Zero Sets of Random Sections of Vector Bundles

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Preliminary Notes

Contents

1. Introduction
2. Formulas for the expected $(n-k)$-dimensional area of $\mathcal{Z}(F_\omega)$
3. The Gaussian measure Γ_x on $E_x \oplus \mathcal{L}(T_x, E_x)$
4. Heat asymptotics and zero set asymptotics
5. Other directions
A. Remarks on $\gamma(n,k)$

1. Introduction

Let M be a compact, n-dimensional Riemannian manifold. Let $E \to M$ be a smooth, rank k, real vector bundle, such that the fibers E_x are equipped with a smoothly varying inner product. Let $L : C^\infty(M,E) \to C^\infty(M,E)$ be a strongly elliptic, self adjoint differential operator. We assume L has order 2 and is positive semi-definite (though other assumptions can be used). An example would be $L = -\Delta$, where Δ is the Hodge Laplacian on ℓ-forms, $E_x = \Lambda^\ell T_x^*$, and $k = \binom{n}{\ell}$. We will want to restrict attention to cases where $k \le n$, which would require $\ell \in \{0, 1, n-1, n\}$.

The space $L^2(M,E)$ has an orthonormal basis $\{f_j : j \ge 0\}$, consisting of eigenfunctions of L:

$$Lf_j = \lambda_j^2 f_j. \tag{1.1}$$

We take a function $\varphi : [0,\infty) \to \mathbb{R}$, assumed to be rapidly decreasing at infinity, and form the following random field:

$$F_\omega(x) = \sum_{k\ge 0} \varphi(\lambda_k) X_k(\omega) f_k(x), \tag{1.2}$$

where $\{X_k\}$ are independent, identically distributed Gaussian random variables, on some auxiliary probability space (Ω, μ), with mean 0 and variance 1. Note that we can take

$$\|F_\omega\|^2_{H^s(M)} = \sum_{k\ge 0} (1+\lambda_k^2)^s \varphi(\lambda_k)^2 |X_k(\omega)|^2, \tag{1.3}$$

hence

$$\mathbb{E}\Big(\|F\|^2_{H^s(M)}\Big) = \sum_{k\ge 0}(1+\lambda_k^2)^s\varphi(\lambda_k)^2 < \infty, \quad \forall\, s\in\mathbb{R}, \tag{1.4}$$

so ω-a.e. F_ω is in $C^\infty(M,E)$.

Our goal is to study the set

$$\mathcal{Z}(F_\omega) = \{x\in M : F_\omega(x)=0\}. \tag{1.5}$$

We claim that, for a.e. $\omega\in\Omega$ (and suitable φ), this has Hausdorff dimension $n-k$, and we seek a formula for the expectation of its $(n-k)$-dimensional Hausdorff measure. Once in possession of such a formula, we take a one parameter family of functions φ_τ and consider asymptotics in τ.

One key ingredient in our calculation will be the identity

$$\sum_{k\ge 0}\psi(\lambda_k) f_k(x)\otimes f_k(x) = K_\psi(x,y), \tag{1.6}$$

where $K_\psi(x,y)$ is the integral kernel of the operator $\psi(\sqrt{L})$, i.e.,

$$\psi(\sqrt{L})g(x) = \int_M K_\psi(x,y)g(y)\,dV(y). \tag{1.7}$$

Note that $K_\psi(x,y)\in E_x\otimes E_y \approx \mathcal{L}(E_y,E_x)$, the latter isomorphism via the inner product on E_y.

2. Formulas for the expected $(n-k)$-dimensional area of $\mathcal{Z}(F_\omega)$

Assume $F_\omega \in C^\infty(M,E)$ and that 0 is a regular value of F_ω.

