# NOTES ON THE WEYL CALCULUS

BY
Michael E. Taylor

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## Introduction

These notes are divided into parts. The first part is an extremely sketchy outline of those aspects of the Weyl calculus having to do with products of operators with nice symbols. Emphasis is placed on contact with the representation theory of the Heisenberg group and with explicit formulas, particularly involving the harmonic oscillator.

The second part uses the Weyl calculus to give a "naive" heat equation proof of the index formula for first order elliptic differential operators of Dirac type on 2-dimensional manifolds. The advantage of the Weyl calculus here is that, if $A(X,D)$ is a second order elliptic operator and $p(x,\xi) = \varphi(A(x,\xi))$ has order $m$, then $(pA)(X,D)$ (which has order $m+2$) differs from $A(X,D)p(X,D)$ by an operator of order $m$, rather than one of order $m+1$, which is what you have using the Kohn-Nirenberg calculus. This enables one to shorten by an order of magnitude the number of calculations required to determine, in a straightforward fashion, the second term in the expansion (on the diagonal) of the heat kernel.

As an illustration, we include a proof of the Riemann-Roch formula. Incidentally (though the point is hardly important) we show how the proof of the Gauss-Bonnet formula drops out as a special case.

Most of the material here is discussed in further detail in at least one of the references [T1]–[T3].

### 1.1. The Heisenberg group

The Heisenberg group $\mathbf{H}^n$ is the universal covering group of the group of unitary operators on $L^2(\mathbb{R}^n)$ generated by $e^{iq\cdot X}$ and $e^{ip\cdot D}$, where

$$e^{iq\cdot X}f(x) = e^{iq\cdot x}f(x), \quad e^{ip\cdot D}f(x) = f(x+p). \tag{1}$$

Let us note the relation

$$e^{i(q\cdot X + p\cdot D)} = e^{iq\cdot p/2}\ e^{iq\cdot X}\ e^{ip\cdot D}. \tag{2}$$

This leads to the following multiplication law for $e^{i(t+q\cdot X+p\cdot D)} = \pi_1(t,q,p)$ :

$$(t,q,p)\cdot(t',q',p') = (t+t'+\frac{1}{2}(p\cdot q' - q\cdot p'), q+q', p+p'). \tag{3}$$

Here we have the symplectic form

$$p\cdot q' - q\cdot p' = \sigma\big((p,q),(p',q')\big). \tag{4}$$

Thus the action of the symplectic group $Sp(n,\mathbb{R})$ on $(q,p)$ gives a group of automorphisms of the Heisenberg group $\mathbf{H}^n$.

The Lie algebra action of $\mathfrak{h}^n$ is

$$\pi_1(T) = iI, \quad \pi_1(L_j) = iX_j, \quad \pi_1(M_j) = \partial/\partial x_j. \tag{5}$$

Here we identify $\mathfrak{h}^n$ with $T_e\mathbf{H}^n \approx T_0\mathbb{R}^{2n+1}$, with $T = \partial/\partial t,\ L_j = \partial/\partial q_j,\ M_j = \partial/\partial p_j$, at the origin.

There is a family of unitary representations of $\mathbf{H}^n$, on $L^2(\mathbb{R}^n)$, given by

$$\pi_{\pm\lambda}(t,q,p) = e^{(\pm\lambda t \pm \lambda^{\frac{1}{2}} q\cdot X + \lambda^{\frac{1}{2}} p\cdot D)}, \tag{6}$$

for $\lambda \in (0,\infty)$. Explicitly

$$\pi_{\pm\lambda}(t,q,p)f(x) = e^{i(\pm\lambda t \pm \lambda^{\frac{1}{2}} q\cdot x + \lambda q\cdot p/2)}\ f(x+\lambda^{\frac{1}{2}}p). \tag{7}$$

Each one is irreducible. By the Stone-von Neuman Theorem, every irreducible unitary representation representation of $\mathbf{H}^n$ is equivalent to either one of these or to one of the one-dimensional representations

$$\pi_{y,\eta}(t,q,p) = e^{i(y\cdot q+\eta\cdot p)}. \tag{8}$$

Another way to describe the multiplication law on $\mathbf{H}^n$ is by

$$e^{ip\cdot D}e^{iq\cdot X} = e^{iq\cdot p}e^{iq\cdot X}e^{ip\cdot D}.$$

This suggests the following multiplication law for $e^{it}e^{iq\cdot X}e^{ip\cdot D}$ :

$$(t,q,p)\circ(t',q',p') = (t+t'+p\cdot q', q+q', p+p').$$

The disadvantage of this approach is that the symplectic symmetry is hidden here.

### 1.2. The Weyl calculus

Given a symbol $a(x,\xi)$, we define the operator $a(X,D)$, via the Weyl calculus, as

$$a(X,D)u = (2\pi)^{-n} \int \hat{a}(q,p)\ e^{i(q\cdot X+p\cdot D)} u\ dqdp. \tag{1}$$

Recall that $e^{i(q\cdot X+p\cdot D)}u(x) = e^{i(q\cdot x+q\cdot p/2)}u(x+p)$. Then a few simple manipulations yield

$$a(X,D)u = (2\pi)^{-n} \int a\Big(\frac{x+y}{2},\xi\Big)\ e^{i(x-y)\cdot\xi} u(y)\ dy\ d\xi. \tag{2}$$

This can be compared with the Kohn-Nirenberg calculus, which associates to $a(x,\xi)$ the operator $a(x,D)$, defined by

$$a(x,D)u = (2\pi)^{-n} \int \hat{a}(q,p)\ e^{iq\cdot X} e^{ip\cdot D} u\ dqdp, \tag{3}$$

or alternatively as

$$a(x,D)u = (2\pi)^{-n} \int a(x,\xi)\ e^{i(x-y)\cdot\xi} u(y)\ dy\ d\xi. \tag{4}$$

The first fundamental result in the Weyl calculus is the
**Product law:**

$$a(X,D)b(X,D) = (a\circ b)(X,D), \tag{5}$$

with

$$(a\circ b)(x,\xi) = e^{-\frac{1}{2}i(\partial_y\cdot\partial_\xi - \partial_x\cdot\partial_\eta)} a(x,\xi)b(y,\eta)\Big|_{y=x,\eta=\xi}. \tag{6}$$

The proof proceeds by examining

$$\begin{aligned}(a\circ b)(X,D) &= \int \hat{a}(q,p)e^{i(q\cdot X+p\cdot D)}\hat{b}(q',p')e^{i(q'\cdot X+p'\cdot D)}\ dqdpdq'dp' \\ &= \int \hat{a}(q,p)\hat{b}(q',p')e^{\frac{1}{2}i(p\cdot q'-q\cdot p')}e^{i\big((q+q')\cdot X+(p+p')\cdot D\big)}\ dqdpdq'dp'.\end{aligned} \tag{7}$$

Formally:

$$(a\circ b)(x,\xi) \sim ab + \sum_{j\ge 1} \frac{1}{j!}\{a,b\}_j(x,\xi), \tag{8}$$

where

$$\{a,b\}_j(x,\xi) = \Big(-\frac{i}{2}\Big)^j \big(\partial_y\cdot\partial_\xi - \partial_x\cdot\partial_\eta\big)^j a(x,\xi)b(y,\eta)\Big|_{y=x,\eta=\xi}. \tag{9}$$

Note that $\{a,b\}_1 = -\frac{1}{2}i\{a,b\}$, involving the ordinary Poisson bracket.

An important fact is that, if either $a(x,\xi)$ or $b(x,\xi)$ is a polynomial in $(x,\xi)$, then (8) is a finite sum, and is exact.

We record a few consequences of the product rule when the factors have symbols of type (1,0). Recall that

$$p(x,\xi)\in S^m_{1,0} \Longleftrightarrow |D^\beta_x D^\alpha_\xi p(x,\xi)| \le C_{\alpha\beta}\langle\xi\rangle^{m-|\alpha|}. \tag{10}$$

**Proposition.** *If* $p_j(x,\xi)\in S^{m_j}_{1,0}$, *then*

$$p_1\circ p_2 = p_1p_2 - \frac{i}{2}\{p_1,p_2\} \ \textit{mod}\ S^{m_1+m_2-2}_{1,0}, \tag{11}$$

*and, mod* $S^{m_1+m_2+m_3-2}_{1,0}$,

$$p_1\circ p_2\circ p_3 = p_1p_2p_3 - \frac{i}{2}\big(\{p_1,p_3\}p_2 + \{p_2,p_3\}p_1 + \{p_1,p_2\}p_3\big). \tag{12}$$

As a consequence of (12), we have

$$q\circ p\circ q = q^2 p \mod S^{m+2\mu-2}_{1,0}, \tag{13}$$

if $p\in S^m_{1,0}$ and $q\in S^\mu_{1,0}$. More generally, if $p_{jk}\in S^m_{1,0}$, $p_{jk}=p_{kj}$, and if $q_j\in S^\mu_{1,0}$, then

$$\sum_{j,k} q_j\circ p_{jk}\circ q_k = \sum_{j,k} q_j p_{jk} q_k \mod S^{m+2\mu-2}_{1,0}. \tag{14}$$

