Eigenfunction Concentration Results

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Abstract. This note has three vignettes dealing with spectral behavior of Laplacians and sub-Laplacians. In §1 we show that spherical harmonics $\varphi_{k\ell}$ with angular momentum ℓ satisfying $|\ell| \le (1-\beta^2)^{1/2}k$ concentrate on a strip of latitude-width 2β about the equator, as $k \to \infty$. In §2 we have a generalization on concentration of the square integrals of joint eigenfunctions of Δ and a commuting vector field.

In §3 we show that if L is a positive, self-adjoint, second-order differential operator on a compact, n-dimensional manifold M, hypoelliptic with loss of < 2 derivatives, with principal symbol L_2, satisfying

$$\int\limits_{S^*M} L_2(x,\omega)^{-n/2}\, dS(x,\omega) = \infty,$$

then the orthonormal eigenfunctions $\{\varphi_k\}$ of L concentrate microlocally on the characteristic set $\Sigma \subset T^*M \setminus 0$, given by

$$\Sigma = \{(x,\xi) \in T^*M \setminus 0 : L_2(x,\xi) = 0\},$$

(except for a sparse subset), as $k \to \infty$.

These results are studies for other works. Results of §§1–2 have been folded into [T3], where stronger results appear. This note appears on this website because §3 has not yet found a home, and I have grown weary of having to dig it up.

Contents

1. Concentration of eigenfunctions on S^n

Here we work on S^n, the unit sphere in $\mathbb{R}^{n+1}$, with its standard metric. Then the geodesic flow $\{\mathcal{G}_t\}$ is periodic of period 2π. It is convenient to take

$$\Lambda = \sqrt{-\Delta + \Bigl(\frac{n-1}{2}\Bigr)^2} - \frac{n-1}{2}, \tag{1.1}$$

so $e^{it\Lambda}$ is also periodic of period 2π (cf. (1.8) below). Then, given $A \in \mathcal{L}(L^2(S^n))$,

$$\Pi(A) = \frac{1}{2\pi}\int_0^{2\pi} e^{-it\Lambda} A e^{it\Lambda}\, dt. \tag{1.2}$$

In case $a \in C^\infty(S^*S^n)$, we have

$$Pa(x,\xi) = \frac{1}{2\pi}\int_0^{2\pi} a(\mathcal{G}_t(x,\xi))\, dt, \tag{1.3}$$

and it is a straightforward consequence of Egorov's theorem that, if $A = \mathrm{op}_F(a)$,

$$\Pi(A) - \mathrm{op}_F(Pa) \in OPS^{-1}(S^n). \tag{1.4}$$

We now specialize to the case where A is a multiplication operator,

$$Au(x) = a(x)u(x), \quad a \in C^\infty(S^n), \tag{1.5}$$

and, to keep things simple, assume that

$$n = 2, \text{ and } a(x) \text{ is invariant under } R(t), \tag{1.6}$$

where $R(t)$ is the group of rotations about the x_3-axis. Then A commutes with the associated unitary group $R(t)$ on $L^2(S^2)$, which we write as

$$R(t) = e^{itX}, \tag{1.7}$$

where $iX = Y$ is the real vector field on S^2 generating the rotation. This group is also periodic, of period 2π. We note that

$$\operatorname{Spec}\Lambda = \{k \in \mathbb{Z} : k \ge 0\}, \tag{1.8}$$

and if V_k denotes the k-eigenspace of Λ, then

$$\dim V_k = 2k + 1, \tag{1.9}$$

and

$$\operatorname{Spec} X\big|_{V_k} = \{\ell \in \mathbb{Z} : -k \le \ell \le k\}. \tag{1.10}$$

Let us note that Λ and X commute, and that the pair $\{\Lambda, X\}$ has simple spectrum. Also, under the hypothesis (1.5)–(1.6), $\Pi(A)$ commutes with X as well as with Λ. Hence $\Pi(A)$ is a function of (Λ, X),

$$\Pi(A) = F(\Lambda, X). \tag{1.11}$$

Also, given $a \in C^\infty(S^*S^2)$, we have

$$\Pi(A) \in OPS^0(S^2), \tag{1.12}$$

with principal symbol given by (1.3).

