WORKSHEETS for MATH 521
Introduction to Analysis in One Variable
Chapter 4, Calculus, plus Section 5.4

Instructor: Michael Taylor

Contents

Introduction

These worksheets were produced as an aid for the study of Chapter 4, "Calculus" (plus one section from Chapter 5) in the text for the course, *Introduction to Analysis in One Variable*, by M. Taylor. They were designed so that each worksheet covers the material of one lecture. Each worksheet deals with material in a designated section of the text, and the idea is that a student can do the exercises in a worksheet, in consultation with the text, and in that manner master the material in the text.

These worksheets were produced in response to the health crisis of 2020. They are dated, to correspond to a class meeting three times a week. In addition to the dated worksheets, there are three supplementary worksheets, covering material that fills out the development of calculus.

Worksheet 1, Monday, 03/23

§4.1, The Derivative (review)

1. Take $f : [a, b] \to \mathbb{R}$. Define what it means for f to be differentiable at $x \in (a, b)$, with derivative $f'(x)$.

2. State the product rule, and use it to show that

$$\frac{d}{dx} x^n = n x^{n-1}, \quad n \in \mathbb{N}.$$

3. State the chain rule, and use it to show that

$$\frac{d}{dx} f(x)^n = n f(x)^{n-1} f'(x), \quad n \in \mathbb{N}.$$

4. State the Mean Value Theorem, and use it to show that, for $f : [a, b] \to \mathbb{R}$,

$$f'(x) \equiv 0 \Longrightarrow f \text{ constant}.$$

What role does the notion of compactness play in the proof of the Mean Value Theorem?

5. State the Inverse Function Theorem, and use it to show that

$$\frac{d}{dx} x^{1/n} = \frac{1}{n} x^{1/n-1}, \quad n \in \mathbb{N},\ x > 0.$$

6. Going further, show that

$$\frac{d}{dx} x^r = r x^{r-1}, \quad r \in \mathbb{Q},\ x > 0.$$

Worksheet 2, Wednesday, 03/25

§4.2, The Integral (part 1)

1. Given $f : [a,b] \to \mathbb{R}$, define what it means for f to be Riemann integrable, i.e., $f \in \mathcal{R}([a,b])$, and define

$$\int_a^b f(x)\,dx,$$

for $f \in \mathcal{R}([a,b])$.

2. What role does uniform continuity play in the proof that

$$f \in C([a,b]) \Longrightarrow f \in \mathcal{R}([a,b])?$$

3. State the Darboux theorem, and use it to show that

$$\int_0^1 (x^2 - x)\,dx = \lim_{\nu\to\infty} \sum_{k=1}^{\nu} \Big(\frac{k^2}{\nu^2} - \frac{k}{\nu}\Big) \cdot \frac{1}{\nu}.$$

By the way, how do you know that $f(x) = x^2 - x$ is Riemann integrable?

4. Give an example of a bounded function $f : [0,1] \to \mathbb{R}$ that is not Riemann integrable.

5. State the Fundamental Theorem of Calculus. This has two parts. Which part makes use of the Mean Value Theorem?

Worksheet 3, Friday, 03/27

§4.2, The Integral (part 2)

1. Given a set $S \subset I = [a, b]$, define its outer measure,

$$m^*(S).$$

State the Riemann inegrability criterion, Proposition 4.2.12.
Show that

$$S \text{ countable } \Longrightarrow m^*(S) = 0.$$

2. Which of the following classes of functions $f : I \to \mathbb{R}$ are contained in $\mathcal{R}(I)$?
(a) $f : I \to \mathbb{R}$ bounded, with at most countably many points of discontinuity,
(b) $f : I \to \mathbb{R}$ bounded and monotone,
(c) $f : I \to \mathbb{R}$ discontinuous at each point,
(d) $f = gh, \quad g, h \in \mathcal{R}(I)$.

3. Let $\varphi : [a, b] \to [A, B]$ be C^1 on a neighborhood of $[a, b]$, with $\varphi'(x) > 0$ for $x \in [a, b]$. Assume $\varphi(a) = A, \ \varphi(b) = B$. Show that the identity

$$\int_A^B f(y)\, dy = \int_a^b f(\varphi(t))\varphi'(t)\, dt$$

for each $f \in C([A, B])$, follows from the chain rule and the fundamental theorem of calculus. This identity is called the *change of variable formula* for the integral.
Hint. Replace b by x, B by $\varphi(x)$, and differentiate. Compare the proof of Theorem 4.2.7.

