u^ε to (2.28), with $R = 1/\varepsilon$, such that $Xu^\varepsilon \ne 0$.*

3 Mach 1 NLKG traveling waves on S^2

Let S^2 denote the unit sphere centered at the origin in $\mathbb{R}^3$, and let X, Y, and Z generate 2π-periodic rotations of S^2 about the x, y, and z-axis, respectively. The condition (1.11) fails for X. We have $\langle X, X\rangle < 1$ except on the great circle $\gamma = S^2 \cap \{(0, y, z)\}$, but $\langle X, X\rangle = 1$ on γ. The operator

$$L_0 = \Delta - X^2 = Y^2 + Z^2 \tag{3.1}$$

satisfies Hörmander's condition for hypoellipticity with loss of one derivative. We have

$$\mathcal{D}(L_0) \subset H^1(S^2), \quad \text{hence} \quad \mathcal{D}((-L_0)^{1/2}) \subset H^{1/2}(S^2), \tag{3.2}$$

and in fact

$$-(L_0 u, u) \geq C\|u\|^2_{H^{1/2}}, \quad \text{if} \quad \int\limits_{S^2} u\, dS = 0. \tag{3.3}$$

We can write (1.7) as

$$-L_{2\lambda} u + (m^2 - \lambda^2) u = K|u|^{p-1} u, \tag{3.4}$$

where

$$L_\alpha = L_0 - i\alpha X. \tag{3.5}$$

For $\alpha \in \mathbb{R}$, L_α is self adjoint. Well known results on spherical harmonics (cf., e.g., [11], Chapters 2 and 4) imply $-L_\alpha$ is positive semidefinite with 1-dimensional kernel if $|\alpha| < 1$, positive semidefinite with infinite-dimensional kernel if $|\alpha| = 1$, and not semibounded if $|\alpha| > 1$. Also, classical results on subellipticty (cf. [10], Chapter 15) imply that

$$\begin{gathered} |\alpha| < 1 \Rightarrow \mathcal{D}(L_\alpha) = \mathcal{D}(L_0), \ \text{hence } \mathcal{D}((-L_\alpha)^{1/2}) = \mathcal{D}((-L_0)^{1/2}), \ \text{and} \\ -(L_\alpha u, u) \geq C_\alpha \|u\|^2_{H^{1/2}}, \ \text{if} \int\limits_{S^2} u\, dS = 0. \end{gathered} \tag{3.6}$$

To proceed, we bring in the following variant of (1.20)–(1.21). Namely, there exists $q_* > 2$ such that

$$\mathcal{D}((-L_0)^{1/2}) \subset L^q(S^2), \quad \forall\, q \in [2, q_*], \tag{3.7}$$

and hence

$$\mathcal{D}((-L_0)^{1/2}) \hookrightarrow L^q(S^2) \ \text{ is compact, } \forall\, q \in [2, q_*). \tag{3.8}$$

It would be of interest to know the sharp value of q_* for which (3.7) holds. From the Sobolev embedding result

$$H^{1/2}(S^2) \subset L^4(S^2), \tag{3.9}$$

we have a lower bound, $q_* \geq 4$. We will obtain a better bound below. First, we formulate an existence result.

Proposition 3.1 *Let X be as in the first paragraph of this section. With q_* as in (3.8), assume*

$$2 < p + 1 < q_*. \tag{3.10}$$

Also assume $\lambda \in \mathbb{R}$ satisfies

$$|\lambda| < \frac{1}{2}, \tag{3.11}$$

and

$$m^2 > \lambda^2. \tag{3.12}$$

Then, given $K > 0$, (3.4) has a nonzero solution $u \in \mathcal{D}((-L_0)^{1/2})$, obtained by the method previewed in §1.

Proof. Under these hypotheses, with $\mathcal{F}_{m,\lambda,X}$ as in (1.14), we have

$$\mathcal{F}_{m,\lambda,X}(u) \approx \|u\|^2_{\mathcal{D}((-L_0)^{1/2})}. \tag{3.13}$$

Via (3.8), we can pick $A \in (0, \infty)$ and minimize $\mathcal{F}_{m,\lambda,X}(u)$, under the constraint

$$\int\limits_{S^2} |u|^{p+1}\, dS = A. \tag{3.14}$$

Such a minimizer solves (1.7), for some $K = K_0 > 0$, and replacing such u by au gives solutions for general $K > 0$. $\square$

Remark. Also (3.4) has a constant solution. Arguments as in §2 imply that the solution obtained above is not constant on orbits of X, at least for some values of λ, m, and K.