Given these facts, we can use results of Chapter 12 of [T1] to analyze F in (1.11). These results yield

$$F \in S^0(\mathbb{R}^2) \Longrightarrow F(\Lambda, X) = B \in OPS^0(S^2), \tag{1.13}$$

with principal symbol

$$b(x, \xi) = F(|\xi|, \langle Y, \xi \rangle). \tag{1.14}$$

Recall that $Y = iX$ is a real vector field. Note that it suffices to specify F on $\{(\lambda_1, \lambda_2) : \lambda_1 \ge 0,\ |\lambda_2| \le \lambda_1\}$, in light of (1.8)–(1.10), and also taking into account that $|Y| \le 1$ on S^2. We want the principal part of (1.14) to match up with (1.3) on S^*S^2.

Thus, we want to define $F_0(\lambda_1, \lambda_2)$, homogeneous of degree 0 in (λ_1, λ_2), so that

$$F_0(1, \langle Y, \xi \rangle) = Pa(x, \xi), \quad \text{for } (x, \xi) \in S^*S^2. \tag{1.15}$$

Now $F_0(1, \lambda_2)$ is a function of $\lambda_2 \in [-1, 1]$, while Pa is a function on S^*S^2, which has dimension 3. However, Pa is invariant under the flows $\mathcal{G}_t$ and $R(t)$, and in fact it is uniquely specified by its behavior on $S^*_{x_0}S^2$, where x_0 is an arbitrarily chosen point on the equator of S^2. At x_0, Y is a unit vector parallel to the equator, and (1.15) becomes

$$F_0(1, \lambda_2) = Pa\big(x_0, (\lambda_2, \sqrt{1 - \lambda_2^2})\big). \tag{1.16}$$

At first glance, this looks non-smooth at $\lambda_2 = \pm 1$, but in fact we have

$$Pa(x_0, (\xi_1, \xi_2)) = Pa(x_0, (\xi_1, -\xi_2)). \tag{1.17}$$

Such an identity is clear if $a(x)$ is even under $x_3 \mapsto -x_3$. On the other hand, if $a(x)$ is odd under this transformation its invariance under $R(t)$ guarantees that (1.3) vanishes, so we have (1.17) for general $R(t)$-invariant $a \in C^\infty(S^2)$. From (1.17) we have that (1.16) defines a smooth function of $\lambda_2 \in [-1, 1]$. Then

$$\begin{aligned} &F_0(\Lambda, X) \in OPS^0(S^2), \quad \text{and} \\ &\Pi(A) - F_0(\Lambda, X) \in OPS^{-1}(S^2). \end{aligned} \tag{1.18}$$

Note that

$$F_0(\Lambda, X) = g(\Lambda^{-1} X), \tag{1.19}$$

where $g(\lambda) = F_0(1, \lambda)$, i.e.,

$$g(\lambda) = Pa(x_0, (\lambda, \sqrt{1 - \lambda^2})). \tag{1.20}$$

Results just described have implications for concentration of spherical harmonics. In fact, we can take an orthonormal basis

$$\{\varphi_{k\ell} : k, \ell \in \mathbb{Z},\ k \geq 0,\ |\ell| \leq k\} \tag{1.21}$$

of $L^2(S^2)$, satisfying

$$\Lambda \varphi_{k\ell} = k \varphi_{k\ell}, \quad X \varphi_{k\ell} = \ell \varphi_{k\ell}. \tag{1.21A}$$

Then

$$\begin{aligned} \int_{S^2} a(x) |\varphi_{k\ell}(x)|^2 \, dS(x) &= (A \varphi_{k\ell}, \varphi_{k\ell})_{L^2} \\ &= (\Pi(A) \varphi_{k\ell}, \varphi_{k\ell})_{L^2} \\ &= (F_0(\Lambda, X) \varphi_{k\ell}, \varphi_{k\ell})_{L^2} + R_{k\ell}, \end{aligned} \tag{1.22}$$

where

$$R_{k\ell} \longrightarrow 0, \quad \text{as} \ \ k \to \infty. \tag{1.25}$$

Hence

$$\int_{S^2} a(x) |\varphi_{k\ell}(x)|^2 \, dS(x) = g\Big(\frac{\ell}{k}\Big) + R_{k\ell}, \tag{1.24}$$

with $g(\lambda)$ given by (1.20).