4. Show that, if $f, g \in C^1$ on a neighborhood of $[a, b]$, then

$$\int_a^b f(s)g'(s)\, ds = -\int_a^b f'(s)g(s)\, ds + \big[f(b)g(b) - f(a)g(a)\big].$$

This transformation is called *integration by parts*.
Apply Theorem 4.2.7 to $G(x) = f(x)g(x)$.

Supplementary Worksheet A, Week ending 03/27

§4.2, Supplements on the integral

1. Use the fundamental theorem of calculus and results of Worksheet 1 to compute

$$\int_a^b x^r\,dx, \quad r \in \mathbb{Q} \setminus \{-1\},$$

where $-\infty < a < b < \infty$ if $r \in \mathbb{N}$ and $0 < a < b < \infty$ otherwise.

2. Use the change of variable formula to compute

$$\int_0^1 x\sqrt{1+x^2}\,dx.$$

3. Use the change of variable formula to show that, for $N > 0$,

$$\int_1^2 x^{-1}\,dx = \int_N^{2N} x^{-1}\,dx.$$

4. We say $f \in \mathcal{R}(\mathbb{R})$ provided $f|_{[-k,k]} \in \mathcal{R}([-k,k])$ for each $k \in \mathbb{N}$ and there exists $A < \infty$ such that

$$\int_{-k}^k |f(x)|\,dx \le A, \quad \forall\, k.$$

or equivalently, if and only if

$$\sum_{k=-\infty}^{\infty} \int_k^{k+1} |f(x)|\,dx < \infty.$$

If $f \in \mathcal{R}(\mathbb{R})$, we set

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{k\to\infty} \int_{-k}^k f(x)\,dx.$$

Formulate basic properties of the integral over $\mathbb{R}$ of elements of $\mathcal{R}(\mathbb{R})$, parallel to properties of the integral over intervals $[a,b]$ given in this section.

Similarly define $\mathcal{R}(\mathbb{R}^+)$.

5. This exercise discusses the *integral test* for absolute convergence of an infinite series, which goes as follows. Let f be a positive, monotonically decreasing, continuous function on $[0, \infty)$, and suppose $|a_k| = f(k)$. Then

$$\sum_{k=0}^{\infty} |a_k| < \infty \Longleftrightarrow \int_0^{\infty} f(x)\, dx < \infty.$$

Prove this.
Hint. Use

$$\sum_{k=1}^{\infty} |a_k| \le \int_0^{\infty} f(x)\, dx \le \sum_{k=0}^{\infty} |a_k|.$$

6. Use the integral test to show that, if $p > 0$,

$$\sum_{k=1}^{\infty} \frac{1}{k^p} < \infty \Longleftrightarrow p > 1.$$

For now, take $p \in \mathbb{Q}^+$. Results of §4.5 allow one to take $p \in \mathbb{R}^+$.
Hint. Use Exercise 1 to evaluate $I_N(p) = \int_1^N x^{-p}\, dx$, for $p \neq -1$, and let $N \to \infty$. See if you can show that $\int_1^{\infty} x^{-1}\, dx = \infty$ without knowing about $\log N$, making use of Exercise 3.

Worksheet 4, Monday, 03/30

§4.3, Power series (part 1)

1. Given a power series

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k,$$

with $a_k, z, z_0 \in \mathbb{C}$,

(a) Define the radius of convergence R of this series.

For $S > 0$, set

$$D_S(z_0) = \{z \in \mathbb{C} : |z - z_0| < S\}.$$

(b) On what disks $D_S(z_0)$ does the power series above converge uniformly?

(c) How does the result of (b) imply that f is continuous on $D_R(z_0)$?

2. In Exercise 1, take $z_0 = 0$ and restrict attention to

$$f(t) = \sum_{k=0}^{\infty} a_k t^k, \quad t \in (-R, R).$$

Still allow $a_k \in \mathbb{C}$. What can you say about the derivative

$$f'(t), \quad t \in (-R, R)?$$

In particular, what power series is it given by? Discuss how to establish this by integrating a power series

$$g(t) = \sum_{k=0}^{\infty} b_k t^k$$

term by term.

3. Give the power series for

$$f(t) = \frac{1}{1-t}, \quad t \in (-1, 1),$$

and write down the power series for $f'(t)$.

