Joint Spectra of Riemannian Manifolds with Rotational Symmetry

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Abstract

We study the joint spectra and joint eigenfunctions of a family of commuting self-adjoint operators $(\Lambda, X_1, \dots, X_\ell)$ on a compact, n-dimensional Riemannian manifold M. Here $\Lambda = \sqrt{-\Delta}$ (or a convenient perturbation), where Δ is the Laplace operator on M, and X_j are first-order, self-adjoint, differential operators, or more generally pseudodifferential operators, on M. We concentrate on cases where M has a group G of isometries, especially when $G = SO(n)$, where we say M has rotational symmetry.

Two basic cases are the flat 2D torus $\mathbb{T}^2$ and the 2D sphere S^2, each with a natural $SO(2)$ action, yielding two commuting self-adjoint operators (Λ, X). Classical analyses of their joint spectra and eigenfunctions, with emphasis on their differences, are reviewed in §1, and these results provide a springboard for more general studies pursued in subsequent sections, bringing in techniques from microlocal analysis to elucidate various spectral clustering and eigenfunction concentration effects that first appear in these two paradigm cases. Contact is made with earlier work, particularly [2], [5], and [9].

Contents

1 Introduction

Let M be a compact, connected, n-dimensional Riemannian manifold. Assume a compact Lie group G acts effectively on M, as a group of isometries. Denote the action by $(g, x) \mapsto gx,\ g \in G,\ x \in M$.

A case of particular interest will be $G = SO(n)$, under the assumption that there exists $q \in M$ whose G-orbit $\mathcal{O}_q = \{gq : g \in G\}$ is a smooth submanifold of M, diffeomorphic to the standard sphere $S^{n-1} \subset \mathbb{R}^n$, via

$$\varphi_q : \mathcal{O}_q \longrightarrow S^{n-1}, \tag{1.1}$$

in such a way that φ_q intertwines the $SO(n)$ action on $\mathcal{O}_q$ with the standard action of $SO(n)$ on S^{n-1}. Then we say M has rotational symmetry.

Let Δ denote the Laplace-Beltrami operator on M. Then $L^2(M)$ has an orthonormal basis consisting of eigenfunctions of Δ, belonging to eigenspaces

$$V_\lambda = \{u \in C^\infty(M) : \Delta u = -\lambda^2 u\}, \quad \lambda^2 \in \operatorname{Spec}(-\Delta). \tag{1.2}$$

The operator Δ commutes with the $SO(n)$ action on functions, given by

$$L(g)u(x) = u(g^{-1}x). \tag{1.3}$$

Hence $L(g)$ leaves each eigenspace V_λ invariant. We get a unitary representation π_λ of G on V_λ.

Typically the space V_λ is not one-dimensional. We aim to bring in self-adjoint differential (or pseudodifferential) operators that commute with Δ (and with each other), arising from the G-action, and look at the joint spectrum of such a family of commuting self-adjoint operators, and also look at the behavior of the joint eigenfunctions of these operators.

To frame the study, we start with a look at two paradigm cases, when $n = 2$, namely

$$M = S^2, \quad M = \mathbb{T}^2, \tag{1.4}$$

where $S^2 \subset \mathbb{R}^3$ is the standard unit sphere and $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$ is a flat torus. In these cases, $G = SO(2) \approx \mathbb{T}^1$. The group $SO(2)$ acts on S^2 rotation about the x_3-axis, and $SO(2) \approx \mathbb{T}^1$ acts on $\mathbb{T}^2$ via

$$\varphi \cdot (\theta_1, \theta_2) = (\theta_1 + \varphi, \theta_2), \tag{1.5}$$

with addition in $\mathbb{R}/2\pi\mathbb{Z}$. We will describe results on eigenfunctions of Δ here, referring to Chapters 3 and 8 of [11] or Chapter 7 of [12] for details.

We start with $M = \mathbb{T}^2$, and take the L^2-inner product

$$(u, v)_{L^2} = (2\pi)^{-2} \int_{\mathbb{T}^2} u(\theta)\overline{v(\theta)}\, d\theta. \tag{1.6}$$

Here an orthonormal basis is given by

$$e_k(\theta) = e^{ik\cdot\theta}, \quad k = (k_1, k_2) \in \mathbb{Z}^2, \tag{1.7}$$

and we have

$$\Delta e_k = -|k|^2 e_k, \quad |k|^2 = k_1^2 + k_2^2. \tag{1.8}$$

We have differential operators

$$\partial_j : C^\infty(\mathbb{T}^2) \longrightarrow C^\infty(\mathbb{T}^2), \quad j = 1, 2, \tag{1.9}$$

and

$$X_j = \frac{1}{i}\partial_j \Longrightarrow X_j e_k = k_j e_k, \quad k \in \mathbb{Z}^2. \tag{1.10}$$

The operator X_1 is derived from the $SO(2)$ action on $\mathbb{T}^2$ given by (1.5). The fact that X_2 commutes with both Δ and X_1 is in some sense serendipitous. We have $-\Delta = X_1^2 + X_2^2$. For our considerations of joint spectra, we first have

$$\mathrm{Spec}(X_1, X_2) = \{(k_1, k_2) : k_j \in \mathbb{Z}\} = \mathbb{Z}^2. \tag{1.11}$$

In place of considering the joint spectrum of $-\Delta$ and X_1, it is convenient to set

$$\Lambda = (-\Delta)^{1/2}, \tag{1.12}$$

so

$$\Lambda e_k = |k| e_k, \tag{1.13}$$

and

$$\mathrm{Spec}(\Lambda, X_1) = \{(\sqrt{k_1^2 + k_2^2}, k_1) : (k_1, k_2) \in \mathbb{Z}^2\}. \tag{1.14}$$

In addition, we have

$$\mathrm{Spec}(\Lambda, X_1, X_2) = \{(\sqrt{k_1^2 + k_2^2}, k_1, k_2) : (k_1, k_2) \in \mathbb{Z}^2\}. \tag{1.15}$$

The joint spectrum of X_1 and X_2, specified in (1.11), is the very regular integer lattice in $\mathbb{R}^2$. The triple joint spectrum specified in (1.15) is the lift of this lattice to a cone in $\mathbb{R}^3$. The set $\mathrm{Spec}(\Lambda, X_1)$, specified in (1.14), is an edge-on view of this spotted cone, depicted in Figure 1.1.

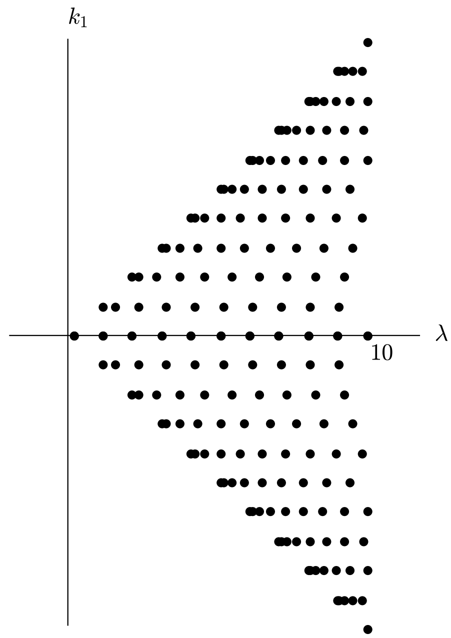

Figure 1.1: Spec(Λ, X_1) on $\mathbb{T}^2$

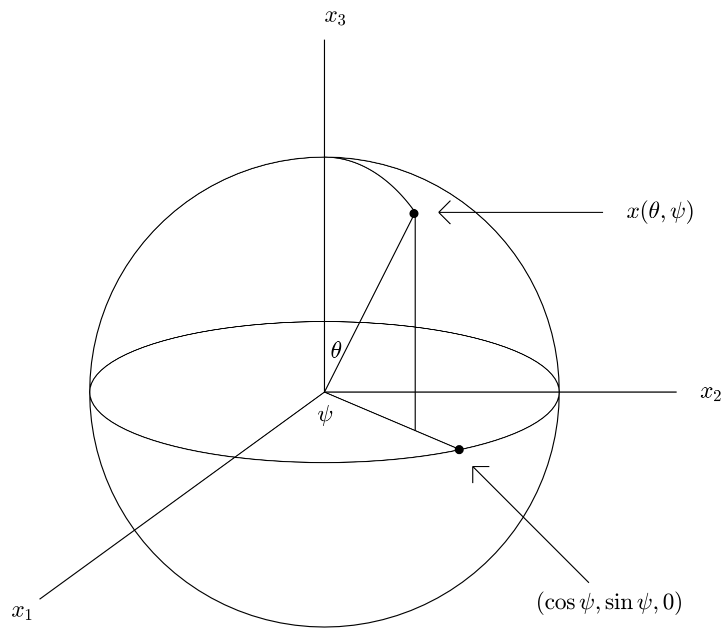

Figure 1.2: Spherical coordinates on S^2

We now look at S^2. We use spherical coordinates (θ, ψ), defined by

$$\begin{gathered} x(\theta,\psi) = (\sin\theta\,\cos\psi, \sin\theta\,\sin\psi, \cos\theta), \\ 0 \le \theta \le \pi, \quad 0 \le \psi \le 2\pi, \end{gathered} \tag{1.16}$$

illustrated in Figure 1.2.

In this case (cf. [12], (7.4.30)),

$$\operatorname{Spec}(-\Delta) = \{k^2 + k : k \in \mathbb{Z}^+\}. \tag{1.17}$$

It is convenient to set

$$\Lambda = \Bigl(-\Delta + \frac{1}{4}\Bigr)^{1/2} - \frac{1}{2}, \tag{1.18}$$

so

$$\operatorname{Spec} \Lambda = \{k : k \in \mathbb{Z}^+\}, \tag{1.19}$$

and denote $-\lambda^2$-eigenspace of Δ by V_k, for $\lambda = \lambda_k = \sqrt{k^2+k}$:

$$V_k = \{u \in C^\infty(S^2) : \Lambda u = ku\}. \tag{1.20}$$

Each eigenspace V_k is seen to contain a 1-dimensional space of zonal harmonics,

$$\mathcal{Z}_k = \{u \in V_k : X_1 u = 0\}, \tag{1.21}$$

where X_1 is a first order differential operator with the property that $Y_1 = iX_1$ is a real vector field generating the $SO(2)$ action on S^2, i.e., rotation about the x_3-axis (of period 2π). The fact that $\dim \mathcal{Z}_k = 1$ is established in Proposition 7.4.18 of [12]. (A different argument will be presented later on here, in §2.) Further calculations presented in [12] yield

$$\mathcal{Z}_k = \operatorname{Span}(Z_k), \tag{1.22}$$

where

$$Z_k(\omega) = P_k(\cos\theta), \tag{1.23}$$

for $\omega = x(\theta,\psi) \in S^2$, and $P_k(t)$ are Legendre polynomials, given by the generating function

$$(1 - 2tr + r^2)^{-1/2} = \sum_{k=0}^\infty P_k(t) r^k. \tag{1.24}$$

To normalize this eigenfunction to have unit L^2-norm, one takes

$$Y_k^0(\omega) = \Bigl(\frac{2k+1}{4\pi}\Bigr)^{1/2} P_k(\cos\theta). \tag{1.25}$$

