Bôcher's Theorem

Michael Taylor

Contents

1. Introduction

Let $\mathcal{O} \subset \mathbb{R}^n$ be a connected open set, $p \in \mathcal{O}$. Assume

$$u \in C^2(\mathcal{O} \setminus p), \quad \Delta u = 0, \quad u \geq 0, \quad \text{on } \mathcal{O} \setminus p. \tag{1.1}$$

Here $\Delta = \partial_1^2 + \cdots + \partial_n^2$ is the Laplace operator on Euclidean space $\mathbb{R}^n$. Examples of such functions include

$$\begin{aligned} V(x) &= |x-p|^{2-n}, \quad n \geq 3, \\ &\log \frac{1}{|x-p|}, \quad n = 2, \end{aligned} \tag{1.2}$$

the latter holding provided $\mathcal{O} \subset B_1(p)$ (add a constant if $\mathcal{O}$ is a larger bounded planar domain). Bôcher's theorem says the following.

Theorem 1.1. *If u satisfies (1.1), then there exist a function $h \in C^\infty(\mathcal{O})$, harmonic on $\mathcal{O}$, and a constant $A \in [0, \infty)$, such that*

$$u(x) = AV(x) + h(x), \tag{1.3}$$

with $V(x)$ as in (1.2).

Since this result appeared in [B], other proofs have been given, including proofs in [H], [ABR], and [ABR2]. There have also been extensions, both to variable-coefficient Laplacians and to higher order operators, in [EP], [KE], [C], and [L]. In fact, [B] discussed variable coefficients, at least in lower order terms.

Our goal here is to establish a variable coefficient extension of Theorem 1.1, involving generalized Laplace operators whose coefficients possess rather little smoothness. The following is our main result. Set

$$Lu = \sum_{j,k} \partial_j (a^{jk}(x) \partial_k u). \tag{1.4}$$

We assume the coefficients a^{jk} are real-valued functions, satisfying

$$a^{jk} = a^{kj}, \quad \sum_{j,k} a^{jk}(x) \xi_j \xi_k \geq \lambda |\xi|^2. \tag{1.5}$$

Theorem 1.2. *Let $\mathcal{O} \subset \mathbb{R}^n$ be a connected, open subset, $p \in \mathcal{O}$. Assume $\mathcal{O}$ is bounded, with smooth boundary. Assume*

$$u \in C^1(\mathcal{O} \setminus p), \quad Lu = 0, \quad u \geq 0 \quad \text{on} \quad \mathcal{O} \setminus p. \tag{1.6}$$

In addition to (1.5), assume the coefficients a^{jk} have the Sobolev space regularity

$$\begin{aligned} &\nabla a^{jk} \in H^{\varepsilon,n}(\mathcal{O}), \quad \varepsilon > 0, \quad \text{hence} \\ &\nabla a^{jk} \in L^r(\mathcal{O}), \quad r > n. \end{aligned} \tag{1.7}$$

Let V_p be given by

$$LV_p = -\delta_p \text{ on } \mathcal{O}, \quad V_p\big|_{\partial\mathcal{O}} = 0. \tag{1.8}$$

Then there exist $h \in C^1(\mathcal{O})$ and $A \in [0, \infty)$ such that

$$Lh = 0 \quad \text{on} \quad \mathcal{O}, \tag{1.9}$$

and

$$u(x) = AV_p(x) + h(x). \tag{1.10}$$

Here δ_p is the unit point mass ("delta function") supported at p. See §§3–4 for material on the existence, uniqueness, and positivity of such V_p.

We approach the proof of Theorem 1.2 in stages. We begin in §2 with a short proof of Theorem 1.1, taking an approach that is designed to extend to variable coefficient situations. We follow this in §3 with a short proof of Theorem 1.2 in case the coefficients a^{jk} belong to $C^\infty(\mathcal{O})$. In §4 we tackle Theorem 1.2 in the case of low regularity, specified in (1.7).

In outline, our argument goes as follows. First we establish an upper bound on u that implies u is locally integrable on a neighborhood of p, in fact in L^s for some $s > 1$. Using this, we can define Lu as a distribution on $\mathcal{O}$,

$$Lu = \mu \in \mathcal{D}'(\mathcal{O}), \tag{1.11}$$

satisfying

$$\mu \in H^{-2,s}(\mathcal{O}), \quad \operatorname{supp} \mu \subset \{p\}. \tag{1.12}$$

We combine this information on the support and regularity of μ with the positivity of u to show that $\mu = -A\delta_p$, for some $A \in [0, \infty)$, hence obtaining (1.10). Substantial technical issues arise in §4, including some local elliptic regularity results, which we prove in §5. In the course of proving these regularity results, we also show that Theorem 1.2 holds with the hypothesis (1.7) replaced by

$$a^{jk} \in C^1(\mathcal{O}). \tag{1.13}$$

2. Proof of the classic result

We start the proof of Theorem 1.1 with the following estimate.

Proposition 2.1. *Take $q \in \mathbb{R}^n$ and assume*

$$\Delta v = 0, \quad v \geq 0 \ \text{on} \ \ B_R(q), \tag{2.1}$$

where

$$B_R(q) = \{x \in \mathbb{R}^n : |x - q| < R\}. \tag{2.2}$$

Then, for $x \in B_R(q)$,

$$v(x) \leq 2v(q)\Bigl(1 - \frac{|x-q|}{R}\Bigr)^{-(n-1)}. \tag{2.3}$$

Proof. Translating and dilating, we can assume $q = 0$ and $R = 1$. So we work on $B = B_1(0)$. Let us temporarily assume also that $v \in C(\overline{B})$. We have the Poisson integral formula:

$$v(x) = \operatorname{PI} f(x) = \frac{1 - |x|^2}{A_{n-1}} \int\limits_{S^{n-1}} \frac{f(y)}{|x-y|^n}\, dS(y), \quad f = v\big|_{S^{n-1}}, \tag{2.4}$$

where $S^{n-1} = \partial B$ is the unit sphere in $\mathbb{R}^n$ and A_{n-1} is its area. Then

$$|v(x)| \leq (1 - |x|^2)\Bigl(\max_{|y|=1} |x-y|^{-n}\Bigr) \frac{1}{A_{n-1}} \int\limits_{S^{n-1}} f(y)\, dS(y). \tag{2.5}$$

Since $\min_{|y|=1} |x - y| = 1 - |x|$ for $x \in B$, this gives

$$|v(x)| \leq 2v(0)(1 - |x|)^{-(n-1)}, \tag{2.6}$$

for $v \in C(\overline{B})$ satisfying (2.1). Replacing $v(x)$ by $v(\rho x)$ and letting $\rho \nearrow 1$ removes the extra hypothesis and establishes the proposition.

Returning to Theorem 1.1, take $R > 0$ such that $\overline{B_{2R}(p)} \subset \mathcal{O}$ and apply Proposition 2.1 to u, restricted to $B_R(q)$, as q ranges over $\partial B_R(p)$. We get the following.

Lemma 2.2. *In the setting of Theorem 1.1, there exists* $C \in (0, \infty)$ *such that, for* $0 < |x - p| \leq R$,

$$u(x) \leq C|x - p|^{-(n-1)}. \tag{2.7}$$

It follows from (2.7) that the restriction of u to $\Omega = B_R(p)$ satisfies

$$u \in L^s(\Omega), \quad \forall\, s \in \Big[1, \frac{n}{n-1}\Big), \tag{2.8}$$

hence

$$\Delta u = \mu \in H^{-2,s}(\Omega), \quad \operatorname{supp} \mu \subset \{p\}. \tag{2.9}$$

The support condition on μ implies (cf. Proposition 4.5 in Chapter 3 of [T])

$$\mu = P\delta_p, \tag{2.10}$$

where P is a constant-coefficient differential operator, and the L^s-Sobolev space regularity condition in (2.9) implies that P is a first order differential operator. Consequently, u differs from

$$XV + AV \tag{2.11}$$

by a function that is harmonic on Ω, where X is a constant coefficient vector field and A is a constant, and V has the form (1.2). Since u is real valued, X must be a real vector field. Rotating coordinates, we can assume X is a multiple of ∂_1. Then a calculation gives

$$\partial_1 V(x) = c_n(x_1 - p_1)|x - p|^{-n}, \tag{2.12}$$

so the hypothesis $u \geq 0$ implies $X = 0$. Then $A \geq 0$ in (2.11), and we have that $u - AV$ is harmonic on a neighborhood of p, hence on all of $\mathcal{O}$. This proves Theorem 1.1.