4. Write down the power series for

$$\frac{1}{1+t}, \quad \frac{1}{1-t^2}, \quad \frac{1}{1+t^2}.$$

5. State the ratio test. Use it to find the radius of convergence of

$$e(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!}.$$

Find the power series of $e'(t)$.

6. Extend the treatment of power series in Exercise 2 to

$$f(t) = \sum_{k=0}^{\infty} a_k (t-t_0)^k, \quad t \in (t_0 - R, t_0 + R).$$

Show that, if this holds, then

$$a_k = \frac{f^{(k)}(t_0)}{k!}.$$

Worksheet 5, Wednesday, 04/01

§4.3, Power series (part 2)

1. Taking

$$f(t) = (1-t)^{-r}, \quad t \in (-1,1), \quad r \in \mathbb{Q},$$

show that if $f(t)$ is given by a convergent power series

$$\sum_{k=0}^{\infty} b_k t^k,$$

then

$$b_k = \frac{1}{k!} r(r+1)\cdots(r+k-1).$$

Show that if

$$g(t) = \sum_{k=0}^{\infty} b_k t^k,$$

with b_k given as above, then the ratio test implies this power series has radius of convergence $R = 1$, so it converges for $t \in (-1,1)$.

2. Given $f \in C^{n+1}((a,b))$, $y \in (a,b)$, write

$$f(x) = f(y) + f'(y)(x-y) + \cdots + \frac{f^{(n)}(y)}{n!}(x-y)^n + R_n(x,y).$$

Apply d/dy to both sides, and observe massive cancellation, to deduce that

$$\frac{\partial R_n}{\partial y}(x,y) = -\frac{1}{n!} f^{(n+1)}(y)(x-y)^n, \quad R_n(x,x) = 0.$$

Observe in §4.3 of the text how applying the fundamental theorem of calculus yields

$$R_n(x,y) = \frac{1}{n!}\int_y^x (x-s)^n f^{(n+1)}(s)\, ds.$$

This is called the *integral formula for the remainder* in the power series for f about y.

3. State the Cauchy formula and the Lagrange formula for the remainder $R_n(x,y)$, defined in Exercise 2.

(a) Which formula is shorter and neater?

(b) Which formula is more powerful?

4. Read the analysis in §4.3 of how one can use the Cauchy formula for the remainder to show that the power series produced in Exercise 1 above actually converges to $(1-t)^{-r}$, for $t \in (-1,1)$.

See also the last exercise at the end of §4.3 for an alternative approach, avoiding remainder formulas.

5. Given the results of Exercises 1 and 4, produce power series for

$$\frac{1}{\sqrt{1-t}}, \quad \frac{1}{\sqrt{1-t^2}}, \quad \frac{1}{\sqrt{1+t}}, \quad \frac{1}{\sqrt{1+t^2}}, \quad t \in (-1,1).$$

Hint.

$$f(t) = \sum_{k=0}^{\infty} b_k t^k \Longrightarrow f(t^2) = \sum_{k=0}^{\infty} b_k t^{2k}.$$

Worksheet 6, Friday, 04/03

§4.4, Curves and arc length

1. Let $\gamma : I \to \mathbb{R}^n$ be a C^1 curve, $I = [a, b]$. Write down the integral formula for the length $\ell(\gamma)$ of this curve.

2. Suppose $u : [\alpha, \beta] \to [a, b]$ is a C^1 map with C^1 inverse, and consider the curve

$$\sigma = \gamma \circ u : [\alpha, \beta] \longrightarrow \mathbb{R}^n.$$

Write down the integral formula for $\ell(\sigma)$. Use the change of variable formula for integrals to show that

$$\ell(\sigma) = \ell(\gamma).$$

We say that σ is a reparametrization of the curve γ.

3. We say that σ is a parametrization by arc length (or a unit speed parametrization) if

$$|\sigma'(t)| \equiv 1.$$

Discuss the reparametrization of a curve γ by arc length, when γ' is nowhere vanishing. Highlight the role of the Inverse Function Theorem in this reparametrization.

4. Consider the unit circle

$$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

Show that the upper half of this circle is parametrized by

$$\gamma_+ : (-1, 1) \longrightarrow \mathbb{R}^2, \quad \gamma_+(t) = (t, \sqrt{1 - t^2}).$$

If $\ell(t)$ denotes the length of the arc $\gamma_+([0, t])$, show that

$$\ell(t) = \int_0^t \frac{ds}{\sqrt{1 - s^2}}, \quad \text{for} \quad 0 < t < 1.$$

5. In the context of Exercise 4 above, make use of Exercise 5 in Worksheet 5 to write $\ell(t)$ as a power series

$$\ell(t) = \sum_{k=0}^{\infty} c_k t^k, \quad 0 < t < 1.$$

Write down the coefficients c_k.

