Proof of the Weber integral formula

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We desire to prove the identity

$$\int_0^\infty e^{-t\lambda^2} J_\nu(r_1\lambda) J_\nu(r_2\lambda)\lambda\,d\lambda = \frac{1}{2t} e^{-(r_1^2+r_2^2)/4t} I_\nu\Bigl(\frac{r_1 r_2}{2t}\Bigr), \tag{1}$$

for $t, r_1, r_2 > 0$, where $J_\nu(z)$ is the standard Bessel function and $I_\nu(y) = e^{-\pi i\nu/2} J_\nu(iy),\ y > 0$, so

$$I_\nu(z) = \Bigl(\frac{z}{2}\Bigr)^\nu \sum_{k=0}^\infty \frac{1}{k!\Gamma(\nu+k+1)} \Bigl(\frac{z}{2}\Bigr)^{2k}. \tag{2}$$

To begin, one can expand $J_\nu(r_j\lambda)$ in power series (similar to (2)) and integrate term by term, to see that the left side of (1) is equal to

$$\frac{1}{2t}\Bigl(\frac{r_1 r_2}{4t}\Bigr)^\nu \sum_{j,k\ge 0} \frac{\Gamma(\nu+j+k+1)}{\Gamma(\nu+j+1)\Gamma(\nu+k+1)} \frac{1}{j!k!} \Bigl(-\frac{r_1^2}{4t}\Bigr)^j \Bigl(-\frac{r_2^2}{4t}\Bigr)^k. \tag{3}$$

Meanwhile, by (2), the right side of (1) is equal to

$$\sum_{\ell,m\ge 0} \frac{1}{\ell!m!} \Bigl(-\frac{r_1^2}{4t}\Bigr)^\ell \Bigl(-\frac{r_2^2}{4t}\Bigr)^m \sum_{n=0}^\infty \frac{1}{n!\Gamma(\nu+n+1)} \Bigl(\frac{r_1 r_2}{4t}\Bigr)^{2n}. \tag{4}$$

If we set $y_j = -r_j^2/4t$, we see that the asserted identity (1) is equivalent to the identity

$$\begin{aligned} &\sum_{j,k\ge 0} \frac{\Gamma(\nu+j+k+1)}{\Gamma(\nu+j+1)\Gamma(\nu+k+1)} \frac{1}{j!k!}\, y_1^j y_2^k \\ &\qquad = \sum_{\ell,m,n\ge 0} \frac{1}{\ell!m!} \frac{1}{n!\Gamma(\nu+n+1)}\, y_1^{\ell+n} y_2^{m+n}. \end{aligned} \tag{5}$$

This approach was taken in §8, Chapter 8 of [T], but no explicit proof of (5) was given. We fill in the details here.

We compare coefficients of $y_1^j y_2^k$ in (5). Since both sides of (5) are symmetric in (y_1, y_2), it suffices to treat the case

$$j \le k, \tag{6}$$

which we assume henceforth. Then we take $\ell + n = j,\ m + n = k$ and sum over $n \in \{0, \dots, j\}$, to see that (5) is equivalent to the validity of

$$\sum_{n=0}^j \frac{1}{(j-n)!(k-n)!n!\Gamma(\nu+n+1)} = \frac{\Gamma(\nu+j+k+1)}{\Gamma(\nu+j+1)\Gamma(\nu+k+1)} \frac{1}{j!k!}, \tag{7}$$

whenever $0 \le j \le k$. Using the identity

$$\Gamma(\nu+j+1) = (\nu+j)\cdots(\nu+n+1)\Gamma(\nu+n+1)$$

and its analogues for the other Γ-factors in (7), we see that (7) is equivalent to the validity of

$$\sum_{n=0}^{j} \frac{j!k!}{(j-n)!(k-n)!n!}(\nu+j)\cdots(\nu+n+1) = (\nu+j+k)\cdots(\nu+k+1), \tag{8}$$

for $0 \le j \le k$. Note that the right side of (8) is a polynomial of degree j in ν, and the general term on the left side of (8) is a polynomial of degree $j-n$ in ν.

In order to establish (8), it is convenient to set

$$\mu = \nu + j \tag{9}$$

and consider the associated polynomial identity in μ. With

(10)

$$p_0(\mu) = 1, \quad p_1(\mu) = \mu, \quad p_2(\mu) = \mu(\mu-1), \quad \dots \quad , p_j(\mu) = \mu(\mu-1)\cdots(\mu-j+1),$$

we see that $\{p_0, p_1, \dots, p_j\}$ is a basis of the space $\mathcal{P}_j$ of polynomials of degree j in μ, and our task is to write

$$p_j(\mu+k) = (\mu+k)(\mu+k-1)\cdots(\mu+k-j+1) \tag{11}$$

as a linear combination of $p_0, \dots, p_j$. To this end, define

$$T : \mathcal{P}_j \longrightarrow \mathcal{P}_j, \quad Tp(\mu) = p(\mu+1). \tag{12}$$

By explicit calculation,

$$\begin{aligned} p_1(\mu+1) &= p_1(\mu) + p_0(\mu), \\ p_2(\mu+1) &= (\mu+1)\mu = \mu(\mu-1) + 2\mu = p_2(\mu) + 2p_1(\mu), \end{aligned} \tag{13}$$

and an inductive argument gives

$$Tp_i = p_i + ip_{i-1}. \tag{14}$$

By convention we set $p_i = 0$ for $i < 0$. Our goal is to compute $T^k p_j$. Note that

$$T = I + N, \quad Np_i = ip_{i-1}, \tag{15}$$

and

$$T^k = \sum_{n=0}^{j} \binom{k}{n} N^n, \tag{16}$$

if $j \le k$. By (15),

$$N^n p_i = i(i-1)\cdots(i-n+1)p_{i-n}, \tag{17}$$

so we have

$$\begin{aligned} T^k p_j &= \sum_{n=0}^{j} \binom{k}{n} j(j-1)\cdots(j-n+1)p_{j-n} \\ &= \sum_{n=0}^{j} \frac{k!}{(k-n)!n!} \frac{j!}{(j-n)!} p_{j-n}. \end{aligned} \tag{18}$$

This verifies (8) and completes the proof of (1).

Reference

[T] M. Taylor, Partial Differential Equations, Vol. 2, Springer-Verlag, New York, 1996.