Vanishing Viscosity Limit for Navier-Stokes Flows: No-slip Boundary Condition

Navier-Stokes Equations

$$\text{(NS)}\qquad \begin{aligned} &\partial_t u^\nu + \nabla_{u^\nu} u^\nu + \nabla p^\nu = \nu \Delta u^\nu,\\ &\operatorname{div} u^\nu = 0,\\ &u^\nu(t) = 0 \ \text{ on } \ \partial\Omega,\\ &u^\nu(0,x) = u_0(x) \ \text{ given.} \end{aligned}$$

More generally might take $u^\nu(t) = B(t)$ on $\partial\Omega$, $B(t) \parallel \partial\Omega$ (moving boundary). Assume

$$u_0 \parallel \partial\Omega.$$

Circularly symmetric 2D flow

Joint with M. Lopes Filho, A. Mazzucato, and H. Nussenzveig Lopes

$$\Omega = D, \text{disk, or annulus, centered at } \ 0 \in \mathbb{R}^2.$$

$$u^\nu(t,x) = \frac{\alpha(t)}{2\pi} x^\perp, \quad x \in \partial D.$$

Circular symmetry:

$$u_0(R_\theta x) = R_\theta u_0(x).$$

Implies circular symmetry for all $t > 0$.
Get detailed analysis of convergence

$$u^\nu(t) \to u^0(t) \equiv u_0,$$

steady solution to the Euler equation.

3D plane parallel channel flow
And related singular perturbation problems

Joint with A. Mazzucato

$$\begin{aligned} &\Omega = \{(x,y,z) : 0 \le z \le 1\}\\ &u^\nu(t,x,y,z) = (v^\nu(t,z), w^\nu(t,x,z), 0) \end{aligned}$$

For such flows, we have

$$\begin{aligned} &\nabla_{u^\nu} u^\nu = (0, v^\nu(t,z)\partial_x w^\nu(t,x,z), 0)\\ &\Rightarrow \operatorname{div} \nabla_{u^\nu} u^\nu = 0\\ &\Rightarrow p^\nu \equiv 0 \text{ (WLOG).} \end{aligned}$$

So NS equations become

$$\begin{aligned} \frac{\partial v^\nu}{\partial t} &= \nu \frac{\partial^2 v^\nu}{\partial z^2} \\ \frac{\partial w^\nu}{\partial t} + v^\nu \frac{\partial w^\nu}{\partial x} &= \nu \Big(\frac{\partial^2 w^\nu}{\partial x^2} + \frac{\partial^2 w^\nu}{\partial z^2} \Big). \end{aligned} \tag{1}$$

Boundary conditions:

$$v^\nu(t,z) = 0, \quad w^\nu(t,x,z) = 0 \text{ for } z = 0,\, 1.$$

Initial data

$$\begin{aligned} &v^\nu(0,z) = V(z), \quad w^\nu(0,x,z) = W(x,z) \\ &V \in C^\infty(I), \quad W \in C^\infty(\overline{\mathcal{O}}), \quad I = [0,1], \quad \overline{\mathcal{O}} = (\mathbb{R}/\mathbb{Z}) \times [0,1], \end{aligned}$$

(assume periodicity in x, for convenience).

Euler equations:

$$\begin{aligned} &\partial_t u^0 + \nabla_{u^0} u^0 + \nabla p^0 = 0, \\ &\operatorname{div} u^0 = 0, \quad u^0(t) \parallel \partial\Omega. \end{aligned} \tag{E}$$

Plane parallel case:

$$u^0(t,x,y,z) = (v^0(t,z), w^0(t,x,z), 0).$$

Euler equations become

$$\begin{aligned} \frac{\partial v^0}{\partial t} &= 0 \Rightarrow v^0(t,z) \equiv V(z) \\ \frac{\partial w^0}{\partial t} + v^0 \frac{\partial w^0}{\partial x} &= 0 \Rightarrow w^0(t,x,z) = W(x - tV(z), z). \end{aligned}$$

Desire convergence results:

$$v^\nu \to v^0, \quad w^\nu \to w^0, \quad \text{as } \nu \searrow 0.$$

Convergence $v^\nu \to v^0$ elementary.
Convergence $w^\nu \to w^0$ much more subtle.