3. Variable coefficients, smooth case

As in §1, let $\mathcal{O} \subset \mathbb{R}^n$ be a connected open set, $p \in \mathcal{O}$. For simplicity, assume $\mathcal{O}$ is bounded and $\partial\mathcal{O}$ is smooth. Assume

$$u \in C^2(\mathcal{O} \setminus p), \quad Lu = 0, \quad u \geq 0 \text{ on } \mathcal{O} \setminus p. \tag{3.1}$$

Here

$$Lu = \sum_{j,k} \partial_j (a^{jk}(x) \partial_k u), \tag{3.2}$$

where

$$a^{jk} = a^{kj} \in C^\infty(\overline{\mathcal{O}}), \quad \sum_{j,k} a^{jk}(x) \xi_j \xi_k \geq \lambda |\xi|^2. \tag{3.3}$$

Our main object of interest is

$$a^{jk}(x) = g(x)^{1/2} g^{jk}(x), \tag{3.4}$$

where $G = (g_{jk})$ is a smooth metric tensor on $\overline{\mathcal{O}}$, $(g^{jk}) = G^{-1}$, and $g = \det G$. Then $Lu = 0$ says u is harmonic with respect to the Laplace-Beltrami operator associated with this metric tensor. An example of (3.1) is

$$V_p(x) = E(x, p), \tag{3.5}$$

satisfying

$$LV_p = -\delta_p, \quad V_p\big|_{\partial\mathcal{O}} = 0. \tag{3.6}$$

Existence and uniqueness of such V_p is well known in the smooth setting. See Chapter 5 of [T].

The following result extends Theorem 1.1.

Theorem 3.1. *If u satisfies (3.1), then there exist $h \in C^\infty(\mathcal{O})$ satisfying $Lh = 0$ on $\mathcal{O}$ and a constant $A \in [0, \infty)$ such that*

$$u(x) = AV_p(x) + h(x), \tag{3.7}$$

with $V_p(x)$ as in (3.5)–(3.6).

Ingredients in the proof of Theorem 3.1 are parallel to those used in §2. To start, assume $\overline{B_{3R}(p)} \subset \mathcal{O}$, and let B be a ball of radius $S \in [R/2, 2R]$ such that $B \subset B_{3R}(p)$. The following result parallels Proposition 2.1.

Proposition 3.2. *Take such a ball $B = B_S(q)$, and assume*

$$Lv = 0, \quad v \geq 0 \quad on \quad B. \tag{3.8}$$

Then there exists $C < \infty$ such that, for $x \in B$,

$$v(x) \leq Cv(q)\Bigl(1 - \frac{|x-q|}{S}\Bigr)^{-(n-1)}. \tag{3.9}$$

Proof. Temporarily assume that also $v \in C(\overline{B})$. We have a Poisson integral formula:

$$v(x) = \mathrm{PI}\, f(x) = \int\limits_{\partial B} p_B(x,y) f(y)\, dS(y), \quad f = v\big|_{\partial B}. \tag{3.10}$$

Consequently,

$$v(x) \leq \Bigl(\max_{y \in \partial B} p_B(x,y)\Bigr) \int\limits_{\partial B} f(y)\, dS(y). \tag{3.11}$$

Furthermore, as we discuss below, there are estimates

$$0 < p_B(x,y) \leq C|x-y|^{-(n-1)}, \quad x \in B,\ y \in \partial B, \tag{3.12}$$

and

$$0 < \alpha \leq p_B(q,y), \quad y \in \partial B_S(q). \tag{3.13}$$

These two estimates lead from (3.11) to (3.9) when, in addition to (3.8), we have $v \in C(\overline{B})$. We can apply such a conclusion, with B replaced by $B_T(q)$, and let $T \nearrow S$ to finish the proof of Proposition 3.2 (given the estimates (3.12)–(3.13)).

Before discussing the estimates (3.12)–(3.13), we show how Proposition 3.2 leads to Theorem 3.1. The next step is to note that Proposition 3.2 leads to the following. Take $R > 0$ as described above the statement of Proposition 3.2.

Lemma 3.3. *In the setting of Theorem 3.1, there exists $C \in (0, \infty)$ such that, for $0 < |x-p| \leq R$,*

$$u(x) \leq C|x-p|^{-(n-1)}. \tag{3.14}$$

To proceed, we have from (3.14) that the restriction of u to $\Omega = B_R(p)$ satisfies

$$u \in L^s(\Omega), \quad \forall\, s \in \Bigl[1, \frac{n}{n-1}\Bigr), \tag{3.15}$$

hence

$$Lu = \mu \in H^{-2,s}(\Omega), \quad \operatorname{supp} \mu \subset \{p\}. \tag{3.16}$$

As in §2, this leads to the conclusion that there exist a constant coefficient vector field X and a constant A such that $\mu = -X\delta_p - A\delta_p$, i.e.,

$$Lu = -X\delta_p - A\delta_p, \quad \text{on } \Omega. \tag{3.17}$$

We want to compare u with $w = XV_p + AV_p$, with V_p given by (3.5)–(3.6). Note that

$$\begin{aligned} Lw &= LXV_p + LAV_p \\ &= -X\delta_p - A\delta_p + [L,X]V_p, \end{aligned} \tag{3.18}$$

where the commutator $[L,X]$ is a differential operator of order 2. We have

$$L(w-u) = [L,X]V_p. \tag{3.19}$$

Now a parametrix construction leads to an expansion of $V_p(x)$ for x near p,

$$V_p(x) \sim c\Bigl(\sum_{j,k} g_{jk}(p)(x_j - p_j)(x_k - p_k)\Bigr)^{(2-n)/2} + \cdots, \tag{3.20}$$

whose succeeding terms are progressively less singular at $x = p$. (This holds if $n \geq 3$. For $n = 2$, log terms arise.) In particular, $L(w-u)$ is a conormal distribution whose singularity has (at most) the same order as δ_p, so $w - u$ has (at most) the same order of singularity as V_p. Consequently, if $X \neq 0$,

$$u(x) = XV_p + R_p(x), \tag{3.21}$$

where XV_p has leading singularity homogeneous of degree $-(n-1)$ in $x - p$, while $R_p(x)$ has leading singularity homogeneous of degree $-(n-2)$ in $x - p$ (if $n > 2$, with a logarighmic singularity for $n = 2$). An inspection of the application of X to $V_p(x)$, satisfying (3.20), shows that

$$u(x) \geq 0 \text{ on } \Omega \setminus p \Longrightarrow X = 0. \tag{3.22}$$

Thus (3.17) becomes

$$Lu = -A\delta_p, \tag{3.23}$$

which leads to the conclusion (3.7), proving Theorem 3.1, given the estimates (3.12)–(3.13).

In the case of smooth coefficients, a parametrix construction of PI is available, from which the upper estimate in (3.12) follows. See Chapter 7, §§11 and 12, of [T]. As for the lower estimates in (3.12) and (3.13), we have the strong maximum principle. This yields $p_B(x,y) > 0$ for each $x \in B,\ y \in \partial B$. As for the positive lower bound, uniform in q and $S \in [R/2, 2R]$, in (3.13), this then follows from the continuous dependence of $p_B(x,y)$ on these parameters.

4. Variable coefficients, rough cases

As in §3, let $\mathcal{O} \subset \mathbb{R}^n$ be a connected open set, and assume for simplicity that $\mathcal{O}$ is bounded and $\partial\mathcal{O}$ is smooth. Take $p \in \mathcal{O}$. Assume

$$u \in C^1(\mathcal{O} \setminus p), \quad Lu = 0, \quad u \geq 0 \ \text{ on } \ \mathcal{O} \setminus p. \tag{4.1}$$

Here

$$Lu = \sum_{j,k} \partial_j (a^{jk}(x) \partial_k u). \tag{4.2}$$

In the setting of Theorem 1.2, the coefficients a^{jk} are assumed to satisfy (1.5) and (1.7).

We will want to extend Proposition 3.2 to this setting, which will be harder than what we did in §3. We start with a cruder estimate, which works in a more general setting.

