Uniformization of Compactly Perturbed Planes, And Related Green Function Constructions

Preliminary Notes
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1. Introduction

This work is motivated by an issue in geometrical optics, concerning the null bicharacteristics of a variable speed d'Alembertian

$$\partial_t^2 - a(x)^2\Delta, \tag{1.1}$$

with $t \in \mathbb{R},\ x \in \mathbb{R}^n,\ \Delta = \partial_1^2 + \cdots + \partial_n^2$. We assume

$$a \in C^\infty(\mathbb{R}^n), \quad a > 0, \quad a(x) = 1 \ \text{ for } \ |x| \ge R, \tag{1.2}$$

for some $R \in (0, \infty)$. To leading order, the operator (1.1) agrees with

$$\partial_t^2 - \Delta_g, \tag{1.3}$$

where Δ_g is the Laplace-Beltrami operator on $M = \mathbb{R}^n$, endowed with the metric tensor

$$g_{jk} = a(x)^{-2}\delta_{jk}, \tag{1.4}$$

and in particular the two operators have the same null bicharacteristics, and hence propagate singularities along the same rays. These rays correspond naturally to orbits of the geodesic flow on S^*M, with metric tensor g_{jk}.

When it comes to constructing examples that have periodic orbits with prescribed geometric properties, the setting (1.3) is quite convenient, as it allows one's geometrical intuition to take hold. We take g_{jk} to be an arbitrary compactly supported perturbation of the flat metric on $\mathbb{R}^n$:

$$g_{jk} \in C^\infty(\mathbb{R}^n), \quad \text{positive definite}, \quad g_{jk}(x) = \delta_{jk} \ \text{ for } \ |x| \ge R. \tag{1.5}$$

For example, we can take a sphere S^n, cut out a disk about its south pole, cut out a disk about the origin in $\mathbb{R}^n$, and attach these two spaces by a tube, obtaining a Riemannian manifold, diffeomorphic to $\mathbb{R}^n$, with closed geodesics of a certain type. This leads to the question of what such a construction might say about (1.1). That is to say, does there exist a function $a(x)$, satisfying, not quite (1.2), but

$$a \in C^\infty(\mathbb{R}^n), \quad a > 0, \quad a(x) \sim 1 \ \text{ as } \ |x| \to \infty, \tag{1.6}$$

such that $(\mathbb{R}^n, a(x)^{-2}\delta_{jk})$ is isometric to (M, g)? Certainly this will fail in general if $n \geq 3$, since (M, g) will typically not be locally conformally flat. As we will see, it does succeed when $n = 2$.

Here is our first task, in case $n = 2$. Let g_{jk} be a metric tensor on $\mathbb{R}^2$, satisfying (1.5). We desire to find

$$u \in C^\infty(\mathbb{R}^2), \quad u(x) \to 0 \ \text{ as } \ |x| \to \infty, \tag{1.7}$$

such that the new metric tensor

$$\tilde{g}_{jk} = e^{2u} g_{jk} \ \text{ has zero curvature.} \tag{1.8}$$

Then $(\mathbb{R}^2, \tilde{g}_{jk})$ is flat, complete, and simply connected, and it is well known that such a space is isometric to $(\mathbb{R}^2, \delta_{jk})$. Generally, if $k(x)$ denotes the Gauss curvature of $(\mathbb{R}^2, g_{jk})$, then the Gauss curvature $K(x)$ of $(\mathbb{R}^2, e^{2u} g_{jk})$ is given by

$$K(x) = (-\Delta_g u + k(x)) e^{-2u}. \tag{1.9}$$

If we want $K \equiv 0$, we want to solve the linear equation

$$\Delta_g u = k, \tag{1.10}$$

and we want a solution satisfying (1.7). In case $g_{jk} = \delta_{jk}$, we would solve (1.10) for a general function $k \in C_0^\infty(\mathbb{R}^2)$ by convolving k with the fundamental solution

$$E_0(x) = \frac{1}{2\pi} \log |x|. \tag{1.11}$$

Typically, $k * E_0(x)$ has a log blow-up as $|x| \to \infty$, unless k integrates to zero. Fortunately, k in (1.10) has this property. In fact, the Gauss-Bonnet theorem implies

$$\int_M k(x)\, dV(x) = 0, \tag{1.12}$$

where $M = \mathbb{R}^2$ and $dV(x) = \sqrt{g(x)}\, dx$ is the area element associated to the metric tensor g_{jk}, with $g(x) = \det(g_{jk})$.

In §2 we will show that if (M, g_{jk}) is a compactly supported perturbation of $(\mathbb{R}^2, \delta_{jk})$ and $k \in C_0^\infty(M)$ satisfies (1.12), then (1.10) has a solution u satisfying (1.7). We will not require M to be diffeomorphic to $\mathbb{R}^2$; we could add handles to the plane. Of course, in such a case, the Gauss-Bonnet theorem implies that (1.12) fails if $k(x)$ is the Gauss curvature of M, but it is still of intrinsic interest to have this solvability result. In §2 we also show there is a Green function, behaving like $\log |x|$ at infinity.

It would be desirable to loosen the hypothesis that the perturbation be compactly supported. The remaining material attempts to deal with this.

In §3 we study (1.10) when M is an n-dimensional, asymptotically Euclidean, Riemannian manifold and k has an asymptotic expansion in terms of powers r^{-k-2}, $k \in \mathbb{N}$, and obtain u, with a more complicated asymptotic expansion, such that (1.10) holds asymptotically. This leads to the problem of solving (1.10) when $k \in \mathcal{S}(M)$, i.e., k and all its covariant derivatives vanish rapidly at infinity.

In Appendix A, we give a more general criterion for $(\mathbb{R}^2, g_{jk})$ to be conformally equivalent to $(\mathbb{R}^2, \delta_{jk})$ than done in §2. The proof uses the uniformization theorem and a Liouville theorem, and provides less information about the resulting conformal factor $a(x)^{-2}$.

Appendices B–E tackle (1.10) where M is a general complete, n-dimensional Riemannian manifold, assuming $k \in C_0^\infty(M)$. (In some of these appendices, we require M to have nonempty boundary.) It remains to see when we can pass to the more interesting case $k \in \mathcal{S}(M)$, at least when M is asymptotically Euclidean.

2. Solving $\Delta_g u = f$ on compactly perturbed planes

Let (M, g) be a two-dimensional Riemannian manifold. We assume M is connected and that there exist a compact $K \subset M$ and $R \in (0, \infty)$ such that $M \setminus K$ is isometric with $\mathbb{R}^2 \setminus \overline{B_R(0)}$. We denote the Laplace-Beltrami operator of (M, g) by Δ_g. We aim to prove the following.

Proposition 2.1. *Given $f \in C_0^\infty(M)$ such that*

$$\int_M f(x)\, dV(x) = 0, \tag{2.1}$$

there exists a unique solution u to

$$\Delta_g u = f, \tag{2.2}$$

satisfying

$$u \in C^\infty(M), \quad u(x) \to 0 \quad \text{as} \quad x \to \infty. \tag{2.3}$$

To start, we can take a compact, smoothly bounded $\overline{\Omega} \subset M$ such that

$$K \subset \Omega, \quad \operatorname{supp} f \subset \Omega, \quad \text{and } M \setminus \Omega \text{ isometric to } \mathbb{R}^2 \setminus B_S(0), \tag{2.4}$$

for some $S \in (R, \infty)$. Rescaling, we can assume $S = 1$. We will simply identify $M \setminus \Omega$ with $\mathbb{R}^2 \setminus B_1(0)$. We will construct u on Ω to solve a certain nonlocal boundary problem (see (2.8) below). With $v = u|_{\partial\Omega}$ (and $\partial\Omega$ identified with $\partial B_1(0) = S^1$) we define u on $\mathbb{R}^2 \setminus B_1(0)$ to be

$$u(x) = \sum_{k=-\infty}^{\infty} \hat{v}(k) r^{-|k|} e^{ik\theta}, \quad x = re^{i\theta}, \quad r > 1, \tag{2.5}$$

where

$$\hat{v}(k) = \frac{1}{2\pi} \int_{S^1} v(\theta) e^{-ik\theta}\, d\theta. \tag{2.6}$$

Note that, for $|x| > 1$, and $x = (x_1, x_2)$ identified with $z = x_1 + ix_2$,

$$u(x) = \sum_{k=0}^{\infty} \hat{v}(k) \overline{z}^{-k} + \sum_{k=1}^{\infty} \hat{v}(-k) z^{-k} \tag{2.7}$$

is harmonic. To fit this function together with a function on Ω and solve (2.2), we want u on Ω to solve

$$\Delta_g u = f \ \text{ on } \ \Omega, \quad \partial_\nu u = -\Lambda u \ \text{ on } \ \partial\Omega, \tag{2.8}$$

where ν is the outward-pointing unit normal to $\partial\Omega$, and Λ is the operator defined on functions on $\partial\Omega = S^1$ by

$$\Lambda v(\theta) = \sum_{k=-\infty}^{\infty} |k| \hat{v}(k) e^{ik\theta}. \tag{2.9}$$

Note that if u is given on $\mathbb{R}^2 \setminus B_1(0)$, then $\partial_r u = -\Lambda v$ on S^1. If we can solve (2.8), then using (2.5) with $v = u|_{\partial\Omega}$ produces a function that solves (2.6) on $M \setminus \partial\Omega$ and has the property that neither u nor ∇u have a jump across $\partial\Omega$, so in fact u solves (2.6) on all of M.