Let us pick $\beta \in (0,1)$ and take $a \in C^\infty(S^2)$, invariant under $R(t)$, and satisfying

$$a(x) = 0, \quad \text{for } \ |x_3| \le \beta. \tag{1.25}$$

It follows from (1.20) and (1.3) that

$$g(\lambda) = 0, \quad \text{for } \ \sqrt{1-\lambda^2} \le \beta, \tag{1.26}$$

i.e., for $|\lambda| \ge \sqrt{1-\beta^2}$. Hence

$$\begin{aligned} \int\limits_{S^2} a(x)|\varphi_{k\ell}(x)|^2 \, dS(x) = R_{k\ell} \to 0, \quad \text{as} \ \ k \to 0, \\ \text{for } \ |\ell|/k \ge \sqrt{1-\beta^2}. \end{aligned} \tag{1.27}$$

Conclusion. The orthonormal eigenfunctions $\varphi_{k\ell}$ concentrate on the strip $|x_3| \le \beta$ as $k \to \infty$, for $|\ell|/k \ge \sqrt{1-\beta^2}$.

2. More general concentration results for manifolds with a continuous symmetry group

Let M be a compact, connected Riemannian manifold, and assume M has a nonzero Killing field Y, generating a 1-parameter family of isometries of M. We will also make the hypothesis that

$$A_0 = \min_{x\in M} |Y(x)| < \max_{x\in M} |Y(x)| = A_1. \tag{2.1}$$

The operator $X = iY$ is self adjoint on $L^2(M)$ and commutes with $\Lambda = \sqrt{-\Delta}$. Thus there is an orthonormal basis $\{\varphi_k\}$ of $L^2(M)$ consisting of joint eigenfunctions,

$$\Lambda\varphi_k = \lambda_k\varphi_k, \quad X\varphi_k = \mu_k\varphi_k, \tag{2.2}$$

with $\lambda_k \nearrow +\infty$. Note that

$$\begin{aligned} \mu_k^2 = \|X\varphi_k\|_{L^2}^2 &\le A_1^2\|\nabla\varphi_k\|_{L^2}^2 = A_1^2(-\Delta\varphi_k,\varphi_k) \\ &= A_1^2\|\Lambda\varphi_k\|_{L^2}^2 = A_1^2\lambda_k^2, \end{aligned} \tag{2.3}$$

i.e.,

$$|\mu_k| \le A_1\lambda_k. \tag{2.4}$$

We can define a function $F(\Lambda, X)$ by

$$F(\Lambda, X)\varphi_k = F(\lambda_k,\mu_k)\varphi_k. \tag{2.5}$$

Then, as shown in Chapter 12 of [T1],

$$\begin{aligned} F \in S^0(\mathbb{R}^2) \Longrightarrow F(\Lambda, X) &\in OPS^0(M), \quad \text{and} \\ \sigma_{F(\Lambda,X)}(x,\xi) &= F(|\xi|, \langle Y,\xi\rangle). \end{aligned} \tag{2.6}$$

From here on, we assume $F \in C^\infty(\mathbb{R}^2 \setminus 0)$ is homogeneous of degree 0, and note that only its behavior on the wedge $\{(\lambda,\mu) : |\mu| \le A_1\lambda\}$ is significant for the behavior of $F(\Lambda, X)$. We set

$$\varphi(\mu) = F(1,\mu), \quad \text{so} \ \ F(\Lambda, X) = \varphi(\Lambda^{-1}X). \tag{2.7}$$

Note that only the behavior of φ on $\mu \in [-A_1, A_1]$ is significant. The Weyl law **(Recall?)** yields