Proposition 4.1. *Assume $B = B_S(q) \subset \mathcal{O}$ and $v \in H^{1,2}_{\text{loc}}(B)$ satisfies*

$$Lv = 0, \quad v \geq 0 \ \ on \ \ B. \tag{4.3}$$

In place of (1.5) and (1.7), assume $a^{jk} = a^{kj}$ are real valued and measurable, and that we have

$$\lambda |\xi|^2 \leq \sum_{j,k} a^{jk}(x) \xi_j \xi_k \leq \Lambda |\xi|^2, \tag{4.4}$$

with $0 < \lambda \leq \Lambda < \infty$. Then there exist C and $M = M(n, \Lambda/\lambda)$ such that for $x \in B$,

$$v(x) \leq C v(q)\, S^M \ \operatorname{dist}(x, \partial B)^{-M}. \tag{4.5}$$

Proof. The fact that each $v \in H^{1,2}_{\text{loc}}(B)$ satisfying $Lv = 0$ is continuous (even Hölder continuous) on B follows from the DeGiorgi-Nash theory (cf. [T], Chapter 14, §9, or [GT], §8.9). In addition, there is the following Moser Harnack inequality (cf. [GT], §8.8). If $B_{2\rho}(y) \subset B$, then

$$\sup_{B_\rho(y)} v \leq C_1 \inf_{B_\rho(y)} v, \quad C_1 = C_1(n, \Lambda/\lambda). \tag{4.6}$$

Iterating this gives

$$\sup_{B_{S_\nu}(q)} v \leq v(q)\, C_1^\nu, \quad S_\nu = S\Bigl(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^\nu}\Bigr) = S(1 - 2^{-\nu}), \tag{4.7}$$

from which (4.5) follows.

Before deriving a result closer to Proposition 3.2, we record a result on the existence and uniqueness of a positive solution V_p to

$$LV_p = -\delta_p \ \text{ on } \ \mathcal{O}, \quad V_p\big|_{\partial\mathcal{O}} = 0. \tag{4.8}$$

Proposition 4.2. *Take $\mathcal{O}$ and L as above, and assume the coefficients $a^{jk} = a^{kj}$ are real valued and measurable and satisfy (4.4) on $\mathcal{O}$. Then, given $p \in \mathcal{O}$, there is a unique*

$$V_p \in \bigcap_{\varepsilon>0} H^{1,2}(\mathcal{O} \setminus B_\varepsilon(p)) \cap H_0^{1,1}(\mathcal{O}) \tag{4.9}$$

satisfying (4.8). Furthermore, $V_p \ge 0$ on $\mathcal{O} \setminus p$.

This result is part of Theorem 1.1 in [GW].

We now formulate our extension of Proposition 3.2.

Proposition 4.3. *Take $B = B_S(q) \subset \mathcal{O}$, L, and v as in Proposition 4.1, and add the hypothesis that*

$$a^{jk} \in C^\alpha(\overline{B}), \quad \textit{for some} \ \ \alpha \in (0,1). \tag{4.10}$$

Then there exists $C = C(n, \Lambda/\lambda, \|a^{jk}\|_{C^\alpha}) < \infty$ such that, for $x \in B$,

$$v(x) \le C v(q) S^{n-1} \operatorname{dist}(x, \partial B)^{-(n-1)}. \tag{4.11}$$

The proof of Proposition 4.3 is formally parallel to that of Proposition 3.2. We temporarily assume that also $v \in C(\overline{B})$. We have a Poisson integral formula,

$$v(x) = \mathrm{PI}\, f(x) = \int_{\partial B} p_B(x,y) f(y)\, dS(y), \quad f = v\big|_{\partial B}, \tag{4.12}$$

hence

$$v(x) \le \Bigl(\max_{y\in\partial B} p_B(x,y)\Bigr) \int_{\partial B} f(y)\, dS(y). \tag{4.13}$$

Furthermore, as we will show, there are estimates

$$0 < p_B(x,y) \le C|x-y|^{-(n-1)}, \quad x \in B,\ y \in \partial B, \tag{4.14}$$

and

$$0 < \alpha \le p_B(q,y), \quad y \in \partial B_S(q). \tag{4.15}$$

These two estimates lead from (4.13) to (4.11) when, in addition to (4.3), we also have $v \in C(\overline{B})$. We can then apply such a conclusion, with B replaced by $B_T(q)$, and let $T \nearrow S$ to finish the proof of Proposition 4.3.

The key difference between the proofs of Proposition 3.2 and Proposition 4.3 is that (4.14) and (4.15) are a bit harder to establish than their counterparts in §3. For this task, results of [GW] will be useful.

To proceed, we bring in the Green function $G_B(x,y)$, defined as follows, in analogy with (4.9). First, for $y \in B$, there is (under the hypotheses of Proposition 4.3) a unique

$$G_B(\cdot,y) \in C^{1+\alpha}(\overline{B}\setminus y) \cap H_0^{1,1}(B) \tag{4.16}$$

satisfying

$$LG_B(\cdot,y) = -\delta_y, \quad G_B(\cdot,y)\big|_{\partial B} = 0. \tag{4.17}$$

Furthermore, $G(x,y) \ge 0$. Also, one has

$$G_B(x,y) = G_B(y,x), \tag{4.18}$$

for $x,y \in B$, and this allows us to extend $G_B(x,y)$ to $y \in \partial B$, for $x \in B$. One has from Theorem 3.3 of [GW] that, for $x \in B,\ y \in \overline{B}$,

$$\begin{aligned} G_B(x,y) &\le K_1 \delta(x) |x-y|^{-(n-1)}, \\ G_B(x,y) &\le K_1 |x-y|^{-(n-2)}, \\ |\nabla_y G_B(x,y)| &\le K_1 \delta(x) |x-y|^{-n}, \\ |\nabla_y G_B(x,y)| &\le K_1 |x-y|^{-(n-1)}, \\ |\nabla_x \nabla_y G_B(x,y)| &\le K_1 |x-y|^{-n}, \end{aligned} \tag{4.19}$$

with $K_1 = K_1(n,\lambda,\Lambda,\|a^{jk}\|_{C^\alpha})$ and $\delta(x) = \operatorname{dist}(x,\partial B)$. Furthermore, by part (ii) of Theorem 3.5 in [GW],

$$|\nabla_y G(x,y_1) - \nabla_y G(x,y_2)| \le K_1 |y_1-y_2|^\alpha \sum_{\ell=1}^{2} |x-y_\ell|^{1-n-\alpha}. \tag{4.20}$$

Now an application of Green's formula gives, for $\mathrm{PI} : C(\partial B) \to C(\overline{B})$, the formula

$$\mathrm{PI}\, f(x) = \int_{\partial B} f(y)\, \nu(y) \cdot \mathcal{A}(y) \nabla_y G(x,y)\, dS(y), \tag{4.21}$$

first for $f \in C^1(\partial B)$, then by extension for $f \in C(\partial B)$. Here $\nu(y)$ is the unit outward pointing normal to ∂B at y, and $\mathcal{A}(y) = (a^{jk}(y))$. In other words,

$$p_B(x,y) = \nu(y) \cdot \mathcal{A}(y) \nabla_y G(x,y), \tag{4.22}$$

for $x \in B,\ y \in \partial B$. Hence the estimates (4.19) yield

$$\begin{aligned} p_B(x,y) &\le \Lambda K_1 \delta(x)|x-y|^{-n}, \\ p_B(x,y) &\le \Lambda K_1 |x-y|^{-(n-1)}, \\ |\nabla_x p_B(x,y)| &\le \Lambda K_1 |x-y|^{-n}, \end{aligned} \tag{4.23}$$

and (4.20) yields

$$|p_B(x,y_1) - p_B(x,y_2)| \le K_1 \|\nu \cdot \mathcal{A}\|_{C^\alpha} |y_1 - y_2|^\alpha \sum_{\ell=1}^{2} |x - y_\ell|^{1-n-\alpha}. \tag{4.24}$$

The first (or second) part of (4.23), together with the strong maximum principle, yields (4.14). We next tackle the

Proof of (4.15). Assume to the contrary that there are balls $B_\nu = B_{S_\nu}(q_\nu) \subset \mathcal{O}$ and points $y_\nu \in \partial B_\nu$ such that

$$p_{B_\nu}(q_\nu, y_\nu) \longrightarrow 0. \tag{4.25}$$

It is convenient to apply translations, dilations, and rotations to map $B_{S_\nu}(q_\nu)$ to $B_S(q)$ and y_ν to y_0, so we have Poisson kernels $p_\nu(x,y)$ for solutions to Dirichlet problems

$$L_\nu w = 0 \text{ on } B, \quad w = f \text{ on } \partial B, \tag{4.26}$$

with

$$L_\nu w = \sum_{j,k} \partial_j (a_\nu^{jk}(x) \partial_k w), \tag{4.27}$$

the solution given by

$$w(x) = \int_{\partial B} p_\nu(x,y) f(y)\, dS(y), \tag{4.28}$$

and (with q the center of B and $y_0 \in \partial B$ fixed)

$$p_\nu(q, y_0) \longrightarrow 0. \tag{4.29}$$

The hypothesis (4.10) then yields

$$\|a_\nu^{jk}\|_{C^\alpha(\overline{B})} \le c_0 < \infty, \quad \forall\, \nu. \tag{4.30}$$

Our goal is to show that (4.29) cannot occur.