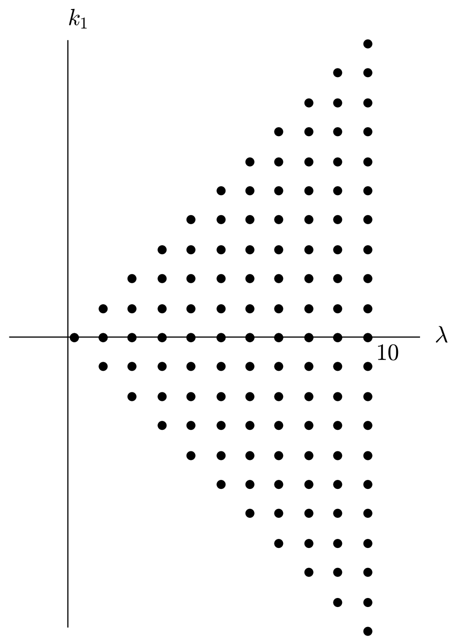

Figure 1.3: $\mathrm{Spec}(\Lambda, X_1)$ on S^2

Then (cf. [12], Proposition 7.4.35), an orthonormal basis of V_k is given by

$$Y_k^\ell(\omega) = \alpha_{k\ell} e^{i\ell\psi} \sin^{|\ell|} \theta \, P_k^{(|\ell|)}(\cos\theta), \quad |\ell| \le k, \tag{1.26}$$

where $\alpha_{k\ell}$ are normalizing constants. An alternative formula for this basis is

$$Y_k^{\pm\ell}(\omega) = \alpha_{k\ell}(\omega_1 \pm i\omega_2)^\ell P_k^{(\ell)}(\omega_3), \quad 0 \le \ell \le k. \tag{1.27}$$

We have

$$X_1 Y_k^\ell = \ell Y_k^\ell, \quad \text{for} \ \ |\ell| \le k, \tag{1.28}$$

and the joint spectrum of Λ and X_1 is

$$\mathrm{Spec}(\Lambda, X_1) = \{(k, \ell) \in \mathbb{Z}^+ \times \mathbb{Z} : |\ell| \le k\}. \tag{1.29}$$

We display this joint spectrum in Figure 1.3. Note similarities and differences in comparison with the display of $\mathrm{Spec}(\Lambda, X_1)$ on $\mathbb{T}^2$, depicted in Figure 1.1.

Both figures display $\operatorname{Spec}(\Lambda, X_1)$ for pairs (λ, k) satisfying $0 \le \lambda \le 10$. The first figure has more spectral points, in large part because the area of $\mathbb{T}^2$ $(4\pi^2)$ exceeds the area of S^2 (4π). Somewhat mitigating the ratio of the number of spectral points is the fact that all the joint spectra for S^2 are simple, as one can deduce from (1.26)–(1.28), while most joint spectral points for $\mathbb{T}^2$ are double. In fact, the joint spectra in (1.11) are simple, but the points (k_1, k_2) and $(k_1, -k_2) \in \mathbb{Z}^2$ from (1.11) have the same image in $\operatorname{Spec}(\Lambda, X_1)$ in (1.14). Hence all the points $(\lambda, k_1) \in \operatorname{Spec}(\Lambda, X_1)$ in (1.14) have multiplicity 2 except for those for which $k_1 = \lambda$. We will investigate the geometrical roots of this difference between having simple spectra and double spectra later on.

Another noticeable distinction between the two sets of joint spectra is their degree of regularity. For S^2, $\operatorname{Spec}(\Lambda, X_1)$ is simply that part of the lattice $\mathbb{Z}^2$ lying within the quadrant $\{(x, y) : x \ge 0, |y| \le x\}$. For $\mathbb{T}^2$, $\operatorname{Spec}(\Lambda, X_1)$ lies in the same quadrant, but its points form a somewhat more elaborate pattern. One can make out families of points lying on branches of hyperbolas, but the spacing of the points is not even. For example, there is substantial clustering near the edges $y = \pm x$.

The more elaborate behavior of $\operatorname{Spec}(\Lambda, X_1)$ on $\mathbb{T}^2$ is related to the notorious difficulty of the "lattice point problem," i.e., to the difficulty of precisely specifying the spectral counting function of $-\Delta$ on $\mathbb{T}^2$. In this connection, we observe that one has a similarly intricate spectral counting function for

$$\Lambda^2 + X_1^2 \quad \text{on} \quad S^2. \tag{1.30}$$

Note that $\Lambda^2 + X_1^2$ is equal to

$$-\Delta + X_1^2, \tag{1.31}$$

modulo an element of $OPS^1(S^2)$. One might check out the spectral counting function of this operator.

We turn to the differences in the behavior of the joint eigenfunctions in these two cases. For $\mathbb{T}^2$, the eigenfunctions e_k, $k \in \mathbb{Z}^2$, all have absolute value 1 everywhere. As mentioned above, each joint eigenspace of (Λ, X_1) has dimension 1 or 2. In case $k_2 \neq 0$, the $(\sqrt{k_1^2 + k_2^2}, k_1)$ eigenspace is

$$\operatorname{Span}(e_{k_1,k_2}, e_{k_1,-k_2}) = \{e^{ik_1\theta_1}(c_1 e^{ik_2\theta_2} + c_2 e^{-ik_2\theta_2}) : c_j \in \mathbb{C}\}, \tag{1.32}$$

which does not display concentration or spiking effects. As for the eigenspaces V_λ of Λ, it is the case that they can have arbitrarily large dimension (this is largely why the lattice counting problem is so hard), so a bit of spiking can occur. But such spiking does not occur for joint eigenfunctions of (Λ, X_1).

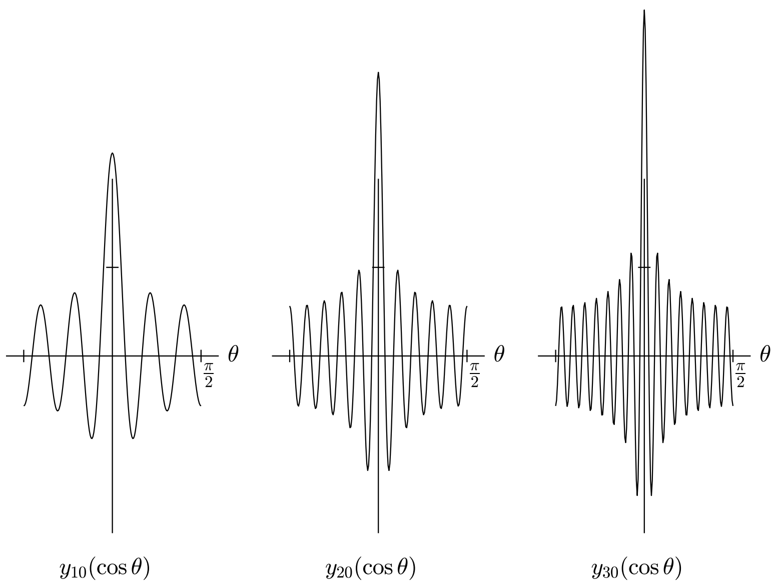

Figure 1.4: Graphs of $y_k(\cos\theta) = Y_k^0(\omega)$, $-\pi/2 \le \theta \le \pi/2$

We now look at joint eigenspaces of (Λ, X_1) for S^2. As noted above, in this case each joint eigenspace has dimension 1, and the (k,ℓ)-eigenspace is spanned by the unit-norm eigenfunction Y_k^ℓ. We start with the zonal harmonics Y_k^0, given by (1.25). One can produce graphs of these functions from computations of the Legendre polynomials $P_k(t)$. In turn, these polynomials satisfy the recursion relation

$$\frac{k+1}{2k+1} P_{k+1}(t) = tP_k(t) - \frac{k}{2k+1} P_{k-1}(t), \tag{1.33}$$

cf. (7.4.292) of [12]. This is convenient for such a computation. Figure 1.4 illustrates the graphs of $Y_k^0(\omega)$ for $\omega_3 = \cos\theta$, in cases $k = 10, 20$, and 30.

Referring to Figure 1.2, we see that the graphs in Figure 1.4 yield graphs of Y_k^0 on the "northern hemisphere" of S^2. Now the polynomials $P_k(t)$ are even in t for k even and odd in t for k odd, so the zonal harmonics $Y_k^0(\omega)$ have the corresponding parity with respect to the inversion $\omega_3 \mapsto -\omega_3$ about the "equator" of S^2.

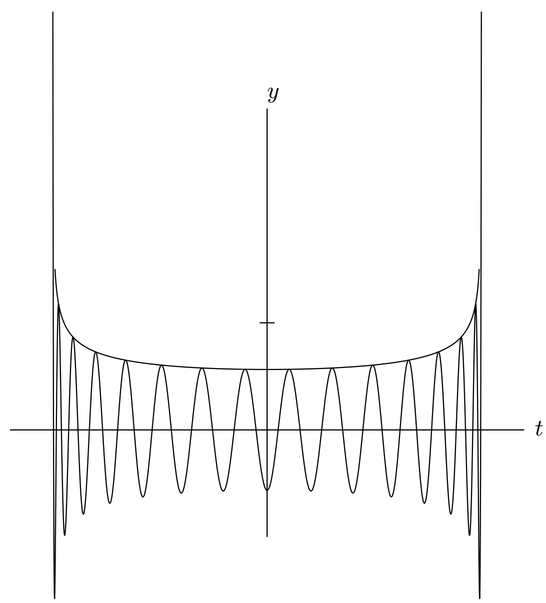

Figure 1.5: $y_{30}(t)$ and the upper envelope $y = (1/\pi)(1-t^2)^{-1/4}$

The graphs in Figure 1.4 illustrate the fact that the zonal harmonics Y_k^0 spike at the north and south poles of S^2. On the other hand, they do not *concentrate* at the poles; that is, their amplitudes do not tend to zero on any strip in S^2. In fact, in the limit as $k \to \infty$. the sequence of functions $y_k(t)$ has the upper envalope

$$y = \frac{1}{\pi}(1-t^2)^{-1/4}. \tag{1.34}$$

See [7], §4.8 for a derivation of the large k asymptotics of $P_k(t)$ that yield this. This phenomenon is illustrated in Figure 1.5.

We have distinguished between "spiking" and "concentration" and noted that the zonal harmonics Y_k^0 exhibit spiking but not concentration. Other eigenfunctions do exhibit concentration, as we will now illustrate with the eigenfunctions of highest "angular momentum"

$$Y_k^k(\omega) = \alpha_k(\omega_1 + i\omega_2)^k, \tag{1.35}$$

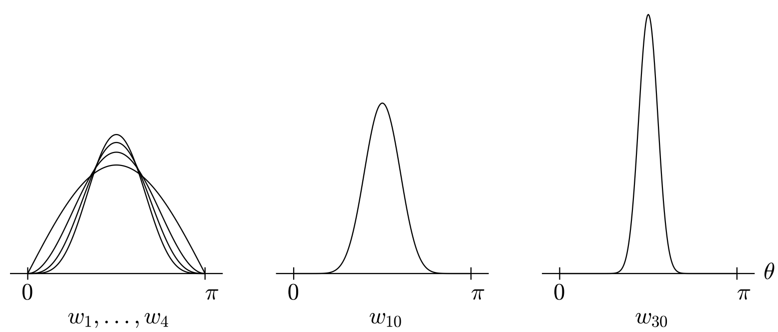

Figure 1.6: Graphs of $w_k(\cos\theta) = |Y_k^k(\omega)|,\ \omega_3 = \cos\theta,\ 0 \le \theta \le \pi$

satisfying

$$|Y_k^k(\omega)|^2 = |\alpha_k|^2(1-\omega_3^2)^k, \tag{1.36}$$

or equivalently

$$|Y_k^k(\omega)| = w_k(\cos\theta), \quad w_k(t) = |\alpha_k|(1-t^2)^{k/2}. \tag{1.37}$$

These eigenfunctions concentrate at the equator, $\theta = \pi/2$, for k large, as illustrated in Figure 1.6.