To proceed, take $k \in \mathbb{N}$ and define a family of operators

$$L_\tau : H^{k+2}(\Omega) \longrightarrow H^k(\Omega) \oplus H^{k+1/2}(\partial\Omega), \tag{2.10}$$

for $\tau \in \mathbb{C}$, by

$$L_\tau u = (\Delta_g u, \partial_\nu u + \tau \Lambda u). \tag{2.11}$$

Lemma 2.2. *When $\tau \neq -1$, L_τ in (2.10) is Fredholm, of index zero.*

Proof. We show that L_τ defines a regular, elliptic boundary problem when $\tau \neq -1$. Standard methods reduce this to studying solutions to

$$\Delta_g u = 0 \ \text{ on } \ \Omega, \quad \partial_\nu + \tau\Lambda u = h \ \text{ on } \ \partial\Omega, \tag{2.12}$$

and looking for w on $\partial\Omega$ such that (2.12) is solved (mod C^∞) by

$$u = \mathrm{PI}\, w, \tag{2.13}$$

where $\mathrm{PI}\, w$ solves the Dirichlet problem for Δ_g on $\overline{\Omega}$, with boundary data w. If $M = \mathbb{R}^2$ with its flat metric, then $\partial_\nu \mathrm{PI}\, w = \Lambda w$. In the current setting, local regularity results for the Dirichlet problem imply that if (2.13) holds, then

$$\partial_\nu u = \Lambda_1 w, \quad \Lambda_1 - \Lambda \in OPS^0(\partial\Omega), \tag{2.14}$$

so

$$\partial_\nu u + \tau\Lambda u = (\Lambda_1 + \tau\Lambda)u, \quad \Lambda_1 + \tau\Lambda = (1+\tau)\Lambda \bmod OPS^0(\partial\Omega), \tag{2.15}$$

so $\Lambda_1 + \tau\Lambda$ is elliptic in $OPS^1(\partial\Omega)$ whenever $\tau \neq -1$. Such ellipticity implies L_τ in (2.10) is Fredholm whenever $\tau \neq -1$. Since $\mathbb{C} \setminus \{-1\}$ is connected, the index is constant on this set. When $\tau = 0$ we have the Neumann boundary problem, which is known to be Fredholm of index 0.

Of course the case of direct interest in (2.8) is $\tau = +1$.

Lemma 2.3. *Given* $u \in H^2(\Omega)$,

$$u \in \mathcal{N}(L_1) \Longrightarrow u \quad \text{is constant.} \tag{2.16}$$

Proof. Without loss of generality, we can assume u is real valued. Green's formula gives, for $u \in H^2(\Omega)$,

$$\int_\Omega |\nabla_g u|^2 \, dV = -\int_\Omega u \, \Delta_g u \, dV + \int_{\partial\Omega} u \, \frac{\partial u}{\partial \nu} \, dS. \tag{2.17}$$

If $u \in \mathcal{N}(L_1)$, then

$$\int_\Omega |\nabla_g u|^2 \, dV = -(u, \Lambda u)_{L^2(\partial\Omega)}. \tag{2.18}$$

The left side of (2.18) is ≥ 0 and the right side is ≤ 0, so both sides must vanish, implying u is constant.

From Lemmas 2.2–2.3 we have

$$\mathcal{R}(L_1) \text{ has codimension 1 in } \; H^k(\Omega) \oplus H^{k+1/2}(\partial\Omega). \tag{2.19}$$

Taking $k = 0$, we want to identify the annihilator of $\mathcal{R}(L_1)$ in $L^2(\Omega) \oplus H^{-1/2}(\partial\Omega)$, a space we know has dimension 1. To say (w, h) belongs to the annihilator of $\mathcal{R}(L_1)$ is to say that

$$(\Delta_g u, w) + (\partial_\nu u + \Lambda u, h) = 0, \quad \forall \, u \in H^2(\Omega). \tag{2.20}$$

We note that $(w, h) = (1, -1)$ satisfies this condition. In fact, Green's theorem implies

$$(\Delta_g u, 1) = \int_{\partial\Omega} (\partial_\nu u) \, dS, \tag{2.21}$$

and

$$(\Lambda u, 1) = (u, \Lambda 1) = 0. \tag{2.22}$$

The dimension count implies

$$(w, h) = (1, -1) \; \text{ spans the annihilator of } \; \mathcal{R}(L_1). \tag{2.23}$$

Corollary 2.4. *If $f \in L^2(\Omega)$ satisfies (2.1), then $(f, 0) \in \mathcal{R}(L_1)$, hence there exists $u \in H^2(\Omega)$ satisfying (2.8).*

If $f \in C_0^\infty(\Omega)$ satisfies (2.1), elliptic regularity yields $u \in C^\infty(\overline{\Omega})$. Fitting in the construction (2.5)–(2.7), we have a smooth solution to (2.2), which tends to a constant limit at infinity. Subtracting this constant gives a solution satisfying (2.3). Uniqueness follows from the maximum principle.

Strengthening the uniqueness result, we have the following Liouville theorem.

Proposition 2.5. *In the setting of Proposition 2.1, if $u \in C^\infty(M)$ is bounded and solves*

$$\Delta_g u = 0 \quad \text{on} \quad M, \tag{2.24}$$

then u is constant.

Proof. On $\mathbb{R}^2 \setminus B_1(0)$, u must have the form (2.5), with $v = u|_{S^1}$, and on Ω, u must solve (2.8), with $f = 0$, so $u \in \mathcal{N}(L_1)$. Hence, by Lemma 2.3, u is constant on Ω, hence on $\partial\Omega = S^1$, and the representation (2.5) implies u is equal to the same constant on $\mathbb{R}^2 \setminus B_p(0)$.

See Appendix A for a much more general Liouville theorem.

We now extend the scope of Proposition 2.1.

Proposition 2.6. *In the setting of Proposition 2.1, replace (2.1) by*

$$\int\limits_M f(x)\, dV(x) = a. \tag{2.25}$$

Then there exists a unique solution u to $\Delta_g u = f$, satisfying

$$u(x) - \frac{a}{2\pi} \log|x| \to 0 \quad \text{as} \quad x \to \infty. \tag{2.26}$$

Proof. Pick $\varphi \in C^\infty(\mathbb{R}^2)$ such that $\varphi(x) = 0$ for $|x| \le 2,\ 1$ for $|x| \ge 3$. Define $G \in C^\infty(M)$ by

$$G(x) = \begin{matrix} \dfrac{\varphi(x)}{2\pi} \log|x|, & x \in \mathbb{R}^2 \setminus B_1(0), \\ 0, & x \in \Omega. \end{matrix} \tag{2.27}$$

Then (with E_0 as in (1.11))

$$\begin{aligned} \int\limits_M \Delta_g G(x)\, dV(x) &= \int\limits_{\mathbb{R}^2 \setminus B_1(0)} \Delta G(x)\, dx \\ &= \int\limits_{\mathbb{R}^2} \Delta E_0(x)\, dx - \int\limits_{\mathbb{R}^2} \Delta\big((1-\varphi)E_0\big)\, dx \\ &= 1. \end{aligned} \tag{2.28}$$

Thus, if we set

$$F(x) = \Delta_g G(x), \tag{2.29}$$

then $F \in C_0^\infty(M)$ and $\int_M (f - aF)\, dV = 0$, so Proposition 2.1 applies, to give $w \in C^\infty(M)$ satisfying

$$\Delta_g w = f - aF \quad \text{on} \quad M, \qquad w(x) \to 0 \quad \text{as} \quad x \to \infty. \tag{2.30}$$

Hence

$$\Delta_g(w + aG(x)) = f, \tag{2.31}$$

and $u = w + aG$ is the desired solution.

3. Asymptotic solutions to $\Delta_g u = f$

Here we look at

$$\Delta_g u = f, \tag{3.1}$$

when the n-dimensional Riemannian manifold M is asymptotically flat, so that, for some compact $K \subset M$,

$$M \setminus K \sim (1, \infty) \times S, \tag{3.2}$$

and, on $M \setminus K$,

$$\Delta_g u = \partial_r^2 u + M(r) \partial_r u + r^{-2} \Delta_{S(r)} u, \tag{3.3}$$

where, as $r \to \infty$,

$$M(r) \sim \frac{n-1}{r} + \sum_{\ell \geq 1} a_\ell(\omega) r^{-1-\ell}, \tag{3.4}$$

and

$$\Delta_{S(r)} \sim \Delta_S + \sum_{\ell \geq 1} r^{-\ell} L_\ell. \tag{3.5}$$

Here $\omega \in S$, $a_\ell \in C^\infty(S)$, Δ_S is the Laplace-Beltrami operator on S, and L_ℓ are second-order differential operators on S. Cf. [Ch], p. 18. We take $S = S^{n-1}$, so

$$\mathrm{Spec}(-\Delta_S) = \{\ell^2 + (n-2)\ell : \ell = 0, 1, 2, \dots\}, \tag{3.6}$$

though extensions to other compact, $(n-1)$-dimensional Riemannian manifolds S are possible.