We next establish a result of the form (3.8), better than what follows from (3.9).

Proposition 3.2 *For L_0 as in (3.1), we have*

$$\mathcal{D}((-L_0)^{1/2}) \hookrightarrow L^q(S^2), \quad \forall\, q \in [2,6), \tag{3.15}$$

and the inclusion is compact.

Proof. It suffices to show that there is a continuous inclusion of the form (3.15). In fact, from there, interpolation implies that, for each $q \in [2,6)$, there exists $s \in (0,1)$ such that $\mathcal{D}((-L_0)^{s/2}) \hookrightarrow L^q(S^2)$ is continuous, and we can compose this with the compact inclusion $\mathcal{D}((-L_0)^{1/2}) \hookrightarrow \mathcal{D}((-L_0)^{s/2})$.

To proceed, we complement (3.2) with the characterization

$$\mathcal{D}((-L_0)^{1/2}) = \{u \in L^2(S^2) : Yu, Zu \in L^2(S^2)\}. \tag{3.16}$$

Since $Y(\varphi u) = (Y\varphi)u + \varphi(Yu)$, with a similar identity for $Z(\varphi u)$, we see that $\mathcal{D}((-L_0)^{1/2})$ is stable under $u \mapsto \varphi u$, when $\varphi \in C^\infty(S^2)$. Also, ellipticity away from γ yields

$$u \in \mathcal{D}((-L_0)^{1/2}) \Longrightarrow \varphi u \in H^1(S^2), \tag{3.17}$$

whenever $\operatorname{supp} \varphi$ is disjoint from γ. It remains to show that if $u \in \mathcal{D}((-L_0)^{1/2})$ has support in a small neighborhood of a point $z_0 \in \gamma$, then $u \in L^q(S^2)$ for all $q < 6$. Using a coordinate chart that takes a neighborhood of z_0 in γ to the x-axis in $\mathbb{R}^2$, we obtain a compactly supported u on $\mathbb{R}^2$ such that

$$u \in H^{1/2}(\mathbb{R}^2), \quad \partial_y u \in L^2(\mathbb{R}^2). \tag{3.18}$$

It remains to show that this implies $u \in L^q(\mathbb{R}^2)$, $\forall\, q < 6$.

To establish this, we note that the conditions in (3.18) imply $(\xi^2 + \eta^2)^{1/4}\hat{u}(\xi,\eta)$ and $\eta\hat{u}(\xi,\eta)$ belong to $L^2(\mathbb{R}^2)$, hence

$$\hat{f} = (\xi^2 + \eta^4)^{1/4}\hat{u}(\xi,\eta) \in L^2(\mathbb{R}^2). \tag{3.19}$$

We have

$$u = k * f, \tag{3.20}$$

with

$$\hat{k}(\xi,\eta) = (\xi^2 + \eta^4)^{-1/4}. \tag{3.21}$$

Now, away from $(\xi,\eta) = (0,0)$, $\hat{k}(\xi,\eta)$ satisfies symbol estimates for membership in $S^{-1/2}_{1/2,0}(\mathbb{R}^2)$, hence

$$k \in C^\infty(\mathbb{R}^2 \setminus 0). \tag{3.22}$$

Also,

$$\hat{k}(\delta^{-2}\xi, \delta^{-1}\eta) = \delta\hat{k}(\xi,\eta) \ \ \forall\, \delta > 0 \Longrightarrow k(\delta^2 x, \delta y) = \delta^{-2} k(x,y). \tag{3.23}$$

To continue, for $j \in \mathbb{Z}$, let

$$\mathcal{O}_j = \{(2^{-j}x, 2^{-j/2}y) : x^2 + y^2 < 1\}, \quad \Omega_j = \mathcal{O}_j \setminus \mathcal{O}_{j+1}. \tag{3.24}$$

Then $\mathbb{R}^2 = \cup_{j\in\mathbb{Z}}\Omega_j$ is a partition of $\mathbb{R}^2$ into disjoint sets, and

$$(x,y) \mapsto (2^{-1}x, 2^{-1/2}y) \ \text{ takes } \ \Omega_j \ \text{ onto } \ \Omega_{j+1}. \tag{3.25}$$