Extending this result, we will show that, for each $\beta \in (0,1)$, the set of eigenfunctions

$$\begin{gathered}\Big\{Y_k^\ell(\omega) : \frac{|\ell|}{k} \ge \sqrt{1-\beta^2}\Big\} \\ \text{concentrates on the strip } |\omega_3| \le \beta \\ \text{as } k \longrightarrow \infty.\end{gathered} \tag{1.38}$$

Our approach to this will involve, not a study of the special functions arising in the formula (1.26), but rather general considerations, applicable to other

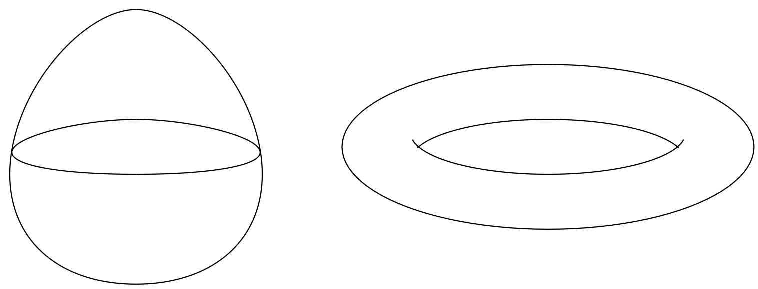

Figure 1.7: Surfaces of revolution, with and without poles

classes of n-dimensional manifolds with $SO(n)$-symmetry, bringing in tools from microlocal analysis. One such result, applicable specifically to the spherical harmonics on S^2, is given in Proposition 3.2 and the accompanying formulas (3.29)–(3.37). Concentration results for joint eigenfunctions in a much more general setting are given in Propositions 4.1–4.3.

The following observation illustrates a limitation on what sorts of sets spherical harmonics can concentrate on. Let $S_\pm$ denote the hemispheres $\{\omega \in S^2 : \pm\omega_3 \geq 0\}$. Then, as we will show in §2,

$$u \in V_k \Longrightarrow \int\limits_{S_\pm} |u|^2 \, dS = \frac{1}{2} \|u\|_{L^2}^2. \tag{1.39}$$

In fact, thanks to the $SO(3)$-invariance of each eigenspace V_k, such an identity holds for all hemispheres $S_\pm$ of S^2. We will establish several generalizations of (1.39), both for higher dimensional spheres S^n and for other classes of manifolds with $SO(n)$-symmetry. See Propositions 2.3–2.4.

We point out a couple of phenomena that drive the differences between the spectral behaviors of $\mathbb{T}^2$ and S^2. One is that the real vector fields $Y_1 = iX_1$ have constant length on $\mathbb{T}^2$ but variable length on S^2. In fact, Y_1 vanishes at two points of S^2, the "north and south poles." This observation motivates a concept that will play a role in investigating conditions that yield simple joint spectra. Namely, let M be a compact, connected, n-dimensional Riemannian manifold with $SO(n)$-symmetry, as introduced at the beginning of this introduction. We say M has a *pole* at $p \in M$ provided

$$gp = p, \quad \forall\, g \in SO(n), \tag{1.40}$$

and the derived action

$$D(g) : T_pM \longrightarrow T_pM \tag{1.41}$$

on the n-dimensional inner-product space T_pM is equivalent to the standard action of $SO(n)$ on $\mathbb{R}^n$. In Figure 1.7 we display two surfaces of revolution in $\mathbb{R}^3$, one with a pair of poles, the other without poles.

The rest of this paper is organized as follows. In §2 we decompose $L^2(M)$ into mutually orthogonal pieces on which G acts like copies of π_α, where $\{\pi_\alpha : \alpha \in \mathcal{A}\}$ is a complete set of irreducible unitary representations of G. These pieces are the images of orthogonal projections Q_α. We use these projections to decompose the eigenspaces V_λ of Δ,

$$V_\lambda = \bigoplus_\alpha V_{\lambda\alpha}, \tag{1.42}$$

noting that $Q_\alpha : V_\lambda \to V_\lambda$. We observe that, for $u \in L^2(M)$,

$$\int_M f|u|^2\, dV = \sum_\beta \int_M f|Q_\beta u|^2\, dV, \tag{1.43}$$

whenever $f \in L^\infty(M)$ is invariant under the G-action. This leads to an extension of (1.39), from $M = S^2$ (indeed, from $M = S^n$), to the setting where M has an isometric involution ι, commuting with the G-action, yielding

$$M = M_+ \cup M_-, \quad \iota : M_\pm \longrightarrow M_\mp, \tag{1.44}$$

with $M_\pm$ invariant under the G-action. Then (1.39) extends to

$$u \in V_\lambda \Longrightarrow \int_{M_\pm} |u|^2\, dV = \frac{1}{2} \int_M |u|^2\, dV, \tag{1.45}$$

provided one has that

$$G = SO(n) \text{ acts irreducibly on each space } V_{\lambda\alpha}. \tag{1.46}$$

This is Proposition 2.3. We show in Proposition 2.4 that

$$\text{if } M \text{ has a pole, then (1.46) holds.} \tag{1.47}$$

In Proposition 2.8 we show that, if M does not have a pole, but there is a G-orbit $\mathcal{O}_q$ as in (1.1), then, for each nonzero $V_{\lambda\alpha}$,

$$\begin{aligned}&\text{the action of } SO(n) \text{ on } V_{\lambda\alpha} \text{ contains}\\ &\text{at most two irreducible components.}\end{aligned} \tag{1.48}$$

In §3, we focus our attention back on $M = S^2$, and examine the asymptotic behavior of

$$\int\limits_{S^2} f|u|^2\,dS, \quad u \in V_k, \quad f \in C^\infty(S^2), \text{ zonal.} \tag{1.49}$$

In Proposition 3.2 we show that, if

$$u = \sum_{|\ell|\le k} a_\ell Y_k^\ell, \tag{1.50}$$

then

$$\int\limits_{S^2} f|u|^2\,dS = \sum_{|\ell|\le k} |a_\ell|^2 g\Bigl(\frac{\ell}{k}\Bigr) + R_k(u), \tag{1.51}$$

where

$$|R_k(u)| \le \frac{C}{k}\|u\|_{L^2}^2, \quad C = C(f), \tag{1.52}$$

and

$$g(\lambda) = \frac{1}{\pi}\int_{-1}^1 f_0\bigl(s\sqrt{1-\lambda^2}\bigr)\,\frac{ds}{\sqrt{1-s^2}}, \quad f(\omega) = f_0(\omega_3). \tag{1.53}$$

We show that $g \in C^\infty([-1,1])$ (perhaps despite appearances). For the special cases $u = Y_k^0$ and $u = Y_k^k$, one has, respectively,

$$\int\limits_{S^2} f|Y_k^0|^2\,dS = \frac{1}{\pi}\int_{-1}^1 f_0(s)\,\frac{ds}{\sqrt{1-s^2}} + O(k^{-1}), \tag{1.54}$$

and

$$\int_{S^2} f\,|Y_k^k|^2\,dS = f_0(0) + O(k^{-1}), \tag{1.55}$$

which can be compared, respectively, with the statement (1.34) about the upper envelope of $P_k(t)$ and the concentration analysis (1.37). Going further, we establish a version of (1.38), namely, for $\beta \in (0,1),\ \delta > 0$,

$$\frac{|\ell|}{k} \geq \sqrt{1-\beta^2} \Longrightarrow \int_{|\omega_3|\geq\beta+\delta} |Y_k^\ell|^2\,dS \leq \frac{C(\delta)}{k}. \tag{1.56}$$

We obtain sharper estimates, in more general settings, in §4.

To get the results (1.51)–(1.56), we do not delve into the analysis of Y_k^ℓ as special functions (as in (1.25)–(1.27)). Rather, we use methods of microlocal analysis. We start by writing (1.49) as

$$(\Pi(A)u,u)_{L^2}, \quad Au = fu, \tag{1.57}$$

where

$$\Pi(A) = \frac{1}{2\pi}\int_0^{2\pi} e^{-it\Lambda} A e^{it\Lambda}\,dt. \tag{1.58}$$

Egorov's theorem allows us to write $\Pi(A)$ as a pseudodifferential operator, in $OPS^0(S^2)$, and to specify its principal symbol. The operator $\Pi(A)$ commutes with both Λ and the operator X_1 arising in (1.21). Using a functional calculus presented in [9], and developed further in Chapter 12 of [10] (see also [8], [2], for related developments), we show in Proposition 3.1 that

$$\Pi(A) = F_0(\Lambda, X_1) + R, \quad R \in OPS^{-1}(S^2), \tag{1.59}$$

with

$$F_0(\Lambda, X_1) = g(\Lambda^{-1}X_1), \tag{1.60}$$

and $g(\lambda)$ as in (1.53). This gives rise to the results (1.51)–(1.56).

The two-pronged goal of §4 is to extend and sharpen the estimate (1.56). In the expanded setting, M is a compact, connected, n-dimensional Riemannian manifold, on which there is a vector field Y, which generates a 1-parameter group of isometries of M, so $X = iY$ is self adjoint and commutes with $\Lambda = \sqrt{-\Delta}$. Then V_λ splits into joint eigenspaces

$$V_\lambda = \bigoplus_\mu V_{\lambda\mu}, \quad V_{\lambda\mu} = \{u \in V_\lambda : Xu = \mu u\}. \tag{1.61}$$

As for the vector field Y, we assume

$$A_0 = \min_{x \in M} |Y(x)| < \max_{x \in M} |Y(x)| = A_1. \tag{1.62}$$

The role of the set $\{\omega \in S^2 : |\omega_3| \geq \beta + \delta\}$ in (1.56) is expanded to

$$\Omega_A = \{x \in M : |Y(x)| \leq A\}, \quad \text{given} \;\; A \in (A_0, A_1). \tag{1.63}$$

The role played by the condition on (k, ℓ) in (1.56) will be expanded by choosing

$$g \in C^\infty([-A_1, A_1]), \quad g(\mu) = 0 \text{ for } |\mu| \leq A', \text{ given } A' > A. \tag{1.64}$$

The extension and sharpening of (1.56) is then given by the following result, Proposition 4.3:

$$\begin{gathered} \text{if } f \in C^\infty(M), \text{ supp}\, f \subset \Omega_A, \text{ then} \\ u \in V_{\lambda\mu} \Rightarrow g\Bigl(\frac{\mu}{\lambda}\Bigr) \|fu\|_{C^m(M)} \leq \frac{C_m}{\lambda^m} \|u\|_{L^2}, \end{gathered} \tag{1.65}$$

for each $m \in \mathbb{N}$. The key to this estimate again resides in the use of the functional calculus for commuting self-adjoint pseudodifferential operators. It is shown in Proposition 4.2 that, under the hypotheses above,

$$M_f g(\Lambda^{-1} X) \in OPS^{-\infty}(M), \tag{1.66}$$

and this leads to (1.65).

The result (1.65) can be interpreted as implying that, for $u \in V_{\lambda\mu}$, with $|\mu/\lambda| > A'$, u concentrates on the set $M \setminus \Omega_A$, as $\lambda \to \infty$. Alternatively, we say Ω_A is a *shadow region* for such a family of eigenfunctions.