6. Read the material in §4.4 regarding the unit speed parametrization of the circle S^1 given by

$$C(t) = (\cos t, \sin t), \quad C(0) = (1, 0), \quad C'(0) = (0, 1).$$

Another approach to this, with deep connections to the exponential function, will be explored in §4.5.

Supplementary Worksheet B, Week ending 04/03

§4.4, Supplement on curves and arc length

Recall from Worksheet 6 the introduction of the trigonometric functions $\cos t$ and $\sin t$ as providing a unit speed parametrization of the circle S^1,

$$C(t) = (\cos t, \sin t), \quad C(0) = (1, 0), \quad C'(0) = (0, 1).$$

1. Apply d/dt to the identity $C(t) \cdot C(t) \equiv 1$ to get

$$C'(t) \cdot C(t) \equiv 0,$$

and review the argument in §4.4 leading to

$$C'(t) = (-\sin t, \cos t).$$

Using this, compute the derivatives $c^{(k)}(t)$ and $s^{(k)}(t)$, where $c(t) = \cos t$, $s(t) = \sin t$. Evaluate these at $t = 0$.

2. Using Exercise 1 above and Exercise 6 of Worksheet 4, show that if $\cos t$ and $\sin t$ are given by power series, then

$$\cos t = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} t^{2k}, \quad \sin t = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} t^{2k+1}.$$

3. Define the "remainder terms" $C_{2n}^b(t)$ and $S_{2n+1}^b(t)$ by

$$\cos t = \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} t^{2k} + C_{2n}^b(t),$$
$$\sin t = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} t^{2k+1} + S_{2n+1}^b(t).$$

Use the remainder formulas discussed in Exercises 2–3 of Worksheet 5 to show that

$$C_{2n}^b(t) = \pm \frac{t^{2n+1}}{(2n+1)!} \sin \xi_n,$$
$$S_{2n+1}^b(t) = \pm \frac{t^{2n+2}}{(2n+2)!} \sin \zeta_n,$$

for some $\xi_n, \zeta_n \in [-|t|, |t|]$. Deduce that

$$C^b_{2n}(t),\ S^b_{2n+1}(t) \longrightarrow 0, \quad \text{as} \ \ n \to \infty,$$

uniformly for t in a bounded set. Deduce that $\cos t$ and $\sin t$ actually are given by the power series in Exercise 2, for all $t \in \mathbb{R}$.

Remark. A completely different approach to such results for $\cos t$ and $\sin t$ is given in §4.5; see Worksheet 9. The approach there does not require the use of remainder formulas.

Worksheet 7, Monday, 04/06
Review for Test, W 04/08

Covering §§4.1–4.4 and Selected Parts of Chapters 2–3

1. For $x, y \in \mathbb{R}^n$,
   (a) define the dot product, $x \cdot y$,
   (b) define the Euclidean norm $|x|$,
   (c) state the triangle inequality,
   (d) state Cauchy's inequality. What is its relevance to the triangle inequality?

2. Define the following concepts:
   (a) metric space,
   (b) complete metric space,
   (c) compact metric space.

3. Let X and Y be metric spaces, and assume X is compact. Take

$$f : X \longrightarrow Y, \quad \text{continuous.}$$

   (a) Show that f is uniformly continuous.
   (b) Assume f is one-to-one and onto. Show that Y is compact and that

$$f^{-1} : Y \longrightarrow X \quad \text{is continuous.}$$

4. State the Weierstass M-test, and apply it to

$$\sum_{k=1}^{\infty} \frac{z^k}{k^2}, \quad z \in \overline{D_1(0)} = \{z \in \mathbb{C} : |z| \le 1\}.$$

5. Review Worksheets 1–6, for material on calculus.

Worksheet 8, Monday, 04/13

§4.5, Exponential and trigonometric functions (part 1)

1. Use the ratio test to show that

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

is absolutely convergent for all $z \in \mathbb{C}$. We also denote the sum by $\exp(z)$. Deduce that the series converges uniformly on each disk $D_R(0)$, and that $\exp : \mathbb{C} \to \mathbb{C}$ is continuous.