To relate the Weyl calculus to the Heisenberg group $\mathbf{H}^n$, recall the representations $\pi_{\pm\lambda}$ of $\mathbf{H}^n$ on $L^2(\mathbb{R}^n)$. They yield representations of the convolution algebra $L^1(\mathbf{H}^n)$ on $L^2(\mathbb{R}^n)$,

$$\pi_{\pm\lambda}(k) = \int_{\mathbf{H}^n} k(t,q,p)\ \pi_{\pm\lambda}(t,q,p)\ dtdqdp. \tag{15}$$

Using (1), we obtain the formula

$$\pi_{\pm\lambda}(k) = \tilde{k}(\pm\lambda, \pm\lambda^{\frac12}X, \lambda^{\frac12}D). \tag{16}$$

## 1.3. The metaplactic representation, infinitesimally

The crucial fact about compositions of operators in the Weyl calculus which gives rise to the metaplactic representation is that, if either $a(x,\xi)$ or $b(x,\xi)$ is a polynomial of degree $\le 2$ in $(x,\xi)$, then

$$(a\circ b)(x,\xi) = ab - \frac{i}{2}\{a,b\} + \frac{1}{2}\{a,b\}_2, \tag{1}$$

and consequently

$$[a(X,D),b(X,D)] = c(X,D), \text{ with } c(x,\xi) = -i\{a,b\}(x,\xi). \tag{2}$$

**Proposition.** *If $Q(x,\xi)$ is a polynomial homogeneous of degree 2 in $(x,\xi)$, (we say $Q\in\mathcal{P}_2$), then*

$$e^{-isQ(X,D)}\ a(X,D)\ e^{isQ(X,D)} = a_{sQ}(X,D), \tag{3}$$

*with*

$$a_{sQ}(x,\xi) = a\big((exp\ sH_Q)(x,\xi)\big). \tag{4}$$

*Proof.* The identity (3) is equivalent to the operator equation

$$\partial_s\ a_{sQ}(X,D) = i[a_{sQ}(X,D),Q(X,D)], \tag{5}$$

so

$$\partial_s\ a_{sQ}(x,\xi) = \{Q,a_{sQ}\}(x,\xi), \tag{6}$$

which gives (4).

Note that $\mathcal{P}_2$, with the Poisson bracket, is the Lie algebra of $Sp(n,\mathbb{R})$.

**Corollary.** *We get a representation $\tilde\omega$ of $\widetilde{Sp(n,\mathbb{R})}$, such that, for $g\in\widetilde{Sp(n,\mathbb{R})}$,*

$$a_g(X,D) = \tilde\omega(g)^{-1}\ a(X,D)\ \tilde\omega(g), \tag{7}$$

*where, given the covering map $j:\widetilde{Sp(n,\mathbb{R})}\to Sp(n,\mathbb{R})$,*

$$a_g(x,\xi) = a\big(j(g)(x,\xi)\big). \tag{8}$$

Here $\widetilde{Sp(n,\mathbb{R})}$ denotes the universal covering group of $Sp(n,\mathbb{R})$. In fact, one gets a representation of the <u>double cover</u> of $Sp(n,\mathbb{R})$, as will be shown in §5.

### 1.4. The harmonic oscillator

Our main goal here is an explicit formula for $e^{-tH}$ when $H = Q(X, D)$, with $Q(x, \xi) = |x|^2 + |\xi|^2$. In this case, $e^{-tH}$ is called the Hermite semigroup, and $H = |X|^2 + |D|^2$ is called the Hermite operator. It is the Schrodinger hamiltonian associated to the harmonic oscillator.

**Proposition.** *We have*

$$e^{-tH} = h_t(X, D), \tag{1}$$

*with*

$$h_t(x, \xi) = (cosh\ t)^{-n}\ e^{-(\tanh\ t)(|x|^2+|\xi|^2)}. \tag{2}$$

Let us first note that metaplectic covariance implies that

$$h_t(x, \xi) = g(t, Q), \quad Q = |x|^2 + |\xi|^2. \tag{3}$$

Thus, if $h_t(x, \xi)$ is defined by (1), then

$$\begin{aligned} \frac{\partial}{\partial t}\ h_t(x, \xi) &= -(Q \circ h_t)(x, \xi) \\ &= -Q(x, \xi)h_t(x, \xi) - \frac{1}{2}\{Q, h_t\}_2(x, \xi) \\ &= -\big(|x|^2 + |\xi|^2\big)h_t(x, \xi) + \frac{1}{4}\sum_k \big(\partial_{x_k}^2 + \partial_{\xi_k}^2\big)h_t(x, \xi). \end{aligned} \tag{4}$$

Thus we have

$$\frac{\partial g}{\partial t} = -Qg + Q\frac{\partial^2 g}{\partial Q^2} + n\frac{\partial g}{\partial Q}. \tag{5}$$

If we make the 'guess' $g(t, Q) = a(t)e^{-b(t)Q}$, with $a(t)$ and $b(t)$ to be determined, then we obtain

$$\frac{a'(t)}{a(t)} = -n\, b(t), \quad b'(t) = 1 - b(t)^2. \tag{6}$$

The initial condition $h_0(x, \xi) = 1$ implies $a(0) = 1$ and $b(0) = 0$. Hence we get

$$b(t) = \tanh\ t, \quad a(t) = (\cosh\ t)^{-n}, \tag{7}$$

establishing (2).

We can obtain a formula for

$$e^{-tQ(X,D)}=h_t^Q(X,D), \tag{8}$$

for a general positive quadratic form $Q(x,\xi)$. First, in the case

$$Q(x,\xi)=\sum_{j=1}^n \mu_j(x_j^2+\xi_j^2),\quad \mu_j>0, \tag{9}$$

it follows easily from (2) that

$$h_t^Q(x,\xi)=\prod_{j=1}^n\bigl(\cosh t\mu_j\bigr)^{-1}\cdot\ \exp\Bigl\{-\sum_{j=1}^n(\tanh t\mu_j)(x_j^2+\xi_j^2)\Bigr\}. \tag{10}$$

Now any positive quadratic form $Q(x,\xi)$ can be put in the form (9) via a linear symplectic transformation, so to get the general formula we need only rewrite (10) in a symplectically invariant fashion. This is accomplished using the 'Hamilton map' $F_Q$, a skew symmetric transformation on $\mathbb{R}^{2n}$ defined by

$$Q(u,v)=\sigma(u,F_Qv),\quad u,v\in\mathbb{R}^{2n}, \tag{11}$$

where $Q(u,v)$ is the bilinear form polarizing $Q$. When $Q$ has the form (9), $F_Q$ is a sum of $2\times 2$ blocks $\begin{pmatrix}0 & \mu_j\\ -\mu_j & 0\end{pmatrix}$, and we have

$$\prod_{j=1}^n\bigl(\cosh t\mu_j\bigr)^{-1}=\Bigl(\det\cosh itF_Q\Bigr)^{-\frac{1}{2}}. \tag{12}$$

Passing from $F_Q$ to

$$A_Q=\bigl(-F_Q^2\bigr)^{\frac{1}{2}}, \tag{13}$$

the unique positive definite square root, means passing to blocks $\begin{pmatrix}\mu_j & 0\\ 0 & \mu_j\end{pmatrix}$, and, when $Q$ has the form (9), then

$$\sum_{j=1}^n(\tanh t\mu_j)(x_j^2+\xi_j^2)=tQ\bigl(\vartheta(tA_Q)\zeta,\zeta\bigr), \tag{14}$$

where $\zeta=(x,\xi)$, and

$$\vartheta(t)=\frac{\tanh t}{t}. \tag{15}$$

Thus the general formula for (8) is

$$h_t^Q(x,\xi) = \Big(\cosh\, tA_Q\Big)^{-\frac{1}{2}}\ e^{-tQ(\vartheta(tA_Q)\zeta,\zeta)}. \tag{16}$$

Analytic continuation and other arguments give, for generic real $Q(x,\xi) \in \mathcal{P}_2$,

$$e^{iQ(X,D)}u(x) = a(Q)\int e^{i\varphi(Q,\frac{1}{2}(x+y),\xi)+i(x-y)\cdot\xi}\ u(y)\ dy\ d\xi, \tag{17}$$

where

$$a(Q) = (2\pi)^{-n}\big(\det\ \cos A_Q\big)^{-\frac{1}{2}} \tag{18}$$

and

$$\varphi(Q,x,\xi) = -Q(\theta(A_Q)\zeta,\zeta), \quad \theta(t) = \frac{\tan\, t}{t}. \tag{19}$$

In particular, analytic continuation of (2) gives

$$e^{it(H-n)} = E_t(X,D), \quad E_t(x,\xi) = \Big(\frac{e^{-it}}{\cos t}\Big)^n e^{i(\tan t)(|x|^2+|\xi|^2)}. \tag{20}$$

Note that the right side is periodic in $t$ of period $\pi$, consistent with the fact that spec $H = \{n+2k : k = 0,1,2,\dots\}$. We deduce that

$$\begin{aligned} f(H-n) &= \psi_f(X,D), \quad \psi_f(x,\xi) = \varphi_f\big(|x|^2+|\xi|^2\big), \\ \varphi_f(\lambda) &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \hat{f}(t)\Big(\frac{e^{-it}}{\cos t}\Big)^n e^{i\lambda\tan t}\ dt. \end{aligned} \tag{21}$$

