# The Weierstrass $\wp$-function as a Distribution on the Complex Torus $\mathbb{C}/\Lambda$, and its Fourier Series

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**Abstract.** We treat the Weierstrass $\wp$ function associated to a lattice $\Lambda \subset \mathbb{C}$ as a principal value distribution on the torus $\mathbb{C}/\Lambda$ and compute its Fourier coefficients. The computation of these coefficients for nonzero frequencies is straightforward, but quite pretty. The "constant term" is more mysterious. It leads to a non-absolutely convergent doubly infinite series, which we denote $\sigma_1$. This can be regarded as a version of an Eisenstein series, though as we discuss in §4 it differs from the "Eisenstein summation" of the series, as treated in [W]. Material from §3 on the Fourier series of elliptic functions arising from the Weierstrass zeta function leads to a formula connecting $\sigma_1$ with the Eisenstein series treated in [W], and thereby yields a rapidly convergent approximation to $\sigma_1$.

## Contents

## 1. Introduction and basic calculations

The Weierstrass $\wp$-function associated with a lattice $\Lambda = \{j\omega_1 + k\omega_2 : j, k \in \mathbb{Z}\}$ in the complex plane $\mathbb{C}$ is given by

$$(1.1) \qquad \wp_\Lambda(z) = \frac{1}{z^2} + \sum_{0 \neq \omega \in \Lambda} \Big( \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \Big),$$

which is absolutely convergent on $\mathbb{C} \setminus \Lambda$, thanks to

$$(1.2) \qquad \Big| \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \Big| \leq C \frac{|z|}{|\omega|^3}, \quad \text{for} \quad |\omega| \geq 2|z|.$$

It has the perdiodicity property

$$(1.3) \qquad \wp_\Lambda(z+\omega) = \wp_\Lambda(z), \quad \forall\, \omega \in \Lambda.$$

See, e.g., [T2], §30. We can regard $\wp_\Lambda(z)$ as a distribution on $\mathbb{C}$, in the principal value sense. Then $(\partial/\partial\overline{z})\wp_\Lambda$ is a distribution supported on $\Lambda$, which we can identify as follows.

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**Proposition 1.1.** *We have*

$$(1.4) \qquad \frac{\partial}{\partial \overline{z}} \wp_\Lambda = -\pi \sum_{\omega \in \Lambda} \frac{\partial}{\partial z} \delta_\omega.$$

*Proof.* By the periodicity (1.3), we see that

$$(1.5) \qquad \frac{\partial}{\partial \overline{z}} \wp_\Lambda(z) = \sum_{\omega \in \Lambda} \gamma(z - \omega),$$

where

$$(1.6) \qquad \begin{aligned} \gamma &= \frac{\partial}{\partial \overline{z}} \, \mathrm{PV} \, \frac{1}{z^2} \\ &= -\frac{\partial}{\partial \overline{z}} \frac{\partial}{\partial z} \frac{1}{z}. \end{aligned}$$

As is well known (cf. [T], Chapter 3, (4.53)),

$$(1.7) \qquad \frac{\partial}{\partial \overline{z}} \frac{1}{z} = \pi \delta,$$

so

$$(1.8) \qquad \gamma = -\pi \frac{\partial}{\partial z} \delta,$$

yielding (1.4).

Now a $\Lambda$-periodic function on $\mathbb{C}$ can be regarded as a function on the torus

$$(1.9) \qquad \mathbb{T}_\Lambda = \mathbb{C}/\Lambda,$$

and similarly a $\Lambda$-periodic distribution on $\mathbb{C}$ can be regarded as an element of $\mathcal{D}'(\mathbb{T}_\Lambda)$. Such objects have Fourier series, defined as follows. First, we have the dual lattice to $\Lambda$,

$$(1.10) \qquad \Gamma = \{\nu \in \mathbb{C} : \langle \nu, \omega \rangle \in 2\pi\mathbb{Z}, \ \forall \, \omega \in \Lambda\},$$

where $\langle \ , \ \rangle$ is the standard real inner product on $\mathbb{R}^2 \approx \mathbb{C}$, i.e.,

$$(1.11) \qquad \langle \nu, \omega \rangle = \mathrm{Re}(\overline{\nu}\omega).$$

Then the functions $e_\nu$, defined for $\nu \in \Gamma$ by

$$(1.12) \qquad e_\nu(z) = e^{i\langle \nu, z \rangle}, \quad z \in \mathbb{C},$$

satisfy

(1.13)
$$e_\nu(z + \omega) = e_\nu(z), \quad \forall \omega \in \Lambda,$$

and form an orthonormal basis for $L^2(\mathbb{T}_\Lambda)$, with inner product

(1.14)
$$(f, g)_{L^2} = \frac{1}{A(\Lambda)} \int_{\mathbb{T}_\Lambda} f(z)\overline{g(z)} \, dx \, dy,$$

with $z = x + iy$, $A(\Lambda)$ the area of $\mathbb{T}_\Lambda$. Given $u \in \mathcal{D}'(\mathbb{T}_\Lambda)$, we set

(1.15)
$$\hat{u}(\nu) = \frac{1}{A(\Lambda)} \langle u, \bar{e}_\nu \rangle, \quad \text{for} \ \ \nu \in \Gamma,$$

so, if $u \in L^2(\mathbb{T}_\Lambda)$,

(1.16)
$$\hat{u}(\nu) = (u, e_\nu)_{L^2},$$

and we have

$$\|u\|_{L^2}^2 = \sum_{\nu \in \Gamma} |\hat{u}(\nu)|^2,$$

and

(1.17)
$$u = \sum_{\nu \in \Gamma} \hat{u}(\nu) e_\nu,$$

with convergence in $L^2$-norm. More generally, if $u \in \mathcal{D}'(\mathbb{T}_\Lambda)$, then (1.17) holds, with convergence in the topology of $\mathcal{D}'(\mathbb{T}_\Lambda)$.