We assume f has the form

$$f \sim \sum_{k \geq 1} r^{-k-2} f_k(\omega), \tag{3.7}$$

as $r \to \infty$, with $f_k \in C^\infty(S)$, and look for

$$u \sim \sum_{k \geq 1} u_k(r, \omega) \tag{3.8}$$

such that

$$\Delta_g u \sim f, \tag{3.9}$$

in the sense that $\Delta_g u - f$ vanishes rapidly, with all derivatives, as $r \to \infty$. In (3.8), we want $u_k(r, \omega)$ to decay roughly like r^{-k} as $r \to \infty$, though as we will see, formulas for $u_k(r, \omega)$ can have a more complicated form than $r^{-k} u_k(\omega)$.

Plugging (3.7)–(3.9) into (3.3)–(3.5) gives

$$\begin{aligned} &\sum_{k \geq 1} \Bigl(\partial_r^2 + \frac{n-1}{r}\partial_r + \frac{1}{r^2}\Delta_S\Bigr) u_k(r, \omega) \\ &\sim \sum_{k \geq 1} r^{-k-2} f_k(\omega) - \sum_{k, \ell \geq 1} \Bigl(a_\ell(\omega) r^{-\ell-1}\partial_r + r^{-\ell-2} L_\ell\Bigr) u_k(r, \omega). \end{aligned} \tag{3.10}$$

We find it convenient to make a change of variable,

$$v_k(s, \omega) = u_k(r, \omega), \quad r = e^s, \tag{3.11}$$

so

$$\begin{aligned} u_k(r, \omega) &= v_k(\log r, \omega), \\ \partial_r u_k(r, \omega) &= \frac{1}{r}\partial_s v_k(\log r, \omega), \\ \partial_r^2 u_k(r, \omega) &= \frac{1}{r^2}\partial_s^2 v_k(\log r, \omega) - \frac{1}{r^2}\partial_s v_k(\log r, \omega), \end{aligned} \tag{3.12}$$

and (3.10) becomes

$$\begin{aligned} &\sum_{k \geq 1} (\partial_s^2 + (n-2)\partial_s + \Delta_S) v_k(s, \omega) \\ &\sim \sum_{k \geq 1} e^{-ks} f_k(\omega) - \sum_{k, j \geq 1} e^{-js}(a_j(\omega)\partial_s + L_j) v_k(s, \omega). \end{aligned} \tag{3.13}$$

We seek $v_k(s, \omega)$ in the form

$$v_k(s, \omega) = p_k(s, \omega) e^{-ks}, \tag{3.14}$$

where $p_k(s, \omega)$ is a polynomial in s, with coefficients in $C^\infty(S)$ (functions of ω).

The case $k = 1$ of (3.13) is

$$(\partial_s^2 + (n-2)\partial_s + \Delta_S) v_1(s, \omega) = e^{-s} f_1(\omega). \tag{3.15}$$

We expand both sides in terms of eigenfunctions of Δ_S In case $S = S^{n-1}$ and (3.6) holds, let

$$V_\ell = \bigl\{h \in C^\infty(S) : -\Delta_S h = [\ell^2 + (n-2)\ell] h\bigr\}. \tag{3.16}$$

If $f_{1\ell}$ is the component of f_1 in V_ℓ, we want to solve

$$(\partial_s^2 + (n-2)\partial_s + \nu_\ell^2)v_{1\ell}(s) = e^{-s}, \quad \nu_\ell^2 = \ell^2 + (n-2)\ell. \tag{3.17}$$

Then

$$v_1(s,\omega) = \sum_\ell v_{1\ell}(s) f_{1\ell}(\omega). \tag{3.18}$$

We rewrite (3.17) as

$$(\partial_s - \ell)(\partial_s + \ell + n - 2)v_{1\ell}(s) = e^{-s}. \tag{3.19}$$

At this point, let us pause and consider solving

$$(\partial_s - \ell)v = p(s)e^{-ks}, \tag{3.20}$$

when $p(s)$ is a polynomial in s and $k \in \mathbb{Z}^+$. In (3.6), $\ell \in \mathbb{Z}^+$, but let us more generally take $\ell \in \mathbb{R}$. We write $v(s) = q(s)e^{-ks}$, so (3.20) becomes

$$(\partial_s - \ell - k)q(s) = p(s), \tag{3.21}$$

with solution

$$q(s) = J_{k+\ell}\, p(s), \tag{3.22}$$

where the operators J_m are given as follows, for $m \in \mathbb{R}$. First,

$$J_0 p(s) = \int_0^s p(\sigma)\, d\sigma. \tag{3.23}$$

If $m \neq 0$, we take

$$\begin{aligned} J_m p(s) &= (\partial_s - m)^{-1} p(s) \\ &= -\frac{1}{m}\Big(1 - \frac{1}{m}\partial_s\Big)^{-1} p(s) \\ &= -\frac{1}{m}\sum_{j\geq 0}\Big(\frac{1}{m}\partial_s\Big)^j p(s), \end{aligned} \tag{3.24}$$

the last sum being over $j \leq K$ if $p(s)$ is a polynomial of degree K. Then (3.20) is solved by

$$v(s) = J_{k+\ell}\, p(s) \cdot e^{-ks}. \tag{3.25}$$

Returning to (3.18), we have the solution

$$v_{1\ell}(s) = J_{1+2-\ell-n}J_{1+\ell}(1)\cdot e^{-s} \\ = q_{\ell n}(s)e^{-s}, \tag{3.26}$$

where $q_{\ell n}(s)$ is a polynomial in s. Note that

$$\ell \geq 0 \Longrightarrow J_{1+\ell}(1) = -\frac{1}{\ell+1}, \tag{3.27}$$

and

$$J_{3-\ell-n}(1) \tag{3.28}$$

is constant if $\ell + n \neq 3$, and a constant multiple of s if $\ell + n = 3$. In this way, we have a solution $v_1(s,\omega)$ to (3.15).

From here, we find $v_k(s,\omega)$ in (3.17) by induction, for $k \geq 2$. It solves

$$(\partial_s^2 + (n-2)\partial_s + \Delta_S)v_k(s,\omega) = e^{-ks}\varphi_k(s,\omega), \tag{3.29}$$

where $\varphi_k(s,\omega)$ is a polynomial in s, with coefficients in $C^\infty(S)$. Let

$$\{f_\ell^\mu : 1 \leq \mu \leq \dim V_\ell\} \ \text{ be an orthonormal basis of } \ V_\ell.$$

Write

$$\varphi_k(s,\omega) = \sum_{\ell,\mu} \varphi_{k\ell}^\mu(s) f_\ell^\mu(\omega). \tag{3.30}$$

Then we want to solve

$$(\partial_s^2 + (n-2)\partial_s + \nu_\ell^2)v_{k\ell}^\mu(s) = \varphi_{k\ell}^\mu(s)e^{-ks}, \tag{3.31}$$

to obtain

$$v_k(s,\omega) = \sum_{\ell,\mu} v_{k\ell}^\mu(s) f_\ell^\mu(\omega). \tag{3.36}$$

Equivalently, we solve

$$(\partial_s - \ell)(\partial_s + \ell + n - 2)v_{k\ell}^\mu(s) = \varphi_{k\ell}^\mu(s)e^{-ks}, \tag{3.33}$$

so we take

$$v_{k\ell}^\mu(s) = q_{k\ell}^\mu(s)e^{-ks}, \quad q_{k\ell}^\mu(s) = J_{k+2-\ell-n}J_{k+\ell}\varphi_{k\ell}^\mu(s). \tag{3.34}$$

Thus $q_{k\ell}^\mu(s)$ is a polynomial in s of degree at most 1 more than that of $\varphi_{k\ell}^\mu(s)$. That $v_j(s,\omega)$ in (3.32) is e^{-ks} times a polynomial in s with coefficients in $C^\infty(S)$ is a straightforward consequence of the formulas (3.23)–(3.24). Let us formalize this:

$$v_k(s,\omega) = q_k(s,\omega)e^{-ks}, \tag{3.35}$$

where $q_k(s,\omega)$ is a polynomial in s with coefficients in $C^\infty(S)$. Rewinding (3.8)–(3.11), we have an asymptotic solution to (3.9) of the form

$$u(r,\omega) \sim \sum_{k\geq 1} q_k(\log r,\omega)r^{-k}. \tag{3.36}$$

Borel's theorem on summing asymptotic series yields the following.

Proposition 3.1. *Let M be an asymptotically Euclidean, Riemannian manifold, of dimension n. Take $f \in C^\infty(M)$ having the asymptotic expansion (3.7), with $f_k \in C^\infty(S)$. Then there exists $u \in C^\infty(M)$, having an asymptotic expansion of the form (3.36), where each q_k is a polynomial in $\log r$ with coefficients in $C^\infty(S)$, such that*

$$\Delta_g u - f = h \in \mathcal{S}(M), \tag{3.37}$$

i.e., h and all its covariant derivatives vanish at infinity.

Given this, we are highly motivated to establish solvability of

$$\Delta_g u = f \tag{3.38}$$

given $f \in \mathcal{S}(M)$, perhaps integrating to 0, and investigate asymptotic properties of the solution. Appendices B–C have results on this for quite general M, but they require $f \in C_0^\infty(M)$. They obtain $u \in C^\infty(M) \cap L^\infty(M)$, but they do not get finer asymptotic results.

A. Harnack estimates, Liouville theorems, and uniformization

The first Liouville theorem we establish is the following.