In §5 we retain the setting of §4, involving M, Λ, X, and examine Weyl asymptotics. In its basic form, this involves the counting function for the eigenvalues of Λ, repeatd according to multiplicity. This can be expressed in the form $\operatorname{Tr} \varphi_\nu(\Lambda)$, for a sequence of functions $\varphi_\nu : \mathbb{R}^+ \to \mathbb{R}$, such as characteristic functions of $[0, \nu]$. It is convenient to take smoother functions, and deduce information on the counting functions via a Tauberian theorem. For example, one might start with heat asymptotics, $\operatorname{Tr} e^{t\Delta}$, using $\varphi_t(\Lambda) = e^{-t\Lambda^2}$, $t \searrow 0$. More precise results arise via the use of wave equation techniques. One takes

$$\varphi \in \mathcal{S}(\mathbb{R}), \quad \varphi \geq 0, \quad \operatorname{supp} \hat{\varphi} \subset (-r, r), \quad r < \operatorname{Inj} M, \tag{1.67}$$

where $\operatorname{Inj} M$ is the injectivity radius, and considers

$$N_\varphi(R) = \operatorname{Tr} \varphi(\Lambda - R). \tag{1.68}$$

One studies the asymptotic behavior of this as $R \to +\infty$. More generally, there are microlocal Weyl asymptotics, involving the behavior of

$$\operatorname{Tr} B\varphi(\Lambda - R), \quad B \in OPS^0(M). \tag{1.69}$$

As in the classical work [6], this is analyzed by writing

$$B\varphi(\Lambda - R) = \int_{-\infty}^{\infty} Be^{it(\Lambda - R)} \hat{\varphi}(t)\, dt, \tag{1.70}$$

and using a parametrix for the wave evolution operator $e^{it\Lambda}$, for $|t| < r$. In particular, results of [6] give

$$N_\varphi(R) \sim C(\varphi, M) R^{n-1}, \quad R \to \infty. \tag{1.71}$$

Going further, we take

$$B = M_f h(\Lambda^{-1} X), \quad f \in C^\infty(M),\ h \in C^\infty([-A_1, A_1]), \tag{1.72}$$

and obtain the following, in Proposition 5.2:

$$\begin{aligned} &\lim_{R\to\infty} \frac{1}{N_\varphi(R)} \sum_\lambda \varphi(\lambda - R) \sum_{\mu, j} h\Big(\frac{\mu}{\lambda}\Big) \int\limits_M f |u_{\lambda\mu j}|^2 \, dV \\ &= \int\limits_{S^*M} f(x) h\big(\langle Y(x), \xi\rangle\big)\, dS(x, \xi), \end{aligned} \tag{1.73}$$

where, for $(\lambda, \mu) \in \operatorname{Spec}(\Lambda, X)$,

$$\{u_{\lambda\mu j} : 1 \le j \le \dim V_{\lambda\mu}\} \text{ is an orthonormal basis of } V_{\lambda\mu}. \tag{1.74}$$

Specializing (1.74) to $f \equiv 1$ gives

$$\begin{aligned} &\lim_{R\to\infty} \frac{1}{N_\varphi(R)} \sum_\lambda \varphi(\lambda - R) \sum_{\mu} h\Big(\frac{\mu}{\lambda}\Big) \dim V_{\lambda\mu} \\ &= \int\limits_{S^*M} h\big(\langle Y(x), \xi\rangle\big)\, dS(x, \xi). \end{aligned} \tag{1.75}$$

We show in Lemma 5.4 that the last integral is equal to

$$\int_I h(y)\Psi(y)\,dy, \quad I = [-A_1, A_1], \tag{1.76}$$

with Ψ satisfying

$$\Psi = \Psi_{M,X} \in L^1(I, dy), \quad \Psi \geq 0, \quad \int_I \Psi(y)\,dy = 1. \tag{1.77}$$

Having that the left side of (1.75) is given by (1.76), for $h \in C^\infty(I)$, we next have two tasks. First, we want to extend the validity of this identity to a larger class of functions h on I, including functions that are piecewise continuous with a finite number of jumps. In fact, the extension goes further, to the situation where $h : I \to \mathbb{R}$ is a bounded function satisfying

$$h \in \mathcal{R}(I, \gamma), \quad \gamma = \Psi(y)\,dy, \tag{1.78}$$

that is to say, h is Riemann integrable on the measured metric space (I, γ). We expound on this concept in Appendix A.

We call the function Ψ that arises in (1.76) the *joint spectral clustering factor*, and the second task we face after obtaining (1.75)–(1.77) is to analyze the behavior of this factor, and see how it depends on M and X. In §5 we work through the examples $M = \mathbb{T}^2$ and $M = S^2$, obtaining

$$\Psi(y) = \frac{1}{\pi} \frac{1}{\sqrt{1-y^2}}, \quad y = [-1, 1], \quad M = \mathbb{T}^2, \tag{1.79}$$

and

$$\Psi(y) = \frac{1}{2}, \quad y \in [-1, 1], \quad M = S^2. \tag{1.80}$$

This is a quantitative expression of what is behind the difference in appearance of the joint spectra of (Λ, X), pictured in Figures 1.1 and 1.3, in these two cases.

In §6 we apply the results of (1.73)–(1.77), and also the shadowing results of §4, to additional examples of 2D surfaces of revolution:

3. More general convex surfaces of revolution.
4. Symmetric dumbbell.
5. Top-heavy dumbbell.
6. Surface with inflective invariant geodesic.

7. Inner tube.
8. Surface with flattened equator.
9. Capped cylinder.

Illustrations of curves in $\mathbb{R}^2$ that produce such surfaces upon rotation about the x_3-axis are given in Figures 6.1–6.4. The first two of these figures also sketch the graphs of the factor Ψ arising in Examples 4 and 5.

For surfaces in Example 3, the behavior of Ψ is close to that of S^2; one has $\Psi \in C^\infty(I)$, though it is typically not constant. In the other examples, Ψ has singularities, though in Examples 4–7 the singularities occur on the interior of I, rather than at the endpoints. In Examples 4–8, such singularities as occur are weaker than we see for $M = \mathbb{T}^2$ in (1.79). We have logarithmic singularities in Example 4, both log singularities and jumps in Example 5, power singularities with exponent $-1/6$ in Example 6, log singularities for the inner tube in Example 7, and power singularities with exponent $-1/4$ in Example 8. For the capped cylinder in Example 9, Ψ has singularities of the same strength as (1.79).

Regarding the application of results on concentration and shadowing to these examples, we mention that, in Example 4 (the symmetric dumbbell), we show in Proposition 6.1 that there are eigenfunctions in $V_{\lambda\mu}$ that concentrate on small neighborhoods of the union of the two equators about the fattest parts of these dumbbells. However, thanks to results of §2, they cannot concentrate on a small neighborhood of just one of these equators, since each such eigenfunction is either even or odd under the associated involution ι on M that arises here. We also have a result, Proposition 6.2, on the existence of pairs of elements of $\mathrm{Spec}(\Lambda, X)$ that are very close together, one associated to a joint eigenfunction that is even under the action of ι, and one associated to an odd eigenfunction.

There are different conclusions to be reached about concentration and shadowing of joint eigenfunctions in Example 5, which the reader can check out.

This paper has two appendices. As already mentioned, Appendix A treats the notion of Riemann integrable functions on a compact, measured metric space, of relevance to extending (1.75)–(1.76) from $h \in C^\infty(I)$ to cases allowing h to be discontinuous. Appendix B discusses the action of a finite symmetry group on eigenspaces. Results given here are relevant to Proposition 6.2, mentioned above.

2 Decomposition of the eigenspaces of Δ

Let M be a compact, connected, n-dimensional Riemannian manifold, with an $SO(n)$-action by isometries, as described in the opening paragraph of §1. Denote the eigenspaces of the Laplace operator on M by

$$V_\lambda = \{u \in C^\infty(M) : \Delta u = -\lambda^2 u\}, \quad \lambda^2 \in \operatorname{Spec}(-\Delta). \tag{2.1}$$

If π_α is an irreducible unitary representation of $SO(n)$, set

$$Q_\alpha u(x) = d_\alpha \int\limits_{SO(n)} u(g^{-1}x)\overline{\chi_\alpha(g)}\, dg, \tag{2.2}$$

where

$$\chi_\alpha(g) = \operatorname{Tr} \pi_\alpha(g), \quad d_\alpha = \chi_\alpha(I), \tag{2.3}$$

I denoting the identity element of $SO(n)$. The operator Q_α is the orthogonal projection of $L^2(M)$ onto the subspace of $L^2(M)$ on which $L(g)$ acts like copies of π_α, where

$$L(g)u(x) = u(g^{-1}x), \quad g \in SO(n). \tag{2.4}$$

Since $L(g)$ commutes with Δ, so does Q_α, so

$$Q_\alpha : V_\lambda \longrightarrow V_\lambda. \tag{2.5}$$

We set

$$V_{\lambda\alpha} = Q_\alpha(V_\lambda), \tag{2.6}$$

so we have an orthogonal decomposition

$$V_\lambda = \bigoplus_{\alpha \in \mathcal{A}_\lambda} V_{\lambda\alpha}, \tag{2.7}$$

where $\mathcal{A}_\lambda = \{\alpha : V_{\lambda\alpha} \neq 0\}$.

As one use of such a decomposition, we derive a simple but useful identity. Suppose $f \in L^\infty(M)$ is invariant under the $SO(n)$-action, so

$$M_f L(g) = L(g) M_f, \quad \forall\, g \in G = SO(n), \tag{2.8}$$

where $M_f u = fu$. Hence

$$M_f Q_\alpha = Q_\alpha M_f. \tag{2.9}$$

Proposition 2.1 *If $f \in L^\infty(M)$ is $SO(n)$-invariant and $u \in L^2(M)$, then*

$$\int_M f|u|^2\,dV = \sum_\beta \int_M f|Q_\beta u|^2\,dV. \tag{2.10}$$

Proof. The left side of (2.10) is equal to

$$\begin{aligned}(fu,u) &= \sum_{\alpha,\beta} (fQ_\alpha u, Q_\beta u) \\ &= \sum_{\alpha,\beta} (Q_\beta M_f Q_\alpha u, Q_\beta u) \\ &= \sum_{\alpha,\beta} (M_f Q_\beta Q_\alpha u, Q_\beta u) \\ &= \sum_\beta (M_f Q_\beta u, Q_\beta u),\end{aligned} \tag{2.11}$$

where we have used $Q_\beta = Q_\beta^2 = Q_\beta^* Q_\beta$ and $Q_\alpha Q_\beta = 0$ for $\alpha \neq \beta$. The last quantity in (2.11) is equal to the right side of (2.10). □

Corollary 2.2 *In the setting of Proposition 2.1,*

$$u \in V_\lambda \Rightarrow \int_M f|u|^2\,dV = \sum_{\beta\in\mathcal{A}_\lambda} \int_M f|Q_\beta u|^2\,dV. \tag{2.12}$$

One can deduce the identity (1.39) from (2.12). Here is a natural generalization.