Our goal is to obtain a formula for $\widehat{\wp}_\Lambda(\nu)$, making use of (1.4). To do this, we want formulas relating the Fourier coefficients $\hat{u}(\nu)$ of $u \in \mathcal{D}'(\mathbb{T}_\Lambda)$ to those of $(\partial/\partial x)u$ and $(\partial/\partial y)u$, hence of

(1.18)
$$\frac{\partial}{\partial z} u = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right)u, \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} u = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)u.$$

With this in mind, we set

(1.19)
$$z = x + iy, \quad \nu = \alpha + i\beta, \quad x, y, \alpha, \beta \in \mathbb{R},$$

so $\langle \nu, z \rangle = \alpha x + \beta y$, and

(1.20)
$$e_\nu(z) = e^{i(\alpha x + \beta y)}.$$

Hence, for $u \in \mathcal{D}'(\mathbb{T}_\Lambda)$,

$$
\begin{aligned}
\widehat{\partial_x u}(\nu) &= \frac{1}{A(\Lambda)} \langle \partial_x u, \overline{e}_\nu \rangle = -\frac{1}{A(\Lambda)} \langle u, \partial_x \overline{e}_\nu \rangle = i\alpha \hat{u}(\nu), \\
\widehat{\partial_y u}(\nu) &= \frac{1}{A(\Lambda)} \langle \partial_y u, \overline{e}_\nu \rangle = -\frac{1}{A(\Lambda)} \langle u, \partial_y \overline{e}_\nu \rangle = i\beta \hat{u}(\nu).
\end{aligned}
$$
(1.21)

Hence

$$
\begin{aligned}
\widehat{\frac{\partial}{\partial z}} u(\nu) &= \frac{1}{2}(i\alpha + \beta)\hat{u}(\nu) = \frac{i}{2}\overline{\nu}\hat{u}(\nu), \\
\widehat{\frac{\partial}{\partial \overline{z}}} u(\nu) &= \frac{1}{2}(i\alpha - \beta)\hat{u}(\nu) = \frac{i}{2}\nu\hat{u}(\nu).
\end{aligned}
$$
(1.22)

Note that

$$
\hat{\delta}(\nu) = \frac{1}{A(\Lambda)} \langle \delta, \overline{e}_\nu \rangle = \frac{1}{A(\Lambda)}.
$$
(1.23)

Hence (1.4) yields

$$
\nu \widehat{\wp_\Lambda}(\nu) = -\frac{\pi}{A(\Lambda)} \overline{\nu}, \quad \nu \in \Gamma.
$$
(1.24)

We have the following conclusion.

**Proposition 1.2.** *We have the Fourier coefficients*

$$
\widehat{\wp_\Lambda}(\nu) = -\frac{\pi}{A(\Lambda)} \frac{\overline{\nu}}{\nu}, \quad \text{for all nonzero } \nu \in \Gamma.
$$
(1.25)

It remains to compute

$$
\begin{aligned}
\widehat{\wp_\Lambda}(0) &= \frac{1}{A(\Lambda)} \text{ PV} \int_{\mathbb{T}_\Lambda} \wp_\Lambda(z) \, dx \, dy \\
&= \frac{1}{A(\Lambda)} \lim_{\varepsilon \searrow 0} \int_{\mathbb{T}_\Lambda \setminus D_\varepsilon} \wp_\Lambda(x) \, dx \, dy,
\end{aligned}
$$
(1.26)

where $D_\varepsilon = \{z \in \mathbb{C} : \text{dist}(z, \Lambda) < \varepsilon\}$. Understanding this constant term will be a major focus for the rest of this paper.

In §2 we show that $\widehat{\wp_\Lambda}(0) = -\sigma_1$, where

$$
\begin{aligned}
\sigma_1 &= \text{PV} \sum_{\omega \in \Lambda \setminus 0} \frac{1}{\omega^2} \\
&= \lim_{R \to \infty} \sum_{\omega \in \Lambda_R \setminus 0} \frac{1}{\omega^2},
\end{aligned}
$$
(1.27)

where $\Lambda_R$ is (e.g.) the intersection of $\Lambda$ with $\{x + iy : |x|, |y| \leq R\}$. This is not an absolutely convergent series, and convergence in (1.27) is painfully slow. One goal, finally achieved in §5, will be to obtain an identity for $\sigma_1$ that allows for fast computation.

In §3 we consider the Fourier series of functions arising from

$$(1.28) \qquad \zeta_\Lambda(z) = \frac{1}{z} + \sum_{\omega \in \Lambda \setminus 0} \left( \frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right).$$