Proposition 2.1. *In such a case, the $(n-k)$-dimensional Hausdorff measure of $\mathcal{Z}(F_\omega)$ satisfies*

$$\mathcal{H}^{n-k}\mathcal{Z}(F_\omega) = \lim_{\varepsilon\to 0} \int_M \eta_\varepsilon(F_\omega(x)) L(\nabla F_\omega(x))\, dV(x), \tag{2.1}$$

where, for $v \in E_x$,

$$\eta_\varepsilon(v) = \begin{cases} V_k^{-1}\varepsilon^{-k} & \text{if } |v| \le \varepsilon, \\ 0 & \text{if } |v| > \varepsilon, \end{cases} \tag{2.2}$$

with V_k the volume of the unit ball in $\mathbb{R}^k$, and, for $A \in \mathcal{L}(T_x, E_x)$,

$$L(A) = (\det AA^t)^{1/2}. \tag{2.3}$$

Here ∇F_ω is defined by a choice of connection on E. Note however that $\nabla F_\omega(x_0)$ is independent of the choice of such a connection for $x_0 \in \mathcal{Z}(F_\omega)$, so two such connections yield close results for x close to $\mathcal{Z}(F_\omega)$. Hence the right side of (2.1) is independent of such a choice.

Proof of Proposition 2.1. Take $x_0 \in \mathcal{Z}(F_\omega)$ and pick geodesic coordinates centered at x_0. Identify $T_{x_0}\mathcal{Z}(F_\omega)$ with $\mathbb{R}^{n-k}$ and its orthogonal complement $N_{x_0}\mathcal{Z}(F_\omega)$ with $\mathbb{R}^k$. The key is to identify, to leading order in ε, the k-dimensional measure of

$$\{x \in N_{x_0}\mathcal{Z}(F_\omega) : |F_\omega(x)| \le \varepsilon\}, \tag{2.3A}$$

or equivalently (to leading order) the k-dimensional measure of

$$\{x \in N_{x_0}\mathcal{Z}(F_\omega) : |Ax| \le \varepsilon\}, \tag{2.3B}$$

where

$$A = \nabla F_\omega(x_0) : T_{x_0}M \longrightarrow E_{x_0}, \tag{2.3C}$$

can be identified with

$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^k, \quad A = (0\ B), \quad B : \mathbb{R}^k \to \mathbb{R}^k, \tag{2.3D}$$

and we want to evaluate the k-dimensional volume of

$$\{u \in \mathbb{R}^k : |Bu| \le \varepsilon\}. \tag{2.3E}$$

Now applying B multiplies volumes of subsets of $\mathbb{R}^k$ by a factor of

$$|\det B| = (\det AA^t)^{1/2}, \tag{2.3F}$$

so the volume of (2.3E), hence of (2.3B), is $V_k \varepsilon^k |\det B|^{-1}$, and to leading order this is the volume of (2.3A). The factor $L(\nabla F_\omega(x))$ needs to cancel out the extra factor of $|\det B|^{-1}$, to leading order, and this leads to (2.3).

Let us denote the integral on the right side of (2.1) by

$$Z_\varepsilon(F_\omega) = \int_M \eta_\varepsilon(F_\omega(x)) L(\nabla F_\omega(x))\, dV(x). \tag{2.4}$$

From here, we have

$$\mathbb{E}\, Z_\varepsilon(F) = \int_M \mathbb{E}\Big[\eta_\varepsilon(F(x)) L(\nabla F(x))\Big]\, dV(x). \tag{2.5}$$

By (1.2),

$$\begin{aligned} G_\omega(x) &= (F_\omega(x), \nabla F_\omega(x)) \\ &= \sum_k \varphi(\lambda_k) X_k(\omega)(f_k(x), \nabla f_k(x)) \end{aligned} \tag{2.6}$$

is, for each x, a Gaussian random variable, taking values in $E_x \oplus \mathcal{L}(T_x, E_x)$, with mean zero. This Gaussian random variable hence induces a Gaussian probability measure Γ_x on $E_x \oplus \mathcal{L}(T_x, E_x)$, and

$$\mathbb{E}\Big[\eta_\varepsilon(F(x)) L(\nabla F(x))\Big] = \int_{E_x \oplus \mathcal{L}(T_x, E_x)} \eta_\varepsilon(v) L(A)\, d\Gamma_x(v, A). \tag{2.7}$$

Later on we will show that this Gaussian measure has the form

$$d\Gamma_x(v, A) = c_\varphi(x) e^{-\gamma_{\varphi,x}(v,A)}\, dv\, dA, \tag{2.8}$$

where $\gamma_{\varphi,x}(v, A)$ is a positive definite quadratic form in (v, A). Consequently,

$$\begin{aligned} &\lim_{\varepsilon \to 0} \mathbb{E}\Big[\eta_\varepsilon(F(x)) L(\nabla F(x))\Big] \\ &\qquad = c_\varphi(x) \int_{\mathcal{L}(T_x, E_x)} e^{-\gamma_{\varphi,x}(0,A)} L(A)\, dA. \end{aligned} \tag{2.9}$$