To proceed, pick $\alpha_1 \in (0,\alpha)$ and, passing to a subsequence, assume

$$a_\nu^{jk} \longrightarrow a_0^{jk} \text{ in } C^{\alpha_1}(\overline{B})\text{-norm}, \tag{4.31}$$

with $a_0^{jk} \in C^\alpha(\overline{B})$, also satisfying (4.4). Say its Poisson kernel is $p_0(x,y)$. If we can show that (after perhaps passing to a further subsequence), as $\nu \to \infty$,

$$p_\nu(x,y) \longrightarrow p_0(x,y), \quad \text{uniformly on } \ K \times \partial B, \tag{4.32}$$

for each compact $K \subset B$, then (4.29) would imply

$$p_0(q,y_0) = 0, \tag{4.33}$$

which violates the strong maximum principle.

We hence pursue a demonstration of (4.32). To start, we can apply (4.23)–(4.24) to obtain estimates

$$\begin{aligned} p_\nu(x,y) &\le K_2 |x-y|^{-(n-1)}, \\ |\nabla_x p_\nu(x,y)| &\le K_2 |x-y|^{-n}, \\ |p_\nu(x,y_1) - p_\nu(x,y_2)| &\le K_2 \delta(x)^{1-n-\alpha} |y_1-y_2|^\alpha, \end{aligned} \tag{4.34}$$

valid uniformly in ν, for $x \in B,\ y, y_1, y_2 \in \partial B$. We can hence apply the Arzela-Ascoli theorem and, passing to a subsequence, obtain

$$p_\nu(x,y) \longrightarrow P(x,y), \quad \text{uniformly on } \ K \times \partial B, \tag{4.35}$$

for each compact $K \subset B$. In light of this, our task becomes that of showing that

$$P(x,y) = p_0(x,y), \quad \text{on } \ B \times \partial B. \tag{4.36}$$

In preparation for this, let us denote by

$$\mathrm{PI}_\nu : C(\partial B) \longrightarrow C(\overline{B}), \quad \mathrm{PI}_0 : C(\partial B) \longrightarrow C(\overline{B}) \tag{4.37}$$

the solution operators to (4.26) and to its counterpart, with $L_\nu w$ replaced by

$$L_0 w = \sum_{j,k} \partial_j (a_0^{jk}(x) \partial_k w), \tag{4.38}$$

so $\mathrm{PI}_\nu f(x)$ is given by (4.28) and

$$\mathrm{PI}_0 f(x) = \int_{\partial B} p_0(x,y) f(y)\, dS(y). \tag{4.39}$$

We have the following key result.

Lemma 4.4. *Given* $f \in C^1(\partial B)$,

$$\mathrm{PI}_\nu f \longrightarrow \mathrm{PI}_0 f \quad in \quad H^1_0(B). \tag{4.40}$$

Proof. Take $f = F|_{\partial B},\ F \in C^1(\overline{B})$. We have

$$\mathrm{PI}_\nu f = F - w_\nu, \quad \mathrm{PI}_0 f = F - w_0, \tag{4.41}$$

with

$$\begin{aligned} L_\nu w_\nu &= L_\nu F, \quad w_\nu\big|_{\partial B} = 0, \\ L_0 w_0 &= L_0 F, \quad w_0\big|_{\partial B} = 0. \end{aligned} \tag{4.42}$$

We see that $L_\nu F = \sum_{j,k} \partial_j a^{jk}_\nu \partial_k F$ with $a^{jk}_\nu \partial_k F$ bounded in $C(\overline{B})$, hence

$$L_\nu F \ \text{ bounded in } \ H^{-1}(B), \tag{4.43}$$

so

$$w_\nu \ \text{ bounded in } \ H^1_0(B). \tag{4.44}$$

Thus

$$\begin{aligned} L_0(w_\nu - w_0) &= (L_0 - L_\nu)w_\nu + (L_\nu - L_0)F \\ &\to 0 \ \text{ in } \ H^{-1}(B), \end{aligned} \tag{4.45}$$

as $\nu \to \infty$, so

$$w_\nu \longrightarrow w_0 \ \text{ in } \ H^1_0(B), \tag{4.46}$$

and we have (4.40).

We have from (4.40) that, as $\nu \to \infty$,

$$\begin{aligned} &\int_B \int_{\partial B} g(x) p_\nu(x,y) f(y)\, dS(y)\, dx \\ &\longrightarrow \int_B \int_{\partial B} g(x) p_0(x,y) f(y)\, dS(y)\, dx, \end{aligned} \tag{4.47}$$

for each $f \in C^1(\partial B),\ g \in C^1_0(B)$. On the other hand, (4.35) readily implies that

$$\begin{aligned} &\int_B \int_{\partial B} p_\nu(x,y) f(y)\, dS(y)\, dx \\ &\longrightarrow \int_B \int_{\partial B} g(x) P(x,y) f(y)\, dS(y)\, dx, \end{aligned} \tag{4.48}$$

for such f and g. Comparing (4.47) and (4.48), we have (4.36), hence (4.32), and the proof of (4.15) is complete.

At this point, the proof of Proposition 4.3 is complete. Now that we have this, we can deduce the following analogue of Lemma 3.3.

Lemma 4.5. *Take $\mathcal{O} \subset \mathbb{R}^n$, $p \in \mathcal{O}$, and L as in Theorem 1.2, but replace (1.7) by the more general hypothesis*

$$a^{jk} \in C^\alpha(\overline{\mathcal{O}}), \tag{4.49}$$

for some $\alpha \in (0,1)$. In particular, let u satisfy

$$u \in C^1(\mathcal{O} \setminus p), \quad Lu = 0, \quad u \geq 0 \quad on \quad \mathcal{O} \setminus p. \tag{4.50}$$

Assume $B_{2R}(p) \subset \mathcal{O}$. Then there exists $C < \infty$ such that

$$u(x) \leq C|x-p|^{-(n-1)}, \quad for \ \ 0 < |x-p| \leq R. \tag{4.51}$$

Given Lemma 4.5, we have, with $B = B_R(p)$,

$$u \in L^s(B), \quad \forall\, s \in \Big[1, \frac{n}{n-1}\Big), \tag{4.52}$$

hence

$$\partial_k u \in H^{-1,s}(B), \tag{4.53}$$

for all such s. Now, if we strengthen (4.49) to

$$a^{jk} \in H^{1,r}(\mathcal{O}), \quad r > n, \tag{4.54}$$

then multiplication by a^{jk} maps $H_0^{1,\rho}(B)$ to itself, for $\rho \in (n, r]$, so by duality,

$$a^{jk} \partial_k u \in H^{-1,s}(B), \tag{4.55}$$

for s as in (4.52). Therefore

$$Lu = \mu \in H^{-2,s}(B), \tag{4.56}$$

for all such s. We have $Lu = 0$ on $B \setminus p$, so

$$\operatorname{supp} \mu \subset \{p\}. \tag{4.57}$$

In view of the structure of distributions supported at $\{p\}$, we have the following.