2. Differentiate

$$e^{at} = \sum_{k=0}^{\infty} \frac{a^k}{k!} t^k$$

term by term and show that

$$\frac{d}{dt} e^{at} = a e^{at}, \quad t \in \mathbb{R},\ a \in \mathbb{C}.$$

3. Use the product rule to compute

$$\frac{d}{dt} e^{at} e^{-at}, \quad \frac{d}{dt}\left[e^{(a+b)t} e^{-at} e^{-bt}\right].$$

Show that

$$e^{-at} = \frac{1}{e^{at}}, \quad e^{(a+b)t} = e^{at} e^{bt}, \quad \forall\, t \in \mathbb{R},\ a, b \in \mathbb{C}.$$

4. Show that

$$e^t > 0, \quad \frac{d}{dt} e^t > 0, \quad \forall\, t \in \mathbb{R},$$

and that

$$\lim_{t \to +\infty} e^t = +\infty, \quad \lim_{t \to -\infty} e^t = 0.$$

Deduce that

$$\exp : \mathbb{R} \longrightarrow (0, \infty) \ \text{ is one-to-one and onto.}$$

Use the Inverse Function Theorem to deduce that it has an inverse

$$L : (0, \infty) \longrightarrow \mathbb{R},$$

satisfying

$$L'(x) = \frac{1}{x}, \quad \forall\, x > 0.$$

We denote this inverse by

$$\log x = L(x), \quad x > 0.$$

5. Use the fundamental theorem of calculus to show that

$$\log x = \int_1^x \frac{dy}{y}, \quad x > 0,$$

hence

$$\log(1+x) = \int_0^x \frac{dt}{1+t}, \quad x > -1.$$

Look at Exercise 4 of Worksheet 4 and integrate the power series for $1/(1+t)$ term by term to ontain a power series for $\log(1+x)$, valid for $x \in (-1, 1)$.

6. For $x > 0$ and $r \in \mathbb{C}$, define

$$x^r = e^{r \log x}.$$

Show that

$$x^{r+s} = x^r x^s, \quad x^{nr} = (x^r)^n = (x^n)^r,$$

for $x > 0,\ r, s \in \mathbb{C},\ n \in \mathbb{Z}$, and that

$$\frac{d}{dx} x^r = r x^{r-1},$$

Show that the case $r = 1/n$ of x^r defined here, for $n \in \mathbb{N}$, coincides with $x^{1/n}$ as it arose in Worksheet 1. Also show that

$$\int_1^x y^{r-1}\, dy = \frac{x^r - 1}{r}, \quad \text{if } x > 0,\ r \neq 0.$$

Pass to the limit $r \to 0$ to recover the formula for $\log x$ in Exercise 5.

Worksheet 9, Wednesday, 04/15

§4.5, Exponential and trigonometric functions (part 2)

1. Given $z = x + iy,\ x, y \in \mathbb{R}$, show that

$$|e^z|^2 = e^z e^{\overline{z}} = e^{z+\overline{z}} = e^{2x}.$$

2. We aim to analyze the planar curve

$$\gamma(t) = e^{it}, \quad t \in \mathbb{R}.$$

Show that

$$|\gamma(t)| = 1, \quad \forall\, t \in \mathbb{R},$$

and that

$$\gamma'(t) = ie^{it}, \quad \text{hence } |\gamma'(t)| = 1, \ \ \forall\, t \in \mathbb{R}.$$

Deduce that $\gamma(t)$ is a unit speed parametrization of the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, satisfying

$$\gamma(0) = 1, \quad \gamma'(0) = i.$$

3. Deduce from the definitions of the trigonometric functions $\sin t$ and $\cos t$ that

$$e^{it} = \cos t + i \sin t.$$

This is called Euler's formula.

4. Deduce from Euler's formula that

$$\frac{d}{dt} e^{it} = ie^{it}$$

implies

$$\frac{d}{dt} \cos t = -\sin t, \quad \frac{d}{dt} \sin t = \cos t,$$

and that

$$e^{i(s+t)} = e^{is} e^{it}$$

implies

$$\cos(s+t) = \cos s \cos t - \sin s \sin t,$$
$$\sin(s+t) = \sin s \cos t + \cos s \sin t.$$

Deduce from Euler's formula that

$$\cos t = \frac{1}{2}(e^{it} + e^{-it}), \quad \sin t = \frac{1}{2i}(e^{it} - e^{-it}).$$

Also use Euler's formula to give another derivation of the power series for $\cos t$ and $\sin t$ presented in Exercise 2 of Supplementary Worksheet B.