Combining this with (2.1) and (2.5) gives the following variant of the *Kac-Rice formula*:

$$\mathbb{E}\Big[\mathcal{H}^{n-k}\mathcal{Z}(F)\Big] = \int_M \int_{\mathcal{L}(T_x,E_x)} c_\varphi(x) e^{-\gamma_{\varphi,x}(0,A)} L(A)\, dA\, dV(x). \tag{2.10}$$

Our next task, pursued in §§3–4, is to derive information on the integrand on the right side of (2.10), which will follow from information on the Gaussian measure (2.8).

Remark. These results can be localized. If $U \subset M$ is open and smoothly bounded, then

$$\mathbb{E}\Big[\mathcal{H}^{n-k}(U \cap \mathcal{Z}(F))\Big] = \int_U \int_{\mathcal{L}(T_x,E_x)} c_\varphi(x) e^{-\gamma_{\varphi,x}(0,A)} L(A)\, dA\, dV(x). \tag{2.11}$$

3. The Gaussian measure Γ_x on $E_x \oplus \mathcal{L}(T_x, E_x)$

As seen in §2, for each $x \in M$,

$$G_\omega(x) = \sum_k \varphi(\lambda_k) X_k(\omega) u_k(x), \quad u_k(x) = (f_k(x), \nabla f_k(x)), \tag{3.1}$$

is a Gaussian random variable, taking values in $E_x \oplus \mathcal{L}(T_x, E_x)$, with mean 0, and this random variable then induces a Gaussian probability measure Γ_x on $E_x \oplus \mathcal{L}(T_x, E_x)$. Our next goal is to see when Γ_x has the form

$$d\Gamma_x(v, A) = c_\varphi(x) e^{-\gamma_{\varphi,x}(v,A)}\, dv\, dA, \tag{3.1A}$$

and analyze $c_\varphi(x)$ and $\gamma_{\varphi,x}(v, A)$, which is a quadratic form in (v, A). We use the fact that Γ_x is uniquely determined by the covariance of $G_\omega(x)$, which we proceed to analyze. We have

$$\begin{aligned} \mathbb{E}(G(x) \otimes G(y)) &= \sum_{j,k} \mathbb{E}(X_j, X_k) \varphi(\lambda_j) \varphi(\lambda_k)\, u_j(x) \otimes u_k(y) \\ &= \sum_k \varphi(\lambda_k)^2\, u_k(x) \otimes u_k(y). \end{aligned} \tag{3.2}$$

We can expand out $u_k(x) \otimes u_k(y)$ as

$$u_k(x) \otimes u_k(y) = \begin{pmatrix} f_k(x) \otimes f_k(y) & f_k(x) \otimes \nabla f_k(y) \\ \nabla f_k(x) \otimes f_k(y) & \nabla f_k(x) \otimes \nabla f_k(y) \end{pmatrix}. \tag{3.3}$$

Now, as seen in (1.6),

$$\sum_k \varphi(\lambda_k)^2\, f_k(x) \otimes f_k(y) = K_{\varphi^2}(x, y), \tag{3.4}$$

the integral kernel of $\varphi(\sqrt{L})^2$. It follows that

$$\mathbb{E}(G(x) \otimes G(x)) = \begin{pmatrix} K_{\varphi^2}(x,x) & \nabla_2 K_{\varphi^2}(x,x) \\ \nabla_1 K_{\varphi^2}(x,x) & \nabla_1 \nabla_2 K_{\varphi^2}(x,x) \end{pmatrix}, \tag{3.5}$$

where $\nabla_1 K_\psi(x,y) = \nabla_x K_\psi(x,y)$, $\nabla_2 K_\psi(x,y) = \nabla_y K_\psi(x,y)$, etc. Note that (3.5) is an element of

$$\begin{aligned} \operatorname{End}(E_x \oplus \mathcal{L}(T_x, E_x)) \approx\ & \operatorname{End} E_x \oplus \mathcal{L}(\mathcal{L}(T_x, E_x), E_x) \\ & \oplus \mathcal{L}(E_x, \mathcal{L}(T_x, E_x)) \oplus \operatorname{End} \mathcal{L}(T_x, E_x). \end{aligned} \tag{3.6}$$