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^N \|F(\Lambda, X)\varphi_k\|_{L^2}^2 = \int\limits_{S^*M} |\varphi(\langle Y,\xi\rangle)|^2\, dS, \tag{2.8}$$

where dS is the Liouville measure on S^*M, normalized so that $\int_{S^*M} dS = 1$. This gives information on the joint spectrum of the pair (Λ, X), in connection with the classical result

$$\lambda_k \sim (Ck)^{1/n}, \quad \text{as} \quad k \to \infty, \tag{2.9}$$

where $n = \dim M$ and $C = \Gamma(n/2+1)(4\pi)^{n/2}/\mathrm{Vol}\, M$. Another application of the Weyl formula is that, for $a \in C^\infty(M)$,

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \int_M a(x)|F(\Lambda,X)\varphi_k|^2 \, dV \\ = \int_{S^*M} a(x)|\varphi(\langle Y,\xi\rangle)|^2 \, dS. \end{aligned} \tag{2.10}$$

We are ready to obtain some general concentration results, parallel to those of §1, but valid in much greater generality. The key to this result is the observation that, if $A_0 < B < A_1$,

(2.11)

$$\begin{aligned} &\varphi(\mu) = 0 \ \text{ for } \ |\mu| \le B \\ &\Longrightarrow \varphi(\langle Y,\xi\rangle) = 0, \ \ \forall (x,\xi) \in S^*M \text{ such that } x \in M_B = \{x \in M : |Y(x)| \le B\}. \end{aligned}$$

Hence we have the following conclusion.

Proposition 2.1. *With $a \in C^\infty(M)$, set $Au(x) = a(x)u(x)$. Then*

$$\begin{aligned} &\varphi(\mu) = 0 \ \ \textit{for} \ \ |\mu| \le B, \ \textit{supp}\, a \subset M_B \\ &\qquad \Longrightarrow F(\Lambda,X)^* A F(\Lambda,X) \in OPS^{-1}(M). \end{aligned} \tag{2.12}$$

Hence, when these hypotheses hold,

$$\begin{aligned} \lim_{k\to\infty} \int_M a(x)|F(\Lambda,X)\varphi_k|^2 \, dV \\ = \lim_{k\to\infty} (F(\Lambda,X)^* A F(\Lambda,X)\varphi_k, \varphi_k)_{L^2} = 0. \end{aligned} \tag{2.13}$$

Equivalently,

$$\lim_{k\to\infty} |\varphi(\lambda_k^{-1}\mu_k)|^2 \int_M a(x)|\varphi_k(x)|^2 \, dV(x) = 0. \tag{2.14}$$

3. Microlocal concentration of eigenfunctions of subelliptic operators

Let M be a compact, n-dimensional Riemannian manifold, and let $L \in OPS^2(M)$ be a positive, self-adjoint operator. We assume L is not elliptic, but that it is subelliptic, in the sense that there exists $\sigma > 0$ (necessarily $\sigma < 2$) such that

$$(L+1)^{-1} : H^s(M) \longrightarrow H^{s+\sigma}(M), \quad \forall\, s \in \mathbb{R}. \tag{3.1}$$

Let $\{\varphi_k\}$ be an orthonormal basis of $L^2(M)$ consisting of eigenfunctions of L:

$$L\varphi_k = \lambda_k \varphi_k, \quad 0 \le \lambda_1 \le \lambda_2 \le \cdots \nearrow +\infty. \tag{3.2}$$

We aim to prove the following.

Theorem 3.1. *Take L as above, and denote its principal symbol by L_2. Assume*

$$\int\limits_{S^*M} L_2(x,\omega)^{-n/2}\, dS(x,\omega) = \infty. \tag{3.3}$$

Then, except perhaps for a "sparse" subsequence, the sequence $\{\varphi_k\}$ concentrates microlocally on the characteristic set $\Sigma \subset T^*M \setminus 0$, given by

$$\Sigma = \{(x,\xi) \in T^*M \setminus 0 : L_2(x,\xi) = 0\}. \tag{3.4}$$

The proof will involve a study of the semigroup $\{e^{-tL} : t \ge 0\}$, and of products Ae^{-tL}, with $A \in OPS^0(M)$. The hypothesis (3.1) implies

$$e^{-tL} : \mathcal{D}'(M) \longrightarrow C^\infty(M), \tag{3.5}$$

for each $t > 0$. In particular, $\operatorname{Tr} e^{-tL} < \infty$ for each $t > 0$. We will show that, under the hypotheses of Theorem 3.1,

$$t^{n/2}\, \operatorname{Tr} e^{-tL} \longrightarrow +\infty, \quad \text{as} \ \ t \searrow 0. \tag{3.6}$$