5. Define π to be half the length of the unit circle, or equivalently the smallest positive number such that

$$e^{\pi i} = -1.$$

Show that

$$e^{2\pi i} = 1, \quad e^{\pi i/2} = i, \quad e^{\pi i/4} = \frac{1+i}{\sqrt{2}},$$
$$e^{\pi i/3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad e^{\pi i/6} = \frac{\sqrt{3}}{2} + \frac{i}{2}.$$

6. Define the *hyperbolic functions*

$$\cosh u = \frac{1}{2}(e^{u} + e^{-u}), \quad \sinh u = \frac{1}{2}(e^{u} - e^{-u}).$$

Show that

$$\frac{d}{du}\cosh u = \sinh u, \quad \frac{d}{du}\sinh u = \cosh u,$$

and that

$$\cosh^2 u - \sinh^2 u = 1.$$

Show that

$$\sinh : \mathbb{R} \longrightarrow \mathbb{R}$$

is one-to-one and onto. Denote its inverse by

$$\sinh^{-1} : \mathbb{R} \longrightarrow \mathbb{R}.$$

Worksheet 10, Friday, 04/17

§4.5, Exponential and trigonometric functions (part 3)

1. Show that

$$\sin : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow (-1,1)$$

is one-to-one and onto. Denote its inverse by

$$\sin^{-1} : (-1,1) \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Making the change of variable $s = \sin t$, show that

$$\int_0^x \frac{ds}{\sqrt{1-s^2}} = \sin^{-1} x, \quad \text{for} \quad |x| < 1.$$

Deduce from Exercise 5 of Worksheet 9 that $\sin(\pi/6) = 1/2$, and hence

$$\frac{\pi}{6} = \int_0^{1/2} \frac{ds}{\sqrt{1-s^2}}.$$

Referring to Exercises 4–5 of Worksheet 6, obtain from this a rapidly convergent infinite series for π. Show that

$$\pi \approx 3.1415926535 \cdots .$$

2. Define

$$\tan t = \frac{\sin t}{\cos t}, \quad \sec t = \frac{1}{\cos t},$$

for $|t| < \pi/2$. Show that

$$\frac{d}{dt} \tan t = \sec^2 t, \quad \frac{d}{dt} \sec t = \sec t \tan t,$$

and

$$1 + \tan^2 t = \sec^2 t.$$

3. Show that

$$\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow \mathbb{R}$$

is one-to-one and onto. Denote its inverse by

$$\tan^{-1} : \mathbb{R} \longrightarrow \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big).$$

Making the change of variable $s = \tan t$, show that

$$\int_0^x \frac{ds}{1+s^2} = \tan^{-1} x, \quad x \in \mathbb{R}.$$

Deduce from Exercise 6 of Worksheet 7 that $\tan(\pi/6) = 1/\sqrt{3}$, and hence

$$\frac{\pi}{6} = \int_0^{1/\sqrt{3}} \frac{ds}{1+s^2}.$$

Integrate the power series for $1/(1+s^2)$ (valid for $|s| < 1$) term by term to obtain another rapidly convergent infinite series for π. With this in hand, again obtain an approximation to π of the form indicated in Exercise 1.

For a numerical approximation of $\sqrt{3}$, one can use

$$\sqrt{3} = \sqrt{4-1} = 2\sqrt{1-\frac{1}{4}}, \quad \text{or} \quad \sqrt{3} = \sqrt{\frac{49}{16} - \frac{1}{16}} = \frac{7}{4}\sqrt{1-\frac{1}{49}},$$

and a power series expansion of $(1-x)^{1/2}$, or Newton's method, treated in §5.5.

4. In this exercise, we evaluate

$$I(u) = \int_0^u \frac{dv}{\sqrt{1+v^2}}$$

in two ways.