We proceed from (3.5) to a formula for the Gaussian measure Γ_x. First, we place the calculation in a more general setting. Let V be an m-dimensional real inner product space, (Ω, μ) a probability space, and $G : \Omega \to V$ a V-valued random variable, yielding the probability measure $G_*\mu = \Gamma$ on V. Let us assume that G is a Gaussian random variable with mean zero. As is well known, Γ is a Gaussian measure, and it is uniquely determined by the covariance

$$\mathbb{E}(G \otimes G) = \mathcal{C} \in V \otimes V \approx \mathcal{L}(V), \tag{3.7}$$

the latter isomorphism given by the inner product on V. Note that $\mathcal{C} = \mathcal{C}^t$, and this operator is positive semidefinite. If $\mathcal{C}$ is positive definite, then Γ has the form

$$d\Gamma(y) = \alpha(C) e^{-y\cdot Cy}\, dy, \tag{3.8}$$

for some positive definite $C \in \mathcal{L}(V)$, with $\alpha(C)$ chosen so that the right side of (3.8) has mass one. Using orthonormal coordinates on V such that C is diagonal, and computing the Gaussian integrals, via

$$\int_{-\infty}^{\infty} e^{-y^2}\, dy = \sqrt{\pi}, \tag{3.9}$$

we obtain

$$\alpha(C) = \pi^{-m/2} (\det C)^{1/2}. \tag{3.10}$$

Now $\Gamma = G_*\mu$ if and only if

$$\int\limits_V y \otimes y \, d\Gamma(y) = \mathcal{C}. \tag{3.11}$$

To calculate

$$\int\limits_V e^{-y\cdot Cy} y \otimes y\, dy, \tag{3.12}$$

we take an orthonormal basis $\{e_j\}$ of V such that $Ce_j = c_j e_j$, $c_j > 0$. Then $y \otimes y = \sum_{j,k} y_j y_k \, e_j \otimes e_k$, and (3.12) is

$$\sum_{j,k} \int\limits_V e^{-y\cdot Cy} y_j y_k \, dy \, e_j \otimes e_k. \tag{3.13}$$

Symmetry considerations show that each term for which $j \neq k$ vanishes, and we are left to calculate

$$\int\limits_V e^{-y\cdot Cy} y_k^2 \, dy = \prod_{j\neq k} \Bigl(\frac{\pi}{c_j}\Bigr)^{1/2} \int_{-\infty}^{\infty} e^{-c_k y^2} y^2 \, dy, \tag{3.14}$$

making use of the following consequence of (3.9):

$$\int_{-\infty}^{\infty} e^{-cy^2}\, dy = \sqrt{\frac{\pi}{c}}, \tag{3.15}$$

for $c > 0$. Taking the c-derivative of (3.15) yields

$$\int_{-\infty}^{\infty} e^{-cy^2} y^2\, dy = \frac{\sqrt{\pi}}{2}\, c^{-3/2}, \tag{3.16}$$

so (3.13)–(3.14) yield

$$\begin{aligned}
\int\limits_V e^{-y\cdot Cy} y\otimes y\, dy &= \sum_k \prod_{j\neq k} \Bigl(\frac{\pi}{c_j}\Bigr)^{1/2} \frac{\sqrt{\pi}}{2c_k^{3/2}}\, e_k\otimes e_k \\
&= \frac{1}{2}\, \frac{\pi^{m/2}}{(\det C)^{1/2}} \sum_k c_k^{-1}\, e_k\otimes e_k \\
&= \frac{1}{2}\, \frac{\pi^{m/2}}{(\det C)^{1/2}} \sum_k C^{-1} e_k\otimes e_k.
\end{aligned} \tag{3.17}$$

Using (3.10) and taking into account the isomorphism $V\otimes V \approx \mathcal{L}(V)$, we have from (3.11) that

$$\mathcal{C} = \frac{1}{2} C^{-1}, \quad \text{hence} \ \ C = \frac{1}{2}\mathcal{C}^{-1}. \tag{3.18}$$

We record the (well known) conclusion.