This is meromorphic with poles at $\Lambda$, but it is not $\Lambda$-periodic. Rather, we have

$$(1.29) \qquad \zeta_\Lambda(z + \omega) - \zeta_\Lambda(z) = \alpha_\Lambda(\omega), \quad \omega \in \Lambda,$$

for certain constants $\alpha_\Lambda(\omega)$. Consequently the functions

$$(1.30) \qquad \zeta_{a,\Lambda}(z) = \zeta_\Lambda(z - a) - \zeta_\Lambda(z + a)$$

are $\Lambda$-periodic. We show that

$$(1.31) \qquad \hat{\zeta}_{a,\Lambda}(\nu) = -\frac{4\pi}{A(\Lambda)} \frac{\sin\langle \nu, a \rangle}{\nu}, \quad \nu \in \Gamma \setminus 0,$$

and then tackle $\hat{\zeta}_{a,\Lambda}(0)$. We show that

$$(1.32) \qquad \hat{\zeta}_{a,\Lambda}(0) = -2\sigma_1 a - \frac{2\pi}{A(\Lambda)} \overline{a},$$

with $\sigma_1$ as in (1.27). Using this and (1.29), we deduce that

$$(1.33) \qquad \alpha_\Lambda(\omega) = \sigma_1 \omega + \frac{\pi}{A(\Lambda)} \overline{\omega}, \quad \omega \in \Lambda.$$

In §4 we discuss results from [W] on "Eisenstein summation" of $\omega^{-2}$ over $\Lambda \setminus 0$, which produces the quantity

$$(1.34) \qquad \tilde{\sigma}_1(\omega_1, \omega_2) = \lim_{N \to \infty} \sum_{k=-N}^{N} \left( \sum_{j \in \mathbb{Z}, (j,k) \neq (0,0)} (j\omega_1 + k\omega_2)^{-2} \right).$$

One advantage of such summation is that a rapidly convergent expansion is available; see (4.4). However, we show by examples that $\tilde{\sigma}_1$ differs from $\sigma_1$.

In §5 we make use of (1.33) to show that

$$(1.35) \qquad \sigma_1 = \tilde{\sigma}_1(\omega_1, \omega_2) - \frac{\pi}{A(\Lambda)} \frac{\overline{\omega}_1}{\omega_1}.$$

This allows for a fast computation of $\sigma_1$.

In Appendix A er define PV variants $\mathfrak{p}_\Lambda$ and $\mathfrak{z}_\Lambda$ of $\wp_\Lambda$ and $\zeta_\Lambda$, and record analogues of (1.26), (1.29), (1.32), and (1.33) for these functions.

## 2. The constant term

Here we look at $\widehat{\wp}_\Lambda(0)$, defined by (1.26). We start by noting some cases where we can say this is zero.

**Proposition 2.1.** *Assume $\Lambda$ is either a square lattice or a triangular lattice, i.e., either*

$$(2.1) \qquad \omega \in \Lambda \Longleftrightarrow i\omega \in \Lambda, \quad or \ \ \omega \in \Lambda \Longleftrightarrow e^{\pi i/3}\omega \in \Lambda.$$

*Then $\widehat{\wp}_\Lambda(0) = 0$.*

*Proof.* In such cases, we have $\wp_\Lambda(\tau z) = \tau^2 \wp_\Lambda(z)$, with $\tau = i$ or $e^{\pi i/3}$, respectively, and

$$
\begin{aligned}
\mathrm{PV} \int_{\mathbb{T}_\Lambda} \wp_\Lambda(z)\, dx\, dy &= \mathrm{PV} \int_{\mathbb{T}_\Lambda} \wp_\Lambda(\tau z)\, dx\, dy \\
&= \tau^2\, \mathrm{PV} \int_{\mathbb{T}_\Lambda} \wp_\Lambda(z)\, dx\, dy,
\end{aligned}
$$
(2.2)

which implies this integral is 0.

To proceed, let $\Omega \subset \mathbb{C}$ be the parallelogram with vertices

$$(2.3) \qquad -\frac{1}{2}(\omega_1 + \omega_2), \quad \frac{1}{2}(\omega_1 - \omega_2), \quad \frac{1}{2}(\omega_1 + \omega_2), \quad \frac{1}{2}(-\omega_1 + \omega_2),$$

where, recall, $\omega_1$ and $\omega_2$ are generators of $\Lambda$. From (1.1) we have

$$(2.4) \qquad \mathrm{PV} \int_{\mathbb{T}_\Lambda} \wp_\Lambda(z)\, dx\, dy = \mathrm{PV} \int_\Omega \frac{1}{z^2}\, dx\, dy + \sum_{0 \neq \omega \in \Lambda} \int_{\Omega_\omega} \left( \frac{1}{z^2} - \frac{1}{\omega^2} \right) dx\, dy,$$

an absolutely convergent series, by (1.2), where

$$\Omega_\omega = \Omega + \omega.$$

In particular, if $\mathcal{O} \subset \mathbb{C}$ is a neighborhood of 0 with piecewise smooth boundary, then

$$(2.5) \qquad \mathrm{PV} \int_{\mathbb{T}_\Lambda} \wp_\Lambda(z)\, dx\, dy = \lim_{R \to \infty} \left( \mathrm{PV} \int_{U_R} \frac{1}{z^2}\, dx\, dy - A(\Lambda) \sum_{0 \neq \omega \in \Lambda_R} \frac{1}{\omega^2} \right),$$

where

$$(2.6) \qquad \Lambda_R = \Lambda \cap R\mathcal{O}, \quad U_R = \bigcup_{\omega \in \Lambda_R} \Omega_\omega,$$

and $A(\Lambda) = \operatorname{Area} \mathbb{T}_\Lambda = \operatorname{Area} \Omega$. Note that

$$(2.7) \qquad \mathrm{PV} \int_{U_R} \frac{1}{z^2} \, dx \, dy - \mathrm{PV} \int_{R\mathcal{O}} \frac{1}{z^2} \, dx \, dy = O(R^{-1}).$$

Now let us assume that $\mathcal{O}$ satisfies the condition

$$(2.8) \qquad \mathrm{PV} \int_{\mathcal{O}} \frac{1}{z^2} \, dx \, dy = 0.$$

For example, $\mathcal{O}$ could be any neighborhood of 0, with piecewise smooth boundary, satisfying

$$(2.8) \qquad \zeta\mathcal{O} = \mathcal{O}, \quad \zeta = i \ \text{ or } \ e^{\pi i/3}.$$

In particular, $\mathcal{O}$ could be a square or a disk, centered at 0. We then have the following.

