Regularity of Weakly Conformal Maps

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Let $\Omega \subset \mathbb{C}$ be an open set. A C^1 map $f : \Omega \to \mathbb{C}$ is said to be weakly conformal provided

$$|\partial_x f|^2 = |\partial_y f|^2, \quad \langle \partial_x f, \partial_y f \rangle = 0, \tag{1}$$

on Ω, where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^2$, identified with $\mathbb{C}$ in the usual fashion. It is readily verified that such f is holomorphic on the set where $\det Df > 0$, and anti-holomorphic on the set where $\det Df < 0$, and furthermore that $\det Df(z) = 0 \Rightarrow Df(z) = 0$. Hence an equivalent condition is that if

$$\overline{\mathcal{O}}_+ = \{z \in \Omega : \partial_{\overline{z}} f = 0\}, \quad \overline{\mathcal{O}}_- = \{z \in \Omega : \partial_z f = 0\}, \tag{2}$$

then

$$\Omega = \overline{\mathcal{O}}_+ \cup \mathcal{O}_- = \mathcal{O}_+ \cup \overline{\mathcal{O}}_-,$$

where $\mathcal{O}_\pm$ denotes the interior of $\overline{\mathcal{O}}_\pm$. Another equivalent condition is that $\mathcal{O}_+ \cup \mathcal{O}_-$ is dense in Ω.

The following reveals more precisely the structure of a weakly conformal map.

Proposition 1. *If Ω is connected and $f \in C^1(\Omega)$ is weakly conformal, then f is either holomorphic on Ω or anti-holomorphic on Ω.*

Such a result is well known, at least if the hypothesis is strengthened to $f \in C^2(\Omega)$. Proofs are given in [GMS], p. 372, and [Hel], p. 39. Here is a quick variant. Given these hypotheses, we can look at $g = \partial_z f$. We have

$$\partial_{\overline{z}} g = \frac{1}{4} \Delta f = 0 \ \text{ on } \ \mathcal{O}_+ \cup \mathcal{O}_-. \tag{3}$$

Also $f \in C^2(\Omega) \Rightarrow \Delta f \in C(\Omega)$, so $\mathcal{O}_+ \cup \mathcal{O}_-$ dense in $\Omega \Rightarrow \Delta f = 0$ on Ω, so f is real-analytic on Ω, and hence so is g. If $\overline{\mathcal{O}}_-$ has nonempty interior, $g \equiv 0$, hence f is anti-holomorphic on Ω, and otherwise $\overline{\mathcal{O}}_+ = \Omega$, hence f is holomorphic on Ω.

To prove the full strength version of Proposition 1, we continue to look at $g = \partial_z f$. Now we just have $g \in C(\Omega)$, though we still have

$$g \ \text{ holomorphic on } \ \mathcal{O}_+, \quad g = 0 \ \text{ on } \ \overline{\mathcal{O}}_-, \tag{4}$$

and $\mathcal{O}_+ \cup \overline{\mathcal{O}}_- = \Omega$. Hence Proposition 1 is a consequence of the following.

Proposition 2. *Let $\Omega \subset \mathbb{C}$ be a connected open set, $\mathcal{O}_+ \subset \Omega$ an open subset, and $g \in C(\Omega)$. Assume*

$$g \text{ holomorphic on } \mathcal{O}_+, \quad g = 0 \text{ on } K = \Omega \setminus \mathcal{O}_+. \tag{5}$$

Then g is holomorphic on Ω. In particular if K has nonempty interior, then $g \equiv 0$.

We emphasize the last assertion only because this is all that is needed to complete the proof of Proposition 1.

Proposition 2 is a classical result of T. Rado. Thanks to N. Kerzman for bringing this to my attention, and also for discussing the following proof (which is in fact close to a proof presented in [N], pp. 53–54). The method involves the use of subharmonic functions. Background material on subharmonic functions can be found on pp. 16–19 of [H].

Assume g is not $\equiv 0$ on $\mathcal{O}_+$, since otherwise the result is trivial. Then $\log |g|$ is subharmonic on $\mathcal{O}_+$ ([H], Corollary 1.6.6). Given $N \in \mathbb{Z}^+$, set

$$\begin{aligned} \Phi_N(z) = \max(-N, \log|g(z)|) &\quad \text{on } \mathcal{O}_+, \\ -N &\quad \text{on } K. \end{aligned} \tag{6}$$

Then Φ_N is subharmonic on $\mathcal{O}_+$ ([H], Theorem 1.6.2). Also $\Phi_N = -N$ on a neighborhood of K. Hence Φ_N is subharmonic on Ω ([H], Corollary 1.6.5). Now, as $N \nearrow +\infty$,

$$\begin{aligned} \Phi_N(z) \searrow \Phi(z) = \log|g(z)| &\quad \text{on } \mathcal{O}_+, \\ -\infty &\quad \text{on } K, \end{aligned} \tag{7}$$

and again by ([H], Theorem 1.6.2), Φ is subharmonic on Ω. Next, if Ω is connected and Φ is not $\equiv -\infty$, then Φ is locally integrable ([H], Theorem 1.6.9). This implies that K has measure zero, hence empty interior, unless $g \equiv 0$.

As mentioned, this is enough to prove Proposition 1, but we push on with the proof of Proposition 2. At this point, the fact that (7) holds and Φ is subharmonic implies that K is a (closed) *polar* set. In this case, given

$$g \in C(\Omega), \quad g \text{ holomorphic (or even harmonic) on } \Omega \setminus K, \tag{8}$$

it is a well known result of Rado [R] that g is holomorphic (resp., harmonic) on Ω, so Proposition 2 is proven.

References

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