Proposition A.1. *Let $G(x) = (g_{ij}(x))$ be a continuous symmetric $n \times n$ matrix function, defining a metric tensor on $\mathbb{R}^n$. Assume there exist $B_0, B_1 \in (0, \infty)$ such that*

$$B_0 I \le G(x) \le B_1 I, \quad \forall\, x \in \mathbb{R}^n. \tag{A.1}$$

If u is a bounded solution to

$$\Delta_g u = 0 \quad \text{on} \quad \mathbb{R}^n, \tag{A.2}$$

then u is constant.

Before proving this, we deduce the following result.

Proposition A.2. *In the setting of Proposition A.1, if $n = 2$ and g_{jk} is Hölder continuous, then $(\mathbb{R}^2, g_{jk})$ is conformally equivalent to the flat plane $(\mathbb{R}^2, \delta_{jk})$.*

Proof. Under these hypotheses, there are local isothermal coordinates, so $(\mathbb{R}^2, g_{jk})$ has the structure of a Riemann surface. By the uniformization theorem, it is conformally equivalent to

(A.3a) the flat plane, or

(A.3b) the Poincaré disk.

(See [For] for a careful treatment of the uniformization theorem. For a PDE proof, see [MT].) The case (A.3b) holds if and only if there is a nonconstant bounded harmonic function on $(\mathbb{R}^2, g_{jk})$; otherwise the case (A.3a) holds. By Proposition A.1, we know case (A.3b) cannot hold.

Remark. While the setting of Proposition A.2 is much more general than that of (1.7)–(1.10), as carried out in §2, Proposition A.2 does not imply the results given there, since we have no large x asymptotics on the conformal diffeomorphism of $(\mathbb{R}^2, g_{jk})$ with $(\mathbb{R}^2, \delta_{jk})$ given by Proposition A.2.

The proof of Proposition A.1 (which is perhaps well known) makes use of Harnack's inequality. See [GT], pp. 44–45, for a related argument. We use the following form of Harnack's inequality, which follows from Corollary 8.21 of [GT].

Proposition A.3. *Let $A(x) = (a^{jk}(x))$ be a continuous, symmetric, $n \times n$ matrix function on $B_2(0) \subset \mathbb{R}^n$. Assume there exist $A_0, A_1 \in (0, \infty)$ such that*

$$A_0 I \le A(x) \le A_1 I, \quad \forall\, x \in B_2(0). \tag{A.4}$$

There exists $C = C(A_0, A_1, n)$ with the property that, if u is a solution to

$$\partial_j a^{jk}(x) \partial_k u = 0 \ \ on \ \ B_2(0), \quad u \ge 0 \ \ on \ \ B_2(0), \tag{A.5}$$

then

$$\sup_{B_1(0)} u(x) \le C \inf_{B_1(0)} u(x). \tag{A.6}$$

Proof of Proposition A.1. To begin, adding a constant to u, we can arrange

$$u \ge 0 \ \text{ on } \ \mathbb{R}^n, \quad \inf_{\mathbb{R}^n} u = 0. \tag{A.7}$$

Then the goal is to show that $u \equiv 0$. Note that (A.2) is equivalent to

$$\partial_j a^{jk}(x) \partial_k u = 0, \quad a^{jk}(x) = g(x)^{1/2} g^{jk}(x), \tag{A.8}$$

where $(g^{jk}(x)) = (g_{jk}(x))^{-1}$, $g = \det G$. The hypothesis (A.1) implies (A.4), for all $x \in \mathbb{R}^n$. Now, for $R > 0$, define v_R in $B_2(0)$ by

$$v_R(x) = u(Rx). \tag{A.9}$$

We have

$$\partial_j a^{jk}(Rx) \partial_k v_R(x) = 0 \ \text{ on } \ B_2(0). \tag{A.10}$$

Now this replacement of (A.5) has the same ellipticity constants as in (A.4), so Proposition A.3 implies that there exists $C = C(A_0, A_1, n)$ (independent of R) such that

$$\sup_{B_1(0)} v_R \le C \inf_{B_1(0)} v_R, \tag{A.11}$$

hence

$$\sup_{B_R(0)} u \le C \inf_{B_R(0)} u. \tag{A.12}$$

Taking $R \to \infty$ yields $\sup_{\mathbb{R}^n} u = 0$, hence $u \equiv 0$, as desired.

Note that Proposition A.1 does not imply Proposition 2.5, since the latter allows for nontrivial topology. The following extension of Proposition A.1 is strictly stronger than Proposition 2.5.

Proposition A.4. *In the setting of Proposition A.1, cut out $B_1(0)$ from $\mathbb{R}^n$ and glue in Ω, a compact Riemannian manifold with boundary $\partial\Omega \approx S^{n-1}$, to form a Riemannian manifold with continuous metric tensor (M, g_{jk}), agreeing with $(\mathbb{R}^n, g_{jk})$ on $|x| \geq 1$. If u is a bounded solution of*

$$\Delta_g u = 0 \quad on \quad M, \tag{A.13}$$

then u is constant.

Proof. As in the proof of Proposition A.1, we can add a constant to u and arrange

$$u \geq 0 \quad \text{on} \quad M, \quad \inf_M u = 0. \tag{A.14}$$

Then the goal is to show $u \equiv 0$. Note that there must exist $x_\nu \in M \setminus \Omega \approx \mathbb{R}^n \setminus B_1(0)$ such that

$$|x_\nu| = R_\nu + 1 \to \infty, \quad u(x_\nu) = \varepsilon_\nu \to 0 \tag{A.15}$$

($|x_\nu|$ denotes the Euclidean norm on $\mathbb{R}^n$), since otherwise u would have to assume its minimum at a point of M (hence $u \equiv 0$). Now a Harnack inequality argument like that used in the proof of Proposition A.1 gives

$$\sup_{B_{R_\nu/2}(x_\nu)} u \leq C\varepsilon_\nu. \tag{A.16}$$

Then (assuming $R_\nu > 2$) we can cover

$$\mathcal{A}_\nu = \{x \in \mathbb{R}^n : R_\nu \leq |x| \leq R_\nu + 1\} \tag{A.17}$$

by M_n balls of radius $R_\nu/2$, and invoke the Harnack estimate repeatedly to get

$$\sup_{\mathcal{A}_\nu} u \leq \widetilde{C}\varepsilon_\nu. \tag{A.18}$$

That $u \equiv 0$ then follows by the maximum principle.

B. Poisson integral on a complete manifold with compact boundary

Let $\overline{X}$ be a complete, n-dimensional Riemannian manifold with compact boundary ∂X, and interior X. We assume X is connected. We want to establish the existence of a map

$$\mathrm{PI} : C^\infty(\partial X) \longrightarrow C^\infty(\overline{X}) \cap L^\infty(X) \tag{B.1}$$

and record properties of the Dirichlet-to-Neumann map Λ, given by

$$\Lambda f = -\partial_\nu \,\mathrm{PI}\, f\big|_{\partial X}, \tag{B.2}$$

where ν is the unit normal to ∂X, pointing inside X. We also define PI on other function spaces on ∂X. We may as well assume $\overline{X}$ is not compact. Let X_k be an increasing sequence of bounded open subsets of X, such that

$$X_k \supset \{x \in X : \mathrm{dist}(x, \partial X) \le k\}. \tag{B.3}$$

Write $\partial X_k = \partial X \cup S_k$. We define

$$P_k : C^\infty(\partial X) \longrightarrow C^\infty(\overline{X}_k) \tag{B.4}$$

by

$$\Delta_g P_k f = 0 \;\text{ on }\; X_k, \quad P_k f = f \text{ on } \partial X,\; P_k f = 0 \text{ on } S_k. \tag{B.5}$$

We then extend $P_k f$ by 0 on $\overline{X} \setminus \overline{X}_k$, defining $P_k : C^\infty(\partial X) \to C(\overline{X})$. If $C^\infty_+(\partial X)$ denotes the class of $f \ge 0$ in $C^\infty(\partial X)$, we have

$$f \in C^\infty_+(\partial X),\; u_k = P_k f \Longrightarrow 0 \le u_k \le u_{k+1} \le \sup f, \tag{B.6}$$

by the maximum principle, and from here and local elliptic regularity results, we have

$$u_k \longrightarrow u \in C^\infty(\overline{X}) \cap L^\infty(X), \tag{B.7}$$

solving

$$\Delta_g u = 0 \;\text{ on }\; X, \quad u\big|_{\partial X} = f. \tag{B.8}$$

We denote the limit by $\mathrm{PI}\, f$. The construction (B.4)–(B.5) gives

$$f, g \in C^\infty_+(\partial X) \Longrightarrow \mathrm{PI}(f+g) = \mathrm{PI}(f) + \mathrm{PI}(g). \tag{B.9}$$

Given a general (real valued) $f \in C^\infty(\partial X)$, set

$$f = f_1 - f_2, \;\; f_j \in C_+^\infty(\partial X), \tag{B.10}$$

and

$$\mathrm{PI}\, f = \mathrm{PI}\, f_1 - \mathrm{PI}\, f_2. \tag{B.11}$$

It follows from (B.9) that this is independent of the choice of f_j such that (B.10) holds. Note that if $\widetilde{X}_k \nearrow \overline{X}$ also satisfies (B.3) and $\widetilde{P}_k$ is defined analogously to (B.5), then $f \in C_+^\infty(\partial X), \; X_j \subset \widetilde{X}_k \subset X_\ell \Rightarrow P_j f \le \widetilde{P}_k f \le P_\ell f$, so PI is well defined, independently of the choice of $\{X_k\}$.