First attack

Equation for w^ν in (1) has the form

$$\frac{\partial w^\nu}{\partial t} = \nu \Delta w^\nu - X_\nu w^\nu, \quad w^\nu\big|_{\mathbb{R}^+ \times \partial\mathcal{O}} = 0.$$

Here

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}, \quad X_\nu = v^\nu(t,z)\frac{\partial}{\partial x}, \quad v^\nu(t,z) = e^{\nu t \partial_z^2} V(z). \tag{2}$$

Rewrite (2) as

$$\frac{\partial w^\nu}{\partial t} = (\nu\Delta - X)w^\nu + (X - X_\nu)w^\nu,$$

where

$$X = V(z)\partial_x.$$

Apply Duhamel's formula:

$$w^\nu(t) = e^{t(\nu\Delta - X)}W + \int_0^t e^{(t-s)(\nu\Delta - X)}[(X - X_\nu)w^\nu(s)]\, ds. \tag{3}$$

Note:

$$(X - X_\nu)w^\nu(s) = (V(z) - v^\nu(s,z))\partial_x w^\nu(s,x,z).$$

Note that $w_k^\nu = \partial_x^k w^\nu$ satisfies

$$\frac{\partial w_k^\nu}{\partial t} + v^\nu \frac{\partial w_k^\nu}{\partial x} = \nu \Delta w_k^\nu,$$

since ∂_x commutes with X_ν. Also $w_k^\nu|_{\mathbb{R}^+ \times \partial\mathcal{O}} = 0$, so maximum principle applies:

$$\|\partial_x w^\nu(s)\|_{L^\infty(\mathcal{O})} \le \|\partial_x W\|_{L^\infty(\mathcal{O})} \le K \text{ (say)}.$$

Using positivity of $e^{(t-s)(\nu\Delta - X)}$,

$$\begin{aligned}
&\left| e^{(t-s)(\nu\Delta - X)}[(V - v^\nu(s))\partial_x w^\nu(s)] \right| \\
&\le K e^{(t-s)(\nu\Delta - X)} |V(z) - v^\nu(s,z)| \\
&= K e^{(t-s)\nu\Delta} |V(z) - v^\nu(s,z)|.
\end{aligned}$$

Conclusion:

$$\begin{aligned}
&\left| w^\nu(t,x,z) - e^{t(\nu\Delta - X)} W(x,z) \right| \\
&\le K \int_0^t e^{(t-s)\nu\Delta} |V(z) - v^\nu(s,z)|\, ds \\
&= R^\nu(t,z).
\end{aligned} \tag{4}$$

Note: We could replace $e^{(t-s)\nu\Delta}$ in (4) by $e^{(t-s)\nu\partial_z^2}$. There are elementary estimates on $R^\nu(t,z)$.

Second attack

Estimates on $e^{t(\nu\Delta - X)}$
Expand the setting:

$$\begin{aligned}
&\overline{\mathcal{O}} \ \text{ compact Riemannian manifold with smooth boundary } \ \partial\mathcal{O},\\
&\Delta \ \text{ Laplace-Beltrami operator on } \ \overline{\mathcal{O}},\\
&X \ \text{ smooth vector field,} \quad \operatorname{div} X = 0, \quad X \parallel \partial\mathcal{O}.
\end{aligned}$$

Note: $\mathcal{D}(\nu\Delta - X)^k) = \mathcal{D}(\Delta^k)$ for $k = 1, 2$.

$$\begin{aligned}
\mathcal{D}(\Delta) &= H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}),\\
\mathcal{D}(\Delta^2) &= \{u \in H^4(\mathcal{O}) : u, \Delta u \in H_0^1(\mathcal{O})\}.
\end{aligned}$$

Lemma 1. *There exists K, independent of $\nu \in (0,1]$, such that for $f \in \mathcal{D}(\Delta^2)$, $u^\nu(t) = e^{t(\nu\Delta - X)} f$,*

$$\|\Delta u^\nu(t)\|_{L^2}^2 \le e^{2Kt} \|\Delta f\|_{L^2}^2. \tag{5}$$

Proof. Estimate

$$\frac{d}{dt} \|\Delta u^\nu(t)\|_{L^2}^2 = 2 \operatorname{Re} (\Delta \partial_t u^\nu, \Delta u^\nu)_{L^2}.$$

Convergence result: write $u^\nu(t) = e^{t(\nu\Delta - X)} f$ as solving

$$\partial_t u^\nu = -X u^\nu + \nu \Delta u^\nu, \quad u^\nu(0) = f.$$

Duhamel's formula gives

$$u^\nu(t) = e^{-tX} f + \nu \int_0^t e^{-(t-s)X} \Delta u^\nu(s)\, ds.$$

Using Lemma 1, one shows:

Proposition 2. *Given $p \in [1, \infty)$, $f \in L^p(\mathcal{O})$, we have*

$$e^{t(\nu\Delta - X)} f \to f \quad \textit{as} \quad \nu \searrow 0,$$

in L^p-norm.