Proposition 4.6. *In the setting of Theorem 1.2, there exist a constant coefficient vector field X and a constant A such that*

$$Lu = -X\delta_p - A\delta_p. \tag{4.58}$$

We aim to prove that $X = 0$. Parallel to the analysis in §3, we compare u with

$$w = XV_p + AV_p, \tag{4.59}$$

with V_p as in Proposition 4.2. As long as the coefficients a^{jk} are Hölder continuous, we have the following parallel to estimates in (4.19):

$$\begin{aligned} V_p(x) &\le C|x-p|^{-(n-2)}, \\ |\nabla V_p(x)| &\le C|x-p|^{-(n-1)}. \end{aligned} \tag{4.60}$$

We will find it useful to have the following more precise information, established in Proposition 2.4 of [MT2] (which improves (2.70)–(2.71) of [MT1]). Namely,

$$V_p(x) = E_p(x) + r_p(x), \tag{4.61}$$

with

$$E_p(x) = c_n(p)\Bigl(\sum_{j,k} a_{jk}(p)(x_j-p_j)(x_k-p_k)\Bigr)^{-(n-2)/2}, \tag{4.62}$$

where (a_{jk}) is the matrix inverse to (a^{jk}), and

$$\begin{aligned} |r_p(x)| &\le C|x-p|^{-(n-2-\alpha)}, \\ |\nabla r_p(x)| &\le C|x-p|^{-(n-1-\alpha)}. \end{aligned} \tag{4.63}$$

To compare u with w, we compare Lu, given by (4.58), with

$$\begin{aligned} Lw &= LXV_p + LAV_p \\ &= XLV_p + ALV_p + [L,X]V_p \\ &= -X\delta_p - A\delta_p + [L,X]V_p, \end{aligned} \tag{4.64}$$

obtaining

$$L(w-u) = [L,X]V_p. \tag{4.65}$$

Note that

$$[L,X]V_p = \sum_{j,k} \partial_j(b^{jk}(x)\partial_k V_p), \tag{4.66}$$

with

$$b^{jk} = -Xa^{jk} \in L^r(\mathcal{O}), \tag{4.67}$$

given the hypothesis (1.7). We have from (4.60) that

$$\partial_k V_p \in L^s(\mathcal{O}), \quad \forall\, s \in \Big[1, \frac{n}{n-1}\Big), \tag{4.68}$$

so, given $r > n$,

$$b^{jk}\partial_k V_p \in L^\sigma(\mathcal{O}), \quad \text{for some } \ \sigma > 1, \tag{4.69}$$

and we have

$$[L, X]V_p \in H^{-1,\sigma}(\mathcal{O}). \tag{4.70}$$

To recap, we have

$$\begin{aligned} &w, u \in L^s(B), \quad \forall\, s \in \Big[1, \frac{n}{n-1}\Big), \\ &L(w-u) \in H^{-1,\sigma}(B). \end{aligned} \tag{4.71}$$

We bring in the following local elliptic regularity result.

Proposition 4.7. *Take L as in Theorem 1.2, $B = B_R(p)$, with $B_{2R}(p) \subset \mathcal{O}$. Assume*

$$v \in L^s(B), \quad \forall\, s \in \Big[1, \frac{n}{n-1}\Big), \quad Lv \in H^{-1,\sigma}(B), \tag{4.72}$$

for some $\sigma \in (1, n/(n-1))$. Then

$$v \in H^{1,\sigma}(B_{R/2}(p)). \tag{4.73}$$

We will prove this result in §5. Here we apply it to finish the proof of Theorem 1.2.

Noting that $V_p \in H^{1,s}(\mathcal{O})$ for all $s < n/(n-1)$, we see from (4.71) that Proposition 4.7 implies

$$u - XV_p \in H^{1,\sigma}(B_{R/2}(p)), \tag{4.74}$$

for some $\sigma > 1$. Since $H^{1,\sigma}(\mathbb{R}^n) \subset L^{n\sigma/(n-\sigma)}(\mathbb{R}^n)$, we obtain that

$$u - XV_p \in L^{s_1}(B_{R/2}(p)), \quad \text{for some } \ s_1 > \frac{n}{n-1}. \tag{4.75}$$

Meanwhile, by (4.61)–(4.63), we have

$$XV_p - XE_p = Xr_p \in L^{s_1}(B_{R/2}(p)), \quad \text{for some } \ s_1 > \frac{n}{n-1}. \tag{4.76}$$

so

$$u - XE_p \in L^{s_1}(B_{R/2}(p)), \quad \text{for some} \;\; s_1 > \frac{n}{n-1}. \tag{4.77}$$

On the other hand, it is clear from (4.62) that

$$XE_p(x) \;\; \text{is homogeneous of degree } -(n-1) \text{ in } x - p, \tag{4.78}$$

so it just fails to belong to $L^{n/(n-1)}$ on $B_{R/2}(p)$, if $X \neq 0$. Consequently, if $X \neq 0$, the singularity of XE_p cannot be cancelled by the difference $u - XE_p$, so

$$u \geq 0 \Longrightarrow XE_p \geq 0 \text{ on } B_{R/2}(p). \tag{4.79}$$

On the other hand, if $X \neq 0$, one can rotate coordinates so X is a multiple of ∂_1, and a straightforward computation from (4.62) shows that XE_p must change sign. This proves that

$$X = 0, \tag{4.80}$$

and completes the proof of Theorem 1.2, modulo the proof of Proposition 4.7.

5. Some local elliptic regularity theorems

To recall our setting, we have

$$L = \sum_{j,k} \partial_j a^{jk} \partial_k, \tag{5.1}$$

where $a^{jk} = a^{kj}$ are real valued and satisfy

$$\nabla a^{jk} \in H^{\varepsilon,n}(\mathcal{O}), \quad \varepsilon > 0, \tag{5.2}$$

hence

$$\nabla a^{jk} \in L^r(\mathcal{O}), \quad r > n, \tag{5.3}$$

and where $\mathcal{O}$ is an open set in $\mathbb{R}^n$. For current purposes, we may as well take $\mathcal{O} = \mathbb{R}^n$ and assume $a^{jk}(x)$ is constant for $|x|$ large. We make the ellipticity hypothesis

$$\lambda |\xi|^2 \le \sum_{j,k} a^{jk}(x) \xi_j \xi_k \le \Lambda |\xi|^2, \tag{5.4}$$

with $0 < \lambda \le \Lambda < \infty$. The content of Proposition 4.7, which we aim to prove here, is that if $B = B_R(p) \subset\subset \mathcal{O}$ is a ball and

$$\begin{aligned} v \in L^s(B), \quad Lv \in H^{-1,\sigma}(B), \quad &\text{for all } s \in \Bigl[1, \frac{n}{n-1}\Bigr), \\ &\text{and some } \sigma \in \Bigl(1, \frac{n}{n-1}\Bigr), \end{aligned} \tag{5.5}$$

then

$$v \in H^{1,\sigma}(B_{R/2}(p)). \tag{5.6}$$

Let us note that the analysis in §4 made direct use of (5.3), but not of the stronger hypothesis (5.2) (except to invoke it in the statement of Proposition 4.7). Hence one has the conclusion of Theorem 1.2 whenever one has (5.3) and the implication (5.5) $\Rightarrow$ (5.6).

To proceed, assume v satisfies (5.5) and take $\varphi \in C_0^\infty(B)$ so that $\varphi = 1$ on $B_{R/2}(p)$. Then

$$L(\varphi v) = \varphi L v + (L\varphi) v + 2 \sum_{j,k} (\partial_j \varphi) a^{jk} (\partial_k v). \tag{5.7}$$

The hypotheses on v in (5.5) imply $(L\varphi)v \in L^s(B)$ and, given (5.3),

$$\sum_{j,k} (\partial_j \varphi) a^{jk} (\partial_k v) \in H^{1,r} \cdot H^{-1,s} = H^{-1,s}, \tag{5.8}$$

provided $s = \rho',\ n < \rho \le r$, i.e., provided $r' \le s < n/(n-1)$, which, by (5.5), we can take to be the case. We deduce that

$$\varphi v \in L^s(B), \quad L(\varphi v) \in H^{-1,\sigma}(B). \tag{5.9}$$

In order to establish (5.6), we need only show that (5.9) $\Rightarrow \varphi v \in H^{1,\sigma}(B)$.

Note that this reduction involved the hypothesis (5.3), but not (5.2). In conclusion, it suffices to prove the following global regularity result:

$$v \in L^s(\mathbb{T}^n), \quad Lv \in H^{-1,\sigma}(\mathbb{T}^n) \Longrightarrow v \in H^{1,\sigma}(\mathbb{T}^n), \tag{5.10}$$

where we form the flat torus $\mathbb{T}^n$ by putting $B_{2R}(p)$ in a box and identifying opposite faces.