The convergence (B.7) holds in $C^\infty(\overline{\Omega})$ for each compact $\overline{\Omega} \subset \overline{X}$, given $f \in C^\infty(\partial X)$. The maximum principle yields an extension

$$\mathrm{PI} : C(\partial X) \longrightarrow C(\overline{X}) \cap L^\infty(X). \tag{B.12}$$

Also, standard elliptic regularity results yield

$$\mathrm{PI} : H^s(\partial X) \longrightarrow H^{s+1/2}(\overline{X}_1) \cap C^\infty(X) \cap L^\infty(X), \tag{B.13}$$

for $s > (n-1)/2$. Shortly, we will extend (B.13) to a larger range of s.

Note that, for each $f \in C(\partial X)$, elliptic regularity implies

$$\mathrm{PI}\, f - P_2 f \in C^\infty(\overline{X}_1). \tag{B.14}$$

Also, a parametrix construction yields

$$\Lambda \in OPS^1(\partial X), \tag{B.15}$$

elliptic, with

$$\Lambda - \sqrt{-\Delta_S} \in OPS^0(\partial X), \tag{B.16}$$

where Δ_S denotes the Laplace-Beltrami operator on $S = \partial X$.

We pause to consider the family of special cases

$$\overline{X} = \mathbb{R}^n \setminus B_1, \quad B_1 = \{x \in \mathbb{R}^n : |x| < 1\}. \tag{B.17}$$

Take $n \ge 2$. In spherical polar coordinates $x = r\omega, \; r \in [1,\infty), \; \omega \in S^{n-1}$, we have

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{n-1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\Delta_S u, \tag{B.18}$$

where Δ_S is the Laplace-Beltrami operator on S^{n-1}. If we set

$$A = \Big(-\Delta_S + \frac{(n-1)^2}{4}\Big)^{1/2}, \tag{B.19}$$

we have (cf. [T], Chapter 8, §4)

$$\operatorname{Spec} A = \Big\{\frac{n-2}{2} + k : k = 0, 1, 2 \dots\Big\}. \tag{B.20}$$

Separation of variables applied to (B.8) yields

$$\begin{aligned} \mathrm{PI}\, f(r\omega) &= r^{-A-(n-2)/2} f(\omega) \\ &= r^{-B} f(\omega), \end{aligned} \tag{B.21}$$

with

$$\operatorname{Spec} B = \{n - 2 + k : k = 0, 1, 2, \dots\}. \tag{B.22}$$

The definition (B.2) gives

$$\Lambda = B = \Big(-\Delta_s + \frac{(n-2)^2}{4}\Big)^{1/2} + \frac{n-2}{2}, \tag{B.23}$$

a result consistent with (B.16). Note that this is a self-adjoint, positive semi-definite operator, with discrete spectrum, whose smallest eigenvalue is

$$\lambda_0 = n - 2, \tag{B.24}$$

which vanishes if $n = 2$ but is strictly positive if $n \geq 3$. It follows that

$$\mathrm{PI} : C(\partial X) \longrightarrow C_*(\overline{X}) \tag{B.25}$$

if $X = \mathbb{R}^n \setminus B_1$ with $n \geq 3$, where

$$C_*(\overline{X}) = \{u \in C(\overline{X}) : \lim_{x \to \infty} u(x) = 0\}. \tag{B.26}$$

However,

$$\overline{X} = \mathbb{R}^2 \setminus B_1 \Longrightarrow \mathrm{PI}(1) \equiv 1. \tag{B.27}$$

In [T2] it is shown that (B.25) holds whenever $\overline{X}$ is asymptotically Euclidean and has dimension $n \geq 3$.

Back to generalities, take $f, g \in C^\infty(\partial X)$, set $u_k = P_k f$ as in (B.4)–(B.5), and set $v_k = P_k g$. Green's formula gives

$$\int_{X_k} \nabla u_k \cdot \nabla v_k \, dV = -\int_{\partial X_k} u_k (\partial_\nu v_k) \, dS = -\int_{\partial X} u_k (\partial_\nu v_k) \, dS, \tag{B.28}$$

the negative sign because ν points into X. The smooth convergence of u_k to $u = \mathrm{PI}\, f$ and of v_k to $v = \mathrm{PI}\, g$ implies that the right side of (B.28) converges to

$$-\int_{\partial X} u(\partial_\nu v) \, dS = \int_{\partial X} f(\Lambda g) \, dS. \tag{B.29}$$

Since the left side of (B.28) is symmetric in u_k and v_k, we have

$$\int_{\partial X} f(\Lambda v) \, dS = \int_{\partial X} (\Lambda f) g \, dS, \tag{B.30}$$

for $f, g \in C^\infty(\partial X)$. In concert with (B.15)–(B.16), we deduce that Λ is self-adjoint, with domain $H^1(\partial X)$. Taking $g = \overline{f}$ gives $v_k = \overline{u}_k$, and hence

$$\int_{X_k} |\nabla u_k|^2 \, dV = -\int_{\partial X} u_k (\partial_\nu \overline{u}_k) \, dS. \tag{B.31}$$

Taking $k \to \infty$ and applying Fatou's lemma to the left side of (B.31) gives

$$\int_X |\nabla u|^2 \, dV \le (f, \Lambda f), \tag{B.32}$$

for $u = \mathrm{PI}\, f$. This implies Λ is positive semidefinite. Also, by (B.15), $(f, \Lambda f) \le C \|f\|^2_{H^{1/2}(\partial X)}$. This leads to the following result.

Proposition B.1. *The map* PI *extends uniquely from* $C^\infty(\partial X)$ *to*

$$\mathrm{PI} : H^{1/2}(\partial X) \longrightarrow \Bigl\{ u \in C^\infty(X) : \int_X |\nabla u|^2 \, dV < \infty \Bigr\}. \tag{B.33}$$

Proof. Given $f \in H^{1/2}(\partial X)$, we take $f_j \in C^\infty(\partial X)$ such that $f_j \to f$ in $H^{1/2}$ norm, and set $u_j = \mathrm{PI}\, f_j$. Also set $u_{jk} = P_k f_j$. We have

$$\|\nabla u_j\|^2_{L^2(X)} \le (f_j, \Lambda f_j) \le C_0 \|f\|^2_{H^{1/2}(\partial X)}. \tag{B.34}$$

Also

$$\|\nabla(u_j - u_{jk})\|_{L^2(X_k)} \le C_k \|f\|_{H^{1/2}(\partial X)}, \tag{B.35}$$

and

$$u_j - u_{jk}\big|_{\partial X} = 0, \tag{B.36}$$

so, by Poincaré's inequality,

$$\|u_j - u_{jk}\|_{L^2(X_k)} \le C_k \|f\|_{H^{1/2}(\partial X)}. \tag{B.37}$$

These uniform estimates readily yield the extension (B.33).

An interpolation argument then extends (B.13) from $s > (n-1)/2$ to $s \ge 1/2$, with $L^\infty(X)$ replaced by $L^\infty(X^\#)$, where $X^\# = \{x \in X : \operatorname{dist}(x, \partial X) \ge 1\}$. Further extensions are possible, as we will see in Appendix D.

Remark 1. The result (B.32) suggests the following problem.

Determine when one has equality in (B.32).

Remark 2. As we have seen in (B.17)–(B.27), when $\overline{X} = \mathbb{R}^n \setminus B_1$,

$$\mathrm{PI} : C(\partial X) \longrightarrow C_*(\overline{X}), \tag{B.38}$$

when $n \ge 3$, but not when $n = 2$. Also, $\mathcal{N}(\Lambda) = 0$ when $n \ge 3$, but $\mathcal{N}(\Lambda) = \mathrm{Span}(1)$ when $n = 2$. In general, we can deduce the following, from (B.32).

Proposition B.2. *If $f \in \mathcal{N}(\Lambda)$, then $\mathrm{PI}\, f$ is constant.*

The conclusion implies f is constant. The converse need not hold, i.e., $\mathrm{PI}\, 1$ might not be constant. It is constant if $\overline{X} = \mathbb{R}^2 \setminus B_1$; cf. (B.27). Perhaps $\mathrm{PI}\, 1 = 1$ whenever $\overline{X}$ is asymptotically Euclidean, of dimension $n = 2$. By Proposition B.2,

$$\mathrm{PI}(1) \ne 1 \Longrightarrow \mathcal{N}(\Lambda) = 0. \tag{B.39}$$

The implication

$$\mathrm{PI}(1) = 1 \Longrightarrow \mathcal{N}(\Lambda) \supset \mathrm{Span}(1) \tag{B.40}$$

follows directly from the definition (B.2). This together with Proposition B.2 gives

$$\mathrm{PI}(1) = 1 \Longrightarrow \mathcal{N}(\Lambda) = \mathrm{Span}(1). \tag{B.41}$$

C. Solving $\Delta_g u = f$ on a complete Riemannian manifold

Let M be a complete Riemannian manifold, of dimension n. Assume M is connected. Given $f \in C_0^\infty(M)$, we desire to find u such that

$$\Delta_g u = f, \quad u \in C^\infty(M) \cap L^\infty(M). \tag{C.1}$$

This is easily done if $M = \mathbb{R}^n$, for all such f, if $n \geq 3$; for $n = 2$ one can find such u provided

$$\int_M f\, dV = 0. \tag{C.2}$$

We will study solvability of (C.1) under the general hypothesis stated above, and look into when (C.2) is required.