**Proposition 2.2.** *If $\Lambda_R$ is given by (2.6) and $\mathcal{O}$ contains 0 and satisfies (2.8), then*

$$(2.10) \qquad \widehat{\wp}_\Lambda(0) = -\lim_{R \to \infty} \sum_{0 \neq \omega \in \Lambda_R} \frac{1}{\omega^2}.$$

In connection with (2.10), we mention the quantities

$$(2.11) \qquad \sigma_n = \sum_{\omega \in \Lambda \backslash 0} \frac{1}{\omega^{2n}}.$$

For $n \geq 2$, this is an absolutely convergent series, and these numbers are significant in the theory of $\wp_\Lambda(z)$. Cf. (31.20)–(31.21) and (31.25)–(31.27) in [T2]. Many treatments of elliptic function theory say nothing about the case $n = 1$ of (2.11), except to remark that then the series is not absolutely convergent. A notable exception is [W], to which we will return in §4. The calculations above point to the intrinsic interest of the case $n = 1$. We write

$$(2.12) \qquad \sigma_1 = \mathrm{PV} \sum_{\omega \in \Lambda \backslash 0} \frac{1}{\omega^2} = \lim_{R \to \infty} \sum_{0 \neq \omega \in \Lambda_R} \frac{1}{\omega^2},$$

with $\Lambda_R$ as in Proposition 2.2, which then says

$$(2.13) \qquad \widehat{\wp}_\Lambda(0) = -\sigma_1.$$

## 3. Fourier series of Weierstrass zetas

The Weierstrass zeta function (not to be confused with the Riemann zeta function) associated with a lattice $\Lambda \subset \mathbb{C}$ is given by

$$(3.1) \qquad \zeta_\Lambda(z) = \frac{1}{z} + \sum_{\omega \in \Lambda \backslash 0} \left( \frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right).$$

See (30.14) of [T2]. The extra terms in the sum serve to make the series absolutely convergent on $\mathbb{C} \setminus \Lambda$, defining a meromorphic function, and we have

$$(3.2) \qquad \zeta_\Lambda'(z) = -\wp_\Lambda(z).$$

As a result,

$$(3.3) \qquad \zeta_\Lambda(z + \omega) - \zeta_\Lambda(z) = \alpha_\Lambda(\omega), \quad \forall \, \omega \in \Lambda,$$

but $\alpha_\Lambda(\omega)$ is not zero, though of course

$$(3.4) \qquad \alpha_\Lambda(\omega + \omega') = \alpha_\Lambda(\omega) + \alpha_\Lambda(\omega'), \quad \forall \, \omega, \omega' \in \Lambda.$$

We are led to consider, for $a, b \in \mathbb{C}$,

$$(3.5) \qquad \zeta_{a,b,\Lambda}(z) = \zeta_\Lambda(z - a) - \zeta_\Lambda(z - b),$$

obtaining a $\Lambda$-periodic function on $\mathbb{C}$, meromorphic with poles at $(a + \Lambda) \cup (b + \Lambda)$, all simple, if $a - b \notin \Lambda$. We have, by (1.7),

$$(3.6) \qquad \frac{\partial}{\partial \overline{z}} \zeta_{a,b,\Lambda} = \pi \sum_{\omega \in \Lambda} (\delta_{\omega + a} - \delta_{\omega + b}).$$

Parallel to (1.23),

$$(3.7) \qquad \hat{\delta}_{\omega + a}(\nu) = \frac{1}{A(\Lambda)} e^{-i \langle \nu, a \rangle},$$

so, via (1.22),

$$(3.8) \qquad \nu \hat{\zeta}_{a,b,\Lambda}(\nu) = -\frac{2\pi i}{A(\Lambda)} \left( e^{-i \langle \nu, a \rangle} - e^{-i \langle \nu, b \rangle} \right).$$

This yields the following analogue of Proposition 1.2.

**Proposition 3.1.** *We have the Fourier coefficients*

$$(3.9) \qquad \hat{\zeta}_{a,b,\Lambda}(\nu) = -\frac{2\pi i}{A(\Lambda)} \frac{1}{\nu} \left( e^{-i\langle \nu, a \rangle} - e^{-i\langle \nu, b \rangle} \right), \quad \forall \nu \in \Gamma \setminus 0.$$

It remains to compute

$$(3.10) \qquad \hat{\zeta}_{a,b,\Lambda}(0) = \frac{1}{A(\Lambda)} \iint\limits_{\Omega} \zeta_{a,b,\Lambda}(z) \, dx \, dy,$$

where $\Omega \subset \mathbb{C}$ is the period parallelogram, centered at 0, with vertices as in (2.3). It suffices to do this for $b = -a$, and from here on we will work with

$$(3.11) \qquad \begin{aligned} \zeta_{a,\Lambda}(z) &= \zeta_{a,-a,\Lambda}(z) \\ &= \frac{1}{z-a} - \frac{1}{z+a} + \sum_{\omega \in \Lambda \setminus 0} \left( \frac{1}{z-a-\omega} - \frac{1}{z+a-\omega} - \frac{2a}{\omega^2} \right). \end{aligned}$$

