

Linear Algebra

MICHAEL TAYLOR

Linear algebra is an important gateway connecting elementary mathematics to more advanced subjects, such as multivariable calculus, systems of differential equations, differential geometry, and group representations. The purpose of this work is to provide a treatment of this subject in sufficient depth to prepare the reader to tackle such further material.

We start with vector spaces over the set \mathbb{R} of real numbers or the set \mathbb{C} of complex numbers, and linear transformations between such vector spaces. We treat these two cases simultaneously, and use the label \mathbb{F} to apply to either \mathbb{R} or \mathbb{C} .

Later on we consider vector spaces over general *fields*, denoted \mathbb{F} , and the reader can appreciate the early material on this more general level with minimal effort. Going further, we extend the theory of vector spaces over a field to that of modules over a ring, and consider new phenomena that arise in this expanded setting.

Features of our development include a clean treatment of determinants, based on three simple rules as a definition, rather than on a relatively inscrutable defining formula. We also emphasize contact between linear algebra and geometry and analysis, including such topics as spectral theory of self-adjoint, skew-adjoint, orthogonal, and unitary transformations, and the matrix exponential.

Contents

Chapter 1. Vector spaces, linear transformations, and matrices

1. Vector spaces
2. Linear transformations and matrices
3. Basis and dimension
4. Matrix representation of a linear transformation
5. Determinants and invertibility
6. Applications of row reduction and column reduction

Chapter 2. Eigenvalues, eigenvectors, and generalized eigenvectors

1. Eigenvalues and eigenvectors
2. Generalized eigenvectors and the minimal polynomial
3. Triangular matrices and upper triangularization
4. The Jordan canonical form

Chapter 3. Linear algebra on inner product spaces

1. Inner products and norms
2. Norm, trace, and adjoint of a linear transformation

3. Self-adjoint and skew-adjoint transformations
4. Unitary and orthogonal transformations
5. Schur's upper triangular representation
6. Polar decomposition and singular value decomposition
7. The matrix exponential
8. The discrete Fourier transform

Chapter 4. Further basic concepts: duality, convexity, positivity

1. Dual spaces
2. Convex sets
3. Quotient spaces
4. Positive matrices and stochastic matrices

Chapter 5. Multilinear algebra

1. Multilinear mappings
2. Tensor products
3. Exterior algebra
4. Isomorphism $\text{Skew}(V) \approx \Lambda^2 V$ and the Pfaffian

Chapter 6. Linear algebra over more general fields

1. Vector spaces over more general fields
2. Rational matrices and algebraic numbers

Chapter 7. Rings and modules

1. Rings and modules
2. Modules over principal ideal domains
3. The Jordan canonical form revisited
4. Integer matrices and algebraic integers
5. Noetherian rings and Noetherian modules
6. Polynomial rings over UFDs

Chapter 8. Special structures in linear algebra

1. Quaternions and matrices of quaternions
2. Algebras
3. Clifford algebras
4. Octonions

Appendix A. Complementary results

1. The fundamental theorem of algebra
2. Averaging rotations
3. Groups
4. Finite fields and other field extensions