To start, given $f \in C_0^\infty(M)$, pick a smoothly bounded, connected, open set Ω such that

$$\operatorname{supp} f \subset \Omega \tag{C.3}$$

and $\overline{\Omega}$ is compact. Set

$$\overline{X} = M \setminus \Omega. \tag{C.4}$$

We want to find $v \in C^\infty(\overline{\Omega})$ such that

$$\Delta_g v = f \ \text{ on } \ \Omega, \quad \partial_\nu v = -\Lambda v \ \text{ on } \ \partial\Omega, \tag{C.5}$$

where ν is the unit normal to $\partial\Omega = \partial X$ pointing out of Ω (and into X), and Λ is the Dirichlet-to-Neumann map associated to X, discussed in Appendix B (cf. (B.2)). If we have such a solution to (C.5), a solution to (C.1) is given by

$$\begin{aligned} u(x) = v(x), \qquad & x \in \overline{\Omega}, \\ \operatorname{PI} v\big|_{\partial X}, \quad & x \in X, \end{aligned} \tag{C.6}$$

with PI as in (B.1). This clearly solves $\Delta_g u = f$ on $M \setminus \partial\Omega$, and has the property that neither u nor ∇u have a jump across $\partial\Omega$, so in fact (C.1) holds.

There is one minor point to address. In Appendix B, we assumed X was connected. Here, we do not want to impose this restriction. We allow X to have connected components $X_j,\ 1 \leq j \leq K$. Then we have

$$\begin{aligned} \operatorname{PI}_j : C^\infty(\partial X_j) &\longrightarrow C^\infty(\overline{X}_j) \cap L^\infty(X_j), \\ \Lambda_j f = -\partial_\nu \operatorname{PI}_j f, \quad & \Lambda_j \in OPS^1(\partial X_j), \end{aligned} \tag{C.7}$$

and, in the obvious sense,

$$\mathrm{PI} = \mathrm{PI}_1 \oplus \cdots \oplus \mathrm{PI}_K, \quad \Lambda = \Lambda_1 \oplus \cdots \oplus \Lambda_K. \tag{C.8}$$

We also bring in

$$\begin{gathered} \mathrm{PI}_0 : C^\infty(\partial\Omega) \longrightarrow C^\infty(\overline{\Omega}), \\ \Lambda_0 f = \partial_\nu \, \mathrm{PI}_0 \, f, \quad \Lambda_0 \in OPS^1(\partial\Omega). \end{gathered} \tag{C.9}$$

Note the absence of a minus sign, since ν points out of Ω. As in (B.16), we have

$$\Lambda_0 - \sqrt{-\Delta_S} \in OPS^0(\partial\Omega), \tag{C.10}$$

where Δ_S is the Laplace-Beltrami operator on $\partial\Omega = \partial X$. Hence

$$\Lambda_0 - \Lambda \in OPS^0(\partial\Omega). \tag{C.11}$$

To proceed, take $k \in \mathbb{Z}^+$ and define a family of operators

$$L_\tau : H^{k+2}(\Omega) \longrightarrow H^k(\Omega) \oplus H^{k+1/2}(\partial\Omega) \tag{C.12}$$

for $\tau \in \mathbb{C}$, by

$$L_\tau v = (\Delta_g v, \partial_\nu v + \tau \Lambda v). \tag{C.13}$$

Lemma C.1. *When $\tau \neq -1$, L_τ in (C.12) is Fredholm, of index zero.*

Proof. We show that L_τ defines a regular, elliptic boundary problem when $\tau \neq -1$. Standard methods reduce this to studying solutions to

$$\Delta_g v = 0 \text{ on } \Omega, \quad \partial_\nu v + \tau \Lambda v = h \text{ on } \partial\Omega, \tag{C.14}$$

and looking for w on $\partial\Omega$ such that (C.14) is solved (mod C^∞) by

$$v = \mathrm{PI}_0 \, w. \tag{C.15}$$

In such a case, we have

$$\partial_\nu v = \Lambda_0 w, \tag{C.16}$$

and (C.11) holds, so

$$\partial_\nu v + \tau \Lambda v = (\Lambda_0 + \tau \Lambda) v, \quad \Lambda_0 + \tau \Lambda = (1 + \tau) \Lambda \bmod OPS^0(\partial\Omega), \tag{C.17}$$

so $\Lambda_0 + \tau\Lambda$ is elliptic in $OPS^1(\partial\Omega)$ whenever $\tau \neq -1$. Such ellipticity implies L_τ in (C.12) is Fredholm whenever $\tau \neq -1$. Since $\mathbb{C} \setminus \{-1\}$ is connected, the index is constant on this set. When $\tau = 0$ we have the Neumann boundary problem, which is known to be Fredholm of index 0.

Of course, the case of direct interest in (C.13) is $\tau = +1$.

Lemma C.2. *Given* $v \in H^2(\Omega)$*,*

$$v \in \mathcal{N}(L_1) \Longrightarrow v \quad \textit{is constant.} \tag{C.18}$$

Proof. Without loss of generality, we can assume v is real valued. Green's formula gives, for $v \in H^2(\Omega)$,

$$\int_{\Omega} |\nabla_g v|^2 \, dV = -\int_{\Omega} v \Delta_g v \, dV + \int_{\partial\Omega} v \, \frac{\partial v}{\partial \nu} \, dS. \tag{C.19}$$

If $v \in \mathcal{N}(L_1)$, then

$$\int_{\Omega} |\nabla_g v|^2 \, dV = -(v, \Lambda v)_{L^2(\partial\Omega)}. \tag{C.20}$$

The left side of (C.20) is ≥ 0 and the right side is ≤ 0, so both sides must vanish, implying v is constant.

For the constant function 1 to belong to $\mathcal{N}(L_1)$, it is necessary and sufficient that $\Lambda 1 = 0$, i.e.,

$$\Lambda_j 1 = 0, \quad \forall \, j \in \{1, \dots, K\}, \tag{C.21}$$

with Λ_j as in (C.7). This leads to the following.

Proposition C.3. *If (C.21) holds, then* $\mathcal{N}(L_1) = \operatorname{Span}(1)$*. If (C.21) fails, then* $\mathcal{N}(L_1) = 0$*.*

Remark. In light of (B.39)–(B.41), we see that (C.21) is equivalent to

$$\mathrm{PI}_j(1) = 1, \quad \forall \, j \in \{1, \dots, K\}. \tag{C.21A}$$

We are ready for our first existence result.

Proposition C.4. *If (C.21) fails, then (C.1) has a solution for all* $f \in C_0^\infty(\Omega)$*.*

Proof. By Lemmas C.1–C.3, L_1 is an isomorphism in (C.12). Hence, for each $f \in C_0^\infty(\Omega)$, there is a unique $v \in H^{k+1}(\Omega)$ such that $L_1 v = (f, 0)$. Elliptic regularity implies $v \in C^\infty(\overline{\Omega})$. Then the construction (C.6) produces the desired solution u.

The following result complements Proposition C.4.

Proposition C.5. *If (C.21) holds, then (C.1) has a solution for all $f \in C_0^\infty(\Omega)$ satisfying (C.2).*

Proof. By Lemmas C.1–C.3, L_1 is Fredholm of index 0 in (C.12), and

$$\mathcal{N}(L_1) = \text{Span}(1). \tag{C.22}$$

Hence

$$\mathcal{R}(L_1) \text{ has codimension one in } H^k(\Omega) \oplus H^{k+1/2}(\partial\Omega). \tag{C.23}$$

Taking $k = 0$, we want to identify the annihilator of $\mathcal{R}(L_1)$ in $L^2(\Omega) \oplus H^{-1/2}(\partial\Omega)$, a space we know has dimension 1. To say (w, h) belongs to the annihilator of $\mathcal{R}(L_1)$ is to say that

$$(\Delta_g v, w) + (\partial_\nu v + \Lambda v, h) = 0, \quad \forall\, v \in H^2(\Omega). \tag{C.24}$$

We note that $(w, h) = (1, -1)$ satisfies this condition. In fact, Green's theorem implies

$$(\Delta_g v, 1) = \int\limits_{\partial\Omega} \partial_\nu v \, dS, \tag{C.25}$$

and

$$(\Lambda v, 1) = (v, \Lambda 1) = 0, \tag{C.26}$$

the latter identity by (C.21). The dimension count implies

$$(w, h) = (1, -1) \text{ spans the annihilator of } \mathcal{R}(L_1). \tag{C.27}$$

Hence, given $f \in C_0^\infty(\Omega)$,

$$\int\limits_{\Omega} f \, dV = 0 \Longrightarrow (f, 0) \in \mathcal{R}(L_1), \tag{C.28}$$

so there exists $v \in H^2(\Omega)$ such that $L_1 v = (f, 0)$. The end of the proof follows as in Proposition C.4.

D. Back to manifolds with compact boundary

Let $\overline{M}$ be a complete, n-dimensional Riemannian manifold with compact boundary ∂M, and interior M. We assume M is connected. As shown in Appendix B, we have

$$\text{PI} : H^s(\partial M) \longrightarrow H^{s+1/2}(M^b) \cap C^\infty(M) \cap L^\infty(M^\#), \tag{D.1}$$

provided $s \geq 1/2$. Here,

$$M^b = \{x \in M : \text{dist}(x, \partial M) < 1\}, \quad M^\# = M \setminus M^b. \tag{D.2}$$

Our goal is to extend (D.1) to all $s \in \mathbb{R}$.