Lemma 3.1. *If $G : \Omega \longrightarrow V$ is a Gaussian random variable with mean 0 and covariance $\mathcal{C}$, given by (3.7), and if $\mathcal{C}$ is positive definite, then $\Gamma = G_*\mu$ has the form (3.8), with C given by (3.18) and $\alpha(C)$ by (3.10).*

Regarding the condition that $\mathcal{C}$ be positive definite, note from (3.7) that, for $v \in V$,

$$v\cdot \mathcal{C}v = \mathbb{E}(|G\cdot v|^2) = \int\limits_\Omega |G(\omega)\cdot v|^2\, d\mu(\omega). \tag{3.19}$$

Thus $\mathcal{C}$ is positive definite unless there is a proper linear subspace $V_0 \subset V$ such that

$$G(\omega) \in V_0, \quad \text{for } \mu\text{-a.e. } \omega\in\Omega. \tag{3.20}$$

In the case of main interest to us, $\mathcal{C} = \mathcal{C}_x$ is given by (3.5), as a continuous section of $\mathrm{End}(E\oplus \mathcal{L}(TM,E))$. As long as this is positive definite on $E_x \oplus \mathcal{L}(T_x, E_x)$ for each $x \in M$, we have the results (2.8)–(2.10). We turn to a closer look at such $\mathcal{C}_x$ in the next section, for $\varphi(\lambda) = e^{-t\lambda^2/2}$, and examine asymptotics as $t \searrow 0$.

4. Heat asymptotics and zero set asymptotics

Here we assume that the second order operator L has a scalar principal symbol, equal to that of $-\Delta$, where Δ is the Laplace-Beltrami operator on M. Such holds when L is the negative of the Hodge Laplacian on ℓ-forms. Then, for $t \searrow 0$,

$$e^{-tL}u(x) = \int_M K_t(x,y)u(y)\,dV(y), \tag{4.1}$$

where $K_t(x,y) \in \mathcal{L}(E_y, E_x)$ has the form, for x and y close,

$$K_t(x,y) \sim (4\pi t)^{-n/2} e^{-\rho(x,y)/4t}\Big(A_0(x,y) + A_1(x,y)t + \cdots\Big), \tag{4.1}$$

with $A_k \in \mathcal{L}(E_y, E_x)$, depending smoothly on x and y, and

$$A_0(x,x) = I. \tag{4.8}$$

Here,

$$\rho(x,y) = \mathrm{dist}(x,y)^2. \tag{4.4}$$

In particular, if we pick exponential coordinates centered at x,

$$\rho(x,y) = |x-y|^2, \tag{4.5}$$

the square norm being determined by the inner product on T_xM. In such a case, if we take

$$\varphi(\lambda) = \varphi_t(\lambda) = e^{-t\lambda^2/2}, \tag{4.6}$$

then (3.5)–(3.7) give $\mathcal{C} = \mathcal{C}_{t,x}$, with

$$\mathcal{C}_{t,x} = \begin{pmatrix} K_t(x,x) & \nabla_2 K_t(x,x) \\ \nabla_1 K_t(x,x) & \nabla_1\nabla_2 K_t(x,x) \end{pmatrix}. \tag{4.7}$$

We have

$$K_t(x,x) \sim (4\pi t)^{-n/2}\Big(I + A_1(x,x)t + \cdots\Big). \tag{4.8}$$

Since

$$\nabla_1 e^{-|x-y|^2/4t} = -\frac{x-y}{2t} e^{-|x-y|^2/4t}, \tag{4.9}$$

we have

$$\nabla_1 K_t(x,x) = (4\pi t)^{-n/2}\Big(\nabla_1 A_0(x,x) + O(t)\Big). \tag{4.10}$$

Similarly,

$$\nabla_2 K_t(x,x) = (4\pi t)^{-n/2}\Big(\nabla_2 A_0(x,x) + O(t)\Big). \tag{4.11}$$

Furthermore, since

$$\nabla_1\nabla_2 e^{-|x-y|^2/4t} = -\frac{(x-y)\otimes(x-y)}{4t^2}e^{-|x-y|^2/4t} + \frac{1}{2t}e^{-|x-y|^2/4t}I, \tag{4.12}$$

we have

$$\nabla_1\nabla_2 K_1(x,x) = (4\pi t)^{-n/2}\Big(\frac{1}{2t}I + O(1)\Big). \tag{4.13}$$