Set

$$k_1 = k \ \text{ on } \ \bigcup_{j\ge 0} \Omega_j, \quad 0 \quad \text{elsewhere}, \tag{3.26}$$

and

$$k_2 = k \ \text{ on } \ \bigcup_{j< 0} \Omega_j, \quad 0 \quad \text{elsewhere}, \tag{3.27}$$

so $k = k_1 + k_2$. We estimate

$$u_\ell = k_\ell * f, \quad \ell = 1, 2, \tag{3.28}$$

by determining which L^r-spaces k_1 and k_2 belong to. To get this, note that, by (3.22)–(3.23),

$$|k| \le C2^j \ \text{ on } \ \Omega_j. \tag{3.29}$$

Meanwhile, by (3.25),

$$\operatorname{Vol}\Omega_j = 2^{-3/2} \operatorname{Vol}\Omega_{j-1} = C2^{-(3/2)j}, \tag{3.30}$$

so

$$\begin{aligned} \int |k_1|^r \, dx\, dy &\le C \sum_{j\ge 0} 2^{jr-(3/2)j} \\ &< \infty, \quad \forall\, r < \frac{3}{2}, \end{aligned} \tag{3.31}$$

and

$$\begin{aligned} \int |k_2|^r \, dx\, dy &\le \sum_{j< 0} 2^{jr-(3/2)j} \\ &< \infty, \quad \forall\, r > \frac{3}{2}. \end{aligned} \tag{3.32}$$

To use this to estimate u_j in (3.28), we interpolate

$$L^2 * L^2 \to L^\infty \ \text{ and } \ L^2 * L^1 \to L^2. \tag{3.33}$$

Noting that

$$[L^2, L^1]_{1/3} = L^{3/2} \text{ and } [L^\infty, L^2]_{1/3} \subset L^6, \tag{3.34}$$

hence $L^2 * L^{3/2} \to L^6$, we see that

$$u_1 \in L^q(\mathbb{R}^2), \quad \forall\, q \in [2,6), \quad \text{and} \quad u_2 \in L^q(\mathbb{R}^2), \quad \forall\, q > 6. \tag{3.35}$$

Since $u = u_1 + u_2$ has compact support, this gives $u \in L^q(\mathbb{R}^2)$ for all $q < 6$, and completes the proof of Proposition 3.2. □

We record the existence result that follows from Propositions 3.1–3.2.

Proposition 3.3 *Take X as in Proposition 3.1. Assume λ and m satisfy (3.11)–(3.12), and*

$$1 < p < 5. \tag{3.36}$$

Then the equation (3.4) has a solution $u \in \mathcal{D}((-L_0)^{1/2})$, obtained via the minimization process described in §1. This leads, via (1.3), to a traveling wave solution to (1.2), sonic along the "equator" $\gamma \subset S^2$.

It is natural to investigate higher order regularity of such a solution. We treat one case, namely $p = 3$.

Proposition 3.4 *In the setting of Proposition 3.3, take $p = 3$, and let $u \in \mathcal{D}((-L_0)^{1/2})$ solve*

$$-L_{2\lambda} u + (m^2 - \lambda^2) u = K|u|^2 u. \tag{3.37}$$

Then $u \in C^\infty(S^2)$.

Proof. Set

$$P = -L_{2\lambda} + (m^2 - \lambda^2), \quad F(u) = K|u|^2 u. \tag{3.38}$$

Standard subelliptic regularity results (cf., e.g., [10], Chapter 15, Theorem 1.8) yield

$$P^{-1} : H^s(S^2) \longrightarrow H^{s+1}(S^2), \quad \forall\, s \in \mathbb{R}. \tag{3.39}$$

Alternatively, (3.39) can be deduced from the behavior of P on spherical harmonics (cf. [11], Chapters 2 and 4). If (3.37) holds, then

$$u = P^{-1} F(u). \tag{3.40}$$

Now

$$
\begin{aligned}
& u \in \mathcal{D}((-L_0)^{1/2}) \subset \bigcap_{q<6} L^q(S^2) \\
& \Rightarrow F(u) \in \bigcap_{p<2} L^p(S^2) \subset \bigcap_{\varepsilon>0} H^{-\varepsilon,2}(S^2) \\
& \Rightarrow u = P^{-1}F(u) \in \bigcap_{\varepsilon>0} H^{1-\varepsilon,2}(S^2) \\
& \Rightarrow u \in \bigcap_{q<\infty} L^q(S^2) \\
& \Rightarrow F(u) \in \bigcap_{p<\infty} L^p(S^2) \subset L^2(S^2) \\
& \Rightarrow u \in H^1(S^2).
\end{aligned}
\tag{3.41}
$$

