Uniform Convergence of Fourier Series

Michael Taylor

Given $f \in L^1(\mathbb{T}^1)$, we consider the partial sums of the Fourier series of f:

$$S_N f(\theta) = \sum_{k=-N}^{N} \hat{f}(k) e^{ik\theta}. \tag{1}$$

A calculation gives the Dirichlet formula

$$S_N f(\theta) = \frac{1}{2\pi} \int_{\mathbb{T}^1} f(\theta - \varphi) D_N(\varphi)\, d\varphi, \tag{2}$$

where

$$\begin{aligned} D_N(\varphi) &= \sum_{k=-N}^{N} e^{ik\varphi} \\ &= e^{-iN\varphi} \sum_{k=0}^{2N} e^{ik\varphi} \\ &= \frac{\sin(N+1/2)\varphi}{\sin \varphi/2}, \end{aligned} \tag{3}$$

the last identity by virtue of

$$x^{-N} \sum_{k=0}^{2N} x^k = x^{-N} \frac{1 - x^{2N+1}}{1 - x},$$

using $e^{i\varphi}$ for x, and multiplying numerator and denominator by $e^{-i\varphi/2}$. Using

$$\sin\Bigl(N + \frac{1}{2}\Bigr)\varphi = \cos\frac{\varphi}{2} \sin N\varphi + \sin\frac{\varphi}{2} \cos N\varphi,$$

we deduce that

$$\begin{aligned} S_N f(\theta) - f(\theta) &= \frac{1}{2\pi} \int_{\mathbb{T}^1} [f(\theta - \varphi) - f(\theta)] D_N(\varphi)\, d\varphi \\ &= \frac{1}{2\pi} \int_{\mathbb{T}^1} g_\theta(\varphi)\, \sin N\varphi\, d\varphi + \frac{1}{2\pi} \int_{\mathbb{T}^1} h_\theta(\varphi)\, \cos N\varphi\, d\varphi, \end{aligned} \tag{4}$$

where

$$g_\theta(\varphi) = \frac{f(\theta - \varphi) - f(\theta)}{\tan \varphi/2}, \quad h_\theta(\varphi) = f(\theta - \varphi) - f(\theta). \tag{5}$$

Clearly, for $N \neq 0$,

$$f \in L^1(\mathbb{T}^1) \Longrightarrow |\hat{h}_\theta(\pm N)| = |\hat{f}(\pm N)| \to 0 \ \text{ as } \ N \to \infty, \tag{6}$$

the convergence to 0 by the Riemann-Lebesgue lemma.

Applying the Riemann-Lebesgue lemma to $\hat{g}_\theta(\pm N)$ gives the following.

Proposition 1. *Let $f \in L^1(\mathbb{T}^1)$. Let $K \subset \mathbb{T}^1$ be compact. Then*

$$S_N f(\theta) \longrightarrow f(\theta), \quad \text{uniformly for } \theta \in K, \tag{7}$$

provided that

$$\{g_\theta : \theta \in K\} \text{ is a relatively compact subset of } L^1(\mathbb{T}^1). \tag{8}$$

Proof. The Riemann-Lebesgue lemma plus the compactness hypothesis (8) implies that $|\hat{g}_\theta(N)|$ goes to 0 as $|N| \to \infty$, uniformly in $\theta \in K$. In more detail, take $\varepsilon > 0$. Pick a finite set $\{\theta_j : 1 \le j \le M(\varepsilon)\}$ such that, with $g_j(\varphi) = g_{\varepsilon_j}(\varphi)$,

$$\forall\, \theta \in K, \quad \|g_j - g_\theta\|_{L^1} \le \varepsilon, \text{ for some } j \le M(\varepsilon). \tag{8A}$$

The compactness hypothesis (8) guarantees you can do this. The Riemann-Lebesgue lemma says that, for each $j \in \{1, \dots, M(\varepsilon)\}$, there exists N_j such that

$$|\hat{g}_j(N)| < \varepsilon, \quad \forall\, N \text{ such that } |N| > N_j. \tag{8B}$$

Now set $\widetilde{N}(\varepsilon) = \max\{N_j : 1 \le j \le M(\varepsilon)\}$. By (8A) we have, for all $\theta \in K$,

$$\begin{aligned} |\hat{g}_\theta(N)| &\le \min_j \Big(|\hat{g}_j(N)| + |\hat{g}_j(N) - \hat{g}_\theta(N)| \Big) \\ &\le \varepsilon + \varepsilon, \end{aligned} \tag{8C}$$

provided $|N| > \widetilde{N}(\varepsilon)$. The desired conclusion (7) follows from this, in concert with (4)–(6).