Using $y = \tan t$, we can write

$$\begin{aligned} \pi\varphi_f(\lambda) &= \int_{-\infty}^{\infty} \hat{f}(\tan^{-1} y)\frac{(1-iy)^n}{1+y^2}e^{i\lambda y}\ dy \\ &= (1-\partial_\lambda)^n\int_{-\infty}^{\infty}\frac{\hat{f}(\tan^{-1} y)}{1+y^2}e^{i\lambda y}\ dy. \end{aligned} \tag{22}$$

In particular, if $P_k$ is the orthogonal projection on the $(n+2k)$-eigenspace of $H$, then

$$P_k = \Pi_k(X,D), \quad \Pi_k(x,\xi) = \tau_k\big(|x|^2+|\xi|^2\big), \tag{23}$$

with

$$\begin{aligned} \pi\tau_k(\lambda) &= \int_{-\infty}^{\infty}\Big(\frac{(1-iy)^2}{1+y^2}\Big)^k\frac{(1-iy)^n}{1+y^2}e^{i\lambda y}\ dy \\ &= (1-\partial_\lambda)^{n+2k}\int_{-\infty}^{\infty}(1+y^2)^{-k-1}e^{i\lambda y}\ dy. \end{aligned} \tag{24}$$

An alternative approach to formulas for $P_k$ is to use

$$e^{-tH}=\sum_{k\ge 0} e^{-(2k+n)t} P_k, \tag{25}$$

which, together with (2), gives

$$\sum_{k\ge 0} e^{-2kt}\Pi_k(x,\xi)=\Bigl(\frac{2}{1+e^{-2t}}\Bigr)^n e^{-(\tanh\, t)(|x|^2+|\xi|^2)}. \tag{26}$$

Taking $t\to\infty$ gives

$$\Pi_0(x,\xi)=2^n e^{-|x|^2-|\xi|^2}. \tag{27}$$

Also, using $\Pi_k(x,\xi)=\tau_k\bigl(|x|^2+|\xi|^2\bigr)$, we can write the relation above as

$$\sum_{k\ge 0}\tau_k(\lambda)e^{-2kt}=\Bigl(\frac{2}{1+e^{-2t}}\Bigr)^n e^{-\lambda\,\tanh\, t}. \tag{28}$$

If we set $z=e^{-2t}$, and also set

$$\tau_k(\lambda)=\sigma_k(\lambda)e^{-\lambda}, \tag{29}$$

we get

$$\sum_{k=0}^{\infty}\sigma_k(\lambda)z^k=\Bigl(\frac{2}{1+z}\Bigr)^n e^{\frac{2\lambda z}{1+z}}, \tag{30}$$

a generating function for the polynomials $\sigma_k(\lambda)$.

J. Derezinski has pointed out that you can extend some of these formulas, in a way we will illustrate by example. With $H$ as in (1), consider

$$a_t(X,D)=e^{itH}a(X,D). \tag{31}$$

Then $a_t(x,\xi)$ satisfies

$$\begin{gathered}\frac{\partial}{\partial t}a_t(x,\xi)=-i\Bigl\{\frac{1}{4}(\Delta_x+\Delta_\xi)+(\xi\cdot\partial_x-x\cdot\partial_\xi)+(|x|^2+|\xi|^2)\Bigr\}a_t(x,\xi),\\ a_0(x,\xi)=a(x,\xi),\end{gathered} \tag{32}$$

i.e.,

$$a_t(x,\xi)=e^{-it\mathfrak{H}}a(x,\xi), \tag{33}$$

where

$$\mathfrak{H} = \tfrac{1}{4}(\Delta_x + \Delta_\xi) + (\xi \cdot \partial_x - x \cdot \partial_\xi) + |x|^2 + |\xi|^2. \tag{34}$$

In turn, $e^{-it\mathfrak{H}}$ is subject to the same sort of analysis as $e^{itH}$. We remark that

$$\mathfrak{H} = \mathfrak{H}_0 + \mathfrak{L}, \quad \mathfrak{H}_0 = \tfrac{1}{4}(\Delta_x + \Delta_\xi) + |x|^2 + |\xi|^2, \ \mathfrak{L} = \xi \cdot \partial_x - x \cdot \partial_\xi,$$

and $\mathfrak{H}_0$ and $\mathfrak{L}$ commute.

A similar analysis applies to

$$b_t(X, D) = a(X, D)e^{itH}. \tag{35}$$

In fact, $\overline{b_t(x, \xi)} = \overline{a}_{-t}(x, \xi)$.

We conpute a special case of (35) directly. Namely, let

$$\alpha f(x) = f(-x). \tag{36}$$

Then

$$\begin{aligned} a(X, D)\alpha f(x) &= (2\pi)^{-n} \iint a\Big(\frac{x+y}{2}, \xi\Big) e^{i(x-y)\cdot\xi} f(-y) \, dy \, d\xi \\ &= (2\pi)^{-n} \iint a\Big(\frac{x-y}{2}, \xi\Big) e^{i(x+y)\cdot\xi} f(y) \, dy \, d\xi \\ &= \int K(x, y) f(y) \, dy, \end{aligned} \tag{37}$$

with

$$K(x, y) = (2\pi)^{-n} \int a\Big(\frac{x-y}{2}, \xi\Big) e^{i(x+y)\cdot\xi} \, d\xi.$$

We want to write

$$a(X, D)\alpha = b(X, D), \tag{38}$$

i.e., we need to find $b(x, \xi)$ so that

$$K(x, y) = (2\pi)^{-n} \int b\Big(\frac{x+y}{2}, \xi\Big) e^{i(x-y)\cdot\xi} \, d\xi.$$

Let us set $x + y = u, \ x - y = v$. Thus, we want

$$K\Big(\frac{u+v}{2}, \frac{u-v}{2}\Big) = (2\pi)^{-n} \int b(\tfrac{1}{2}u, \xi) e^{iv\cdot\xi} \, d\xi,$$

hence

$$b(\tfrac{1}{2}u, \xi) = \int K\Big(\frac{u+v}{2}, \frac{u-v}{2}\Big) e^{-iv\cdot\xi} \, dv.$$

Therefore, the desired formula is

$$b(x, \xi) = 2^{-\frac{n}{2}} \hat{a}(-2\xi, 2x). \tag{39}$$

## 1.5. The Bargmann-Fock and Weil representations

In §1 we described a representation $\pi_1$ of $\mathbf{H}^n$ on $L^2(\mathbb{R}^n)$, which is often called the Schrödinger repesentation. The Bargmann-Fock representation $\beta_1$ is a unitarily equivalent representation, on the Hilbert space

$$\mathcal{H} = \{u(\zeta) \text{ holomorphic on } \mathbb{C}^n : \int_{\mathbb{C}^n} |u(\zeta)|^2 e^{-|\zeta|^2/2} d\zeta < \infty\}. \tag{1}$$

On the Lie algebra level, we have

$$\beta_1(T) = iI, \quad \beta_1(L_j) = \frac{i}{\sqrt{2}}\Big(\frac{\partial}{\partial\zeta_j} + \zeta_j\Big), \quad \beta_1(M_j) = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial\zeta_j} - \zeta_j\Big). \tag{2}$$

We note that

$$(\partial/\partial\zeta_j)^* = \zeta_j \text{ on } \mathcal{H}.$$

On the Lie group level, if we identify $(t,q,p) \in \mathbf{H}^n$ with $(t,z)$, $z = q + ip \in \mathbb{C}^n$, we have

$$\beta_1(t,z)u(\zeta) = e^{it+(i/\sqrt{2})\zeta\cdot z - |z|^2}\ u(\zeta + i\overline{z}/\sqrt{2}). \tag{3}$$

The unitary equivalence of $\beta_1$ and $\pi_1$ is implemented by a unitary map $K : L^2(\mathbb{R}^n) \to \mathcal{H}$, defined by

$$Kf(\zeta) = \int_{\mathbb{R}^n} K(x,\zeta)\ f(x)\ dx. \tag{4}$$

where

$$K(x,\zeta) = \pi^{-n/4}\ \exp\Big(\sqrt{2}\zeta\cdot x - \frac{1}{2}\big(\zeta\cdot\zeta + |x|^2\big)\Big). \tag{5}$$

Recall that the Lie algebra of $Sp(n,\mathbb{R})$ is isomorphic to $\mathcal{P}_2$. Now we can regard $Sp(n,\mathbb{R})$ as the group of (real) linear transformations on $\mathbb{C}^n$, leaving invariant the symplectic form $\sigma(z,z') = \operatorname{Im} z\cdot\overline{z'}$, i.e., the imaginary part of the natural Hermitian form on $\mathbb{C}^n$. Then we naturally have $U(n) \subset Sp(n,\mathbb{R})$. The inclusion of the Lie algebra $\mathfrak{u}(n)$ in $\mathcal{P}_2$ can be described as follows; $\mathfrak{u}(n)$ is spanned by

$$\lambda_{jk}(x,\xi) = x_j x_k + \xi_j\xi_k, \quad \mu_{jk}(x,\xi) = x_j\xi_k - x_k\xi_j. \tag{6}$$