We emphasize two conclusions:

$$
u \in H^1(S^2), \quad u \in \bigcap_{q<\infty} L^q(S^2). \tag{3.42}
$$

If also v has such properties, then since

$$
\nabla(uv) = (\nabla u)v + u(\nabla v), \tag{3.43}
$$

we have $\nabla(uv) \in L^q(S^2)$ for all $q < 2$. Iteration gives

$$
\nabla F(u) \in \bigcap_{q<2} L^q(S^2), \quad \text{hence} \quad F(u) \in \bigcap_{\varepsilon>0} H^{1-\varepsilon,2}(S^2). \tag{3.44}
$$

Hence, by (3.39),

$$
u \in \bigcap_{\varepsilon>0} H^{2-\varepsilon,2}(S^2). \tag{3.45}
$$

Now $H^{\sigma,2}(S^2)$ is a Banach algebra for $\sigma > 1$, so we have

$$
F(u) \in \bigcap_{\varepsilon>0} H^{k-\varepsilon,2}(S^2), \quad \text{hence} \quad u \in \bigcap_{\varepsilon>0} H^{k+1-\varepsilon}(S^2), \tag{3.46}
$$

first for $k = 2$, then, inductively, for all $k > 2$. This gives the asserted regularity. $\square$

4 Mach 1 traveling waves on S^3

The group $SU(2)$, with its bi-invariant Riemannian metric, is isometric to S^3. It covers $SO(3)$. There are left-invariant vector fields X, Y, and Z on $SU(2)$, covering vector fields on $SO(3)$ that generate 2π-periodic riattions of $\mathbb{R}^3$ about the x, y, and z-axes, respectively. With appropriate scaling of the metric on S^3, we have

$$\langle X, X\rangle \equiv \langle Y, Y\rangle \equiv \langle Z, Z\rangle \equiv 1 \quad \text{on} \quad S^3, \tag{4.1}$$

and

$$\Delta = X^2 + Y^2 + Z^2. \tag{4.2}$$

Furthermore, the operator

$$L_0 = \Delta - X^2 = Y^2 + Z^2 \tag{4.3}$$

satisfies Hörmander's condition for hypoellipticity with loss of one derivative. Parallel to (3.2)–(3.3), we have

$$\mathcal{D}(L_0) \subset H^1(S^3), \quad \text{hence} \quad \mathcal{D}((-L_0)^{1/2}) \subset H^{1/2}(S^3), \tag{4.4}$$

and

$$-(L_0 u, u) \ge C\|u\|^2_{H^{1/2}}, \quad \text{if} \quad \int\limits_{S^3} u\, dV = 0. \tag{4.5}$$

Parallel to (3.4), we can write (1.7) as

$$-L_{2\lambda} u + (m^2 - \lambda^2) u = K|u|^{p-1} u, \tag{4.6}$$

where

$$L_\alpha = L_0 - i\alpha X. \tag{4.7}$$

Parallel to §3, nontrivial solutions to (4.6) give traveling waves on S^3, via (1.3), but in the current setting these waves travel at the speed of sound, not just on an equator, but everywhere on S^3. We proceed to see when (4.6) has nontrivial solutions. Up to a point, this analysis will parallel that of §3, and then details will diverge.

For $\alpha \in \mathbb{R}$, L_α is self adjoint. Results on the representation theory of $SU(2)$ (cf. [11], Chapter 2) imply $-L_\alpha$ is positive semidefinite with 1-dimensional kernel if $|\alpha| < 1$, positive semidefinite with infinite-dimensional kernel if $|\alpha| = 1$, and not semi-bounded if $|\alpha| > 1$. Also, parallel to (3.6),

$$\begin{gathered} |\alpha| < 1 \Rightarrow \mathcal{D}(L_\alpha) = \mathcal{D}(L_0), \text{ hence } \mathcal{D}((-L_\alpha)^{1/2}) = \mathcal{D}((-L_0)^{1/2}), \text{and} \\ -(L_\alpha u, u) \ge C_\alpha \|u\|^2_{H^{1/2}}, \quad \text{if} \quad \int\limits_{S^3} u\, dV = 0. \end{gathered} \tag{4.8}$$

To proceed, we bring in the following variant of (3.8). Namely, there exists $q_* > 2$ such that

$$\mathcal{D}((-L_0)^{1/2}) \hookrightarrow L^q(S^3) \text{ is compact, } \forall\, q \in [2, q_*). \tag{4.9}$$

From the Sobolev embedding result

$$H^{1/2}(S^3) \subset L^3(S^3), \tag{4.10}$$

we have (4.9) with $q_* = 3$. We will obtain a better result below. First we formulate an existence result, parallel to Proposition 3.1.