To begin, a standard parametrix construction (cf. [T3], Chapter 9, §2) yields

$$\widetilde{P} : H^s(\partial M) \longrightarrow H^{s+1/2}(M) \cap C^\infty(M), \tag{D.3}$$

defined simultaneously for all $s \in \mathbb{R}$, such that

$$\begin{aligned} h \in H^s(\partial M) \Longrightarrow \ \text{supp}\, \widetilde{P}h \subset \overline{M}^b, \ \text{ and} \\ f = \Delta_g \widetilde{P}h \in C^\infty(\overline{M}). \end{aligned} \tag{D.4}$$

We construct PI in the form

$$\text{PI}\, h = \widetilde{P}h - Qh, \tag{D.5}$$

where $u = Qh$ satisfies

$$\Delta_g u = f, \quad u \in C^\infty(\overline{M}) \cap L^\infty(M), \quad u = 0 \ \text{ on } \ \partial M. \tag{D.6}$$

This is like (C.1) except that now M has a boundary and we impose a Dirichlet boundary condition. We parallel the construction of Appendix C. In (D.6), we can take arbitrary $f \in C_0^\infty(\overline{M})$ (enlarging M^b).

Let $\Omega \subset M$ be a smoothly bounded, connected open set that contains M^b, with compact closure $\overline{\Omega}$. Set $X = \overline{M} \setminus \overline{\Omega}$. We have $\partial\Omega = \partial M \cup \partial X$, and the construction of Appendix B gives

$$\text{PI}_1 : C^\infty(\partial X) \longrightarrow C^\infty(\overline{X}) \cap L^\infty(X), \tag{D.7}$$

extending to $H^s(\partial X)$ for $s \geq 1/2$. We also have

$$\text{PI}_0 : C^\infty(\partial X) \longrightarrow C^\infty(\overline{\Omega}), \tag{D.8}$$

given by

$$u = \mathrm{PI}_0\, h \ \text{ solves } \ \Delta_g u = 0 \ \text{ on } \ \Omega, \quad u\big|_{\partial M} = 0, \quad u\big|_{\partial X} = h. \tag{D.9}$$

We define Λ_0 and Λ_1 by

$$\Lambda_1 h = -\partial_\nu PI_1 h, \quad \Lambda_0 h = \partial_\nu \,\mathrm{PI}_0\, h, \tag{D.10}$$

where ν is the unit normal to ∂X pointing into X (out of Ω). Then $\Lambda_0,\ \Lambda_1 \in OPS^1(\partial X)$ are elliptic, and

$$\Lambda_0 - \Lambda_1 \in OPS^0(\partial X). \tag{D.11}$$

We want to find $v \in C^\infty(\overline{\Omega})$ such that

$$\Delta_g v = f, \quad v\big|_{\partial M} = 0, \quad \partial_\nu v = -\Lambda_1 v \ \text{ on } \ \partial X, \tag{D.12}$$

given $f \in C^\infty(\overline{M})$, supported on $\overline{M}^b$. If we have such a solution to (D.12), a solution to (D.6) is given by

$$u(x) = \begin{cases} v(x), & x \in \overline{\Omega}, \\ \mathrm{PI}_1\, v\big|_{\partial X}, & x \in X. \end{cases} \tag{D.13}$$

To proceed, take $k \in \mathbb{Z}^+$ and define a family of maps

$$L_\tau : H_b^{k+1}(\Omega) \longrightarrow H^k(\Omega) \oplus H^{k+1/2}(\partial X) \tag{D.14}$$

for $\tau \in \mathbb{C}$, by

$$L_\tau v = (\Delta_g v, \partial_\nu v + \tau \Lambda_1 v). \tag{D.15}$$

Here,

$$H_b^{k+2}(\Omega) = \{v \in H^{k+2}(\Omega) : v = 0 \text{ on } \partial M\}. \tag{D.16}$$

The argument used in Lemma C.1 gives the following.

Lemma D.1. *When $\tau \neq -1$, L_τ in (D.14) is Fredholm, of index zero.*

Then the argument used in Lemma C.2 gives the following.

Lemma D.2. *For $k \geq 0$,*

$$\mathcal{N}(L_1) = 0. \tag{D.17}$$

Hence L_1 is an isomorphism in (D.14).

Proof. The argument in Lemma C.2 works here to show that any $v \in \mathcal{N}(L_1)$ must be constant. Then the constraint $v|_{\partial M} = 0$ yields (D.17). The isomorphism property follows from the index 0 property and (D.17).

Solvability of (D.12) follows directly from Lemma D.2 and elliptic regularity. Then (D.13) produces the solution to (D.6), and the extension of (D.1) to all $s \in \mathbb{R}$ is complete.

E. Convergence and positivity results for manifolds with compact boundary

As in §D, let $\overline{M}$ be a complete, n-dimensional Riemannian manifold, with compact boundary ∂M, and interior M. We assume M is connected and $\partial M \neq \emptyset$. As shown in §D (cf. (D.6)), we have the following. Given $f \in C_0^\infty(\overline{M})$ (i.e., f smooth and compactly supported on $\overline{M}$, not necessarily vanishing on ∂M), we have u solving

$$\Delta u = f \quad \text{on} \quad M, \quad u\big|_{\partial M} = 0, \quad u \in C^\infty(\overline{M}) \cap L^\infty(M). \tag{E.1}$$

For notational simplicity, we denote Δ_g by Δ. Note that the condition (C.2) is not required.

We briefly recall the construction of the solution to (E.1), to set up notation for this appendix, which will differ slightly from that of §D. Given $f \in C_0^\infty(\overline{M})$, let $\Omega \subset M$ be a smoothly bounded, connected open set such that $\operatorname{supp} f \subset \overline{\Omega}$ and $\overline{\Omega}$ is compact, and set $\overline{X} = M \setminus \Omega$. Note that $\partial\Omega = \partial M \cup \partial X$. We want to find $v \in C^\infty(\overline{\Omega})$ such that

$$\Delta v = f \quad \text{on} \quad \Omega, \quad v\big|_{\partial M} = 0, \quad \partial_\nu v + \Lambda v = 0 \quad \text{on} \quad \partial X, \tag{E.2}$$

where ν is the unit normal to ∂X, pointing out of Ω (and into X), and Λ is the Dirichlet-to-Neumann map associated to X, discussed in §B (extended in §C to the setting where X might have several connected components) and denoted Λ_1 in (D.10). If we have such a solution to (E.2), a solution to (E.1) is given by

$$\begin{aligned} u(x) = v(x), \qquad & x \in \overline{\Omega}, \\ \mathrm{PI}\, v|_{\partial X}, \quad & x \in X, \end{aligned} \tag{E.3}$$

with PI as in (B.1) (denoted PI_1 in (D.13)). Solvability of (E.2) follows from the analysis of L_τ, given by (D.14)–(D.15), including the demonstration in Lemma D.2 that

$$L_1 : H_b^{k+2}(\Omega) \longrightarrow H^k(\Omega) \oplus H^{k+1/2}(\partial X) \tag{E.4}$$

is an isomorphism, with $H_b^{k+2}(\Omega)$ given by (D.16). We formally record the result.

Proposition E.1. *For all $f \in C_0^\infty(\overline{M})$, (E.1) has a solution.*

Proof. Since L_1 is an isomorphism in (E.4), for each such f there exists a unique $v \in H_b^{k+2}(\Omega)$ such that $L_1 v = (f, 0)$. Elliptic regularity implies $v \in C^\infty(\overline{\Omega})$. Then the construction (E.3) produces the desired solution u. In fact, such u clearly solves

$\Delta u = f$ on $M \setminus \partial X$ and has the property that neither u nor ∇u have a jump across ∂X, so (E.1) holds.

We will denote the solution to (E.1) produced by Proposition E.1 as

$$u = Gf. \tag{E.5}$$

(Often, this is not the unique solution to (E.1), but it is well defined.) Our next task is to show that u is equal to the limit of the sequence of functions u_ℓ defined as follows. Take Ω and X as above, and (somewhat parallel to (B.3)) for $\ell \geq 1$, let M_ℓ be an increasing sequence of bounded open subsets of M, with smooth boundary, such that

$$\overline{M}_\ell \supset \overline{\Omega} \cup \{x \in X : \operatorname{dist}(x, \partial X) \leq \ell\}. \tag{E.6}$$

Write $\partial M_\ell = \partial M \cup S_\ell$. We solve

$$\Delta u_\ell = f \quad \text{on} \quad M_\ell, \quad u\big|_{\partial M_\ell} = 0, \tag{E.7}$$

and extend u_ℓ by 0 on $M \setminus M_\ell$. (This defines u_ℓ uniquely.) While the standard variational method yields such a solution, it is useful to note that the following construction also produces u_ℓ. We set $\overline{X}_\ell = \overline{M}_\ell \setminus (\Omega \cup \partial M)$ and define

$$P_\ell : C^\infty(\partial X) \longrightarrow C^\infty(\overline{X}_\ell) \tag{E.8}$$

by

$$\Delta P_\ell f = 0 \quad \text{on} \quad X_\ell, \quad P_\ell f\big|_{\partial X} = f, \quad P_\ell f\Big|_{S_\ell} = 0. \tag{E.9}$$