The following is an important special case.

Corollary 2. *Let $f \in C^\omega(\mathbb{T}^1)$, i.e.,*

$$|f(\theta - \varphi) - f(\theta)| \le C\omega(|\varphi|), \quad \forall\, \theta, \varphi \in \mathbb{T}^1. \tag{9}$$

Assume the modulus of continuity $\omega(t)$ satisfies

$$\int_0^{2\pi} \frac{\omega(t)}{t}\, dt < \infty. \tag{10}$$

Then (7) holds with $K = \mathbb{T}^1$.

Proof. We claim the hypotheses (9)–(10) imply that

$$g_\theta \text{ is a continuous function of } \theta \text{ with values in } L^1(\mathbb{T}^1). \tag{11}$$

Given this, the compactness condition (8) holds, with $K = \mathbb{T}^1$. So let $\theta_\nu, \theta_0 \in \mathbb{T}^1$, $\theta_\nu \to \theta_0$. We see that

$$g_{\theta_\nu}(\varphi) \longrightarrow g_{\theta_0}(\varphi) \quad \text{for all} \quad \varphi \in \mathbb{T}^1 \setminus 0, \tag{12}$$

and that

$$|g_{\theta_\nu}(\varphi)| \le C\frac{\omega(|\varphi|)}{|\varphi|} = H(\varphi). \tag{13}$$

Hence $|g_{\theta_\nu}(\varphi) - g_{\theta_0}(\varphi)| \to 0$ for all $\varphi \in \mathbb{T}^1 \setminus 0$, and

$$|g_{\theta_\nu}(\varphi) - g_{\theta_0}(\varphi)| \le 2H(\varphi). \tag{14}$$

Now (10) implies $H \in L^1(\mathbb{T}^1)$, so the convergence

$$\int_{\mathbb{T}^1} |g_{\theta_\nu}(\varphi) - g_{\theta_0}(\varphi)|\, d\varphi \longrightarrow 0 \tag{15}$$

follows by the Dominated Convergence Theorem.

The following is a version of Riemann localization.

Proposition 3. *Take $f \in L^1(\mathbb{T}^1)$. Assume $f = 0$ on $\mathcal{O}$, an open subset of $\mathbb{T}^1$, and let $K \subset \mathcal{O}$ be compact. Then $S_N f \to f$ uniformly on K.*

Proof. Take an interval $I = (-\varepsilon, \varepsilon)$ so small that

$$\theta \in K, \ \varphi \in I \Longrightarrow \theta - \varphi \in \mathcal{O}, \tag{16}$$

so

$$\theta \in K, \ \varphi \in I \Longrightarrow g_\theta(\varphi) = 0. \tag{17}$$

Then take $\psi \in C(\mathbb{T}^1)$ such that $\psi(\varphi) = 1$ for $|\varphi| < \varepsilon/2$, $\psi(\varphi) = 0$ for $|\varphi| \ge \varepsilon$. Then

$$\begin{aligned} \theta \in K &\Longrightarrow \psi g_\theta \equiv 0 \\ &\Longrightarrow g_\theta(\varphi) \equiv \frac{1-\psi(\varphi)}{\tan \varphi/2}\,[f(\theta - \varphi) - f(\theta)]. \end{aligned} \tag{18}$$

Since $(1 - \psi(\varphi))/\tan(\varphi/2)$ is continuous on $\mathbb{T}^1$, it follows that

$$\theta \mapsto g_\theta \quad \text{is continuous from } K \text{ to } \ L^1(\mathbb{T}^1). \tag{19}$$

Thus (8) holds, and Proposition 3 follows from Proposition 1.

Putting together Corollary 2 and Proposition 3 gives the following.

Corollary 4. *Take $f \in L^1(\mathbb{T}^1)$. Let $\mathcal{O} \subset \mathbb{T}^1$ be open and assume $f|_{\mathcal{O}} \in C^\omega(\mathcal{O})$, with ω satisfying (10). Let $K \subset \mathcal{O}$ be compact. Then $S_N f \to f$ uniformly on K.*

We now produce another strengthening of Corollary 2.