In this case, (3.9) becomes

$$(3.12) \qquad \hat{\zeta}_{a,\Lambda}(0) = -\frac{4\pi}{A(\Lambda)} \frac{\sin\langle \nu, a \rangle}{\nu}, \quad \forall \nu \in \Gamma \setminus 0,$$

and we desire to analyze

$$(3.13) \qquad \hat{\zeta}_{a,\Lambda}(0) = \frac{1}{A(\Lambda)} \iint\limits_{\Omega} \zeta_{a,\Lambda}(z) \, dx \, dy.$$

One route to this calculation is to apply $\partial/\partial a$ and $\partial/\partial \overline{a}$ to $\zeta_{a,\Lambda}(z)$, using

$$(3.14) \qquad \frac{\partial}{\partial z} \zeta_\Lambda = -\wp_\Lambda(z), \quad \frac{\partial}{\partial \overline{z}} \zeta_\Lambda = \pi \sum_{\omega \in \Lambda} \delta_\omega.$$

We get

$$(3.15) \qquad \begin{aligned} \frac{\partial}{\partial a} \zeta_{a,\Lambda}(z) &= \wp_\Lambda(z-a) + \wp_\Lambda(z+a), \\ \frac{\partial}{\partial \overline{a}} \zeta_{a,\Lambda}(z) &= -\pi \sum_{\omega \in \Lambda} (\delta_{\omega+a} + \delta_{\omega-a}), \end{aligned}$$

hence

$$(3.16) \qquad \begin{aligned} \frac{\partial}{\partial a} \hat{\zeta}_{a,\Lambda}(0) &= \frac{1}{A(\Lambda)} \mathrm{PV} \iint\limits_{\mathbb{T}_\Lambda} \left( \wp_\Lambda(z-a) + \wp_\Lambda(z+a) \right) dx \, dy \\ &= 2\widehat{\wp}_\Lambda(0) \\ &= -2\sigma_1, \end{aligned}$$

and

$$(3.17) \qquad \frac{\partial}{\partial \bar{a}} \hat{\zeta}_{a,\Lambda}(0) = -\frac{2\pi}{A(\Lambda)}.$$

Writing $a = \alpha + i\beta$, $\alpha, \beta \in \mathbb{R}$, we have

$$(3.18) \qquad \frac{\partial}{\partial \alpha} = \frac{\partial}{\partial a} + \frac{\partial}{\partial \bar{a}}, \quad \frac{\partial}{\partial \beta} = i\left(\frac{\partial}{\partial a} - \frac{\partial}{\partial \bar{a}}\right).$$

Hence

$$(3.19) \qquad \begin{aligned} \frac{\partial}{\partial \alpha} \hat{\zeta}_{a,\Lambda}(0) &= -2\sigma_1 - \frac{2\pi}{A(\Lambda)}, \\ \frac{\partial}{\partial \beta} \hat{\zeta}_{a,\Lambda}(0) &= i\left(-2\sigma_1 + \frac{2\pi}{A(\Lambda)}\right). \end{aligned}$$

Now $\zeta_{0,\Lambda} \equiv 0$, so

$$(3.20) \qquad \begin{aligned} \hat{\zeta}_{a,\Lambda}(0) &= -\left(2\sigma_1 + \frac{2\pi}{A(\Lambda)}\right)\alpha - i\left(2\sigma_1 - \frac{2\pi}{A(\Lambda)}\right)\beta \\ &= -2\sigma_1(\alpha + i\beta) - \frac{2\pi}{A(\Lambda)}(\alpha - i\beta). \end{aligned}$$

We record the conclusion.

**Proposition 3.2.** *We have*

$$(3.21) \qquad \hat{\zeta}_{a,\Lambda}(0) = -2\sigma_1 a - \frac{2\pi}{A(\Lambda)}\bar{a}.$$

We can relate the term $\alpha_\Lambda(\omega)$ in (3.3) to $\sigma_1$, as follows. From (3.3) we obtain

$$(3.22) \qquad \zeta_{a+\omega,\Lambda}(z) = \zeta_{a,\Lambda}(z) - 2\alpha_\Lambda(\omega), \quad \omega \in \Lambda,$$

which gives

$$(3.23) \qquad \hat{\zeta}_{a+\omega,\Lambda}(0) = \hat{\zeta}_{a,\Lambda}(0) - 2\alpha_\Lambda(\omega).$$

Now using both (3.21) and its analogue with $a$ replaced by $a + \omega$ gives

$$(3.24) \qquad -2\sigma_1(a+\omega) - \frac{2\pi}{A(\Lambda)}(\bar{a} + \bar{\omega}) = -2\sigma_1 a - \frac{2\pi}{A(\Lambda)}\bar{a} - 2\alpha_\Lambda(\omega),$$

and cancelling appropriate terms yields the conclusion

$$(3.25) \qquad \alpha_\Lambda(\omega) = \sigma_1 \omega + \frac{\pi}{A(\Lambda)}\bar{\omega}, \quad \omega \in \Lambda.$$

In connection with this, we mention that

$$(3.26) \qquad \omega \in \Lambda, \ \frac{\omega}{2} \notin \Lambda \Longrightarrow \alpha_\Lambda(\omega) = 2\zeta_\Lambda\left(\frac{\omega}{2}\right).$$

See [T2], §30. Thus computations of $\sigma_1$ and of $\zeta_\Lambda(\omega/2)$, for some $\omega$ satisfying the hypotheses of (3.26), are equivalent problems.