(Note that $\partial X_\ell = \partial X \cup S_\ell$.) Then set

$$\Lambda_\ell f = -\partial_\nu P_\ell f\Big|_{\partial X}. \tag{E.10}$$

As in (B.15)–(B.16), we have

$$\Lambda_\ell \in OPS^1(\partial X), \quad \Lambda_\ell - \sqrt{-\Delta_S} \in OPS^0(\partial X), \tag{E.11}$$

where Δ_S denotes the Laplace-Beltrami operator on $S = \partial X$. Consequently, with Λ as in (E.2),

$$\Lambda - \Lambda_\ell \in OPS^0(\partial X), \tag{E.12}$$

for each ℓ. Parallel to (E.2), we want to find $v_\ell \in C^\infty(\overline{\Omega})$ such that

$$\Delta v_\ell = f \quad \text{on} \quad \Omega, \quad v_\ell\big|_{\partial M} = 0, \quad \partial_\nu v_\ell + \Lambda_\ell v_\ell = 0 \quad \text{on} \quad \partial X. \tag{E.13}$$

If we have such v_ℓ, then a solution to (E.7) is given by

$$u_\ell(x) = v_\ell(x), \qquad x \in \overline{\Omega}, \\ P_\ell v_\ell|_{\partial X}, \quad x \in X_\ell, \tag{E.14}$$

and we already know the solution to (E.7) is unique. The process of solving (E.13) is parallel to that of solving (E.2), as done in §D. Take $k \in \mathbb{Z}^+$ and define a family of operators

$$L_{\ell,\tau} : H_b^{k+2}(\Omega) \longrightarrow H^k(\Omega) \oplus H^{k+1/2}(\partial X) \tag{E.15}$$

for $\tau \in \mathbb{C}$ by

$$L_{\ell,\tau} v = (\Delta v, \partial_\nu v + \tau \Lambda_\ell v). \tag{E.16}$$

Arguments parallel to those in §D show that $L_{\ell,\tau}$ is Fredholm of index 0 for each $\tau \neq -1$, and that $L_{\ell,1}$ is an isomorphism in (E.15). This proves the existence of the desired solution v_ℓ, and hence gives u_ℓ in (E.17), solving (E.7).

We want to show that, with v as in (E.2),

$$v_\ell \longrightarrow v \ \text{ as } \ \ell \to \infty, \tag{E.17}$$

and proceed from there to show that $u_\ell \to Gf$. The key to (E.17) is the following.

Proposition E.2. *As $\ell \to \infty$,*

$$\Lambda_\ell \to \Lambda \ \text{ in } \ \mathcal{L}(H^{s+1}(\partial X), H^s(\partial X)), \tag{E.18}$$

in operator norm, whenever $s + 1 > (n-1)/2$.

Proof. Fix $\sigma \in ((n-1)/2, s+1)$ and take $f \in H^\sigma(\partial X)$. Set

$$Q_\ell f = \mathrm{PI}\, f - P_\ell f. \tag{E.19}$$

Results of §B (using the maximum principle) give

$$\|Q_\ell f\|_{L^\infty(X_1)} \leq 2\|f\|_{L^\infty(\partial X)} \leq C\|f\|_{H^\sigma(\partial X)}, \tag{E.20}$$

and $Q_\ell f \to 0$ in $L^p(X_1)$ for all $p < \infty$. Using $Q_\ell f = 0$ on ∂X and elliptic regularity, we get

$$Q_\ell f \longrightarrow 0 \ \text{ in } \ C^\infty(\overline{X}_1), \tag{E.21}$$

for each such f. In particuler,

$$Q_\ell f \longrightarrow 0 \ \text{ in } \ H^{s+3/2}(X_1) \tag{E.22}$$

for each $f \in H^\sigma(\partial X)$. The uniform boundedness theorem applies. From this, we conclude that such convergence holds uniformly on compact subsets of $H^\sigma(\partial X)$, in particular on the unit ball in $H^{s+1}(\partial X)$, so

$$P_\ell \longrightarrow \mathrm{PI} \ \text{ in } \ \mathcal{L}(H^{s+1}(\partial X), H^{s+3/2}(X_1)), \tag{E.23}$$

in operator norm. Applying ∂_ν gives (E.18).

Corollary E.3. *As $\ell \to \infty$,*

$$L_{\ell,1} \longrightarrow L_1 \quad \text{in} \quad \mathcal{L}(H_b^{k+2}(\Omega), H^k(\Omega) \oplus H^{k+1/2}(\partial X)), \tag{E.24}$$

in operator norm, whenever $k + 3/2 > (n-1)/2$.

Since all the operators in (E.24) are invertible, we have the following.

Corollary E.4. *As $\ell \to \infty$,*

$$L_{\ell,1}^{-1} \longrightarrow L_1^{-1} \quad \text{in} \quad \mathcal{L}(H^k(\Omega) \oplus H^{k+1/2}(\partial X), H_b^{k+2}(\Omega)), \tag{E.25}$$

in operator norm, whenever $k + 3/2 > (n-1)/2$.

Recall that, for $f \in C_0^\infty(\overline{M})$, supported in $\overline{\Omega}$, u_ℓ is given by (E.14) with

$$v_\ell = L_{\ell,1}^{-1}(f, 0), \tag{E.26}$$

and $u = Gf$ is given by (E.3) with

$$v = L_1^{-1}(f, 0). \tag{E.27}$$

Let us write

$$\begin{aligned} G_\ell f = u_\ell &\quad \text{on} \quad X_\ell, \\ 0 &\quad \text{on} \quad X \setminus X_\ell. \end{aligned} \tag{E.28}$$

From (E.25) we get, as $\ell \to \infty$,

$$v_\ell \longrightarrow v \quad \text{in} \quad H_b^{k+2}(\Omega), \quad \forall\, k. \tag{E.29}$$

Then, using (E.3) and (E.14) plus (E.23), naturally extended to

$$P_\ell \longrightarrow \text{PI} \quad \text{in} \quad \mathcal{L}(H^{s+1}(\partial X), H^{s+3/2}(X_j)), \tag{E.30}$$

for $j \le \ell \to \infty$, we get

$$G_\ell f \longrightarrow Gf \quad \text{in} \quad H_b^{k+2}(\Omega) \oplus H^{k+2}(X_j), \tag{E.31}$$

for $j \le \ell \to \infty$, which, via elliptic regularity, implies the following.

Proposition E.5. *Given $f \in C_0^\infty(\overline{M})$, supported in $\overline{\Omega}$,*

$$G_\ell f \longrightarrow Gf \quad \text{in} \quad H_b^{k+2}(M_j), \quad \forall\, k, \tag{E.32}$$

for $j \le \ell \to \infty$.

Note that

$$G_\ell f = -\int_0^\infty H_\ell(t) f \, dt, \tag{E.33}$$

where

$$\begin{aligned} H_\ell(t) f = e^{t\Delta_\ell} f &\quad \text{on} \quad M_\ell, \\ 0 &\quad \text{on} \quad M \setminus M_\ell, \end{aligned} \tag{E.34}$$

Δ_ℓ being the Dirichlet Laplacian on M_ℓ. As is well known,

$$f \ge 0 \Longrightarrow e^{t\Delta_\ell} f \ge 0, \tag{E.35}$$

so

$$f \ge 0 \Longrightarrow -G_\ell f \ge 0. \tag{E.36}$$

This gives the following.

Proposition E.6. *In the setting of Proposition E.5,*

$$f \geq 0 \Longrightarrow -Gf \geq 0. \tag{E.37}$$

Results given in Chapter 6, §2 of [T] imply the following.

Proposition E.7. *For $f \in L^2(M)$, supported in $\overline{\Omega}$,*

$$H_\ell(t)f \longrightarrow e^{t\Delta} f \tag{E.38}$$

in $L^2(M)$, as $\ell \to \infty$.

Remark. The result (E.38) is easier to prove than (E.32). What makes (E.32) hard is the possibility that $0 \in \operatorname{Spec} \Delta$.

Note that

$$f \geq 0 \Longrightarrow H_\ell(t)f \leq H_{\ell+1}(t)f. \tag{E.39}$$

Hence, given $f \in L^2(M)$ (supported in $\overline{\Omega}$),

$$f \geq 0 \Longrightarrow H_\ell(t)f \nearrow e^{t\Delta} f, \tag{E.40}$$

as $\ell \nearrow \infty$. The monotone convergence theorem implies

$$-G_\ell f = \int_0^\infty H_\ell(t) f \, dt \nearrow \int_0^\infty e^{t\Delta} f \, dt, \tag{E.41}$$

given $f \geq 0$ (in $L^2(M)$ and supported in $\overline{\Omega}$). Combining this with (E.32), we have the following.

Proposition E.8. *Given $f \in C_0^\infty(\overline{M})$,*

$$Gf = -\int_0^\infty e^{t\Delta} f \, dt. \tag{E.42}$$

Proof. The results (E.41) and (E.32) give this for $f \in C_0^\infty(\overline{M})$ such that $f \geq 0$. Then the result follows for general $f \in C_0^\infty(\overline{M})$ by writing $f = f_1 - f_2$, $f_j \in C_0^\infty(\overline{M})$, $f_j \geq 0$.

Remark. Again we emphasize that what makes (E.42) nontrivial is the possibility that $0 \in \operatorname{Spec} \Delta$, and we note that (E.42) fails when $M = \mathbb{R}^2$ (a case excluded by the hypotheses made at the beginning of this appendix).

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