Proposition 5. *Take $f \in L^1(\mathbb{T}^1)$, and let $K \subset \mathbb{T}^1$ be compact. Assume $f|_K \in C(K)$ and*

$$|f(\theta - \varphi) - f(\theta)| \le C\omega(|\varphi|), \quad \forall\, \theta \in K,\ \varphi \in \mathbb{T}^1, \tag{20}$$

where ω is measurable and satisfies (10). Then (7) holds.

Proof. Again it suffices to show that

$$\theta \mapsto g_\theta \ \text{ is continuous from } K \text{ to } \ L^1(\mathbb{T}^1). \tag{21}$$

So let $\theta_\nu, \theta_0 \in K$ and $\theta_\nu \to \theta_0$. We continue to have (13)–(14), i.e.,

$$|g_{\theta_\nu}(\varphi) - g_{\theta_0}(\varphi)| \le 2H(\varphi), \quad H \in L^1(\mathbb{T}^1). \tag{22}$$

We need a replacement for (12). In fact, we claim that

$$g_{\theta_\nu} \longrightarrow g_{\theta_0} \ \text{ in measure, as } \nu \to \infty, \tag{23}$$

i.e., if m denotes Lebesgue measure on $\mathbb{T}^1$, then, for each $\varepsilon > 0$,

$$m(E_{\varepsilon\nu}) \longrightarrow 0 \ \text{ as } \ \nu \to \infty, \tag{24}$$

where

$$E_{\varepsilon\nu} = \{\varphi \in \mathbb{T}^1 : |g_{\theta_\nu}(\varphi) - g_{\theta_0}(\varphi)| > \varepsilon\}. \tag{25}$$

In fact, (23) follows directly from the continuity of $f|_K$, together with the fact that, with $f_\theta(\varphi) = f(\theta - \varphi)$,

$$f_{\theta_\nu} \longrightarrow f_{\theta_0} \ \text{ in measure, as } \ \nu \to \infty, \tag{26}$$

itself a consequence of the fact that

$$f_{\theta_\nu} \longrightarrow f_{\theta_0} \ \text{ in } L^1\text{-norm}, \tag{27}$$

i.e.,

$$\int_{\mathbb{T}^1} |f_{\theta_\nu}(\varphi) - f_{\theta_0}(\varphi)|\, d\varphi \longrightarrow 0, \tag{28}$$

together with Chebechev's inequality,

$$m\Big(\{\varphi : |F(\varphi)| > \varepsilon\}\Big) \le \frac{1}{\varepsilon}\|F\|_{L^1}. \tag{29}$$

It is a variant of the Dominated Convergence Theorem (cf. [T], pp. 37–38), that (22)–(23) imply (15). This completes the proof of Proposition 5.

Bringing in an argument used for Riemann localization in Proposition 3, we have the following strengthening of Proposition 5.

Proposition 6. *Take $f \in L^1(\mathbb{T}^1)$ and let $K \subset \mathbb{T}^1$ be compact. Assume $f|_K \in C(K)$. Assume there exists $\varepsilon > 0$ such that*

$$|f(\theta - \varphi) - f(\theta)| < c\omega(|\varphi|), \quad \forall\, \theta \in K,\ |\varphi| < \varepsilon, \tag{30}$$

where ω is measurable on $[0, 2\pi]$ and satisfies (10). Then (7) holds.

Proof. Take ψ as in (18), and set

$$g_\theta(\varphi) = u_\theta(\varphi) + v_\theta(\varphi), \tag{31}$$

where

$$\begin{aligned} u_\theta(\varphi) &= \psi(\varphi)\, \frac{f(\theta - \varphi) - f(\theta)}{\tan \varphi/2}, \\ v_\theta(\varphi) &= \frac{1 - \psi(\varphi)}{\tan \varphi/2} [f(\theta - \varphi) - f(\theta)]. \end{aligned} \tag{32}$$

Clearly $f \in L^1(\mathbb{T}^1)$ and $f|_K$ bounded implies $\{v_\theta : \theta \in K\}$ is relatively compact in $L^1(\mathbb{T}^1)$. Meanwhile an analysis parallel to (21)–(29) applies here, with g_θ replaced by u_θ, under the hypotheses given above.

Reference

[T] M. Taylor, Measure Theory and Integration, AMS, Providence RI, 2006.