If $\omega_1$ and $\omega_2$ generate $\Lambda$ and $\mathrm{Im}(\omega_2/\omega_1) > 0$, then (3.25) implies

$$
\begin{aligned}
(3.27) \qquad \alpha_\Lambda(\omega_1)\omega_2 - \alpha_\Lambda(\omega_2)\omega_1 &= \frac{\pi}{A(\Lambda)}(\overline{\omega}_1\omega_2 - \overline{\omega}_2\omega_1) \\
&= 2\pi i \, \frac{\mathrm{Im}(\overline{\omega}_1\omega_2)}{A(\Lambda)} \\
&= 2\pi i.
\end{aligned}
$$

This result also follows directly from

$$(3.28) \qquad 2\pi i = \int_{\partial\Omega} \zeta_\Lambda(z)\,dz = \alpha_\Lambda(\omega_1)\omega_2 - \alpha_\Lambda(\omega_2)\omega_1,$$

where $\Omega$ is a period parallelogram centered at 0; cf. [T2], §30, Exercise 5. Of course, the calculation (3.27) loses the contribution of $\sigma_1\omega$ to (3.25). On the other hand, we can complement (3.27) with

$$
\begin{aligned}
(3.29) \qquad \alpha_\Lambda(\omega_1)\overline{\omega}_2 - \alpha_\Lambda(\omega_2)\overline{\omega}_1 &= (\omega_1\overline{\omega}_2 - \omega_2\overline{\omega}_1)\sigma_1 \\
&= -2iA(\Lambda)\sigma_1.
\end{aligned}
$$

This can also be shown directly by applying Green's theorem to (3.2), i.e., to

$$\frac{\partial}{\partial z}\zeta_\Lambda(z) = -\wp_\Lambda(z),$$

and using (3.3). Conversely, we can solve (3.27) and (3.29) for $\alpha_\Lambda(\omega_1)$ and $\alpha_\Lambda(\omega_2)$, obtaining another derivation of (3.25), this one not involving the calculations (3.15)–(3.21).

## 4. Eisenstein summation vs PV summation

In [W] there is a discussion of "Eisenstein summation" of $\omega^{-2}$ over $\Lambda \setminus 0$, when $\Lambda \subset \mathbb{C}$ is a lattice generated by $(\omega_1, \omega_2)$. This is given by

$$(4.1) \qquad \tilde{\sigma}_1(\omega_1, \omega_2) = \lim_{N \to \infty} \sum_{k=-N}^{N} \Big( \sum_{j \in \mathbb{Z}, (j,k) \neq (0,0)} (j\omega_1 + k\omega_2)^{-2} \Big),$$

the inner sum being absolutely convergent for each $k$. Convergence as $N \to \infty$ is established on pp. 18–19 of [W]. Unlike $\sigma_1$ in (3.1)–(3.2), this quantity depends on the choice of generators. As shown on p. 21 of [W], if one has another pair of generators, $(\omega_1', \omega_2')$, satisfying

$$(4.2) \qquad (\omega_1' \; \omega_2') = (\omega_1 \; \omega_2)A, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G\ell(2, \mathbb{Z}), \quad \det A = \pm 1,$$

then

$$(4.3) \qquad \tilde{\sigma}_1(\omega_1', \omega_2') = \tilde{\sigma}_1(\omega_1, \omega_2) - \frac{2\pi i}{\omega_1 \omega_1'} \Big( \operatorname{sgn} \operatorname{Im} \frac{\omega_2}{\omega_1} \Big) c.$$

By (11) on p. 20 of [W], one has the series

$$(4.4) \qquad \tilde{\sigma}_1(\omega_1, \omega_2) = \frac{4\pi^2}{\omega_1^2} \Big( \frac{1}{12} - 2 \sum_{N=1}^{\infty} \gamma_1(N) q^N \Big),$$

where

$$(4.5) \qquad q = e^{2\pi i \tau}, \quad \tau = \frac{\omega_2}{\omega_1},$$

provided $\operatorname{Im} \tau > 0$, and

$$(4.6) \qquad \gamma_1(N) = \sum_{k|N} k.$$

The series (4.4) is typically rapidly convergent. For example, if $\omega_1 = 1$, $\omega_2 = ai$, with $a > 0$, then $q = e^{-2\pi a}$, so

$$(4.7) \qquad \begin{aligned} a = 1 &\Longrightarrow q = e^{-2\pi} \approx 1.867433 \times 10^{-3}, \\ a = 2 &\Longrightarrow q = e^{-4\pi} \approx 3.487343 \times 10^{-6}. \end{aligned}$$

From these estimates, we can easily see that

$$(4.8) \qquad\qquad \tilde{\sigma}_1(1, i) \neq 0,$$

whereas, by Propositions 2.1–2.2, $\sigma_1 = 0$ for the lattice generated by $1$ and $i$. Let us pursue this a little further. A direct consequence of (4.1) is that

$$(4.9) \qquad\qquad \tilde{\sigma}_1(-i, 1) = -\tilde{\sigma}_1(1, i),$$

a result that also follows from (4.4). On the other hand, (4.3) yields

$$(4.10) \qquad\qquad \tilde{\sigma}_1(-i, 1) = \tilde{\sigma}_1(1, i) - 2\pi.$$

It follows that

$$(4.11) \qquad\qquad \tilde{\sigma}_1(1, i) = \pi.$$

Further comparison with (4.4), for $\omega_1 = 1, \omega_2 = i$, yields the curious identity

$$(4.12) \qquad\qquad \sum_{N=1}^{\infty} \gamma_1(N) e^{-2\pi N} = \frac{1}{8}\left(\frac{1}{3} - \frac{1}{\pi}\right).$$