Thus, for

$$\widetilde{\mathcal{C}}_{t,x} = (4\pi t)^{n/2}\mathcal{C}_{t,x}, \tag{4.14}$$

we have

$$\widetilde{\mathcal{C}}_{t,x} = \begin{pmatrix} I + O(t) & \nabla_2 A_0(x,x) + O(t) \\ \nabla_1 A_0(x,x) + O(t) & (2t)^{-1}I + O(1) \end{pmatrix}. \tag{4.15}$$

Consequently

$$\begin{pmatrix} 1 & \\ & 2t \end{pmatrix}\widetilde{\mathcal{C}}_{t,x} = \begin{pmatrix} I & \nabla_2 A_0(x,x) \\ 0 & I \end{pmatrix} + O(t). \tag{4.16}$$

It follows that, for $t > 0$ sufficiently small, $\widetilde{\mathcal{C}}_{t,x}$ is invertible (hence positive definite) and

$$\widetilde{\mathcal{C}}_{t,x}^{-1}\begin{pmatrix} 1 & \\ & (2t)^{-1} \end{pmatrix} = \begin{pmatrix} I & \beta(x) \\ 0 & I \end{pmatrix} + O(t), \tag{4.17}$$

with $\beta(x) \in \mathcal{L}(E_x \oplus \mathcal{L}(T_x, E_x))$, depending smoothly on x. Then

$$\widetilde{\mathcal{C}}_{t,x}^{-1} = \begin{pmatrix} I & 2t\beta(x) \\ 0 & 2tI \end{pmatrix} + \begin{pmatrix} O(t) & O(t^2) \\ O(t) & O(t^2) \end{pmatrix}. \tag{4.18}$$

It follows that, when $\varphi(\lambda) = e^{-t\lambda^2/2}$, and $t > 0$ is sufficiently small, then (3.1A) holds for $\Gamma_x = \Gamma_{x,t}$, rewritten as

$$d\Gamma_{x,t}(v,A) = c_t(x)e^{-\gamma_{t,x}(v,A)}\,dv\,dA, \tag{4.19}$$

where

$$
\gamma_{t,x}(v,A) = (v,A)C_{t,x}\begin{pmatrix} v \\ A \end{pmatrix}
= \frac{1}{2}(v,A)\mathcal{C}_{t,x}^{-1}\begin{pmatrix} v \\ A \end{pmatrix}
= \frac{1}{2}(4\pi t)^{n/2}(v,A)\widetilde{\mathcal{C}}_{t,x}^{-1}\begin{pmatrix} v \\ A \end{pmatrix}, \tag{4.20}
$$

hence

$$
\gamma_{t,x}(0,A) = \frac{1}{2}(4\pi t)^{n/2}\Big(2t\|A\|^2 + O(t^2)\Big). \tag{4.21}
$$

Also,

$$
c_t(x) = \alpha(C_{t,x}) = \pi^{-m/2}(\det C_{t,x})^{1/2}
= \pi^{-m/2}\Big(\frac{1}{2}(4\pi t)^{n/2}\Big)^{m/2}(2t + O(t^2))^{\nu/2}, \tag{4.22}
$$

with

$$
m = \dim E_x \oplus \mathcal{L}(T_x, E_x) = k + nk,\quad
\nu = \dim \mathcal{L}(T_x, E_x) = nk. \tag{4.23}
$$

In this setting, (2.10) yields

$$
\mathbb{E}\big[\mathcal{H}^{n-k}\mathcal{Z}(F)\big] = \int_M \kappa(t,x)\,dV(x), \tag{4.24}
$$

where

$$
\kappa(t,x) = c_t(x)\int_{\mathcal{L}(\mathbb{R}^n,\mathbb{R}^k)} e^{-\gamma_{t,x}(0,A)}L(A)\,dA
= (2\pi)^{-m/2}(4\pi t)^{mn/4}(2t+O(t^2))\int_{\mathcal{L}(\mathbb{R}^n,\mathbb{R}^k)} e^{-(4\pi t)^{n/2}(t\|A\|^2+O(t^2))}L(A)\,dA. \tag{4.25}
$$