Now, if we set

$$\begin{gathered} A = (1-\Delta)^{1/2}, \quad R_j = A^{-1}\partial_j, \quad w = Av, \\ B = A^{-1} L A^{-1} = \sum_{j,k} R_j a^{jk} R_k, \end{gathered} \tag{5.11}$$

our task is to show that

$$\begin{aligned} & w \in H^{-1,s}(\mathbb{T}^n), \quad \forall\, s \in \Big[1, \frac{n}{n-1}\Big), \quad Bw \in L^\sigma(\mathbb{T}^n), \\ & \Longrightarrow w \in L^\sigma(\mathbb{T}^n), \end{aligned} \tag{5.12}$$

under the hypothesis (5.2).

The operators R_j are pseudodifferential operators, with symbols $R_j(x,\xi) = i\xi_j \langle\xi\rangle^{-1}$. The operators $A_j = \sum_k a^{jk} R_k$ are pseudodifferential operators with symbols

$$A_j(x,\xi) = i \sum_k a^{jk}(x) \xi_j \langle\xi\rangle^{-1}. \tag{5.13}$$

Generally, we say a function $p(x,\xi)$ is a symbol in $S^m_{1,0}$ provided it is C^∞ and

$$|D^\beta_x D^\alpha_\xi p(x,\xi)| \le C_{\alpha\beta} \langle\xi\rangle^{m-|\alpha|}, \quad \forall\, \alpha, \beta. \tag{5.10}$$

An operator with symbol in $S^m_{1,0}$ is said to belong to $OPS^m_{1,0}$. Thus $R_j \in OPS^0_{1,0}$. To describe a smaller class, we say $p(x,\xi) \in S^m_{\mathrm{cl}}$ provided $p(x,\xi) \in S^m_{1,0}$ and we have an asymptotic expansion

$$p(x,\xi) \sim \sum_{k \ge 0} p_k(x,\xi), \tag{5.14}$$

with $p_k(x,\xi) \in S^{m-k}_{1,0}$ homogeneous of degree $m-k$ in ξ, for $|\xi|$ large. To say this expansion is asymptotic is to say

$$p(x,\xi) - \sum_{0\le k<N} p_k(x,\xi) \in S^{m-N}_{1,0}, \quad \forall\, N. \tag{5.15}$$

An operator with symbol in S^m_{cl} is said to belong to OPS^m_{cl}. Thus

$$R_j \in OPS^0_{\mathrm{cl}}. \tag{5.16}$$

The symbol $A_j(x,\xi)$ in (5.13) does not fit into this framework unless $a^{jk} \in C^\infty$. Instead, we need to consider classes of symbols with limited regularity in x. If X is a Banach space of functions, we say

$$p(x,\xi) \in XS^m_{1,0} \Longleftrightarrow \|D^\alpha_\xi p(\cdot,\xi)\|_X \le C_\alpha \langle\xi\rangle^{m-|\alpha|}, \quad \forall\, \alpha. \tag{5.17}$$

Then we say $p(x,D) \in OPXS^m_{1,0}$. Similarly we say $p(x,\xi) \in XS^m_{\mathrm{cl}}$ (and $p(x,D) \in OPXS^m_{\mathrm{cl}}$) provided $p(x,\xi) \in XS^m_{1,0}$ and there is an asymptotic expansion of the form (5.14) with $p_k(x,\xi) \in XS^{m-k}_{1,0}$, homogeneous of degree $m-k$ in ξ for $|\xi|$ large, and the difference in (5.15) belongs to $XS^{m-N}_{1,0}$. For $A_j(x,\xi)$ as in (5.13), we have

$$A_j(x,\xi) \in H^{1+\varepsilon,n} S^0_{\mathrm{cl}} \tag{5.18}$$

if (5.2) holds, and

$$A_j(x,\xi) \in H^{1,r} S^0_{\mathrm{cl}} \tag{5.19}$$

if (5.3) holds. Another case of interest is

$$a^{jk} \in C^1(\mathbb{T}^n) \Longrightarrow A_j(x,\xi) \in C^1 S^0_{\mathrm{cl}}. \tag{5.20}$$

Elliptic regularity results in this section will make use of pseudodifferential operators with symbols in such classes, as well as further classes, defined below. We start with regularity theorems that can be obtained from results on operators in $OPC^1S^0_{\mathrm{cl}}$, established in Chapter 4 of [T2], building on work in [Ca]. As a first observation, one can use the expansion (4.1.2) of [T2] together with Calderón-Zygmund theory to obtain that

$$\begin{aligned} &p(x,\xi) \in C^1 S^0_{\mathrm{cl}} \Longrightarrow \\ &p(x,D) : H^{r,s}(\mathbb{T}^n) \to H^{r,s}(\mathbb{T}^n), \ \ \forall\, r \in [-1,1], \ s \in (1,\infty). \end{aligned} \tag{5.21}$$

We also have use for the following consequence of Proposition 4.2.A of [T2].

Proposition 5.1. *Let* $p_j(x,\xi) \in C^1 S^0_{\mathrm{cl}}$*, and set*

$$q(x,\xi) = p_1(x,\xi) p_2(x,\xi). \tag{5.22}$$

Then $q(x,\xi) \in C^1 S^0_{\mathrm{cl}}$ *and*

$$p_1(x,D) p_2(x,D) = q(x,D) + R, \tag{5.23}$$

with

$$R : H^{r,s}(\mathbb{T}^n) \to H^{r+1,s}(\mathbb{T}^n), \quad \forall\, r \in [-1,0],\ s \in (1,\infty). \tag{5.24}$$

Let us apply this to the operator B of (5.11);

$$B = \sum_j R_j A_j(x,D), \tag{5.25}$$

with $A_j(x,\xi)$ as in (5.13). We see that, with

$$\widetilde{B}(x,\xi) = -\sum_{j,k} a^{jk}(x) \xi_j \xi_k \langle \xi \rangle^{-2}, \tag{5.26}$$

we have $\widetilde{B}(x,D) \in OPC^1 S^0_{\mathrm{cl}}$ and

$$B - \widetilde{B}(x,D) : H^{r,s}(\mathbb{T}^n) \to H^{r+1,s}(\mathbb{T}^n), \quad \forall\, r \in [-1,0],\ s \in (1,\infty). \tag{5.27}$$

Now the ellipticity hypothesis (5.4) implies that

$$E(x,\xi) = (1 - \varphi(\xi)) \widetilde{B}(x,\xi)^{-1} \in C^1 S^0_{\mathrm{cl}}, \tag{5.28}$$

where $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $\varphi(\xi) = 1$ for $|\xi|$ small. Thus another application of Proposition 5.1 yields

$$E(x,D)\widetilde{B}(x,D) - I : H^{r,s}(\mathbb{T}^n) \to H^{r+1,s}(\mathbb{T}^n), \quad \forall\, r \in [-1,0],\ s \in (1,\infty). \tag{5.29}$$

In conjunction with (5.27), this gives

$$E(x,D)B - I : H^{r.s}(\mathbb{T}^n) \to H^{r+1,s}(\mathbb{T}^n), \quad \forall\, r \in [-1,0],\ s \in (1,\infty). \tag{5.30}$$

This puts us in a position to prove the following global regularity result.

Proposition 5.2. *Take L and B as in (5.1) and (5.11). Assume the ellipticity condition (5.4), and assume $a^{jk} \in C^1(\mathbb{T}^n)$. Then, for each $s, \sigma \in (1, \infty)$,*

$$w \in H^{-1,s}(\mathbb{T}^n),\ Bw \in L^\sigma(\mathbb{T}^n) \Longrightarrow w \in L^\sigma(\mathbb{T}^n). \tag{5.31}$$

Proof. If w satisfies the hypotheses in (5.31), we apply (5.30) to get

$$\begin{aligned} w &= E(x,D)Bw \mod L^s(\mathbb{T}^n) \\ &\in L^\sigma(\mathbb{T}^n) + L^s(\mathbb{T}^n). \end{aligned} \tag{5.32}$$

If $s \geq \sigma$, we have the conclusion in (5.31). If $s < \sigma$ (and $s < n$) we have

$$w \in L^s(\mathbb{T}^n) \subset H^{-1,s_1}(\mathbb{T}^n), \quad s_1 = s\frac{n}{n-s}, \tag{5.33}$$

by the Sobolev embedding result $H^{1,s}(\mathbb{T}^n) \subset L^{ns/(n-s)}(\mathbb{T}^n)$. If $s \geq n$, we have $w \in H^{-1,s_1}(\mathbb{T}^n)$ for all $s_1 \in (1, \infty)$. Now, the argument leading to (5.32) gives

$$w \in L^\sigma(\mathbb{T}^n) + L^{s_1}(\mathbb{T}^n). \tag{5.34}$$

Iterating this eventually gives $w \in L^\sigma(\mathbb{T}^n)$.