Furthermore, if

(3.7) The full symbol of A vanishes on a conic neighborhood of Σ in $T^*M \setminus 0$,

we obtain

$$\operatorname{Tr} Ae^{-tL} \sim C(A_0) t^{-n/2}, \quad \text{as} \ \ t \searrow 0, \tag{3.8}$$

where A_0 is the principal symbol of A. From (3.8) we obtain

$$\sum_{k \ge 0} e^{-t\lambda_k} (A\varphi_k, \varphi_k) \sim C(A_0) t^{-n/2}, \quad t \searrow 0, \tag{3.9}$$

when (3.7) holds. Applying this observation to A^*A yields

$$\lim_{t\to 0} t^{n/2} \sum_{k\geq 0} e^{-t\lambda_k} \|A\varphi_k\|_{L^2}^2 = C(|A_0|^2). \tag{3.10}$$

Meanwhile, (3.6) implies

$$\lim_{t\to 0} t^{n/2} \sum_{k\geq 0} e^{-t\lambda_k} = +\infty. \tag{3.11}$$

In preparation for proving (3.6), we will find it useful to recall some properties of e^{-tM} when $M \in OPS^2(M)$ is an *elliptic*, positive, self-adjoint operator, with principal symbol M_2. In such a case, parametrix constructions yield

$$e^{-tM}u(x) = \int_M H(t,x,y)u(y)\,dV(y), \tag{3.12}$$

with

$$H(t,x,y) = C_n \int_{T_x^*M} e^{-tM_2(x,\xi)} e^{i(x-y)\cdot\xi}\,d\xi + \cdots. \tag{3.12A}$$

In particular,

$$H(t,x,x) = C_n \int_{T_x^*M} e^{-tM_2(x,\xi)}\,d\xi + o(t^{-n/2}), \tag{3.13}$$

as $t \searrow 0$. Now

$$\int_{T_x^*M} e^{-tM_2(x,\xi)}\,d\xi = C_n' t^{-n/2} \int_{S_x^*M} M_2(x,\omega)^{-n/2}\,dS_x(\omega), \tag{3.14}$$

hence

$$\operatorname{Tr} e^{-tM} = (4\pi t)^{-n/2} \frac{1}{A_{n-1}} \int_{S^*M} M_2(x,\omega)^{-n/2}\,dS(x,\omega) + o(t^{-n/2}), \tag{3.15}$$

where A_{n-1} is the area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$, so

$$\frac{1}{A_{n-1}} \int_{S^*M} dS(x,\omega) = \operatorname{Vol} M. \tag{3.16}$$

In particular, if $M = -\Delta$, where Δ is the Laplace operator on M, we have

$$\operatorname{Tr} e^{t\Delta} = (4\pi t)^{-n/2} \operatorname{Vol} M + o(t^{-n/2}). \tag{3.17}$$

Behind (3.12)–(3.13) is a parametrix construction of e^{-tM} as a family of pseudodifferential operators. Then pseudodifferential operator calculus yields, for $A \in OPS^0(M)$, with principal symbol A_0,

$$\operatorname{Tr} Ae^{-tM} = (4\pi t)^{-n/2} \frac{1}{A_{n-1}} \int\limits_{S^*M} A_0(x,\omega) M_2(x,\omega)^{-n/2}\, dS(x,\omega) + o(t^{-n/2}). \tag{3.18}$$