Proposition 2.3 *Take M as above, and assume there is an isometric involution*

$$\iota : M \longrightarrow M, \quad \text{commuting with the action of } G = SO(n), \tag{2.13}$$

such that

$$M = M_+ \cup M_-, \quad \iota : M_\pm \longrightarrow M_\mp, \tag{2.14}$$

with

$$M_\pm \text{ invariant under the } G\text{-action}. \tag{2.15}$$

In addition, assume

$$G = SO(n) \text{ acts irreducibly on } V_{\lambda\alpha}, \text{ for each } \alpha \in \mathcal{A}_\lambda. \tag{2.16}$$

Then

$$u \in V_\lambda \Rightarrow \int\limits_{M_\pm} |u|^2\, dV = \frac{1}{2} \int\limits_M |u|^2\, dV. \tag{2.17}$$

Proof. By (2.13), $\iota^* : V_{\lambda\alpha} \to V_{\lambda\alpha}$, and, if (2.16) holds,

$$\iota^* = \pm 1 \text{ on each space } V_{\lambda\alpha}. \tag{2.18}$$

Now we can apply (2.12) with $f = \chi_{M_\pm}$, to get, for $u \in V_\lambda$,

$$\int\limits_{M_\pm} |u|^2\, dV = \sum_{\alpha \in \mathcal{A}_\lambda} \int\limits_{M_\pm} |Q_\alpha u|^2\, dV. \tag{2.19}$$

But (2.18) implies

$$\int\limits_{M_\pm} |Q_\alpha u|^2\, dV = \frac{1}{2} \int\limits_M |Q_\alpha u|^2\, dV, \quad \forall\, \alpha \in \mathcal{A}_\lambda, \tag{2.20}$$

whenever $u \in V_\lambda$, so (2.19) is equal to

$$\frac{1}{2} \sum_{\alpha \in \mathcal{A}_\lambda} \int\limits_M |Q_\alpha u|^2\, dV = \frac{1}{2} \int\limits_M |u|^2\, dV, \tag{2.21}$$

again by (2.12), with $f \equiv 1$. This gives the asserted conclusion (2.17). $\square$

The applicability of Proposition 2.3 to the identity (1.39) follows from the fact that each joint eigenspace of (Λ, X_1) on S^2 is one-dimensional, spanned by Y_k^ℓ, given by (1.26)–(1.27), so clearly the $SO(2)$ action on such a space is irreducible. Here is a more general irreducibility result, applicable to n-dimensional $SO(n)$-symmetric manifolds with a pole (including, of course, $M = S^n$).

Proposition 2.4 *Let M be a compact, connected, n-dimensional Riemannian manifold ($n \geq 2$) with an $SO(n)$-action by isometries. Assume M has a pole at p. Then*

$$G = SO(n) \text{ acts irreducibly on } V_{\lambda\alpha}, \text{ for each } \alpha \in \mathcal{A}_\lambda. \tag{2.22}$$

We get started with the following special case.

Lemma 2.5 *Take M as in Proposition 2.4, and, for $\lambda^2 \in \operatorname{Spec}(-\Delta)$, consider*

$$V_{\lambda 0} = \{u \in V_\lambda : L(g)u = u,\ \forall\, g \in SO(n)\}. \tag{2.23}$$

Then

$$\dim V_{\lambda 0} \le 1. \tag{2.24}$$

Our approach to the proof of this makes use of the following.

Lemma 2.6 *Take M, λ as in Lemma 2.5. Then there exists $r_1 = r_1(\lambda) > 0$ such that*

$$u \in V_\lambda,\ \ r \in (0, r_1],\ \ u\big|_{\partial B_r(p)} = 0 \Longrightarrow u \equiv 0. \tag{2.25}$$

Proof. If $\mu = \mu(r)$ denotes the smallest eigenvalue of $-\Delta$ on the ball $B_r(p)$, with the Dirichlet boundary condition on its boundary $\partial B_r(p)$, we have the variational characterization

$$\mu(r) = \inf \Big\{ \int\limits_{B_r(p)} |\nabla v|^2\, dV : v \in C_0^\infty(B_r(p)),\ \|v\|_{L^2} = 1 \Big\}, \tag{2.26}$$

and, as is classical,

$$\mu(r) \nearrow +\infty, \quad \text{as} \quad r \searrow 0. \tag{2.27}$$

As soon as $\mu(r_1) > \lambda^2$, we have that the hypotheses of (2.25) imply

$$u = 0 \ \text{ on } \ B_r(p), \tag{2.28}$$

and then unique continuation for solutions to $(\Delta + \lambda^2)u = 0$ implies $u \equiv 0$ on M. □

Proof of Lemma 2.5. Assume $V_{\lambda 0} \neq 0$. Pick $u, v \in V_{\lambda 0}$, both $\neq 0$. Take $r_1 = r_1(\lambda)$ as in Lemma 2.6. Both u and v are constant on $\partial B_{r_1}(p)$, so there exist nonzero constants $a_j \in \mathbb{C}$ such that

$$a_1 u + a_2 v = 0 \ \text{ on } \ \partial B_{r_1}(p). \tag{2.29}$$

By Lemma 2.6, this implies $a_1 u + a_2 v \equiv 0$ on M, hence $\dim V_{\lambda 0} = 1$. □

Before proceeding to Proposition 2.4, we note that Lemma 2.5 leads to the following classical result.

Corollary 2.7 *Take $M = S^n$, and assume $n \ge 2$. Then the isometry group $SO(n+1)$ acts irreducibly on each eigenspace V_λ of Δ.*

Proof. Note that the $SO(n+1)$ action commutes with Δ, so $SO(n+1)$ acts on each eigenspace V_λ. Suppose $W \subset V_\lambda$ is a linear subspace that is invariant under this action, and we take a nonzero $w \in W$. There exists $q \in S^n$ such that $w(q) \neq 0$, and there exists $g \in SO(n+1)$ such that $gq = p$, the "north pole" of S^n. Hence $L(g)w = w_1 \in W$ and $w_1(p) \neq 0$. Now form

$$w_2 = \int_{SO(n)} L(g)w_1 \, dg, \tag{2.30}$$

so

$$w_2 \in W \cap V_{\lambda 0}, \quad w_2(p) = w_1(p) \neq 0. \tag{2.31}$$

Now, if $W \neq V_\lambda$, let $W^\perp \subset V_\lambda$ denote its orthogonal complement, and take a nonzero $v \in W^\perp$. The same argument as above yields

$$v_2 \in W^\perp \cap V_{\lambda 0}, \quad v_2 \neq 0. \tag{2.32}$$

Then Lemma 2.5 implies v_2 is a scalar multiple of w_2, contradicting the fact that $v_2 \perp w_2$. This contradiction implies $W = V_\lambda$, and we have the asserted irreducibility. □

We now tackle Proposition 2.4. We divide the proof into two parts.

Proof of Proposition 2.4 when $n = 2$**.** In this case, the action of $G = SO(2)$ is given by the flow $\mathcal{F}^t_{Y_1}$, generated by a real vector field Y_1, periodic of period 2π. The self adjoint operator $X_1 = (1/i)Y_1$ commutes with $\Lambda = (-\Delta)^{1/2}$. The spaces $V_{\lambda\alpha}$ become

$$V_{\lambda k} = \{u \in V_\lambda : X_1 u = ku\}, \quad k \in \mathcal{A}_\lambda \subset \mathbb{Z}. \tag{2.33}$$

We are asserting that $\dim V_{\lambda k} = 1$ for each $k \in \mathcal{A}_\lambda$. The proof is similar to that of Lemma 2.5. Pick $u, v \in V_{\lambda k}$, both $\neq 0$. Take $r_1 = r_1(\lambda)$ as in Lemma 2.6. If we fix $q \in \partial B_{r_1}(p)$, both u and v, restricted to $\partial B_{r_1}(p)$, are constant multiples of the function φ_k defined by

$$\varphi_k(\mathcal{F}^t_{Y_1} q) = e^{ikt}. \tag{2.34}$$

Thus there exist constants $a_j \in \mathbb{C}$ such that $a_1 u + a_2 v = 0$ on $\partial B_{r_1}(p)$. By Lemma 2.6, this implies $a_1 u + a_2 v \equiv 0$ on M, so $\dim V_{\lambda k} = 1$. □

Proof of Proposition 2.4 when $n \geq 3$**.** Note that the representation π_α

of $SO(n)$ must be contained in the standard action of $SO(n)$ on $L^2(S^{n-1})$. Assume $W \subset V_{\lambda\alpha}$ is a linear subspace, invariant under the $SO(n)$-action, and take a nonzero $u \in W$. Take $r_1 = r_1(\lambda)$ as in Lemma 2.6, and set $S_r = \partial B_r(p)$. We see that for each $r \in (0, r_1]$, $u|_{S_r}$ is not $\equiv 0$. Fix $q \in S_{r_1}$. There exists $g \in SO(n)$ such that $u_1 = L(g)u$ is nonvanishing at q. Also, of course, $u_1 \in W$. Identifying $SO(n-1)$ with the subgroup of $SO(n)$ fixing q, set

$$u_2 = \int_{SO(n-1)} L(g)u_1 \, dg, \tag{2.35}$$

so

$$u_2 \in W, \quad u_2(q) = u_1(q) \neq 0. \tag{2.36}$$

Now $\varphi_2 = u_2|_{S_{r_1}}$ is an element of $C^\infty(S_{r_1}) \approx C^\infty(S^{n-1})$ in the subspace on which

$$\{L(g) : g \in SO(n)\} \text{ acts like } \pi_\alpha, \tag{2.37}$$

and $SO(n-1)$ acts trivially on φ_2. The argument used in the proof of Corollary 2.7 implies the space (2.37) has just a 1-dimensional subspace on which $SO(n-1)$ acts trivially (space of "zonal harmonics"), so φ_2 must span this space.

Now, if $W \neq V_{\lambda\alpha}$, then its orthogonal complement $W^\perp \subset V_{\lambda\alpha}$ contains a nonzero element, and by the argument above we have

$$v_2 \in W^\perp, \quad v_2 \neq 0, \quad v_2 \text{ invariant under the } SO(n-1)\text{-action}. \tag{2.38}$$

Since the restrictions of both w_2 and v_2 to S_{r_1} are both zonal functions in the same irreducible component of the $SO(n)$ action on $L^2(S_{r_1})$, we have $a_j \in \mathbb{C}$ such that

$$a_1 u_2 + a_2 v_2 = 0 \text{ on } S_{r_1} = \partial B_{r_1}(p). \tag{2.39}$$

It follows from Lemma 2.6 that

$$a_1 u_2 + a_2 v_2 \equiv 0 \text{ on } M. \tag{2.40}$$

This contradicts the fact that $u_2 \perp v_2$, so $W = V_{\lambda\alpha}$, and we have the asserted irreducibility. □

In cases where there is not a pole, we have the following variant of Proposition 2.4.

Proposition 2.8 *Let M be a compact, connected, n-dimensional Riemannian manifold, with an $SO(n)$-action by isometries. Assume there is a point $q \in M$ whose orbit*

$$\mathcal{O}_q = \{gq : g \in SO(n)\} \tag{2.41}$$

is a smooth submanifold of M, diffeomorphic to S^{n-1}, as in (1.1)*, in such a way that φ_q intertwines the $SO(n)$ action on $\mathcal{O}_q$ with the standard action of $SO(n)$ on S^{n-1}. Then, for $\lambda^2 \in \operatorname{Spec}(-\Delta)$, $\alpha \in \mathcal{A}_\lambda$,*

$$\begin{gathered}\textit{the action of } SO(n) \textit{ on } V_{\lambda\alpha} \textit{ contains}\\ \textit{at most two irreducible components.}\end{gathered} \tag{2.42}$$

In preparation for proving this, we bring in a variant of Lemma 2.6. For $r > 0$, let

$$\Omega_r = \{x \in M : \operatorname{dist}(x, \mathcal{O}_q) < r\}. \tag{2.43}$$

Then there exists $r_0 > 0$ such that, for $r \in (0, r_0]$, $\partial\Omega_r$ consists of two $SO(n)$ orbits, each diffeomorphic to S^{n-1}, like $\mathcal{O}_q$ in (2.41). The following variant of Lemma 2.6 has a similar proof.