Proposition 4.1 *Let X be as in the first paragraph of this section. With q_* as in (4.9), assume*

$$2 < p + 1 < q_*. \tag{4.11}$$

Also assume $\lambda \in \mathbb{R}$ satisfies

$$|\lambda| < \frac{1}{2}, \tag{4.12}$$

and

$$m^2 > \lambda^2. \tag{4.13}$$

Then, given $K > 0$, (4.6) has a nonzero solution $u \in \mathcal{D}((-L_0)^{1/2})$, obtained by the method previewed in §1.

Proof. Same as that of Proposition 3.1. □

Remark. As with Proposition 3.1, arguments as in §2 imply that the solution obtained above is not constant on orbits of X, at least for some values of λ, m, and K.

The following is a result of the form (4.9), improving what follows from (4.10).

Proposition 4.2 *For L_0 as in (4.3), we have*

$$\mathcal{D}((-L_0)^{1/2}) \hookrightarrow L^q(S^3), \quad \forall\, q \in [2, 4), \tag{4.14}$$

and the inclusion is compact.

Proof. As with Proposition 3.2, it suffices to establish the inclusion (4.14), and the compactness automatically follows. We need to show that

$$(1 - L_0)^{-1/2} : L^2(S^3) \longrightarrow L^q(S^3), \quad \forall\, q \in [2, 4). \tag{4.15}$$

This follows from the analysis of a parametrix for $(1 - L_0)^{-1/2}$ using the theory developed in Folland-Stein [4] (see also [12] for an exposition). After a localization, one has a diffeomorphism of an open set in S^3 to an open set in the 3-dimensional Heisenberg group $\mathbb{H}^3$, with coordinates (p, q, t), such that

$$\begin{aligned}(1 - L_0)^{-1/2}u(z) &= \int\limits_{\mathbb{H}^3} F(z, w)u(w^{-1}z)\, dw \\ &= F(z, \cdot) * u(z),\end{aligned} \tag{4.16}$$

plus lower order terms, where $z \mapsto F(z, \cdot)$ is a smooth function of z with values in the space of functions $F \in C^\infty(\mathbb{H}^3 \setminus 0)$ that have the anisotropic homogeneity

$$F(\delta p, \delta q, \delta^2 t) = \delta^{-3} F(p, q, t), \quad \delta > 0. \tag{4.17}$$

Equivalently, its Euclidean Fourier transform $\widehat{F}$ belongs to $C^\infty(\mathbb{R}^3 \setminus 0)$ and satisfies

$$\widehat{F}(sx, s\xi, s^2\lambda) = s^{-1}\widehat{F}(x, \xi, \lambda), \quad s > 0.$$

The $*$ in the second line of (4.16) denotes the group convolution for $\mathbb{H}^3$. Via standard techniques explained in the references given above, to prove (4.14) it suffices to show that if $F \in C^\infty(\mathbb{H}^3 \setminus 0)$ satisfies (4.17), then

$$u \in L^2_{\text{comp}}(\mathbb{H}^3) \Longrightarrow F * u \in L^q_{\text{loc}}(\mathbb{H}^3), \quad \forall\, q < 4. \tag{4.18}$$

The key to this is the following.

Lemma 4.3 *If $F \in C^\infty(\mathbb{H}^3 \setminus 0)$ satisfies (4.17), then*

$$F \in L^p_{loc}(\mathbb{H}^3), \quad \forall\, p < \frac{4}{3}. \tag{4.19}$$

To see how Lemma 4.3 implies (4.18), we interpolate

$$L^2 * L^2 \subset L^\infty, \quad \text{and} \quad L^1 * L^2 \subset L^2. \tag{4.20}$$

Now

$$[L^2, L^1]_\theta = L^{4/3} \quad \text{for} \quad \frac{3}{4} = \frac{\theta}{1} + \frac{1-\theta}{2}, \quad \text{i.e.,} \quad \theta = \frac{1}{2}, \tag{4.21}$$

and

$$[L^\infty, L^2]_{1/2} \subset L^4, \tag{4.22}$$

so

$$L^{4/3} * L^2 \subset L^4. \tag{4.23}$$

The implication (4.19) $\Rightarrow$ (4.18) follows readily from this.