A numerical check verifies that both sides are

$$(4.13) \qquad\qquad \approx .00187793\cdots.$$

We next consider the triangular lattice, with generators

$$(4.14) \qquad\qquad \omega_1 = 1, \quad \omega_2 = e^{\pi i/3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i.$$

Another set of generators is

$$(4.15) \qquad\qquad \omega_1' = e^{-\pi i/3} = 1 - \omega_2, \quad \omega_2' = 1,$$

related to the first set by

$$(4.16) \qquad\qquad (e^{-\pi i/3} \ 1) = (1 \ e^{\pi i/3}) \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

A direct use of (4.1) gives

$$(4.17) \qquad\qquad \tilde{\sigma}_1(e^{-\pi i/3}, 1) = e^{2\pi i/3}\tilde{\sigma}_1(1, e^{\pi i/3}),$$

while (4.3) yields

$$(4.18) \qquad \tilde{\sigma}_1(e^{-\pi i/3}, 1) = \tilde{\sigma}_1(1, e^{\pi i/3}) + 2\pi i e^{\pi i/3}.$$

Comparison of (4.17) and (4.18) yields

$$(4.19) \qquad \tilde{\sigma}_1(1, e^{\pi i/3}) = \frac{\pi}{\sin \pi/3} = \frac{2}{\sqrt{3}}\,\pi.$$

Again, by Propositions 2.1–2.2, $\sigma_1 = 0$ for the lattice generated by 1 and $e^{\pi i/3}$.

The upshot of these calculations is that the relation between the Eisenstein summation (4.1) and the PV summation (2.12) of $\omega^{-2}$ over $\Lambda \backslash 0$ requires elucidation. In light of the fast approximation (4.4) to (4.1), it is desirable to understand this relation better.

We will analyze the difference between these two sums in §5. First we collect a few more useful facts.

The text [W] also studied the variant of $\zeta_\Lambda$, given by

$$(4.20) \qquad E_1(z) = \lim_{N \to \infty} \sum_{k=-N}^{N} \left( \lim_{M \to \infty} \sum_{j=-M}^{M} \frac{1}{z - j\omega_1 - k\omega_2} \right).$$

As with (4.1), this depends on the choice of generators $\{\omega_1, \omega_2\}$, but we suppress this from the notation here. As noted on p. 16 of [W], we have the periodicity

$$(4.21) \qquad E_1(z + j\omega_1) = E_1(z).$$

Furthermore, by (9) on p. 20 of [W], we have, for $z$ close to 0,

$$E_1(z) = \frac{1}{z} - \tilde{\sigma}_1(\omega_1, \omega_2)z - \sum_{k=2}^{\infty} \sigma_k z^{2k-1}.$$

Meanwhile, $\zeta_\Lambda(z)$ has a similar form, except that the coefficient of $z$ is 0. It follows that

$$(4.22) \qquad \zeta_\Lambda(z) = E_1(z) + \tilde{\sigma}_1(\omega_1, \omega_2)z.$$

These identities will lead us in §5 to the relation between $\sigma_1$ and $\tilde{\sigma}_1(\omega_2, \omega_2)$.

## 5. Formula for $\sigma_1$

We can relate $\sigma_1$ to $\tilde{\sigma}_1(\omega_1, \omega_2)$ via (4.21)–(4.22), which yield

$$
(5.1) \qquad \zeta_\Lambda(z + \omega_1) = \zeta_\Lambda(z) + \tilde{\sigma}_1(\omega_1, \omega_2)\omega_1.
$$

By comparison, we have from (3.3) and (3.25) that

$$
(5.2) \qquad
\begin{aligned}
\zeta_\Lambda(z + \omega_1) &= \zeta_\Lambda(z) + \alpha_\Lambda(\omega_1) \\
&= \zeta_\Lambda(z) + \sigma_1 \omega_1 + \frac{\pi}{A(\Lambda)}\overline{\omega}_1.
\end{aligned}
$$

This yields the following formula for $\sigma_1$, which is useful for computation in light of the expansion (4.4).

**Proposition 5.1.** *If the lattice $\Lambda$ is generated by $\omega_1$ and $\omega_2$, then*

$$
(5.3) \qquad \sigma_1 = \tilde{\sigma}_1(\omega_1, \omega_2) - \frac{\pi}{A(\Lambda)}\frac{\overline{\omega}_1}{\omega_1}.
$$

One readily verifies that (5.3) is consistent with (4.11) and (4.19), plus the observation that $\sigma_1 = 0$ for both lattices involved there. We can also recover the formula (4.3) from (5.3), as follows. If $\{\omega_1', \omega_2'\}$ also generate $\Lambda$, then

$$
(5.4) \qquad \sigma_1 = \tilde{\sigma}_1(\omega_1', \omega_2') - \frac{\pi}{A(\Lambda)}\frac{\overline{\omega}_1'}{\omega_1'},
$$

so

$$
(5.5) \qquad
\begin{aligned}
\tilde{\sigma}_1(\omega_1, \omega_2) - \tilde{\sigma}_1(\omega_1', \omega_2') &= \frac{\pi}{A(\Lambda)}\left(\frac{\overline{\omega}_1}{\omega_1} - \frac{\overline{\omega}_1'}{\omega_1'}\right) \\
&= \frac{2\pi i}{\omega_1 \omega_1'}\frac{\mathrm{Im}(\overline{\omega}_1 \omega_1')}{A(\Lambda)},
\end{aligned}
$$

which is equivalent to (4.3).