Lemma 2.9 *Take M, λ as in Proposition 2.8, and construct Ω_r as above. Then there exists $r_1 = r_1(\lambda) \in (0, r_0]$ such that*

$$u \in V_\lambda,\ r \in (0, r_1],\ u\big|_{\partial\Omega_r} = 0 \Longrightarrow u \equiv 0. \tag{2.44}$$

Proof of Proposition 2.8. We concentrate on the case $n \geq 3$. Suppose

$$V_{\lambda\alpha} = W_1 \oplus W_2 \oplus W_3 \tag{2.45}$$

is an orthogonal decomposition, with each W_j invariant under the $SO(n)$ action, and suppose $u_j \in W_j,\ u_j \neq 0$. Hence, for each j, $u_j|_{\partial\Omega_{r_1}}$ is not $\equiv 0$.

Denote by $SO(n-1)$ the subgroup of $SO(n)$ fixing q. This subgroup also fixes the points on the geodesic γ through q that is orthogonal to $\mathcal{O}_q$, and it hence fixes the nearby points $q_\pm$ on the two components $\partial_\pm$ of $\partial\Omega_{r_1}$ where γ intersects the boundary of Ω_{r_1}. One can take $g_j \in SO(n)$ such that, for each j, $v_j = L(g_j)u_j$ does not vanish identically on $\{q_\pm\}$, so

$$w_j = \int\limits_{SO(n-1)} L(g)v_j\, dg \tag{2.46}$$

satisfies

$$w_j \in W_j, \quad w_j\big|_{\partial\Omega_{r_1}} \neq 0, \tag{2.47}$$

and each w_j is invariant under the $SO(n-1)$ action.

Now we have the trace map

$$\tau : C^\infty(M) \longrightarrow C^\infty(\partial_+) \oplus C^\infty(\partial_-), \tag{2.48}$$

and the image under τ of $\mathrm{Span}(w_1, w_2, w_3)$ is contained in a two-dimensional subspace of $C^\infty(\partial_+) \oplus C^\infty(\partial_-)$, consisting of zonal harmonics. Hence there exist $a_j \in \mathbb{C}$, not all zero, such that

$$\tau(a_1 w_1 + a_2 w_2 + a_3 w_3) = 0. \tag{2.49}$$

It follows from Lemma 2.9 that

$$a_1 w_1 + a_2 w_2 + a_3 w_3 = 0 \quad \text{on} \quad M, \tag{2.50}$$

contradicting the mutual orthogonality of the spaces W_j. This gives the desired conclusion (2.42), for $n \geq 3$. The proof for $n = 2$ is a variant, somewhat like the proof for $n = 2$ in Proposition 2.4. □

3 Amplitude distribution asymptotics for spherical harmonics on S^2

As discussed in §1, $L^2(S^2)$ has an orthonormal basis consisting of the spherical harmonics

$$Y_k^\ell, \quad k \in \mathbb{Z}^+,\ \ell \in \mathbb{Z},\ |\ell| \le k, \tag{3.1}$$

given by (1.26)–(1.27), joint eigenfunctions for the operators Λ, given by (1.18), and X_1, described below (1.21):

$$\Lambda Y_k^\ell = k Y_k^\ell, \quad X_1 Y_k^\ell = \ell Y_k^\ell. \tag{3.2}$$

We see from (1.26) that

$$|Y_k^\ell(\omega)|^2 = |\alpha_{k\ell}|^2 (1-\omega_3^2)^{|\ell|} P_k^{(|\ell|)}(\omega_3)^2, \tag{3.3}$$

and this is a zonal function. Here we examine the asymptotic behavior of

$$\int_{S^2} f(\omega) |Y_k^\ell(\omega)|^2 \, dS(\omega) = (M_f Y_k^\ell, Y_k^\ell)_{L^2}, \tag{3.4}$$

when f is a zonal function on S^2. We use methods of microlocal analysis, rather than an analysis of the special functions P_k.

To begin, given $A : L^2(S^2) \to L^2(S^2)$, we form

$$\Pi(A) = \frac{1}{2\pi} \int_0^{2\pi} e^{-it\Lambda} A e^{it\Lambda} \, dt. \tag{3.5}$$

Note that $\{e^{it\Lambda} : t \in \mathbb{R}\}$ is periodic in t of period 2π, and $\Pi(A)$ commutes with $e^{it\Lambda}$ for all t. If

$$A \in OPS^0(S^2) \tag{3.6}$$

has principal symbol $a \in C^\infty(S^*S^2)$, then Egorov's theorem gives

$$\Pi(A) - \mathrm{op}(Pa) \in OPS^{-1}(S^2), \tag{3.7}$$

where $Pa \in C^\infty(S^*S^2)$ is given by

$$Pa(x,\xi) = \frac{1}{2\pi} \int_0^{2\pi} a(\mathcal{G}_t(x,\xi)) \, dt, \tag{3.8}$$

$\{\mathcal{G}_t : t \in \mathbb{R}\}$ denoting the Hamilton flow on S^*S^2 corresponding to the geodesic flow, which is also periodic of period 2π, and

$$\mathrm{op} : C^\infty(S^*S^2) \longrightarrow OPS^0(S^2) \tag{3.9}$$

is given by an appropriate quantization procedure.

Now we specialize to

$$Au = fu, \quad f \in C^\infty(S^2), \text{ zonal.} \tag{3.10}$$

Then $\Pi(A)$ also commutes with $R(t) = e^{itX_1}$, for all t. Consequently, for A of the form (3.10),

$$\Pi(A) \text{ commutes with both } \Lambda \text{ and } X_1. \tag{3.11}$$

Given that $\operatorname{Spec}(\Lambda, X_1)$ is simple, it follows that $\Pi(A)$ has the form

$$\Pi(A) = F(\Lambda, X_1). \tag{3.12}$$

Since we also know that $\Pi(A)$ is a pseudodifferential operator and we have a formula for its principal symbol, we can deduce information about the function F, using results on functional calculus for commuting self-adjoint pseudodifferential operators given in [9] and in Chapter 12 of [10] (see also [8] and [2], for related developments). These results yield

$$F \in S^0(\mathbb{R}^2) \Longrightarrow F(\Lambda, X_1) = B \in OPS^0(S^2), \tag{3.13}$$

with principal symbol

$$b(x, \xi) = F(|\xi|, \langle Y, \xi\rangle), \tag{3.14}$$

where $Y = iX_1$ is the real vector field generating rotation about the x_3-axis (of period 2π). Note that it suffices to specify F on

$$\{(\lambda_1, \lambda_2) : \lambda_1 \geq 0, \ |\lambda_2| \leq \lambda_1\},$$

in light of the identification of $\operatorname{Spec}(\Lambda, X_1)$, and taking into account that $|Y| \leq 1$ on S^2. We want the principal part of (3.14) to match up with (3.8) on S^*S^2.

In light of this, we are motivated to define $F_0(\lambda_1, \lambda_2)$, homogeneous of degree 0 in (λ_1, λ_2), so that

$$F_0\big(1, \langle Y, \xi\rangle\big) = Pa(x, \xi) \ \text{ for } \ (x, \xi) \in S^*S^2. \tag{3.15}$$

Now $F_0(1, \lambda_2)$ is a function of $\lambda_2 \in [-1, 1]$, while Pa is a function on S^*S^2, which has dimension 3. However, Pa is invariant under the flows $\mathcal{G}_t$ and $\mathcal{F}_Y^t$ (the flow generated by Y), and in fact it is uniquely specified by its behavior on $S^*_{x_0}S^2$, where x_0 is an arbitrarily chosen point on the equator of S^2. At x_0, Y is a unit vector tangent to the equator, and (3.15) becomes

$$F_0(1, \lambda_2) = Pa\big(x_0, (\lambda_2, \sqrt{1 - \lambda_2^2})\big). \tag{3.16}$$

At first glance, this looks non-smooth at $\lambda_2 = \pm 1$, but in fact we have

$$Pa(x_0, (\xi_1, \xi_2)) = Pa(x_0, (\xi_1, -\xi_2)). \tag{3.17}$$

Such an identity is clear if $f(x)$ is even under $x_3 \mapsto -x_3$. On the other hand, if $f(x)$ is odd under this transformation its invariance under $R(t)$ guarantees that (3.8) vanishes, so we have (3.17) for general $R(t)$-invariant $f \in C^\infty(S^2)$. From (3.17) we have that (3.16) defines a smooth function of $\lambda_2 \in [-1, 1]$. We have the following conclusion.

Proposition 3.1 *Let $f \in C^\infty(S^2)$ be a zonal function and let $A = M_f$, as in* (3.10)*. Define Pa as in* (3.8)*, with $a(x, \xi) = f(x)$. Then there exists $F_0 \in S^0(\mathbb{R}^2)$ so that* (3.15)–(3.16) *hold, for $|\lambda_2| \le 1$, and we have*

$$\begin{aligned} &F_0(\Lambda, X_1) \in OPS^0(S^2), \quad \textit{and} \\ &\Pi(A) - F_0(\Lambda, X_1) = R \in OPS^{-1}(S^2). \end{aligned} \tag{3.18}$$

Note that we have

$$F_0(\Lambda, X_1) = g(\Lambda^{-1} X_1), \tag{3.19}$$

where $g(\lambda) = F_0(1, \lambda)$, for $|\lambda| \le 1$, i.e.,

$$g(\lambda) = Pa\big(x_0, (\lambda, \sqrt{1 - \lambda^2})\big). \tag{3.20}$$

We return to (3.4) and write

$$\begin{aligned} \int\limits_{S^2} f(\omega) |Y_k^\ell(\omega)|^2 \, dS(\omega) &= (A Y_k^\ell, Y_k^\ell) \\ &= (\Pi(A) Y_k^\ell, Y_k^\ell) \\ &= (F_0(\Lambda, X_1) Y_k^\ell, Y_k^\ell) + (R Y_k^\ell, Y_k^\ell), \end{aligned} \tag{3.21}$$

hence

$$\int\limits_{S^2} f(\omega) |Y_k^\ell(\omega)|^2 \, dS(\omega) = g\Big(\frac{\ell}{k}\Big) + R_{k\ell}, \tag{3.22}$$

with $g(\lambda)$ given by (3.20), and

$$R_{k\ell} = (R Y_k^\ell, Y_k^\ell) = \frac{1}{k} (R \Lambda Y_k^\ell, Y_k^\ell), \tag{3.23}$$

hence

$$|R_{k\ell}| \le \frac{1}{k} \|R\Lambda\|_{\mathcal{L}(L^2)}, \quad R\Lambda \in OPS^0(S^2). \tag{3.24}$$

We can combine (3.22)–(3.24) with Corollary 2.2 to obtain the following.