Other estimates from (5), plus interpolation:

$$\begin{aligned}
&\|e^{t(\nu\Delta - X)} f\|_{\mathcal{D}(\Delta)} \le e^{Kt} \|f\|_{\mathcal{D}(\Delta)}\\
&\Rightarrow \|e^{t(\nu\Delta - X)} f\|_{\mathcal{D}(-\Delta)^{s/2})} \le e^{Kt} \|f\|_{\mathcal{D}(-\Delta)^{s/2})}, \quad 0 < s < 2\\
&\Rightarrow \|e^{t(\nu\Delta - X)} f\|_{H^s(\mathcal{O})} \le C e^{Kt} \|f\|_{H^s(\mathcal{O})}, \quad 0 < s < 1/2.
\end{aligned}$$

Further interpolate with the elementary inequalities

$$\|e^{t(\nu\Delta-X)}f\|_{L^p(\mathcal{O})}\le\|f\|_{L^p(\mathcal{O})},\quad 1\le p<\infty,$$

to get

$$\begin{aligned}&\|e^{t(\nu\Delta-X)}f\|_{H^{\sigma,q}(\mathcal{O})}\le C_{\sigma,q}e^{Kt}\|f\|_{H^{\sigma,q}(\mathcal{O})},\\ &2\le q<\infty,\quad \sigma q\in[0,1).\end{aligned}\tag{6}$$

Proposition 3. *For σ,q as in (6),*

$$f\in H^{\sigma,q}(\mathcal{O})\Rightarrow\lim_{\nu\to 0}e^{t(\nu\Delta-X)}f=e^{-tX}f,$$

in $H^{\sigma,q}$-norm.

More precise estimates
"Conormal estimates," yielding strong interior convergence. Set

$$\mathcal{V}^k(\mathcal{O})=\{u\in L^2(\mathcal{O}):Y_1\cdots Y_ju\in L^2(\mathcal{O}):\forall\, j\le k,\ Y_\ell\in\mathfrak{X}^1\},$$

where

$$\mathfrak{X}^1=\{Y\text{ smooth vector field on }\overline{\mathcal{O}}:Y\parallel\partial\mathcal{O}\}.$$

Lemma 4. *$C_0^\infty(\mathcal{O})$ is dense in $\mathcal{V}^k(\mathcal{O})$.*

Proposition 5. *For each $k\in\mathbb{Z}^+$, $e^{t(\nu\Delta-X)}$ is a strongly continuous semigroup on $\mathcal{V}^k(\mathcal{O})$, and*

$$\|e^{t(\nu\Delta-X)}f\|_{\mathcal{V}^k}\le e^{tB_k}\|f\|_{\mathcal{V}^k},$$

with B_k independent of $\nu\in(0,1]$.

Proof. Set $u=e^{t(\nu\Delta-X)}f$ and estimate

$$\frac{d}{dt}\|Y^Ju(t)\|^2_{L^2}=2(Y^J\partial_tu,Y^Ju)_{L^2},$$

where $Y^J=Y_{j_1}\cdots Y_{j_k}$. One needs to make a careful study of the commutators

$$[\Delta,Y^J].$$

Proposition 6. *In the setting of Proposition 5,*

$$f \in \mathcal{V}^k(\mathcal{O}) \Rightarrow \lim_{\nu \to 0} e^{t(\nu\Delta - X)} f = e^{-tX} f,$$

in norm, in $\mathcal{V}^k(\mathcal{O})$.

Proof. Use Proposition 5 plus L^2-norm convergence to get convergence weak* in $\mathcal{V}^k(\mathcal{O})$. Use denseness of $\mathcal{V}^{2k}(\mathcal{O})$ in $\mathcal{V}^k(\mathcal{O})$ plus the interpolation result:

$$\mathcal{V}^k(\mathcal{O}) = [L^2(\mathcal{O}), \mathcal{V}^{2k}(\mathcal{O})]_{1/2},$$

(whose proof is not trivial) to finish off Proposition 6.

Remark. The spaces $\mathcal{V}^k(\mathcal{O})$ are special cases of "weighted b-Sobolev spaces," introduced by R. Melrose and used in scattering theory.

Still more precise estimates: exhibiting the boundary layer
Set

$$v^\nu(t) = e^{tX} e^{t(\nu\Delta - X)} f. \tag{7}$$

This solves

$$\frac{\partial v^\nu}{\partial t} = \nu L(t) v^\nu, \quad v^\nu(0) = f, \quad v^\nu\big|_{\mathbb{R}^+ \times \partial\mathcal{O}} = 0, \tag{8}$$

where

$$L(t) = e^{tX} \Delta e^{-tX}$$

is a smooth family of strongly elliptic operators on $\overline{\mathcal{O}}$.