To establish (3.6), we argue as follows. Take $\varepsilon > 0$ and set $M = L - \varepsilon\Delta$. We apply (3.15) to such M. The relevance of such an application arises as follows. Say $\{\psi_k\}$ is an orthonormal basis of $L^2(M)$ consisting of eigenfunctions of M:

$$L\psi_k = \mu_k \psi_k, \quad 0 \le \mu_1 \le \mu_2 \le \cdots \nearrow +\infty. \tag{3.19}$$

Lemma 3.2. *Let L, M be positive, self-adjoint operators with compact resolvents. Assume*

$$\mathcal{D}(M) \subset \mathcal{D}(L), \quad L \le M. \tag{3.20}$$

Let the eigenvalues be $\{\lambda_k\}$, $\{\mu_k\}$, as in (3.2) and (3.19). Then, for each k,

$$\lambda_k \le \mu_k. \tag{3.21}$$

Proof. Pick $\mu \in (0,\infty)$, and let $V_\nu \subset L^2(M)$ be the span of $\{\psi_k : \mu_k < \mu\}$, so $((M - \mu I)v, v) < 0$ for all nonzero $v \in V_\mu$, but not for all v in a linear space of larger dimension. The hypotheses above yield $((L - \mu I)v, v) < 0$, for all nonzero $v \in V_\mu$, so

$$\#\{\lambda_j : \lambda_j < \mu\} \ge \#\{\mu_j : \mu_j < \mu\}.$$

From the lemma, we deduce that

$$\operatorname{Tr} e^{-tL} \ge \operatorname{Tr} e^{-t(L-\varepsilon\Delta)}, \tag{3.22}$$

for each $\varepsilon > 0,\ t > 0$. Applying (3.15) to $M = L - \varepsilon\Delta$, we have

$$\lim_{t\to 0} (4\pi t)^{n/2} \operatorname{Tr} e^{-(L-\varepsilon\Delta)} = \frac{1}{A_{n-1}} \int\limits_{S^*M} (L_2(x,\omega) + \varepsilon)^{-n/2}\, dS(x,\omega). \tag{3.23}$$

Hence

$$\liminf_{t\to 0} (4\pi t)^{n/2} \operatorname{Tr} e^{-tL} \ge \frac{1}{A_{n-1}} \int\limits_{S^*M} (L_2(x,\omega) + \varepsilon)^{-n/2}\, dS(x,\omega), \tag{3.24}$$

for each $\varepsilon > 0$. Hence

$$\liminf_{t\to 0} (4\pi t)^{n/2} \operatorname{Tr} e^{-tL} \geq \frac{1}{A_{n-1}} \int_{S^*M} L_2(x,\omega)^{-n/2}\, dS(x,\omega). \tag{3.25}$$

Thus, given the hypothesis (3.3), we have (3.6).

Next, we bring in the fact that, if $A \in OPS^0(M)$ satisfies (3.7), then the construction of a parametrix for $e^{-tL}A$ is microlocal, and yields, parallel to (3.18),

$$\operatorname{Tr} e^{-tL} A = (4\pi t)^{-n/2} \frac{1}{A_{n-1}} \int_{S^*M} A_0(x,\omega) L_2(x,\omega)^{-n/2}\, dS(x,\omega) + o(t^{-n/2}), \tag{3.26}$$

and, of course,

$$\operatorname{Tr} A e^{-tL} = \operatorname{Tr} e^{-tL} A, \tag{3.27}$$

so we have (3.8).

Examples. Let $M = S^2 \subset \mathbb{R}^3$ be the unit sphere, and let X_j be vector fields generating 2π-periodic rotation about the x_j-axis, for $1 \leq j \leq 3$. Then $\Delta = X_1^2 + X_2^2 + X_3^2$. Now

$$L = -(X_1^2 + X_2^2)$$

satisfies (3.1), with $\sigma = 1$, and we also have (3.3). On the other hand,

$$L = -(X_1^2 + X_2^2 + X_3 M_{x_1}^2 X_3)$$

also satisfies (3.1), with $\sigma = 1$, but (3.3) does not hold. In this case, the integral $\int_{S^*M} L_2(x,\omega)^{-1}\, dS$ is finite.

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