The images under $\beta_1$ (or under $K^{-1}\tilde{\omega}K = \omega^{\#}$) of these are

$$L'_{jk} = i\Big(\zeta_j\frac{\partial}{\partial\zeta_k} + \zeta_k\frac{\partial}{\partial\zeta_j}\Big), \quad M'_{jk} = \zeta_j\frac{\partial}{\partial\zeta_k} - \zeta_k\frac{\partial}{\partial\zeta_j}. \tag{7}$$

In particular the Hermite operator $H$ is intertwined to

$$KHK^{-1} = W, \text{ with } W = 2\sum\zeta_j\frac{\partial}{\partial\zeta_j} + n. \tag{8}$$

Note that

$$e^{itW}f(\zeta) = e^{int}f(e^{2it}\zeta). \tag{9}$$

**Proposition.** *The representation* $\omega^{\#}$ *gives a unitary representation of* $MU(n)$*, the double cover of* $U(n)$*, namely*

$$\omega^{\#}(g)u(z) = (det\ g)^{-\frac{1}{2}}\ u(g^{-1} \cdot z), \quad g \in MU(n). \tag{10}$$

**Corollary.** *The Weil representation* $\tilde{\omega}$ *gives a representation of* $Mp(n, \mathbb{R})$*, the double cover of* $Sp(n, \mathbb{R})$*.*

## 1.6. The Toeplitz representation

This representation was introduced by R. Howe. With $\mathcal{H}$ the Hilbert space for the Bargmann-Fock representation, and $L^2_{\mathcal{H}}(B^n)$ the space of $L^2$ functions holomorphic on the unit ball in $\mathbb{C}^n$, define a unitary map

$$V : \mathcal{H} \longrightarrow L^2_{\mathcal{H}}(B^n), \tag{1}$$

by taking the orthonormal basis of $\mathcal{H}$ :

$$u_\alpha = a_\alpha z^\alpha, \quad a_\alpha = \Bigl(\frac{2}{\alpha!}\Bigr)^{\frac{1}{2}} \tag{2}$$

to the orthonormal basis of $L^2_{\mathcal{H}}(B^n)$ :

$$v_\alpha = b_\alpha z^\alpha, \quad b_\alpha = \Bigl(\frac{(n+|\alpha|)!}{\alpha!}\Bigr)^{\frac{1}{2}}. \tag{3}$$

Thus

$$V z^\alpha = \gamma_\alpha z^\alpha, \quad \gamma_\alpha = \frac{b_\alpha}{a_\alpha} = 2^{\frac{1}{2}}\ \pi^{-\frac{n}{2}} \bigl[(n+|\alpha|)!\bigr]^{\frac{1}{2}} = \gamma_{|\alpha|}. \tag{4}$$

We define (unbounded) operators on $L^2_{\mathcal{H}}(B^n)$ :

$$\mathcal{Z}_j = V\ z_j\ V^{-1}, \quad \mathcal{L}_j = V(\partial/\partial z_j)V^{-1}. \tag{5}$$

We get

$$\mathcal{Z}_j = z_j\bigl[|D| + n + 1\bigr]^{\frac{1}{2}}, \quad \mathcal{L}_j = \mathcal{Z}_j^*, \tag{6}$$

where $D = (1/i)X$, $X$ the real vector field on $\mathbb{R}^{2n} = \mathbb{C}^n$ generating the flow $z \mapsto e^{it}z$. Thus the representation $\nu_1$ of $\mathbf{H}^n$ on $L^2_{\mathcal{H}}(B^n)$ defined by

$$\nu_1(g) = V\ \beta_1(g)\ V^{-1}, \quad g \in \mathbf{H}^n, \tag{7}$$

satisfies

$$\nu_1(T) = iI, \quad \nu_1(L_j) = i\pi\mathcal{X}_j\pi, \quad \nu_1(M_j) = i\pi\mathcal{D}_j\pi, \tag{8}$$

where $\pi$ is the orthogonal projection of $L^2(B^n)$ onto $L^2_{\mathcal{H}}(B^n)$, and

$$\begin{aligned} \mathcal{D}_j &= z_j\bigl[|D|+n+1\bigr]^{\frac{1}{2}} + \bigl[|D|+n+1\bigr]^{\frac{1}{2}}\overline{z}_j, \\ i\mathcal{X}_j &= z_j\bigl[|D|+n+1\bigr]^{\frac{1}{2}} - \bigl[|D|+n+1\bigr]^{\frac{1}{2}}\overline{z}_j. \end{aligned} \tag{9}$$

## 1.7. Differential operators in the Weyl calculus

Recall that

$$a(X,D)u = \int W_a(x,y)u(y)dy \tag{1}$$

where

$$W_a(x,y) = (2\pi)^{-n} \int a\Big(\frac{x+y}{2},\xi\Big)e^{i(x-y)\cdot\xi}d\xi. \tag{2}$$

If $a(x,\xi)$ is a polynomial in $\xi$,

$$a(x,\xi) = \sum_\alpha a_\alpha(x)\xi^\alpha, \tag{3}$$

this gives

$$W_a(x,y) = \sum_\alpha a_\alpha\Big(\frac{x+y}{2}\Big)\delta^{(\alpha)}(x-y),$$

so $a(X,D)$ is a differential operator, namely

$$\begin{aligned} a(X,D)u(x) &= \sum \int a_\alpha\Big(\frac{x+y}{2}\Big)\delta^{(\alpha)}(x-y)u(y)\ dy \\ &= \sum \int \delta(x-y)\ D_y^\alpha\Big[a_\alpha\Big(\frac{x+y}{2}\Big)u(y)\Big]dy, \end{aligned}$$

or

$$a(X,D)u(x) = \sum_\alpha D_y^\alpha\Big[a_\alpha\Big(\frac{x+y}{2}\Big)u(y)\Big]\Big|_{y=x}. \tag{4}$$

Expanding by the Leibniz formula, we get

$$a(X,D)u(x) = \sum_\alpha \sum_{\beta+\gamma=\alpha} \binom{\alpha}{\beta} 2^{-|\gamma|}a_\alpha^{(\gamma)}(x)D^\beta u(x). \tag{5}$$

In particular, if $|\alpha| = 1$ in (3), i.e.,

$$a(x,D)u = \sum a_j(x)\partial_j u(x), \tag{6}$$

we have

$$a(X,D)u(x)=\sum_j a_j(x)\partial_j u(x)+\frac{1}{2}\sum(\partial_j a_j)u(x). \tag{7}$$

If $|\alpha|=2$ in (3), i.e.,

$$a(x,D)u=\sum a_{jk}(x)\partial_j\partial_k u, \tag{8}$$

with $a_{jk}=a_{kj}$, we have

$$a(X,D)u(x)=\sum\Bigl[a_{jk}(x)\partial_j\partial_k u+(\partial_j a_{jk})\partial_k u+\frac{1}{4}(\partial_j\partial_k a_{jk})u\Bigr]. \tag{9}$$

For comparison, note that

$$\sum\partial_j(a_{jk}\partial_k u)=\sum\Bigl[a_{jk}\partial_j\partial_k u+(\partial_j a_{jk})\partial_k u\Bigr], \tag{10}$$

so (9) and (10) differ only in the zero order term:

$$a(X,D)u=\sum\partial_j(a_{jk}\partial_k u)+\frac{1}{4}\sum(\partial_j\partial_k a_{jk})u. \tag{11}$$

Let us note the following related phenomenon. Suppose

$$q_j(x,\xi)\in S^{\mu}_{1,0},\quad p_{jk}(x,\xi)\in S^{m}_{1,0}, \tag{12}$$

with $p_{jk}=p_{kj}$. Then

$$\sum_{j,k} q_j(X,D)p_{jk}(X,D)q_k(X,D)=r(X,D)\mod OPS^{m+2\mu-2}_{1,0}, \tag{13}$$

where

$$r(x,\xi)=\sum q_j(x,\xi)p_{jk}(x,\xi)q_k(x,\xi). \tag{14}$$

### 1.8. The Calderon-Vaillancourt thorem

Here we will show that, for any $r > 0$,

$$a(x,\xi) \in C^{2n,r}(\mathbb{R}^{2n}) \Longrightarrow a(X,D) : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n), \tag{1}$$

where $C^{2n,r}(\mathbb{R}^{2n})$ denotes the space of functions whose derivatives of order $\le 2n$ are bounded, and satisfy a uniform Hölder condition. We use a method of H.O.Cordes. We begin with the following identity. If

$$U(y) = e^{iq\cdot X + ip\cdot D}, \quad y = (q,p), \tag{2}$$

then

$$\int \bigl|\bigl(U(y)f,g\bigr)\bigr|^2 \, dy = C_n \|f\|_{L^2}^2 \|g\|_{L^2}^2. \tag{3}$$

