Alternative Formulation of the Wiener Criterion

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Let Ω be a bounded open set in $\mathbb{R}^n$, $K = \mathbb{R}^n \setminus \Omega$. If $p \in \partial\Omega$, the Wiener criterion states that p is a regular point for the Dirichlet problem if and only if

$$\sum_{j=0}^{\infty} 2^{(n-2)j} \operatorname{cap}(K \cap \overline{B_j(p)}) = \infty, \tag{1}$$

where $\overline{B_j(p)} = \{x \in \mathbb{R}^n : |x - p| \le 2^{-j}\}$. If $n = 2$, you replace $2^{(n-2)j}$ by j in (1). For simplicity, we assume $n \ge 3$. If (1) holds, then for arbitrary $f \in C(\partial\Omega)$, the solution to the Dirichlet problem $\Delta u = 0$ on Ω "$u = f$ on $\partial\Omega$," constructed by the PWB method, has the property that $u(x) \to f(p)$ as $x \to p$, $x \in \Omega$. If (1) fails, there exists $f \in C(\partial\Omega)$ for which such convergence fails.

We desire to reformulate (1) in a way that does not make explicit reference to the complement K of Ω. Note that if $S \subset \mathbb{R}^n$ and we set $rS = \{rx : x \in S\}$, then, for compact S (assuming $n \ge 3$),

$$\operatorname{cap}(rS) = r^{n-2} \operatorname{cap}(S). \tag{2}$$

Hence

$$2^{(n-2)j} \operatorname{cap}(K \cap \overline{B_j(p)}) = \operatorname{cap}(\overline{B_1(rp)} \cap rK), \quad r = 2^{(n-2)j}. \tag{3}$$

Now, if $S \subset \overline{B_1(q)}$ is a compact set, we have

$$\operatorname{cap}(S) \approx \lambda_0^*(\overline{B_1(q)} \setminus S), \tag{4}$$

where

$$\lambda_0^*(\overline{B_1(q)} \setminus S) = \min(1, \lambda_0(\overline{B_1(q)} \setminus S)), \tag{5}$$

and

$$\begin{aligned} \lambda_0(\overline{B_1(q)} \setminus S) = \inf\{&\|\nabla u\|_{L^2} : u \in C^\infty(\overline{B_1(q)}), \\ &u = 0 \text{ on a neighborhood of } S,\ \|u\|_{L^2} = 1\} \end{aligned} \tag{6}$$

is the smallest eigenvalue of $-\Delta$ on $B_1(q) \setminus S$, with the Dirichlet boundary condition of ∂S and the Neumann boundary condition on $\partial B_1(q) \setminus S$. Compare Proposition 6.7 in Chapter 11 of [T].

Consequently, the criterion (1) translates to

$$\sum_{j=0}^{\infty} \lambda_0^*(\Omega_{j,p}) = \infty, \tag{7}$$

where

$$\Omega_{j,p} = 2^j(\Omega \cap \overline{B_j(p)}) \subset \overline{B_1(2^j p)}. \tag{8}$$

Note that

$$\begin{aligned} \lambda_0(\Omega_{j,p}) = \inf\{\|\nabla u\|_{L^2} : u \in C^\infty(\Omega_{j,p}), \\ u = 0 \text{ on a neighborhood of } 2^j\partial\Omega \cap \overline{\Omega}_{j,p},\ \|u\|_{L^2} = 1\}, \end{aligned} \tag{9}$$

which does not explicitly refer to the complement of $\overline{\Omega}$.

Reference

[T], M. Taylor, Partial Differential Equations, Vol. 2, Springer, New York, 1996 (2nd ed., 2011).