If we set

$$
B = (4\pi t)^{n/4}t^{1/2}A, \tag{4.26}
$$

we get

$$
\begin{aligned}
\kappa(t,x) &= (2\pi)^{-m/2}(4\pi t)^{mn/4}(2t+O(t^2))^{\nu/2}(4\pi t)^{-n\nu/4}t^{-\nu/2} \\
&\qquad \times \int\limits_{\mathcal{L}(\mathbb{R}^n,\mathbb{R}^k)} e^{-(\|B\|^2+O(t))}(4\pi t)^{-nk/4}t^{-k/2}L(B)\,dB \\
&= (2\pi)^{-m/2}2^{\nu/2}t^{-k/2}(1+O(t)) \int\limits_{\mathcal{L}(\mathbb{R}^n,\mathbb{R}^k)} e^{-\|B\|^2}L(B)\,dB,
\end{aligned}
\tag{4.27}
$$

which, to leading order, is independent of x. Consequently, with

$$
\gamma(n,k) = \int\limits_{\mathcal{L}(\mathbb{R}^n,\mathbb{R}^k)} e^{-\|B\|^2}(\det BB^t)^{1/2}\,dB,
\tag{4.28}
$$

we have

$$
\mathbb{E}\Big[\mathcal{H}^{n-k}\mathcal{Z}(F)\Big] = (2\pi)^{-m/2}2^{\nu/2}\gamma(n,k)t^{-k/2}(\mathrm{Vol}\,M)(1+O(t)),
\tag{4.29}
$$

as $t \searrow 0$, when F is given by (1.2) with $\varphi(\lambda) = e^{-t\lambda^2/2}$.

Remark. In the formulas above, $\|A\|^2$ and $\|B\|^2$ denote the squared Hilbert-Schmidt norms of these elements of $\mathcal{L}(\mathbb{R}^n,\mathbb{R}^k)$.

5. Other directions

There are various other matters to investigate, such as:

(1) Wave equation techniques, as in [CH]. These typically require restrictions on the principal symbol of L.

(2) Pushing heat equation techniques, which are fairly robust, to such situations as manifolds with boundary, manifolds with rough metrics (and/or rough boundaries), etc. Also try to push to cases where the principal symbol of L is not scalar.

(3) Replace the single operator L by a family of commuting operators, such as arise for $M = \mathbb{T}^n$, $M = S^n$, and other situations.

(4) Take L to be a pseudodifferential operator, such as the Dirichlet-to-Neumann map, when $M = \partial\Omega$.

A. Remarks on $\gamma(n,k)$

The coefficients $\gamma(n,k)$ arose in the asymptotic formula (4.29), and were given by (4.28), which we recall is

$$\gamma(n,k) = \int\limits_{\mathcal{L}(\mathbb{R}^n,\mathbb{R}^k)} e^{-\|B\|^2} (\det BB^t)^{1/2}\, dB. \tag{A.1}$$

Recall that $\|B\|$ denotes the Hilbert-Schmidt norm of B, and we are assuming $1 \le k \le n$. We have the following formulas for the two extreme cases.

First,

$$\begin{aligned} \gamma(n,1) &= \int\limits_{\mathbb{R}^n} e^{-|x|^2} |x|\, dx \\ &= A_{n-1} \int_0^\infty e^{-r^2} r^n\, dr \\ &= \frac{1}{2} A_{n-1} \int_0^\infty e^{-s} s^{(n-1)/2}\, ds \\ &= \frac{1}{2} A_{n-1} \Gamma\Bigl(\frac{n+1}{2}\Bigr) \\ &= \pi^{n/2} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})}, \end{aligned} \tag{A.2}$$

where A_{n-1} denotes the area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$.

At the other extreme,

$$\gamma(n,n) = \int\limits_{\mathcal{L}(\mathbb{R}^n)} e^{-\|B\|^2} |\det B|\, dB, \tag{A.3}$$

and using (15.4.12) of [M], we obtain

$$\begin{aligned} \gamma(n,n) &= \pi^{n/2} \prod_{j=1}^n \frac{\Gamma(\frac{1+j}{2})}{\Gamma(\frac{j}{2})} \\ &= \pi^{n/2} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{1}{2})}. \end{aligned} \tag{A.4}$$

I do not have a calculation of $\gamma(n,k)$ for $1 < k < n$, though one might guess a pattern from (A.2) and (A.4).

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