It remains to prove Lemma 4.3. We proceed as follows. It suffices to prove (4.19) when (4.17) holds and

$$F(p,q,t) = 1 \quad \text{for} \quad p^2 + q^2 + t^2 = 1. \tag{4.24}$$

Set

$$\Omega_0 = \{(p,q,t) : 1 \le F(p,q,t) \le 8\}. \tag{4.25}$$

For $k \ge 1$, define Ω_k by

$$(p,q,t) \mapsto (2^{-1}p, 2^{-1}q, 2^{-2}t) \quad \text{maps} \quad \Omega_{k-1} \to \Omega_k. \tag{4.26}$$

Note that

$$\operatorname{Vol}\Omega_k = 2^{-4}\ \operatorname{Vol}\Omega_{k-1} = C\,2^{-4k}, \tag{4.27}$$

and, by (4.17),

$$F\Big|_{\Omega_k} \approx 2^{3k}. \tag{4.28}$$

Hence, with $B = \cup_{k\ge 0}\Omega_k = \{(p,q,t) : p^2+q^2+t^2 \le 1\}$,

$$\begin{aligned}\int_B |F(p,q,t)|^p\,dV &\le C\sum_{k\ge 0} 2^{3pk}\cdot 2^{-4k} \\ &< \infty \Longleftrightarrow p < \frac{4}{3}.\end{aligned} \tag{4.29}$$

This proves Lemma 4.3, and completes the proof of Proposition 4.2.

Putting together Propositions 4.1 and 4.2, we have the following existence result.

Proposition 4.4 *Take X as in Proposition 4.1. Assume λ and m satisfy (4.12)–(4.13), and assume*

$$1 < p < 3. \tag{4.30}$$

Then, given $K > 0$, (4.6) has a nonzero solution $u \in \mathcal{D}((-L_0)^{1/2})$, obtained by the variational method of §1, and yielding, via (1.3), a solution to (1.2), traveling at sonic speed at every point of S^3.

Remark. In [5] there is a study of a class of nonlinear subelliptic PDE on bounded domains (and some unbounded domains) in the Heisenberg group $\mathbb{H}^m$ (a Lie group of dimension $d = 2m + 1$), for which there are conclusions similar to Proposition 4.4, when $d = 3$.

5 Supersonic traveling waves for NLKG

In this section we let X be an arbitrary (nonzero) Killing field on a compact n-dimensional Riemannian manifold M (tangent to ∂M if the boundary is nonempty), and produce traveling waves of the form (1.3), with u solving (1.7), without a restriction on $\langle X, X\rangle$. Such traveling waves are supersonic wherever $\langle X, X\rangle > 1$. Unlike the situations treated in §§1–4, we cannot expect to obtain u by minimizing $\mathcal{F}_{m,\lambda,X}(u)$ subject only to the constraint (1.19). Thus the solutions constructed here are far from being "ground states." We will obtain our solutions by minimizing $\mathcal{F}_{m,\lambda,X}$ over

$$V_\mu = \{u \in H_0^1(M) : Xu = i\mu u\}, \tag{5.1}$$

subject to the constraint (1.19), assuming $V_\mu \neq 0$. Note that X preserves each eigenspace

$$E_\beta = \{u \in H_0^1(M) : \Delta u = -\beta^2 u\}, \quad \beta^2 \in \operatorname{Spec}(-\Delta), \tag{5.2}$$

and X is skew-adjoint on each of these finite-dimensional spaces. Thus there is a countable set of μ such that $V_\mu \neq 0$, and in fact

$$L^2(M) = \bigoplus_\mu \overline{V}_\mu, \tag{5.3}$$

where $\overline{V}_\mu$ is the closure in $L^2(M)$ of V_μ. Note that

$$\begin{aligned} u \in V_\mu \Longrightarrow \mathcal{F}_{m,\lambda,X}(u) &= (-\Delta u + [m^2 - (\lambda+\mu)^2]u, u) \\ &= \mathcal{F}^{\#}_{m,\lambda,\mu}(u), \end{aligned} \tag{5.4}$$

so our task is to minimize $\mathcal{F}^{\#}_{m,\lambda,\mu}$ over V_μ, subject to the constraint (1.19), i.e.,

$$I_p(u) = \int_M |u|^{p+1}\, dV = A. \tag{5.5}$$

The following is the key to an existence result.