Say $\mathcal{O}$ is a smoothly bounded domain in M, a compact Riemannian manifold without boundary, and X is extended to a smooth vector field on M. Construct a parametrix for the solution on $\mathbb{R}^+ \times M$ to

$$\frac{\partial V^\nu}{\partial t} = \nu L(t) V^\nu, \quad V^\nu(0) = F, \tag{9}$$

valid uniformly for $t \in [0, T_0]$, $\nu \in (0, 1]$, increasingly precise as $\nu \searrow 0$. The construction is a (nontrivial) variant of the heat kernel parametrix construction. It gives

$$V^\nu(t, x) = \int_M H(\nu, t, x, y) F(y)\, dV(y), \tag{10}$$

where

$$\begin{aligned} H(\nu, t, x, y) &\sim \sum_{k \geq 0} H_k(\nu, t, x, y), \\ H_0(\nu, t, x, y) &= \frac{\mathcal{G}(t, x)^{1/2}}{(4\pi\nu t)^{n/2}} e^{-\mathcal{G}(t,x)(x-y)\cdot(x-y)/4\nu t} \end{aligned} \tag{11}$$

(in local coordinates), the terms for $k \geq 1$ being progresively smaller and smoother.

Then solve (8) by the method of layer potentials. Here is one result.

Proposition 7. *Given $I = [0,T]$, $\delta > 0$, the family v^ν in (7)–(8) satisfies*

$$\|v^\nu - (f - 2\mathcal{D}_\nu^0 f^b)\|_{L^\infty(I\times\mathcal{O})} \le C(I)\nu^{1/2}\|f\|_{C^{1,\delta}(\overline{\mathcal{O}})}, \tag{12}$$

where

$$f^b(t,y) = \chi_{\mathbb{R}^+}(t)f(y)$$

and $\mathcal{D}_\nu^0$ is the layer potential operator

$$\mathcal{D}_\nu^0 f^b(t,x) = \nu\int_0^t \int_{\partial\mathcal{O}} f(y)\frac{\partial H_0}{\partial n_{s,y}}(\nu,s,t,x,y)\,dS_s(y)\,ds. \tag{13}$$

Here $\partial/\partial n_{s,y}$ and dS_s are suitable normal derivatives and boundary surface area forms, and H_0 is a variant of H_0 in (11).

Third attack

Return to analysis of

$$\frac{\partial w^\nu}{\partial t} = \nu\Delta w^\nu - X_\nu w^\nu. \tag{14}$$

Here X_ν belongs to a suitable class of t-dependent vector fields, containing the example

$$X_\nu = v^\nu(t,z)\partial_x, \quad v^\nu(t,x) = e^{t\nu\partial_z^2}V(z),$$

when $\overline{\mathcal{O}} = (\mathbb{R}/\mathbb{Z})\times I$. As before,

$$w^\nu\big|_{\mathbb{R}^+\times\partial\mathcal{O}} = 0, \quad w^\nu(0) = W. \tag{15}$$

Proposition 8. *Given $W \in \mathcal{V}^k(\mathcal{O})$, there exists a unique solution to (14)–(15), such that*

$$w^\nu \in C([0,\infty),\mathcal{V}^k(\mathcal{O})) \cap C^\infty((0,\infty)\times\overline{\mathcal{O}}),$$

and we have

$$\|w^\nu(t)\|_{\mathcal{V}^k} \le e^{tB_k}\|W\|_{\mathcal{V}^k},$$

with B_k independent of $\nu \in (0,1]$.

Proof. More elaborate variant of that done for $\partial_t u^\nu = (\nu\Delta - X)u^\nu$, in Proposition 5.

Convergence result

Proposition 9. *In the setting of Proposition 8,*

$$W \in \mathcal{V}^k(\mathcal{O}) \Rightarrow w^\nu(t) \to e^{-tX}W, \tag{16}$$

in $\mathcal{V}^k$-norm, as $\nu \searrow 0$.

Proof. L^2-norm convergence follows from the first two attacks. The $\mathcal{V}^k$-norm bounds in Proposition 8 then imply weak* convergence in $\mathcal{V}^k$. Density of $\mathcal{V}^{2k}(\mathcal{O}) \subset \mathcal{V}^k(\mathcal{O})$ and interpolation then imply norm convergence in (16).