Proposition 3.2 *Assume $u \in V_k$, i.e., $\Lambda u = ku$, so*

$$u = \sum_{|\ell| \le k} a_\ell Y_k^\ell. \tag{3.25}$$

Let $f \in C^\infty(S^2)$ be a zonal function, as in Proposition 3.1. Then

$$\int_{S^2} f(\omega) |u(\omega)|^2 \, dS(\omega) = \sum_{|\ell| \le k} |a_\ell|^2 g\Bigl(\frac{\ell}{k}\Bigr) + R_k(u), \tag{3.26}$$

with g as described in Proposition 3.1, and

$$|R_k(u)| \le \frac{C}{k} \|u\|_{L^2}^2, \quad C = \|R\Lambda\|_{\mathcal{L}(L^2)}. \tag{3.27}$$

We give a geometrical perspective on how the function $g \in C^\infty([-1,1])$ depends on the zonal function $f \in C^\infty(S^2)$. Pick a point on the equator of S^2, say $x_0 = (1,0,0) \in S^2 \subset \mathbb{R}^3$. We have natural identifications of $T_{x_0}S^2$, $T^*_{x_0}S^2$, the (x_1,x_3)-plane, and $\mathbb{R}^2$. Given

$$\lambda \in [-1,1], \quad \text{take} \quad v = (\lambda, \sqrt{1-\lambda^2}) \in S^1 \subset \mathbb{R}^2. \tag{3.28}$$

Let γ be the unit speed geodesic through x_0, with initial velocity v. This is a "great circle," of circumference 2π, starting and ending at x_0. Then

$$g(\lambda) = \text{ mean value of } \; f\bigr|_\gamma. \tag{3.29}$$

Under these circumstances, we see that

$$\gamma(t) = (\cos t, \lambda \sin t, \sqrt{1-\lambda^2}\, \sin t). \tag{3.30}$$

If we (pedantically) set $f(\omega) = f_0(\omega_3)$, then

$$g(\lambda) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} f_0(\sqrt{1-\lambda^2}\, \sin t) \, dt. \tag{3.31}$$

Recovering a smoothness argument made above, we see that, if

$$f_0(s) = f_1(s^2) + s f_2(s^2), \quad s \in [-1,1], \tag{3.32}$$

then

$$g(\lambda) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} f_1((1-\lambda^2)\sin^2 t) \, dt. \tag{3.33}$$

Returning to (3.31), we make a change of variable and write

$$g(\lambda) = \frac{1}{\pi} \int_{-1}^{1} f_0(\sqrt{1-\lambda^2}\, s)\, \frac{ds}{\sqrt{1-s^2}}. \tag{3.34}$$

The extreme cases are

$$g(0) = \frac{1}{\pi} \int_{-1}^{1} f_0(s)\, \frac{ds}{\sqrt{1-s^2}}, \quad g(\pm 1) = f_0(0). \tag{3.35}$$

In these cases, (3.22) yields, for smooth zonal functions f, first

$$\begin{aligned} &\int\limits_{S^2} f(\omega)|Y_k^0(\omega)|^2\, dS(\omega) \\ &= \frac{1}{\pi} \int_{-1}^{1} f_0(s)\, \frac{ds}{\sqrt{1-s^2}} + O(k^{-1}) \\ &= \frac{1}{2\pi^2} \int\limits_{S^2} \frac{f(\omega)}{\sqrt{1-\omega_3^2}}\, dS(\omega) + O(k^{-1}), \end{aligned} \tag{3.36}$$

which one can compare with the statement (1.34) about the upper envelope of $P_k(t)$. Second,

$$\int\limits_{S^2} f(\omega)|Y_k^k(\omega)|^2\, dS(\omega) = f_0(0) + O(k^{-1}), \tag{3.37}$$

which one can compare with the concentration analysis (1.37).

Going further, we can establish a version of (1.38). In fact, from (3.34), we see that, if f is a smooth zonal function,

$$\begin{aligned} &f(\omega) = 0 \text{ for } |\omega_3| \le \beta \\ &\Longrightarrow g(\lambda) = 0 \text{ for } \sqrt{1-\lambda^2} \le \beta \\ &\Longrightarrow \int\limits_{S^2} f(\omega)|Y_k^\ell(\omega)|^2\, dS(\omega) = O(k^{-1}), \text{ for } \frac{|\ell|}{k} \ge \sqrt{1-\beta^2}, \end{aligned} \tag{3.38}$$

the latter implication by (3.22). A convenient choice of f yields the following.

Proposition 3.3 *Take $\beta \in (0,1)$, $\delta > 0$. Then*

$$\frac{|\ell|}{k} \ge \sqrt{1-\beta^2} \Longrightarrow \int\limits_{|\omega_3| \ge \beta+\delta} |Y_k^\ell(\omega|^2\, dS(\omega) \le \frac{C(\delta)}{k}. \tag{3.39}$$

We will obtain sharper estimates, in more general settings, in §4.

4 Shadow regions for families of eigenfunctions

In this section we let M be a compact, connected, n-dimensional Riemannian manifold, and assume M has a nonzero Killing field, generating a 1-parameter group of isometries of M. We will also make the hypothesis that

$$A_0 = \min_{x\in M} |Y(x)| < \max_{x\in M} |Y(x)| = A_1. \tag{4.1}$$

Possibly $A_0 = 0$. The operator $X = iY$ is self adjoint and commutes with $\Lambda = \sqrt{-\Delta}$. If $\lambda \in \operatorname{Spec}\Lambda$, then the λ-eigenspace

$$V_\lambda = \{u \in C^\infty(M) : \Lambda u = \lambda u\} \tag{4.2}$$

splits into joint eigenspaces

$$V_\lambda = \bigoplus_{\mu\in\mathcal{A}_\lambda} V_{\lambda\mu}, \quad V_{\lambda\mu} = \{u \in V_\lambda : Xu = \mu u\}, \tag{4.3}$$

where $\mathcal{A}_\lambda = \{\mu : V_{\lambda\mu} \neq 0\}$. We have

$$\operatorname{Spec}(\Lambda, X) = \{(\lambda,\mu) : \lambda \in \operatorname{Spec}\Lambda,\ \mu \in \mathcal{A}_\lambda\}. \tag{4.4}$$

Note that, if $u \in V_{\lambda\mu}$ and $\|u\|_{L^2} = 1$, then

$$\begin{aligned}\mu^2 = \|Xu\|^2_{L^2} \le A_1^2\|\nabla u\|^2_{L^2} = A_1^2(-\Delta u, u)\\ = A_1^2\|\Lambda u\|^2_{L^2} = A_1^2\lambda^2,\end{aligned} \tag{4.5}$$

i.e.,

$$\mu \in \mathcal{A}_\lambda \Longrightarrow |\mu| \le A_1\lambda. \tag{4.6}$$

Now, given a bounded function $F : \operatorname{Spec}(\Lambda, X) \to \mathbb{R}$, we can define $F(\Lambda, X)$ on $L^2(M)$ by

$$F(\Lambda, X)u = F(\lambda,\mu)u, \quad \text{for } u \in V_{\lambda\mu}. \tag{4.7}$$

As shown in Chapter 12 of [10],

$$F \in S^0(\mathbb{R}^2) \Longrightarrow F(\Lambda, X) \in OPS^0(M), \tag{4.8}$$

and its principal symbol is

$$\sigma_{F(\Lambda,X)}(x,\xi) = F(|\xi|, \langle Y, \xi\rangle). \tag{4.9}$$

We will concentrate on F of the following form: $F \in C^\infty(\mathbb{R}^2 \setminus 0)$, homogeneous 0. Note that only its behavior on the wedge $\{(\lambda, \mu) : |\mu| \le A_1 \lambda\}$ is significant for the behavior of $F(\Lambda, X)$. We set

$$g(\mu) = F(1, \mu), \quad \text{so} \;\; F(\Lambda, X) = g(\Lambda^{-1} X). \tag{4.10}$$

Note that only the behavior of g on $\mu \in [-A_1, A_1]$ is significant.

Using this analysis of $F(\Lambda, X)$, we will study certain "shadow regions" $\Omega \subset M$, and families of unit-norm eigenvectors in $V_{\lambda\mu}$ whose restrictions to Ω decay rapidly as $\lambda \to \infty$. The shadow regions will have the form

$$\Omega_A = \{x \in M : |Y(x)| \le A\}, \tag{4.11}$$

where we take $A \in (A_0, A_1)$. To start, take

$$f \in C^\infty(M), \quad \operatorname{supp} f \subset \Omega_A, \quad A' > A. \tag{4.12}$$

From (4.8)–(4.9) we obtain the following:

$$\begin{aligned} &g \in C^\infty([-A_1, A_1]),\; g(\mu) = 0 \text{ for } |\mu| \le A' \\ &\quad \Longrightarrow \sigma_{F(\Lambda,X)}(x, \xi) = 0, \;\; \forall\, x \in \Omega_A,\; \xi \in T_x^* M \\ &\quad \Longrightarrow M_f g(\Lambda^{-1} X) \in OPS^{-1}(M). \end{aligned} \tag{4.13}$$

Under these circumstances, we have

$$\begin{aligned} \int_M f\, |g(\Lambda^{-1}X) u|^2 \, dV &= (M_f g(\Lambda^{-1}X) u, g(\Lambda^{-1}X) u) \\ &\le \|M_f g(\Lambda^{-1}X) u\|_{L^2} \|g(\Lambda^{-1}X) u\|_{L^2}. \end{aligned} \tag{4.14}$$

Since, for $u \in V_{\lambda\mu}$, we have

$$M_f g(\Lambda^{-1}X) u = \lambda^{-1} M_f g(\Lambda^{-1}X) \Lambda u, \tag{4.15}$$

and $M_f g(\Lambda^{-1}X)\Lambda \in OPS^0(M)$, we have the following conclusion:

Proposition 4.1 *Under the hypotheses on f and g given in* (4.12)–(4.13)*,*

$$u \in V_{\lambda\mu} \Rightarrow g\Big(\frac{\mu}{\lambda}\Big) \int_M f |u|^2 \, dV \le \frac{C}{\lambda} \|u\|_{L^2}^2. \tag{4.16}$$

As it stands, this result generalizes Proposition 3.3. However, as advertised there, our real goal in this section is to produce a much sharper estimate. The foundation for this is the following improvement on the conclusion in (4.13).

Proposition 4.2 *Under the hypotheses on f and g given in* (4.12)–(4.13),

$$M_f g(\Lambda^{-1}X) \in OPS^{-\infty}(M). \tag{4.17}$$

Having this result, we can replace (4.15) by

$$M_f g(\Lambda^{-1}X)u = \lambda^{-m} M_f g(\Lambda^{-1}X)\Lambda^m u \tag{4.18}$$

(provided $u \in V_{\lambda\mu}$), with

$$M_f g(\Lambda^{-1}X)\Lambda^m \in OPS^{-\infty}(M), \tag{4.19}$$

to conclude that, for each $m \in \mathbb{N}$,

$$\|M_f g(\Lambda^{-1}X)u\|_{C^m(M)} \le C_m \lambda^{-m} \|u\|_{L^2}, \tag{4.20}$$

yielding the following improvement of Proposition 4.1.

Proposition 4.3 *Under the hypotheses on f and g given in* (4.12)–(4.13), *we have*

$$u \in V_{\lambda\mu} \Rightarrow g\Bigl(\frac{\mu}{\lambda}\Bigr)\|fu\|_{C^m(M)} \le \frac{C_m}{\lambda^m}\|u\|_{L^2}, \tag{4.21}$$

for each $m \in \mathbb{N}$.

The content of Proposition 4.2 is that, under the stated hypotheses on f and g, the total symbol of $F(\Lambda, X) = g(\Lambda^{-1}X)$ vanishes on $T^*\Omega_A \setminus 0$, not just the principal symbol. One approach would be to analyze the total symbol of $F(\Lambda, X)$ on $T^*M \setminus 0$, but we will pursue an alternative approach, making use of local elliptic regularity.