Translating back to the study of L, we have the following.

Corollary 5.3. *In the setting of Proposition 5.2, for each $s, \sigma \in (1, \infty)$,*

$$v \in L^s(\mathbb{T}^n),\ Lv \in H^{-1,\sigma}(\mathbb{T}^n) \Longrightarrow v \in H^{1,\sigma}(\mathbb{T}^n). \tag{5.35}$$

Remark. If we use the hypothesis (5.5), we can arrange that $s > \sigma$, and skip the second part of the proof of Proposition 5.2, but it is of natural interest to record the sharper result here.

Note that if we replace the hypotheses in (5.35) by

$$v \in L^s(B), \quad Lv \in H^{-1,\sigma}(B), \tag{5.36}$$

then, taking φ as in ((5.7), we have $\sum_{j,k} (\partial\varphi) a^{jk} (\partial_k v) \in H^{-1,s}$, given $a^{jk} \in C^1$, hence $\varphi v \in L^s(B)$ and $L(\varphi v) \in H^{-1,\sigma}(B)$, provided $\sigma \leq s$. With this in hand, we can prove the following.

Proposition 5.4. *Take L as in (5.1), with $a^{jk} = a^{kj} \in C^1(\mathcal{O})$, and assume the ellipticity condition (5.4). Then, for $\sigma, s \in (1, \infty)$,*

$$v \in L^s_{\text{loc}}(\mathcal{O}), \quad Lv \in H^{-1,\sigma}_{\text{loc}}(\mathcal{O}) \Longrightarrow v \in H^{1,\sigma}_{\text{loc}}(\mathcal{O}). \tag{5.37}$$

Proof. The localization described above, in conjunction with Corollary 5.3, yields (5.37) provided $\sigma \leq s$. If $\sigma > s$, we have $v \in H^{1,s}_{\text{loc}}(\mathcal{O})$, hence

$$v \in L^{s_1}_{\text{loc}}(\mathcal{O}), \tag{5.38}$$

with $s_1 = ns/(n-s)$ if $s < n$, $s_1 = \infty$ if $s \geq n$. Replacing the first hypothesis on v by this condition, and iterating the argument, as necessary, gives the conclusion.

We now take up the proof of (5.5) $\Rightarrow$ (5.6), or rather the following refinement.

Proposition 5.5. *Take L as in (5.1). Assume the ellipticity condition (5.4), and the regularity condition (5.2)–(5.3), i.e.,*

$$a^{jk} \in H^{1+\varepsilon,n}(\mathbb{T}^n) \subset H^{1,r}(\mathbb{T}^n), \tag{5.39}$$

with $\varepsilon > 0,\ r > n$. Take

$$s > r'. \tag{5.40}$$

Then, for $\sigma \in (1, \infty)$,

$$v \in L^s(\mathbb{T}^n), \quad Lv \in H^{-1,\sigma}(\mathbb{T}^n) \Longrightarrow v \in H^{1,\sigma}(\mathbb{T}^n). \tag{5.41}$$

We approach this with a sequence of reductions, starting with the following.

Lemma 5.6. *To prove Proposition 5.5, it suffices to establish it for*

$$1 < \sigma < 1 + \gamma, \tag{5.42}$$

for some $\gamma > 0$.

Proof. Indeed, suppose $v \in L^s(\mathbb{T}^n)$ and $Lv \in H^{-1,\tau}(\mathbb{T}^n)$, with $\tau \geq 1 + \gamma$. If we have (5.41) for σ satisfying (5.42), we have $v \in H^{1,\sigma}(\mathbb{T}^n)$. On the other hand, the following implication holds, for $1 < \sigma < \tau < \infty$:

$$v \in H^{1,\sigma}(\mathbb{T}^n),\ Lv \in H^{-1,\tau}(\mathbb{T}^n) \Longrightarrow v \in H^{1,\tau}(\mathbb{T}^n). \tag{5.43}$$

In fact, this implication holds under the following much more general "regularity" condition on the coefficients:

$$a^{jk} \in L^\infty \cap \text{vmo}(\mathbb{T}^n). \tag{5.44}$$

See Proposition 1.10 in Chapter 3 of [T3]. This establishes the lemma.

For another reduction, let us take

$$L_1 v = \sum_{j,k} \partial_j (a^{jk} \partial_k v) - v, \tag{5.45}$$

so, manifestly, we have an isomorphism

$$L_1 : H^{1,2} \xrightarrow{\approx} H^{-1,2}(\mathbb{T}^n). \tag{5.46}$$

Suppose v satisfies the hypotheses of (5.41). Then

$$L_1 v \in H^{-1,\sigma}(\mathbb{T}^n) + L^s(\mathbb{T}^n) \subset H^{-1,\sigma}(\mathbb{T}^n), \tag{5.47}$$

if σ satisfies (5.42), since $L^s(\mathbb{T}^n) \subset H^{-1,\sigma}(\mathbb{T}^n)$ as long as (5.40) and (5.42) hold, with γ taken small enough. Therefore, to prove Proposition 5.5, it suffices to prove the following variant.

Proposition 5.7. *Take L and s as in Proposition 5.5 and L_1 as in (5.45). Then, for $\sigma \in (1, \infty)$,*

$$v \in L^s(\mathbb{T}^n), \quad L_1 v \in H^{-1,\sigma}(\mathbb{T}^n) \Longrightarrow v \in H^{1,\sigma}(\mathbb{T}^n). \tag{5.48}$$

We can make a further reduction of this result, using the fact that, for such L_1, the isomorphism (5.46) generalizes to

$$L_1 : H^{1,\sigma}(\mathbb{T}^n) \xrightarrow{\approx} H^{-1,\sigma}(\mathbb{T}^n), \quad \forall\, \sigma \in (1, \infty). \tag{5.49}$$

In fact, this holds with the regularity hypothesis (5.39) replaced by (5.44). See Proposition 1.9 in Chapter 3 of [T3]. When this holds and v satisfies the hypotheses of (5.48), in particular $L_1 v = f \in H^{1,\sigma}(\mathbb{T}^n)$, we can subtract off $v_1 \in H^{1,\sigma}(\mathbb{T}^n)$ such that $L_1 v_1 = f$, and reduce Proposition 5.7 to the following.

Lemma 5.8. *Take L and s as in Proposition 5.5 and L_1 as in (5.45). Assume*

$$v \in L^s(\mathbb{T}^n), \quad L_1 v = 0. \tag{5.50}$$

Then

$$v \in H^{1,\sigma}(\mathbb{T}^n), \tag{5.51}$$

for some (hence each) $\sigma \in (1, \infty)$, hence $v = 0$.

Parallel to Proposition 5.2, we want to rephrase Lemma 5.8 in terms involving a zero-order pseudodifferential operator, namely

$$B_1 = A^{-1} L_1 A^{-1} = B - A^2. \tag{5.52}$$

The translated result becomes the following. For s as in (5.40),

$$w \in H^{-1,s}(\mathbb{T}^n), \quad B_1 w = 0 \tag{5.53}$$

implies

$$w \in L^\sigma(\mathbb{T}^n), \tag{5.54}$$

for some $\sigma \in (1, \infty)$, hence each $\sigma \in (1, \infty)$. (And this in turn leads to $w = 0$.)