(a) Using $v = \sinh y$, show that

$$I(u) = \sinh^{-1} u.$$

(b) Using $v = \tan t$, show that

$$I(u) = \int_0^{\tan^{-1} u} \sec t \, dt.$$

Deduce that

$$\int_0^x \sec t \, dt = \sinh^{-1}(\tan x), \quad \text{for} \quad |x| < \frac{\pi}{2}.$$

Deduce from this that

$$\cosh\Big(\int_0^x \sec t\,dt\Big) = \sec x,$$

hence

$$\exp\Big(\int_0^x \sec t\,dt\Big) = \sec x + \tan x.$$

5. Parametrize the parabola $y = x^2/2$ by

$$\gamma(v) = (v, v^2/2),$$

and show that the length of the parabolic arc $\gamma([0,u])$ is given by

$$L(u) = \int_0^u \sqrt{1+v^2}\,dv.$$

Using $v = \sinh y$, show that

$$L(u) = \int_0^{\sinh^{-1} u} \cosh^2 y\,dy.$$

Show that $2\cosh^2 y = 1 + \cosh 2y$, hence

$$2L(u) = \sinh^{-1} u + \frac{1}{2}\sinh 2(\sinh^{-1} u).$$

Show that $\sinh 2y = 2\sinh y\cosh y$, hence

$$\begin{aligned} 2L(u) &= \sinh^{-1} u + u\cosh(\sinh^{-1} u) \\ &= \sinh^{-1} u + u\sqrt{1+u^2}. \end{aligned}$$

6. For the integral $L(u)$ in Exercise 5, use $v = \tan t$, to write

$$L(u) = \int_0^x \sec^3 t\,dt, \quad u = \tan x.$$

Deduce from Exercise 5 that

$$\begin{aligned} 2\int_0^x \sec^3 t\,dt &= \sinh^{-1}(\tan x) + \tan x\sqrt{1+\tan^2 x} \\ &= \sinh^{-1}(\tan x) + \sec x\tan x. \end{aligned}$$

Supplementary Worksheet C, Week ending 04/17

§4.6, Unbounded integrable functions

1. Given $I = [a, b]$, $f : I \to \mathbb{R}$, define what it means for f to belong to $\mathcal{R}^{\#}(I)$, and define

$$\int_I f\,dx,$$

for $f \in \mathcal{R}^{\#}(I)$,
(a) first for $f : I \to \mathbb{R}^+$,
(b) then for general $f : I \to \mathbb{R}$.

2. Let $f : [0, 1] \to \mathbb{R}^+$ and assume f is continuous on $(0, 1]$. Show that

$$f \in \mathcal{R}^{\#}([0, 1]) \Longleftrightarrow \int_\varepsilon^1 f\,dx \ \text{ is bounded as } \ \varepsilon \searrow 0.$$

In such a case, show that

$$\int_0^1 f\,dx = \lim_{\varepsilon \searrow 0} \int_\varepsilon^1 f\,dx.$$

3. Let $a > 0$. Define $p_a : [0, 1] \to \mathbb{R}^+$ by $p_a(x) = x^{-a}$ if $0 < x \le 1$. Set $p_a(0) = 0$. Show that

$$p_a \in \mathcal{R}^{\#}([0, 1]) \Longleftrightarrow a < 1.$$

4. Compute

$$\int_0^1 \log t\,dt.$$

Hint. To compute $\int_\varepsilon^1 \log t\,dt$, first compute

$$\frac{d}{dt}(t \log t).$$

5. Given $g(s) = 1/\sqrt{1-s^2}$, show that $g \in \mathcal{R}^{\#}([-1,1])$ and that

$$\int_{-1}^{1} \frac{ds}{\sqrt{1-s^2}} = \pi.$$

Relate this to the arclength of the unit circle. See Worksheet 6, Exercise 4, and Worksheet 10, Exercise 1.

6. Given $f(t) = 1/\sqrt{t(1-t)}$, show that $f \in \mathcal{R}^{\#}([0,1])$ and that

$$\int_{0}^{1} \frac{dt}{\sqrt{t(1-t)}} = \pi.$$

Hint. Set $t = s^2$.

Worksheet 11, Monday, 04/20

§5.4, Fourier series (part 1)

1. Given $f \in C(\mathbb{T}^1)$, or more generally $f \in \mathcal{R}(\mathbb{T}^1),\ k \in \mathbb{Z}$, define

$$\hat{f}(k).$$

2. Define what it means to say

$$f \in \mathcal{A}(\mathbb{T}^1).$$

3. Assuming

$$\sum_{k=-\infty}^{\infty} |\hat{f}(k)| < \infty,$$

set

$$g(\theta) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ik\theta}.$$

Show that $g \in C(\mathbb{T}^1)$ and

$$\hat{g}(k) = \hat{f}(k), \quad \forall\, k \in \mathbb{Z}.$$

4. Two forms of the Fourier inversion formula are given in Proposition 5.4.1 and Proposition 5.4.5. State them.

5. Read the treatment of the Stone-Weierstrass theorem in §5.3 and describe how it leads to the result that the set of finite linear combinations of