This is a simple computation, making use of the Fourier inversion formula. Now, if $\|f\|_{L^2} = 1$, let $\Pi_f u = (u,f)f$, and consider

$$T_{b,f} = \int b(y) U(y)^{-1} \Pi_f U(y) \, dy. \tag{4}$$

We have

$$(T_{b,f}u, v) = \int b(y) \bigl(f, U(y)u\bigr) \, \overline{\bigl(f, U(y)v\bigr)} \, dy, \tag{5}$$

and (3) gives

$$|(T_{b,f}u,v)| \le C\|b\|_{L^\infty} \|u\|_{L^2} \|v\|_{L^2}. \tag{6}$$

Using this, we easily deduce that, if $G$ is a trace-class operator and

$$T_{b,G} = \int b(y) U(y)^{-1} G U(y) \, dy, \tag{7}$$

then

$$\|T_{b,G} u\|_{L^2} \le C\|b\|_{L^\infty} \|G\|_{Tr} \|u\|_{L^2}. \tag{8}$$

We will apply the inequality (8) to the estimation of $a(X,D)$, as follows. One readily verifies that

$$G = g(X,D) \Longrightarrow T_{b,G} = a(X,D), \quad a = b * g. \tag{9}$$

Let us take $g(x,\xi)$ such that

$$\hat{g}(q,p)=\big(1+|q|^2+|p|^2\big)^{-s}. \tag{10}$$

We claim that

$$s>n\Longrightarrow \|g(X,D)\|_{Tr}<\infty. \tag{11}$$

One way to see this is to note that, if $\alpha f(x)=f(-x)$, then, as shown in (38)-(39) of §1.4,

$$p(X,D)\circ\alpha=q(X,D),\quad q(x,\xi)=c\hat{p}(-2\xi,2x). \tag{12}$$

Thus, when $g$ satisfies (10),

$$g(X,D)\circ\alpha=c\varphi(X,D),\quad \varphi(x,\xi)=\big(1+4|x|^2+4|\xi|^2\big)^{-s}, \tag{13}$$

and clearly

$$s>n\Longrightarrow \|\varphi(X,D)\|_{Tr}<\infty. \tag{14}$$

In light of (8), this gives

$$\|a(X,D)u\|_{L^2}\le C_s\|(1-\Delta_x-\Delta_\xi)^s a\|_{L^\infty}\|u\|_{L^2},\quad s>n, \tag{15}$$

so (1) is proved.

We can produce variants of this result, replacing (10) by

$$\hat{g}(q,p)=\big(1+|q|^2+|p|^2\big)^{-n}\big[\log\big(2+|q|^2+|p|^2\big)\big]^{-1-r}, \tag{16}$$

for example. The natural replacement for (13), in concert with (21)-(22) of §1.4, gives that $g(X,D)$ is of trace class provided $r>0$ in (16). Then we can replace the hypothesis on $a(x,\xi)$ in (1) by a weaker modulus of continuity on derivatives of $a(x,\xi)$ of order $\le 2n$. We omit the details.

### 2.1. Heat asymptotics via the Weyl calculus

We use the Weyl calculus to construct a parametrix for a 'heat' equation

$$\frac{\partial u}{\partial t} = -Lu, \quad u(0) = f, \tag{1}$$

with

$$Lu = a(X, D)u + b(x)u. \tag{2}$$

We suppose $a(X, D)$ is a self adjoint second order elliptic differential operator, with positive symbol. We assume $a(x, \xi)$ is scalar; $b(x)$ may be a matrix.

We want to write an approximate solution to (1) as

$$u = E(t, X, D)f. \tag{3}$$

We write

$$E(t, x, \xi) \sim E_0(t, x, \xi) + E_1(t, x, \xi) + \cdots \tag{4}$$

and obtain the various terms recursively. The PDE (1) requires

$$\frac{\partial}{\partial t} E(t, X, D) = -L\ E(t, X, D) = -(L \circ E)(t, X, D), \tag{5}$$

where, by the Weyl calculus,

$$(L \circ E)(t, x, \xi) \sim L(x, \xi)E(t, x, \xi) + \sum_{j \ge 1} \frac{1}{j!} \{L, E\}_j(t, x, \xi). \tag{6}$$

Recall that

$$\{L, E\}_j = \Bigl(-\frac{i}{2}\Bigr)^j \Bigl\{\sum_{k=1}^{n} \Bigl(\frac{\partial^2}{\partial y_k \partial \xi_k} - \frac{\partial^2}{\partial x_k \partial \eta_k}\Bigr)\Bigr\}^j L(x, \xi)E(t, y, \eta)\Big|_{y=x, \eta=\xi}. \tag{7}$$

In particular

$$\{L, E\}_1 = -\frac{i}{2} \sum_k \Bigl(\frac{\partial L}{\partial \xi_k} \frac{\partial E}{\partial x_k} - \frac{\partial L}{\partial x_k} \frac{\partial E}{\partial \xi_k}\Bigr) \tag{8}$$

is a multiple of the usual Poisson bracket.

It is natural to set

$$E_0(t,x,\xi)=e^{-ta(x,\xi)}. \tag{9}$$

Note that the Weyl calculus applied to this term provides a better approximation than the Kohn-Nirenberg calculus, because

$$\{a,e^{-ta}\}_1=0! \tag{10}$$

If we plug (4) into (6) and collect the highest order nonvanishing terms, we are led to define $E_1(t,x,\xi)$ as the solution to the 'transport equation'

$$\frac{\partial E_1}{\partial t}=-aE_1-\frac{1}{2}\{a,E_0\}_2-b(x)E_0,\quad E_1(0,x,\xi)=0. \tag{11}$$

Let us set

$$\Omega_1(t,x,\xi)=-\frac{1}{2}\{a,e^{-ta}\}_2-b(x)e^{-ta(x,\xi)}. \tag{12}$$

Then the solution to (11) is

$$E_1(t,x,\xi)=\int_0^t e^{(s-t)a(x,\xi)}\Omega_1(s,x,\xi)ds. \tag{13}$$

We turn to the evaluation of the integral (13). Clearly

$$\int_0^t e^{(s-t)a(x,\xi)}b(x)e^{-sa(x,\xi)}ds=tb(x)e^{-ta(x,\xi)}. \tag{14}$$

Now, a straightforward calculation yields

$$\{a,e^{-sa}\}_2=\frac{s}{2}Q(\nabla^2 a)e^{-sa}-\frac{s^2}{4}T(\nabla a,\nabla^2 a)e^{-sa}, \tag{15}$$

where

$$Q(\nabla^2 a)=\sum_{k,\ell}\Big\{(\partial_{\xi_k}\partial_{\xi_\ell}a)(\partial_{x_k}\partial_{x_\ell}a)-(\partial_{\xi_k}\partial_{x_\ell}a)(\partial_{x_k}\partial_{\xi_\ell}a)\Big\}, \tag{16}$$

and

$$\begin{aligned}T(\nabla a,\nabla^2 a)=\sum_{k,\ell}\Big\{&(\partial_{\xi_k}\partial_{\xi_\ell}a)(\partial_{x_k}a)(\partial_{x_\ell}a)\\&+(\partial_{x_k}\partial_{x_\ell}a)(\partial_{\xi_k}a)(\partial_{\xi_\ell}a)-2(\partial_{\xi_k}\partial_{x_\ell}a)(\partial_{x_k}a)(\partial_{\xi_\ell}a)\Big\}.\end{aligned} \tag{17}$$

Therefore

$$\int_0^t e^{(s-t)a}\{a,e^{-sa}\}_2ds=\frac{t^2}{4}Q(\nabla^2 a)e^{-ta}-\frac{t^3}{12}T(\nabla a,\nabla^2 a)e^{-ta}. \tag{18}$$

We get $E_1(t,x,\xi)$ in (13) from (14) and (18).

Suppose that

$$a(x,\xi)=\sum a_{jk}(x)\xi_j\xi_k, \tag{19}$$

with $a_{jk}=a_{kj}$. Suppose also that, for some point $x_0$,

$$\nabla_x\ a_{jk}(x_0)=0,\quad a_{jk}(x_0)=\delta_{jk}. \tag{20}$$

Then, at $x_0$,

$$\begin{aligned} Q(\nabla^2 a)&=\sum_{k,\ell}\bigl(\partial_{\xi_k}\partial_{\xi_\ell}a\bigr)\bigl(\partial_{x_k}\partial_{x_\ell}a\bigr)\\ &=2\sum_{j,k,\ell}\frac{\partial^2 a_{jk}}{\partial x_\ell^2}(x_0)\xi_j\xi_k, \end{aligned}\tag{21}$$

and

$$\begin{aligned} T(\nabla a,\nabla^2 a)&=\sum_{k,\ell}\bigl(\partial_{x_k}\partial_{x_\ell}a\bigr)\bigl(\partial_{\xi_k}a\bigr)\bigl(\partial_{\xi_\ell}a\bigr)\\ &=4\sum_{j,k,\ell,m}\frac{\partial^2 a_{jk}}{\partial x_\ell\partial x_m}(x_0)\xi_j\xi_k\xi_\ell\xi_m. \end{aligned}\tag{22}$$