Remark. There is no harm in placing an upper bound on s, so we will strengthen (5.40) to the hypothesis that

$$r' < s < \frac{n}{n-1}. \tag{5.55}$$

To tackle the demonstration that (5.53) $\Rightarrow$ (5.54), we treat B_1 as an elliptic pseudodifferential operator with rough symbol, though the details are necessarily different from those arising in the proof of Proposition 5.2. This time we have (5.18), or, equally pertinent for current purposes,

$$A_j(x,\xi) \in H^{1+\varepsilon,n} S^0_{1,0}. \tag{5.56}$$

We apply a "symbol smoothing" to write

$$A_j(x,\xi) = A_j^{\#}(x,\xi) + A_j^b(x,\xi). \tag{5.57}$$

This process is described in §1.3 of [T2] and in §8 of Chapter 1 in [T3]. It follows from Proposition 8.2 of the latter reference that we can pick $\delta \in (0,1)$ (subject to the condition (5.63) below), and achieve the decomposition (5.57), with

$$A_j^{\#}(x,\xi) \in S^0_{1,\delta}, \tag{5.58}$$

and

$$A_j^b(x,\xi) \in H^{1+\varepsilon,n} S^{-\varepsilon\delta}_{1,\delta}. \tag{5.59}$$

Regarding the symbol classes arising in (5.58)–(5.59), for $m \in \mathbb{R}$, $\delta \in [0,1)$, we say

$$p(x,\xi) \in S^m_{1,\delta} \Longleftrightarrow |D_x^\beta D_\xi^\alpha p(x,\xi)| \le C_{\alpha\beta} \langle \xi \rangle^{m-|\alpha|+\delta|\beta|}, \tag{5.60}$$

for all α, β. Such classes were introduced in classical work of Hörmander. Regarding the operator class $OPS^m_{1,\delta}$, one has, for $m \in \mathbb{R},\ \delta \in [0,1)$,

$$p^{\#}(x,\xi) \in S^m_{1,\delta} \Longrightarrow p^{\#}(x,D) : H^{s+m,p} \to H^{s,p}, \quad \forall\, s \in \mathbb{R},\ p \in (1,\infty). \tag{5.61}$$

Also, $OPS^m_{1,\delta}$ has a complete symbol calculus, reviewed in Chapter 0 of [T2].

Going further, we say

$$\begin{aligned} p(x,\xi) \in H^{\tau,q} S^m_{1,\delta} \Leftrightarrow |D_\xi^\alpha p(x,\xi)| &\le C_\alpha \langle \xi \rangle^{m-|\alpha|}, \\ \|D_\xi^\alpha p(\cdot,\xi)\|_{H^{\tau,q}} &\le C_\alpha \langle \xi \rangle^{m-|\alpha|+\delta\tau}. \end{aligned} \tag{5.62}$$

Here we take $q \in (1,\infty)$ and assume $(1-\delta)\tau > n/q$, which in the setting (5.57)–(5.59) requires

$$(1-\delta)(1+\varepsilon) > 1. \tag{5.63}$$

This latter class of symbols was introduced in [Ma], which also established the following Sobolev-space mapping properties (Theorem 2.2 of [Ma]):

Proposition A. *Given* $p(x,\xi) \in H^{\tau,q}S^m_{1,\delta}$, $\delta \in [0,1)$, $(1-\delta)\tau > n/q$, $q, p \in (1,\infty)$, $s, m \in \mathbb{R}$, *we have*

$$p(x,D) : H^{s+m,p} \longrightarrow H^{s,p}, \tag{5.64}$$

for

$$n\Bigl(\frac{1}{p}+\frac{1}{q}-1\Bigr)^+ - (1-\delta)\tau < s \le \tau - n\Bigl(\frac{1}{q}-\frac{1}{p}\Bigr)^+. \tag{5.65}$$

In particular, if

$$q \ge p \quad \text{and} \quad q \ge p', \quad \text{i.e.,} \quad q' \le p \le q, \tag{5.66}$$

then (5.64) holds for

$$-(1-\delta)\tau < s \le \tau. \tag{5.67}$$

We will use symbol smoothing and Proposition A to prove the implication (5.53) $\Rightarrow$ (5.54). Alternatively, at this point we could deduce this implication from Theorem 9 of [Ma2]. However, showing that Theorem 9 applies would involve some of the same steps as taken below, so we take this slightly longer path.

If we apply Proposition A to $A^b_j(x,\xi)$, satisfying (5.59), then we would take $q = n$, $p = s$, and the condition (5.66) would become

$$\frac{n}{n-1} \le s \le n, \tag{5.68}$$

which is not consistent with our need to allow $s < n/(n-1)$. Thus we need to refine our approach a bit.

To proceed, we refine the inclusion $H^{1+\varepsilon,n} \subset H^{1,r}$ in (5.39) to

$$H^{1+\varepsilon,n}(\mathbb{T}^n) \subset H^{1+\gamma,r}(\mathbb{T}^n), \quad r > n, \ \gamma > 0. \tag{5.69}$$

Then we have

$$A_j(x,\xi) \in H^{1+\gamma,r}S^0_{1,0}, \tag{5.70}$$

which leads to a decomposition of the form (5.57) with

$$A^b_j(x,\xi) \in H^{1+\gamma,r}S^{-\mu\delta}_{1,\delta}, \quad \mu = 1+\gamma-\frac{n}{r} > \gamma. \tag{5.71}$$

The condition $(1-\delta)\tau > n/q$ for applicability of Proposition A becomes

$$(1-\delta)(1+\gamma) > \frac{n}{r}. \tag{5.72}$$

Given this, we can apply Proposition A to get

$$A_j^b(x,D) : H^{\nu-\mu\delta,s}(\mathbb{T}^n) \longrightarrow H^{\nu,s}(\mathbb{T}^n), \tag{5.73}$$

provided

$$-(1-\delta)(1+\gamma) < \nu \le 1+\gamma, \tag{5.74}$$

assuming (per (5.66)) that

$$r' \le s \le r. \tag{5.75}$$

Since

$$r > n \Longrightarrow r' < \frac{n}{n-1}, \tag{5.76}$$

this allows applicability to s in the range $(r', n/(n-1))$.

Considering how we want to apply this result, we now see how to choose δ. We choose $\delta > 0$, depending on $\gamma > 0$, so that

$$(1-\delta)(1+\gamma) > 1, \tag{5.77}$$

which in turn implies (5.72), and also implies that (5.73) holds for

$$-1 \le \nu < 1+\gamma. \tag{5.78}$$

We now tackle the demonstration that (5.53) $\Rightarrow$ (5.54). Write

$$B_1 = B_1^{\#} + B_1^b, \tag{5.79}$$

with

$$B_1^{\#} = \sum_j R_j A_j^{\#}(x,D) \in OPS^0_{1,\delta}, \quad \text{elliptic}, \tag{5.80}$$

and

$$B_1^b = \sum_j R_j A_j^b(x,D) + A^{-2}. \tag{5.81}$$

We have from (5.73)–(5.74) that

$$B_1^b : H^{\nu-\mu\delta,s}(\mathbb{T}^n) \longrightarrow H^{\nu,s}(\mathbb{T}^n), \tag{5.82}$$

for ν in the range (5.78), s satisfying (5.75).

Now take w satisfying (5.53), so

$$w \in H^{-1,s}(\mathbb{T}^n) \ \text{ and } \ B_1^{\#} w = -B_1^b w. \tag{5.83}$$

Given (5.80), we can construct

$$E^{\#} \in OPS^0_{1,\delta} \ \text{ such that } \ E^{\#} B_1^{\#} - I \in OPS^{-\infty}. \tag{5.84}$$

Thus (5.82) implies

$$w = -E^{\#} B_1^b w, \quad \text{mod } C^\infty(\mathbb{T}^n). \tag{5.85}$$

From (5.82) we have

$$\begin{aligned} &w \in H^{-1,s}(\mathbb{T}^n) \Rightarrow E^{\#} B_1^b w \in H^{-1+\mu\delta,s}(\mathbb{T}^n) \\ &\Rightarrow w \in H^{-1+\mu\delta,s}(\mathbb{T}^n) \Rightarrow E^{\#} B_1^b w \in H^{-1+2\mu\delta,s}(\mathbb{T}^n) \\ &\Rightarrow \cdots \\ &\Rightarrow w \in H^{1,s}(\mathbb{T}^n), \end{aligned} \tag{5.86}$$

for s satisfying (5.55). This is well more regularity than is needed for the conclusion in (5.50), though of course it leads to the same ultimate conclusion, namely $w = 0$.

This completes the proof of Lemma 5.8, hence of Proposition 5.7, and therefore of Theorem 1.2.

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