Proposition 5.1 *Take $n \ge 2,\ p \in (1,(n+2)/(n-2)),\ m > 0$, and $\lambda \in \mathbb{R}$. Also, take $\mu \in \mathbb{R}$ such that*

$$V_\mu \neq 0. \tag{5.6}$$

Then there exists $u \in V_\mu$ minimizing $\mathcal{F}^\#_{m,\lambda,\mu}$ over V_μ, subject to the constraint (5.5).

Proof. This goes like the proof of Proposition 1.2, once one notes that there exists $\gamma \in (0,\infty)$ such that

$$\|u\|^2_{H^1} \approx \mathcal{F}^\#_{m,\lambda,\mu}(u) + \gamma \|u\|^2_{L^{p+1}}. \tag{5.7}$$

$\square$

Corollary 5.2 *The minimizer $u \in V_\mu$ given by Proposition 5.1 solves the equation*

$$-\Delta u + [m^2 - (\lambda+\mu)^2]u = K_0|u|^{p-1}u, \tag{5.8}$$

for some $K_0 \in \mathbb{R}$, hence

$$-\Delta u + X^2 u + 2i\lambda X u + (m^2 - \lambda^2)u = K_0|u|^{p-1}u, \tag{5.9}$$

which is (1.7).

Proof. To see this, we record the following analogues of (1.23) and (1.28):

$$\begin{aligned} \frac{d}{d\tau} I_p(u+\tau v)\Big|_{\tau=0} &= (p+1)\ \mathrm{Re} \int |u|^{p-1} u\overline{v}\, dV, \\ \frac{d}{d\tau} \mathcal{F}^\#_{m,\lambda,\mu}(u+\tau v)\Big|_{\tau=0} &= 2\ \mathrm{Re}\,(-\Delta u + [m^2 - (\lambda+\mu)^2]u, v). \end{aligned} \tag{5.10}$$

In place of (1.29), this yields

$$\begin{aligned} &v \in V_\mu,\ \ \mathrm{Re}\,(|u|^{p-1}u, v) = 0 \\ &\Longrightarrow\ \mathrm{Re}\,(-\Delta u + [m^2 - (\lambda+\mu)^2]u, v) = 0, \end{aligned} \tag{5.11}$$

and, of course, we have $u \in V_\mu$. Now the $\mathbb{R}$-bilinear pairing of $H^1_0(M)$ with $H^{-1}(M)$ given by $\mathrm{Re}\,(\ ,\ )$ restricts to a dual pairing of V_μ with

$$V'_\mu = \{w \in H^{-1}(M) : Xw = i\mu w\}. \tag{5.12}$$

It is also useful to note that

$$V_\mu = \{u \in H^1_0(M) : u(g(t)x) = e^{it\mu}u(x),\ \forall\, t \in \mathbb{R}\}, \tag{5.13}$$

where $g(t)$ is the 1-parameter group of isometries of M generated by X, and also

$$V'_\mu = \{w \in H^{-1}(M) : w(g(t)x) = e^{it\mu} w(x),\ \forall\, t \in \mathbb{R}\}. \tag{5.14}$$

Hence

$$u \in V_\mu \Longrightarrow u,\ \Delta u, |u|^{p-1} u \in V'_\mu, \tag{5.15}$$

so the implication (5.11) yields the result that (5.8) holds for some $K_0 \in \mathbb{R}$. This is an identity of elements of V'_μ. Pairing both sides with u yields

$$K_0 A = \|\nabla u\|^2_{L^2} + [m^2 - (\lambda + \mu)^2] \|u\|^2_{L^2} = \mathcal{F}^{\#}_{m,\lambda,\mu}(u). \tag{5.16}$$

Hence K_0 is positive if $\mathcal{F}^{\#}_{m,\lambda,\mu}(u)$ is positive, and negative if $\mathcal{F}^{\#}_{m,\lambda,\mu}(u)$ is negative. $\square$

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