Such a situation as (20) arises if $a_{jk}(x)=g^{jk}(x)$ comes from a metric tensor $g_{jk}(x)$, and one uses geodesic normal coordinates centered at $x_0$. Now the Laplace-Beltrami operator is given by

$$\Delta u=g^{-\frac12}\sum\partial_j g^{jk}g^{\frac12}\partial_k\ u, \tag{23}$$

where $g=\det(g_{jk})$. This is symmetric when one uses the Riemannian volume element $dV=\sqrt{g}\ dx_1\cdots dx_n$. To use the Weyl calculus, we want an operator which is symmetric with respect to the Euclidean volume element $dx_1\cdots dx_n$, so we conjugate $\Delta$ by multiplication by $g^{\frac14}$ :

$$\begin{aligned} -Lu&=g^{\frac14}\ \Delta\bigl(g^{-\frac14}u\bigr)\\ &=g^{-\frac14}\sum\partial_j g^{jk}g^{\frac12}\partial_k\bigl(g^{-\frac14}u\bigr). \end{aligned}\tag{24}$$

Note that the integral kernel $k_L^t(x,y)$ of $e^{tL}$ is equal to $g^{\frac14}(x)k_\Delta^t(x,y)g^{-\frac14}(y)$; in particular of course the two kernels coincide on the diagonal $x=y$. To compare $L$ with $g(X,D)$, where

$$g(x,\xi)=\sum g^{jk}(x,\xi)\xi_j\xi_k, \tag{25}$$

note that

$$-Lu = \sum \partial_j g^{jk} \partial_k u + \Phi(x) u, \tag{26}$$

where

$$\Phi(x) = \sum \partial_j \bigl(g^{jk} g^{\frac{1}{2}} \partial_k g^{-\frac{1}{4}}\bigr) - \sum g^{jk} g^{\frac{1}{2}} \bigl(\partial_j g^{-\frac{1}{4}}\bigr)\bigl(\partial_k g^{-\frac{1}{4}}\bigr). \tag{27}$$

If $g^{jk}(x) = a_{jk}(x)$ satisfies (20), we see that

$$\Phi(x_0) = \sum_j \partial_j^2 g^{-\frac{1}{4}}(x_0) = -\frac{1}{4} \sum_\ell \partial_\ell^2 g(x_0). \tag{28}$$

Since $g(x_0 + he_\ell) = \det\bigl(\delta_{jk} + \frac{1}{2} h^2 \partial_\ell^2 g_{jk}\bigr) + O(h^3)$, we have

$$\Phi(x_0) = -\frac{1}{4} \sum_{j,\ell} \partial_\ell^2 g_{jj}(x_0). \tag{29}$$

By comparison, recall from (7.11) that

$$g(X, D)u = -\sum \partial_j g^{jk} \partial_k u + \Psi(x) u, \tag{30}$$

where

$$\Psi(x) = \frac{1}{4} \sum \partial_j \partial_k g^{jk}(x). \tag{31}$$

If $x_0$ is the center of a normal coordinate system, we can express these results in terms of curvature, using

$$\partial_\ell \partial_m g_{jk}(x_0) = \frac{1}{3} R_{j\ell km}(x_0) + \frac{1}{3} R_{jmk\ell}(x_0), \tag{32}$$

in terms of the components of the Riemann curvature tensor. See Spivak, vol.2, p.193. In particular, we get for (29) and (31):

$$\begin{aligned} \Phi(x_0) &= -\frac{1}{4}\frac{2}{3} \sum_{j,\ell} R_{j\ell j\ell}(x_0) = -\frac{1}{6} S(x_0), \\ \Psi(x_0) &= -\frac{1}{4}\frac{1}{3} \sum_{j,k} \bigl[R_{jjkk}(x_0) + R_{jkkj}(x_0)\bigr] = \frac{1}{12} S(x_0). \end{aligned} \tag{33}$$

Here $S$ is the scalar curvature of the metric $g_{jk}$.

When $a(X, D) = g(X, D)$, we can express the quantities (21)-(22) in terms of curvature:

$$Q(\nabla^2 g) = 2 \cdot \frac{2}{3} \sum_{j,k,\ell} R_{j\ell k\ell}(x_0)\xi_j\xi_k = \frac{4}{3} \sum_{j,k} Ric_{jk}(x_0)\xi_j\xi_k, \tag{34}$$

where $Ric_{jk}$ denotes the components of the Ricci tensor, and

$$T(\nabla g, \nabla^2 g) = 4 \cdot \frac{2}{3} \sum_{j,k,\ell,m} R_{j\ell km}(x_0)\xi_j\xi_k\xi_\ell\xi_m = 0, \tag{35}$$

the cancellation here resulting from the antisymmetry of $R_{j\ell km}$ in $(j, \ell)$ and in $(k, m)$.

Thus the heat kernel for (1) with

$$Lu = g(X, D)u + b(x)u \tag{36}$$

is of the form (3)-(4), with $E_0 = e^{-tg(x,\xi)}$ and

$$\begin{aligned} E_1(t, x, \xi) &= \Bigl(-tb(x) - \frac{t^2}{8} Q(\nabla^2 g) + \frac{t^3}{24} T(\nabla g, \nabla^2 g)\Bigr)e^{-tg} \\ &= -\Bigl(tb(x) + \frac{t^2}{6} Ric(\xi, \xi)\Bigr)e^{-tg(x,\xi)}, \end{aligned} \tag{37}$$

at $x = x_0$. Note that $g(x_0, \xi) = |\xi|^2$.

Now the integral kernel of $E_j(t, X, D)$ is

$$K_j(t, x, y) = (2\pi)^{-n} \int E_j\Bigl(t, \frac{x + y}{2}, \xi\Bigr)e^{i(x-y)\cdot\xi}\, d\xi. \tag{38}$$

In particular, on the diagonal we have

$$K_j(t, x, x) = (2\pi)^{-n} \int E_j(t, x, \xi)\, d\xi. \tag{39}$$

We want to compute these quantities, for $j = 0, 1$, and at $x = x_0$. First,

$$K_0(t, x_0, x_0) = (2\pi)^{-n} \int e^{-t|\xi|^2}\, d\xi = (4\pi t)^{-n/2}, \tag{40}$$

since, as is well known, the Gaussian integral in (40) is equal to $(\pi/t)^{\frac{n}{2}}$. Next,

$$(2\pi)^n K_1(t, x_0, x_0) = -tb(x_0) \int e^{-t|\xi|^2} d\xi - \frac{t^2}{6} \sum Ric_{jk}(x_0) \int \xi_j\xi_k e^{-t|\xi|^2} d\xi. \tag{41}$$

We need to compute some more Gaussian integrals. If $j \neq k$, the integrand is an odd function of $\xi_j$, so the integral vanishes. On the other hand,

$$\begin{aligned}\int \xi_j^2 e^{-t|\xi|^2}\,d\xi &= \frac{1}{n}\int |\xi|^2 e^{-t|\xi|^2}\,d\xi \\ &= -\frac{1}{n}\,\frac{d}{dt}\int e^{-t|\xi|^2}\,d\xi = \frac{1}{2}\pi^{\frac{n}{2}} t^{-\frac{n}{2}-1}.\end{aligned} \tag{42}$$

Thus

$$K_1(t,x_0,x_0) = -(4\pi t)^{-n/2}\Bigl(tb(x_0) + \frac{t}{12}S(x_0)\Bigr), \tag{43}$$

since $\sum Ric_{jj}(x) = S(x)$.

As noted above, the Laplace operator $\Delta$ on scalar functions, when conjugated by $g^{\frac{1}{4}}$, has the form (36), with

$$b(x_0) = \Phi(x_0) - \Psi(x_0) = -\frac{1}{4}S(x_0).$$

Thus, for the keat kernel $e^{t\Delta}$, on scalars, we have

$$K_1(t,x_0,x_0) = (4\pi t)^{-n/2}\frac{t}{6}S(x_0). \tag{44}$$

These computations may allow for an elementary computation of the index of a first order elliptic differential operator

$$D : C^\infty(M,E_0) \longrightarrow C^\infty(M,E_1) \tag{45}$$

between sections of vector bundles $E_j$ over a 2-manifold $M$. Suppose that, with respect to choices of local frame fields on an open cover $U_\nu$ of $M$,

$$D^*D = g(X,D) + B_0(x), \quad DD^* = g(X,D) + B_1(x), \tag{46}$$

with $B_j$ sections over $U_\nu$ of $\mathrm{End}(E_j)$. Then, the heat kernel difference satisfies

$$\lim_{t\to 0}\, K_{D^*D}(t,x,x) - K_{DD^*}(t,x,x) = \frac{1}{4\pi}\bigl[B_1(x) - B_0(x)\bigr], \quad x \in U_\nu. \tag{47}$$

Hence the difference on the right side is globally well defined, and

$$\text{Index } D = \frac{1}{4\pi}\int_M Tr\bigl[B_1(x) - B_0(x)\bigr]\,dV. \tag{48}$$

We can generalize this, setting

$$a(x,\xi) = g(x,\xi) + \ell(x,\xi), \quad \ell(x,\xi) = \sum \ell_j(x)\xi_j. \tag{49}$$