To end this section, we discuss the numerical calculation of $\sigma_1$ in the special case of the lattice with generators

$$
(5.6) \qquad \omega_1 = 1, \quad \omega_2 = 2i.
$$

In such a case, we have $A(\Lambda) = 2$, so

$$
(5.7) \qquad \sigma_1 = \tilde{\sigma}_1(1, 2i) - \frac{\pi}{2},
$$

and furthermore (4.4) holds with $\omega_1 = 1$, $q = e^{-4\pi}$. Since $\gamma_1(1) = 1, \gamma_1(2) = 3, \gamma_1(3) = 4$, we have

$$(5.8) \qquad \tilde{\sigma}_1(1, 2i) \approx \frac{\pi^2}{3}(1 - 24q - 72q^2 - 96q^3),$$

with an error less than $10^{-20}$. A double precision calculation in C on a Mac gives (in the blink of an eye)

$$(5.9) \qquad \sigma_1 \approx 1.7187964545059,$$

most likely accurate to at least 12 digits after the decimal point.

By contrast, one can take the approximation from (2.12),

$$(5.10) \qquad \sigma_1 = \lim_{R \to \infty} S_R, \quad S_R = \sum_{\omega \in \Lambda_R \setminus 0} \frac{1}{\omega^2},$$

with

$$(5.11) \qquad \Lambda_R = \{j + 2ki : |j| \leq R, |2k| \leq R\}.$$

A C program on a Mac yields

$$(5.12) \qquad \begin{aligned} S_{4000} &= 1.718671 \\ S_{8000} &= 1.718734 \\ S_{16000} &= 1.718765. \end{aligned}$$

These figures indicate an error in $S_R$ that is $O(R^{-1})$, as one would expect from (2.7). The calculation of $S_{16000}$ took about half a billion floating point operations, and was performed in a handful of seconds. As we can see from (5.9), it differs from $\sigma_1$ by about .00003. Increasing the accuracy of $S_R$ by $k$ digits would require $10^{2k}$ times as many operations. Clearly the brute force approach to summing (5.10) would not yield 12 digits of accuracy, even after running for many years. Therefore the formulas (5.3) and (4.4) are especially welcome.

## A. PV variants of $\wp_\Lambda$ and $\zeta_\Lambda$

As [W] treated "Eisenstein sum" variants of $\wp_\Lambda$ and $\zeta_\Lambda$, we are motivated to introduce "PV" variants of these functions. Now that we have seen that

$$\text{(A.1)} \qquad \lim_{R\to\infty} \sum_{\omega\in\Lambda_R\backslash 0} \frac{1}{\omega^2} = \text{PV} \sum_{\omega\in\Lambda\backslash 0} \frac{1}{\omega^2} = \sigma_1$$

exists, given

$$\text{(A.2)} \qquad \Lambda_R = \Lambda \cap R\mathcal{O},$$

with $\mathcal{O}$ as described in §2, if we further require that $\mathcal{O} = -\mathcal{O}$, e.g., by requiring

$$\text{(A.3)} \qquad \mathcal{O} = i\mathcal{O},$$

we can write

$$\text{(A.4)} \qquad \begin{aligned} \wp_\Lambda(z) &= \mathfrak{p}_\Lambda(z) - \sigma_1, \\ \zeta_\Lambda(z) &= \mathfrak{z}_\Lambda(z) + \sigma_1 z, \end{aligned}$$

where we define

$$\text{(A.5)} \qquad \mathfrak{p}_\Lambda(z) = \lim_{R\to\infty} \sum_{\omega\in\Lambda_R} \frac{1}{(z-\omega)^2} = \text{PV} \sum_{\omega\in\Lambda} \frac{1}{(z-\omega)^2},$$

and

$$\text{(A.6)} \qquad \mathfrak{z}_\Lambda(z) = \lim_{R\to\infty} \sum_{\omega\in\Lambda_R} \frac{1}{z-\omega} = \text{PV} \sum_{\omega\in\Lambda} \frac{1}{z-\omega}.$$

Given these definitions of $\mathfrak{p}_\Lambda$ and $\mathfrak{z}_\Lambda$, (3.3) and (3.25) are equivalent to

$$\text{(A.7)} \qquad \mathfrak{z}_\Lambda(z+\omega) - \mathfrak{z}_\Lambda(z) = \beta_\Lambda(\omega) = \frac{\pi}{A(\Lambda)}\overline{\omega}, \quad \omega\in\Lambda,$$

Proposition 3.2 is equivalent to the formula

$$\text{(A.8)} \qquad \hat{\mathfrak{z}}_{a,\Lambda}(0) = -\frac{2\pi}{A(\Lambda)}\overline{a},$$

for the Fourier coefficient at 0 of the elliptic function

$$\text{(A.9)} \qquad \begin{aligned} \mathfrak{z}_{a,\Lambda}(z) &= \mathfrak{z}_\Lambda(z-a) - \mathfrak{z}_\Lambda(z+a) \\ &= \zeta_{a,\Lambda}(z) + 2a\sigma_1, \end{aligned}$$

and Proposition 2.2 is equivalent to

$$\text{(A.10)} \qquad \hat{\mathfrak{p}}_\Lambda(0) = 0.$$

Note that the formulas (A.7), (A.8), and (A.10) have a simpler form than their counterparts for $\zeta_\Lambda$ and $\wp_\Lambda$.

## References

[T] M. Taylor, Partial Differential Equations, Vols. 1–3, Springer-Verlag, New York, 1996 (2nd ed., 2011).

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