$$e^{ik\theta}, \quad k \in \mathbb{Z}$$

is dense in $C(\mathbb{T}^1)$. Describe how this result leads to the proof that

$$u \in C(\mathbb{T}^1),\ \hat{u}(k) = 0\ \forall\, k \Longrightarrow u = 0,$$

and how this leads to a proof of Proposition 5.4.1, i.e., to the proof that

$$f = g,$$

given continuous $f \in \mathcal{A}(\mathbb{T}^1)$ and g as in Exercise 3.

6. Consider the function on $\mathbb{T}^1$ defined by

$$f(\theta) = |\theta|, \quad -\pi \le \theta \le \pi.$$

Compute $\hat{f}(k)$ for $k \in \mathbb{Z}$ and verify that $f \in \mathcal{A}(\mathbb{T}^1)$. Evaluate both sides of the resulting Fourier inversion formula

$$f(\theta) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ik\theta}$$

at $\theta = 0$, and use the resulting identity to show that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

Worksheet 12, Wednesday, 04/22

§5.4, Fourier series (part 2)

1. Two forms of the Plancherel identity

$$\sum_k |\hat{f}(k)|^2 = \frac{1}{2\pi} \int_{\mathbb{T}^1} |f(\theta)|^2 \, d\theta$$

are given, in Proposition 5.4.3 and Proposition 5.4.5. State them.

2. Consider the function f on $\mathbb{T}^1$ defined by

$$\begin{aligned} f(\theta) = 0 \quad & \text{for} \quad -\pi \le \theta < 0, \\ 1 \quad & \text{for} \quad 0 \le \theta < \pi. \end{aligned}$$

Compute $\hat{f}(k)$ for each $k \in \mathbb{Z}$. Evaluate the two sides of the Plancherel identity in this case, and use this to give another proof that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

3. Record the Plancherel identity for the function f given by Exercise 6 of Worksheet 11. Use this to establish that

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}.$$

4. Given $f \in \mathcal{R}(\mathbb{T}^1)$, $N \in \mathbb{N}$, set

$$S_N f(\theta) = \sum_{|k| \le N} \hat{f}(k) e^{ik\theta}.$$

Show that

$$S_N f(\theta) = \int_{\mathbb{T}^1} f(\varphi) D_N(\theta - \varphi) \, d\varphi,$$

where

$$D_N(\theta) = \frac{1}{2\pi} \sum_{k=-N}^{N} e^{ik\theta}.$$

Write $D_N(\theta)$ as $(1/2\pi)e^{-iN\theta}$ times a geometric series, and sum the geometric series to obtain

$$D_N(\theta) = \frac{1}{2\pi} \frac{\sin(N+1/2)\theta}{\sin\theta/2}.$$

5. Show that Proposition 5.4.10 implies

$$S_N f(\theta) \longrightarrow f(\theta), \quad \text{as} \quad N \to \infty,$$

for f as in Exercise 2, as long as

$$-\pi < \theta < 0 \quad \text{or} \quad 0 < \theta < \pi.$$

Take $\theta = \pi/2$ and deduce from this the identity

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

Worksheet 13, Friday, 04/24

Review of Course

Task: make sure you are on top of each of the following topics.

1. Real numbers are defined as equivalence classes of Cauchy sequences of rational numbers.

2. $\mathbb{R}, \mathbb{C}$, and $\mathbb{R}^n$ are complete metric spaces.
Nonempty, closed bounded subsets of $\mathbb{R}, \mathbb{C}$, and $\mathbb{R}^n$ are compact.
Intervals in $\mathbb{R}$ are connected.
Such phenomena are worthy of study on more general classes of metric spaces.

3. Continuous functions on a metric space X have a number of special properties:
Uniform continuity, maxima and minima achieved, etc., when X is compact.
Intermediate value theorem when X is connected.
Pathwise connected metric spaces are connected.

4. Uniform limits of continuous functions are continuous.

5. The Weierstrass M-test is a key result about infinite series of $\mathbb{R}^n$-valued functions.
It gives a sufficient condition for uniform convergence.

6. Power series with radius of convergence R converge uniformly on $D_S(0)$ for each $S < R$. The limit is continuous on $D_R(0)$.

7. And then there's calculus! See Worksheets 1–12.