Again assume $a(x,\xi)$ is scalar and consider $L = a(X,D) + b(x)$. We have

$$E_0(t,x,\xi) = e^{-ta(x,\xi)} = e^{-t\ell(x,\xi)}\ e^{-tg(x,\xi)}, \tag{50}$$

and $E_1(t,x,\xi)$ is still given by (11)-(18). A point to keep in mind is that we can drop $\ell(x,\xi)$ from the computation involving $\{a, e^{-ta}\}_2$, altering $K_1(t,x,x)$ only by $o(t^{-\frac{n}{2}+1})$ as $t \searrow 0$. Thus, mod $o(t^{-\frac{n}{2}+1})$, $K_1(t,x_0,x_0)$ is still given by (44). To get $K_0(t,x_0,x_0)$, expand $e^{-t\ell(x,\xi)}$ in (50) in powers of $t$ :

$$E_0(t,x,\xi) \sim \Big[1 - t\ell(x,\xi) + \frac{t^2}{2}\ell(x,\xi)^2 + \cdots\Big]e^{-tg(x,\xi)}. \tag{51}$$

When doing the $\xi$-integral, the term $t\ell(x,\xi)$ is obliterated, of course, while, by (42),

$$\frac{t^2}{2}\int \ell(x_0,\xi)^2 e^{-t|\xi|^2}\ d\xi = \frac{1}{4}\pi^{\frac{n}{2}} t^{-\frac{n}{2}+1} \sum \ell_j(x_0)^2. \tag{52}$$

Hence, in this situation,

$$\begin{aligned} &K_0(t,x_0,x_0) + K_1(t,x_0,x_0) = \\ &\qquad = (4\pi t)^{-\frac{n}{2}}\Big[1 + t\big(\sum \ell_j(x_0)^2 - b(x_0) - \frac{1}{12}S(x_0)\big) + O(t^2)\Big]. \end{aligned} \tag{53}$$

Next, we drop the assumption that $\ell(x,\xi)$ in (49) be scalar. We still assume $g(x,\xi)$ defines the metric tensor. There are several changes whose effects on (53) need to be investigated. In the first place, (10) is no longer quite true. We have

$$\{a, e^{-ta}\}_1 = \frac{i}{2}\sum\Big\{\frac{\partial a}{\partial x_j}\frac{\partial}{\partial \xi_j}e^{-ta} - \frac{\partial a}{\partial \xi_j}\frac{\partial}{\partial x_j}e^{-ta}\Big\}. \tag{54}$$

In this case, with $a(x,\xi)$ matrix valued, we have

$$\begin{aligned} \frac{\partial}{\partial x_j}e^{-ta} &= -te^{-ta}\ \Xi\big(\mathrm{ad}(-ta)\big)\Big(\frac{\partial a}{\partial x_j}\Big) \\ &= -te^{-ta}\ \Xi\big(\mathrm{ad}(-t\ell)\big)\Big(\frac{\partial a}{\partial x_j}\Big), \end{aligned} \tag{55}$$

where $\Xi(z) = (1 - e^{-z})/z$, so

$$\begin{aligned} \frac{\partial}{\partial x_j}e^{-ta} &= te^{-ta}\Big(\frac{\partial a}{\partial x_j} + \frac{t}{2}\big[\ell, \frac{\partial \ell}{\partial x_j}\big] + \cdots\Big) \\ &= -t\frac{\partial a}{\partial x_j} + O(t^2|\xi|)e^{-ta} + \cdots\ , \end{aligned} \tag{56}$$

etc. Hence

$$\{a, e^{-ta}\}_1 = -\frac{i}{2} t \sum \Bigl[\frac{\partial \ell}{\partial x_j}, \frac{\partial \ell}{\partial \xi_j}\Bigr] e^{-ta} + \cdots \tag{57}$$

This is smaller than any of the terms in the transport equation (11) for $E_1$, so it could be put in a higher transport equation. It does not affect (53).

Another change comes from the following modification of (14):

$$\int_0^t e^{(s-t)a(x,\xi)} b(x) e^{-sa(x,\xi)}\, ds = \Bigl[\int_0^t e^{(s-t)\ell(x,\xi)} b(x) e^{-s\ell(x,\xi)} ds\Bigr] \cdot e^{-tg(x,\xi)}. \tag{58}$$

This time, $b(x)$ and $\ell(x,\xi)$ may not commute. We can write the right side as

$$\begin{aligned} &\int_0^t e^{s \text{ ad } \ell(x,\xi)} \bigl[b(x)\bigr] ds\ e^{-t\ell(x,\xi)} e^{-tg(x,\xi)} \\ &= t\Bigl\{b(x) - \frac{t}{2}\bigl(\ell(x,\xi)b(x) + b(x)\ell(x,\xi)\bigr) + \cdots\Bigr\} e^{-tg(x,\xi)}. \end{aligned} \tag{59}$$

Due to the extra power of $t$ with the anticommutator, this does not lead to a change in (53).

The other change in letting $\ell(x,\xi)$ be non-scalar is that

$$\ell(x,\xi)^2 = \sum_{j,k} \ell_j(x)\ell_k(x)\xi_j\xi_k \tag{60}$$

generally has non-commuting factors, but this also does not affect (53). In conclusion, allowing $\ell(x,\xi)$ to be non-scalar does not change (53).

## 2.2. Applications to 2-D index theory

Consider a first order elliptic differential operator $D = A(X, D)$, with

$$A(x,\xi) = \sum A_j(x)\xi_j + C(x), \tag{1}$$

a $K \times K$ matrix valued symbol. Assume that

$$\begin{aligned} D^*D &= g(X,D) + \ell_0(X,D) + B_0(x), \\ DD^* &= g(X,D) + \ell_1(X,D) + B_1(x), \end{aligned} \tag{2}$$

where $g(x,\xi)$ defines a metric tensor, while $\ell_j(x,\xi)$ and $B_j(x)$ are $K \times K$ matrix valued, and

$$\ell_\nu(x,\xi) = \sum_j \ell_j^{(\nu)}(x)\xi_j. \tag{3}$$

By (8.53), extended to the non-scalar case, we have

$$\text{Index } D = \frac{1}{4\pi} \int_M \Big\{ Tr \sum_j \big[\ell_j^{(0)}(x)^2 - \ell_j^{(1)}(x)^2\big] + Tr\big[B_1(x) - B_0(x)\big] \Big\} dV. \tag{4}$$

Of course, the individual terms in the integrand are not generally globally well defined on $M$; only the total is. We want to express these terms directly in terms of the symbol of $D$. We have $D^*D = L_0(X,D)$ and $DD^* = L_1(X,D)$, with

$$\begin{aligned} L_0(x,\xi) &= A(x,\xi)^*A(x,\xi) + \frac{i}{2}\{A^*, A\}, \\ L_1(x,\xi) &= A(x,\xi)A(x,\xi)^* + \frac{i}{2}\{A, A^*\}. \end{aligned} \tag{5}$$

Hence

$$\begin{aligned} \ell_0(x,\xi) &= A_1(x,\xi)^*C(x) + C(x)^*A_1(x,\xi) + \frac{i}{2}\{A_1^*, A_1\} \\ \ell_1(x,\xi) &= A_1(x,\xi)C(x)^* + C(x)A_1(x,\xi)^* + \frac{i}{2}\{A_1, A_1^*\}, \end{aligned} \tag{6}$$

where $A_1(x,\xi) = \sum A_j(x)\xi_j$, and

$$\begin{aligned} B_0(x) &= C(x)^*C(x) + \frac{i}{2}\{C^*, A_1\} + \frac{i}{2}\{A_1^*, C\} \\ B_1(x) &= C(x)C(x)^* + \frac{i}{2}\{C, A_1^*\} + \frac{i}{2}\{A_1, C^*\}. \end{aligned} \tag{7}$$

Suppose that, for a given point $x_0 \in M$, we arrange $C(x_0) = 0$. Then

$$
\tag{8}
\begin{aligned}
\ell_0(x_0,\xi) &= \frac{i}{2}\{A_1^*, A_1\} = \frac{i}{2}\sum_j \Bigl(\frac{\partial A_1^*}{\partial \xi_j}\frac{\partial A_1}{\partial x_j} - \frac{\partial A_1^*}{\partial x_j}\frac{\partial A_1}{\partial \xi_j}\Bigr) \\
\ell_1(x_0,\xi) &= \frac{i}{2}\{A_1, A_1^*\} = \frac{i}{2}\sum_j \Bigl(\frac{\partial A_1}{\partial \xi_j}\frac{\partial A_1^*}{\partial x_j} - \frac{\partial A_1}{\partial x_j}\frac{\partial A_1^*}{\partial \xi_j}\Bigr),
\end{aligned}
$$

and

$$
\tag{9}
\begin{aligned}
B_0(x_0) &= \frac{i}{2}\{C^*, A_1\} + \frac{i}{2}\{A_1^*, C\} = \frac{i}{2}\sum_j \Bigl(-\frac{\partial C^*}{\partial x_j}\frac{\partial A_1}{\partial \xi_j} + \frac{\partial A_1^*}{\partial \xi_j}\frac{\partial C}{\partial x_j}\Bigr) \\
B_1(x_0) &= \frac{i}{2}\{C, A_1^*\} + \frac{i}{2}\{A_1, C^*\} = \frac{i}{2}\sum_j \Bigl(-\frac{\partial C}{\partial x_j}\frac{\partial A_1^*}{\partial \xi_j} + \frac{\partial A_1}{\partial \xi_j}\frac{\partial C^*}{\partial x_j}\Bigr).
\end{aligned}
$$

Note that, if $A_1(x,\xi)$ is scalar, then $\ell_0(x_0,\xi) = -\ell_1(x_0,\xi)$, (granted that $C(x_0) = 0$), and their contributions to the integrand in (4) cancel. Also, if $A_1(x,\xi)$ is scalar, $B_1(x_0) = -B_0(x_0)$. Thus, at $x_0$, the integrand in (4) is equal to

$$
\tag{10}
2\ Tr\ B_1(x_0) = -\ Tr\ \sum_j \Bigl(\overline{A}_j \frac{\partial C}{\partial x_j} - A_j \frac{\partial C^*}{\partial x_j}\Bigr),
$$

in this case. This situation arises for elliptic differential operators on sections of complex line bundles. In such a case, $C(x)$ is also scalar, and we can rewrite (10) as

$$
\tag{11}
-2\ \mathrm{Im}\ \sum_j \overline{A}_j \frac{\partial C}{\partial x_j}.
$$

Let's take a look at the operator $D_L : C^\infty(M,L) \to C^\infty(M, L\otimes\overline{\kappa})$, where $M$ is a Riemann surface, $L \to M$ a complex line bundle, with a Hermitian metric and a metric connection $\nabla$, and, for a vector field $X$,

$$
\tag{12}
\langle D_L u, X\rangle = \nabla_X u + i\nabla_{JX} u.
$$

Here $J$ is the complex structure on $TM$. We can assume $M$ has a Riemannian metric with respect to which $J$ is rotation by $90°$. Pick $x_0 \in M$. Use a geodesic normal coordinate system centered at $x_0$, so the metric tensor $g_{jk}$ satisfies

$$
\tag{13}
\nabla g_{jk}(x_0) = 0.
$$

Let $X(x_0) = \partial/\partial x_1$ and define $X$ by parallel transport radially from $x_0$ (along geodesics). Then

$$
\tag{14}
X(x) = a_1^1(x)\frac{\partial}{\partial x_1} + a_1^2(x)\frac{\partial}{\partial x_2}
$$

with

$$a_1^1(x_0) = 1, \quad a_1^2(x_0) = 0, \quad \nabla a_1^j(x_0) = 0. \tag{15}$$

Furthermore,

$$JX(x) = a_2^1(x)\frac{\partial}{\partial x_1} + a_2^2(x)\frac{\partial}{\partial x_2} \tag{16}$$

with

$$a_2^1(x_0) = 0, \quad a_2^2(x_0) = 1, \quad \nabla a_2^j(x_0) = 0. \tag{17}$$

Next, let $\varphi$ be a local section of $L$ such that $\varphi(x_0)$ has norm 1, and $\varphi(x)$ is obtained from $\varphi(x_0)$ by radial parallel translation. Thus

$$u = v\varphi \Longrightarrow \nabla_{\partial_j} u = (\partial_j v + i\theta_j v)\varphi, \tag{18}$$

where the connection coefficients satisfy

$$\theta_j(x_0) = 0. \tag{19}$$

In such a coordinate system, and with respect to such choices, the operator $D_L$ takes the form

$$D_L(v\varphi) = \frac{1}{i}\sum \Big[A_j \frac{\partial v}{\partial x_j} - A_j\theta_j v\Big]\varphi \otimes \vartheta, \tag{20}$$

where

$$A_j = i\big(a_1^j + i a_2^j\big) \tag{21}$$

and where $\vartheta \in C^\infty(U, \overline{\kappa})$ satisfies

$$\langle X, \vartheta\rangle = 1, \quad \langle JX, \vartheta\rangle = i.$$

Then $D_L^* : C^\infty(M, L\otimes\overline{\kappa}) \to C^\infty(M, L)$ is given by

$$D_L^*(w\ \varphi\otimes\vartheta) = \frac{1}{i}\sum g^{-\frac12}\Big[\overline{A}_j\frac{\partial}{\partial x_j} + (\partial_j\overline{A}_j + \overline{A}_j\overline{\theta}_j)\Big]\big(g^{\frac12} w\big)\varphi. \tag{22}$$

Now we want to take adjoints using $L^2(U, dx)$ rather than $L^2(U, \sqrt{g}dx)$, so we conjugate by $g^{\frac14}$, and replace $D_L$ by

$$\tilde{D}_L = \frac{1}{i}\sum\Big[g^{\frac14}A_j\frac{\partial}{\partial x_j}\big(g^{-\frac14}v\big) - A_j\theta_j v\Big]. \tag{23}$$

Thus we are in the situation of considering an operator of the form (1), with $A_j$ given by (21) and

$$C(x) = \sum \Big[ \frac{i}{2} \frac{\partial A_j}{\partial x_j} - A_j \theta_j - \frac{1}{4} g^{-1} \frac{\partial g}{\partial x_j} A_j \Big]. \tag{24}$$

Thus $C(x_0) = 0$, by (15)-(19), while

$$\partial_k C(x_0) = \sum_j \Big[ -A_j(\partial_k \theta_j) + \frac{i}{2} \partial_k \partial_j A_j - \frac{1}{4} A_j (\partial_k \partial_j g) \Big]. \tag{25}$$

Now $\partial_k \theta_j(x_0)$ is given by the curvature of $\nabla$ on $L$ :

$$\frac{\partial \theta_j}{\partial x_k}(x_0) = \frac{1}{2} F_{jk}(x_0). \tag{26}$$

Meanwhile, via (8.32), $\partial_k \partial_j A_j$ can be expressed in terms of the Riemannian curvature:

$$\partial_j \partial_k a^\ell_m(x_0) = -\frac{1}{6} R_{\ell jmk} - \frac{1}{6} R_{\ell kmj}, \tag{27}$$

and of course so can $\partial_k \partial_j g(x_0)$. Consequently, at $x_0$, the formula (11) for the integrand in (4) becomes

$$\frac{2}{i} F_{12} + \frac{1}{2} S(x_0). \tag{28}$$

Note that $\frac{1}{2} S = K$, the Gauss curvature. Thus the formula (4) becomes

$$\begin{aligned} \text{Index } D_L &= \frac{1}{4\pi} \int_M \Big( \frac{2}{i} F_{12} + K \Big) dV \\ &= \frac{1}{2\pi i} \int_M \omega_L + \frac{1}{4\pi} \int_M K \, dV, \end{aligned} \tag{29}$$

where $\omega_L$ is the curvature form of $L$. We have the identities

$$\frac{1}{2\pi i} \int_M \omega_L = c_1(L)[M], \quad \frac{1}{4\pi} \int_M K \, dV = \frac{1}{2} \chi(M), \tag{30}$$

the latter being the Gauss-Bonnet theorem.

Now, if $L \to M$ is a holomorphic line bundle, then $\frac{1}{2} D_L$ has the same principal symbol, hence the same index, as

$$\overline{\partial}_L : C^\infty(M, L) \longrightarrow C^\infty(M, L \otimes \overline{\kappa}).$$

Hence we obtain the **Riemann-Roch formula:**

$$\text{Index } \overline{\partial}_L = c_1(L)[M] + \frac{1}{2}\chi(M). \tag{31}$$

We finish with a comment on the Gauss-Bonnet formula; $\chi(M)$ is the index of

$$d + \delta : \Lambda^0 M \oplus \Lambda^2 M \longrightarrow \Lambda^1 M, \tag{32}$$

if dim $M = 2$. If $M$ is oriented, both $\Lambda^1 M$ and $(\Lambda^0 \oplus \Lambda^2)M$ get structures of complex line bundles via the Hodge $*$ operator; use

$$J = * \text{ on } \Lambda^1, \quad J = -* : \Lambda^0 \to \Lambda^2, \quad J = * : \Lambda^2 \to \Lambda^0. \tag{33}$$

It follows easily that $(d+\delta)J = J(d+\delta)$, so we get a $\mathbb{C}$-linear differential operator

$$\vartheta : \Lambda_e M \longrightarrow \Lambda_o M, \tag{34}$$

where $\Lambda_e = \Lambda^0 \oplus \Lambda^2,\ \Lambda_o = \Lambda^1$, regarded as complex line bundles, so

$$\text{Index } \vartheta = \frac{1}{2} \text{ Index } (d+\delta).$$

Ker $\vartheta$ is a one dimensional complex vector space:

$$\text{Ker } \vartheta = \text{ span } (1) = \text{ span } (*1).$$

The cokernel of $d+\delta$ in (32) consists of the space $\mathcal{H}^1(M)$ of (real) harmonic 1-forms. This is invariant under $*$, so it becomes a complex vector space:

$$\dim_{\mathbb{C}} \mathcal{H}^1(M) = \frac{1}{2} \dim_{\mathbb{R}} \mathcal{H}^1(M) = g. \tag{35}$$

Thus

$$\text{Index } \vartheta = \frac{1}{2}(2-2g) = 1-g. \tag{36}$$

When one applies an analysis parallel to that above, leading to (29), one gets

$$\text{Index } \vartheta = \frac{1}{4\pi} \int_M K \ dV. \tag{37}$$

Putting together (36) and (37), we have the Gauss-Bonnet formula, for a compact oriented surface.

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