To proceed, pick $h \in C^\infty([-A_1, A_1])$ such that

$$\begin{aligned} h(\mu) = 1 \quad &\text{for } |\mu| \le A, \\ 0 \quad &\text{for } |\mu| \ge A'. \end{aligned} \tag{4.22}$$

Now $\sigma_{h(\Lambda^{-1}X)}(x, \xi) = h(\langle Y(x), \xi/|\xi|\rangle)$, so

$$h(\Lambda^{-1}X) \in OPS^0(M) \text{ is elliptic on } \Omega_A. \tag{4.23}$$

Thus there exists $P \in OPS^0(M)$ such that $Ph(\Lambda^{-1}X)$ is microlocally I on a conic neighborhood of $T^*\Omega_A \setminus 0$, so

$$M_f - M_f P h(\Lambda^{-1}X) = R \in OPS^{-\infty}(M), \tag{4.24}$$

so (since $gh \equiv 0$)

$$\begin{aligned} M_f g(\Lambda^{-1}X) &= M_f P h(\Lambda^{-1}X) g(\Lambda^{-1}X) + R g(\Lambda^{-1}X) \\ &= R g(\Lambda^{-1}X), \end{aligned} \tag{4.25}$$

which belongs to $OPS^{-\infty}(M)$. This proves Proposition 4.2.

5 Weyl asymptotics for joint eigenfunctions

Take M, Λ, X as in §4. For each $(\lambda, \mu) \in \operatorname{Spec}(\Lambda, X)$, take an orthonormal basis

$$u_{\lambda\mu j} \in V_{\lambda\mu}, \quad 1 \le j \le d_{\lambda\mu} = \dim V_{\lambda\mu}. \tag{5.1}$$

Given $B \in OPS^0(M)$, there is the Weyl formula

$$\lim_{R\to\infty} \frac{1}{N(R)} \sum_{\lambda\le R} \sum_{\mu\in\mathcal{A}_\lambda} \sum_{j\le d_{\lambda\mu}} (Bu_{\lambda\mu j}, u_{\lambda\mu j}) = \int\limits_{S^*M} \sigma_B(x,\xi)\, dS(x,\xi), \tag{5.2}$$

where $dS(x, \xi)$ denotes the Liouville measure on S^*M, normalized to have total mass 1, and

$$N(R) = \dim \bigoplus_{\lambda\le R} V_\lambda = \sum_{\lambda\le R} \sum_{\mu\in\mathcal{A}_\lambda} d_{\lambda\mu}. \tag{5.3}$$

There is also the Weyl asymptotic formula

$$N(R) \sim C(M)R^n + O(R^{n-1}), \quad C(M) = \frac{V_n}{(2\pi)^n} \operatorname{Vol}(M), \tag{5.4}$$

where $V_n = \pi^{n/2}/\Gamma(n/2 + 1)$ is the volume of the unit ball in $\mathbb{R}^n$.

To obtain a variant of Proposition 3.2, we take

$$h \in C^\infty([-A_1, A_1]), \quad f \in C^\infty(M), \tag{5.5}$$

and set

$$B = M_f h(\Lambda^{-1}X) \in OPS^0(M), \tag{5.6}$$

with principal symbol

$$\sigma_B(x, \xi) = f(x)h\big(\langle Y(x), \xi/|\xi|\rangle\big), \tag{5.7}$$

to obtain the following.

Proposition 5.1 *For M, Λ, X as in §4, h, f as in (5.5), we have*

$$\begin{aligned} &\lim_{R\to\infty} \frac{1}{N(R)} \sum_{\{(\lambda,\mu,j):\lambda\le R\}} h\Big(\frac{\mu}{\lambda}\Big) \int\limits_M f(x)|u_{\lambda\mu j}(x)|^2\, dV(x) \\ &= \int\limits_{S^*M} f(x)h\big(\langle Y(x), \xi\rangle\big)\, dS(x,\xi). \end{aligned} \tag{5.8}$$

The result (5.2) is typically established using heat equation asymptotics, which yield

$$\operatorname{Tr} Be^{t\Delta} \sim C_1(M)\Bigl(\int\limits_{S^*M} \sigma_B \, dS\Bigr) t^{-n/2}, \quad \text{as} \;\; t \searrow 0, \tag{5.9}$$

together with a Tauberian theorem. Sharper results are obtained via a wave equation approach, presented in [6], which generated a large body of work. This yields an analysis of

$$\operatorname{Tr} Be^{it\Lambda} \in \mathcal{D}'(\mathbb{R}), \tag{5.10}$$

as a distribution having an isolated singularity at $t = 0$ (and typically other singularities, which for current purposes one arranges to ignore). One obtains the following result. Take

$$\varphi \in \mathcal{S}(\mathbb{R}), \quad \varphi \geq 0, \quad \operatorname{supp} \hat{\varphi} \subset (-r, r), \quad r < \operatorname{Inj} M, \tag{5.11}$$

where Inj M denotes the injectivity radius of M. Then, making use of

$$B\varphi(\Lambda - R) = \int_{-\infty}^{\infty} Be^{it(\Lambda - R)} \hat{\varphi}(t) \, dt,$$

one analyzes

$$\operatorname{Tr} B\varphi(\Lambda - R).$$

One obtains, in place of (5.2),

$$\lim_{R\to\infty} \frac{1}{N_\varphi(R)} \sum_{\lambda,\mu,j} \varphi(\lambda - R)(Bu_{\lambda\mu j}, u_{\lambda\mu j}) = \int\limits_{S^*M} \sigma_B(x,\xi) \, dS(x,\xi), \tag{5.12}$$

where

$$N_\varphi(R) = \operatorname{Tr} \varphi(\Lambda - R) = \sum_\lambda \varphi(\lambda - R) \dim V_\lambda, \tag{5.13}$$

which satisfies

$$N_\varphi(R) \sim C(\varphi, M) R^{n-1}. \tag{5.14}$$

Taking B as in (5.6), we have the following.

Proposition 5.2 *For* M, Λ, X *as in* §*4,* h, f *as in* (5.5)*, and* φ *as in* (5.11)*, we have*

$$\begin{aligned} \lim_{R\to\infty} \frac{1}{N_\varphi(R)} \sum_\lambda \varphi(\lambda - R) \sum_{\mu,j} h\Bigl(\frac{\mu}{\lambda}\Bigr) \int\limits_M f(x) |u_{\lambda\mu j}(x)|^2 \, dV(x) \\ = \int\limits_{S^*M} f(x) h\bigl(\langle Y(x), \xi\rangle\bigr) \, dS(x,\xi). \end{aligned} \tag{5.15}$$

Let us take a look at the following case, of 2D surfaces of revolution, with a pole.

Corollary 5.3 *In the setting of Proposition 5.2, assume in addition that*

$$\dim M = 2, \quad M \text{ has a pole}, \quad \mathcal{F}_Y^t \text{ has period } 2\pi, \tag{5.16}$$

so

$$(\lambda, \mu) \in \operatorname{Spec}(\Lambda, X) \Rightarrow \mu = \ell \in \mathbb{Z},\ d_{\lambda\ell} \equiv 1. \tag{5.17}$$

Then

$$\begin{aligned}\lim_{R\to\infty} \frac{1}{N_\varphi(R)} \sum_\lambda \varphi(\lambda - R) \sum_\ell h\Bigl(\frac{\ell}{\lambda}\Bigr) \int\limits_M f(x)|u_{\lambda\ell}(x)|^2\, dS(x) \\ = \int\limits_{S^*M} f(x) h\bigl(\langle Y(x), \xi\rangle\bigr)\, dS(x, \xi).\end{aligned} \tag{5.18}$$

We can interpret (5.18) as saying that, in an "averaged" sense, as $\lambda \to \infty$,

$$\sum_{\ell \in \mathcal{A}_\lambda} h\Bigl(\frac{\ell}{\lambda}\Bigr) \int\limits_M f|u_{\lambda\ell}|^2\, dS \sim \dim V_\lambda \int\limits_{S^*M} f(x) h\bigl(\langle Y(x), \xi\rangle\bigr)\, dS(x, \xi). \tag{5.19}$$

For comparison, in case $M = S^2$, (3.22) implies, for $\lambda = k \in \operatorname{Spec}\Lambda,\ k \to \infty$, and f zonal,

$$\begin{aligned}\sum_{\ell=-k}^{k} h\Bigl(\frac{\ell}{k}\Bigr) \int\limits_{S^2} f|Y_k^\ell|^2\, dS &\sim \sum_{\ell=-k}^{k} h\Bigl(\frac{\ell}{k}\Bigr) g\Bigl(\frac{\ell}{k}\Bigr) \\ &\sim \dim V_k \int_{-1}^{1} h(s) g(s)\, ds,\end{aligned} \tag{5.20}$$

with g given by (3.20). We mention this here simply as a "heuristic," rather than something we will take further. But heuristics sometimes do lead to further interesting results.

We return to the general setting of Proposition 5.2 and take $f \equiv 1$, to obtain further information on the joint spectrum $\operatorname{Spec}(\Lambda, X)$. We deduce from (5.15) that

$$\begin{aligned}\lim_{R\to\infty} \frac{1}{N_\varphi(R)} \sum_\lambda \varphi(\lambda - R) \sum_{\mu \in \mathcal{A}_\lambda} h\Bigl(\frac{\mu}{\lambda}\Bigr) d_{\lambda\mu} \\ = \int\limits_{S^*M} h\bigl(\langle Y(x), \xi\rangle\bigr)\, dS(x, \xi).\end{aligned} \tag{5.21}$$

To proceed, we write

$$\int_{S^*M} h\big(\langle Y(x),\xi\rangle\big)\,dS(x,\xi) = \int_I h(y)\,d\gamma(y), \quad I=[-A_1,A_1], \tag{5.22}$$

where γ is the push-forward of Liouville measure on S^*M under the map

$$\sigma=\sigma_X : S^*M \longrightarrow I, \quad \sigma_X(x,\xi)=\langle Y(x),\xi\rangle. \tag{5.23}$$

The following is a useful observation.

Lemma 5.4 *The measure γ is absolutely continuous with respect to Lebesgue measure on I, so*

$$\int_{S^*M} h\big(\langle Y(x),\xi\rangle\big)\,dS(x,\xi) = \int_I h(y)\Psi(y)\,dy, \tag{5.24}$$

with Ψ (called the joint spectral clustering factor) satisfying

$$\Psi=\Psi_{M,X}\in L^1(I,dy), \quad \Psi\ge 0, \quad \int_I \Psi(y)\,dy=1. \tag{5.25}$$

Proof. What is to be shown is that if $K\subset I$ is a Borel set,

$$\gamma(K)=0 \Longrightarrow \int_{\sigma^{-1}(K)} dS(x,\xi)=0. \tag{5.26}$$

Indeed, for each $x\in M$, we have $\sigma_x : S^*_xM\to\mathbb{R}$ given by $\sigma_x(\xi)=\langle Y(x),\xi\rangle$, and (if $\gamma(K)=0$) $\sigma_x^{-1}(K)$ has measure 0 in the $(n-1)$-sphere S^*_xM for each x for which $Y(x)\neq 0$, which is all but at most 2 values of x. The implication (5.26) then follows via Fubini's theorem. □

From (5.21)–(5.24) we have

$$\begin{aligned}\lim_{R\to\infty} \frac{1}{N_\varphi(R)} \sum_\lambda \varphi(\lambda-R) \sum_{\mu\in\mathcal{A}_\lambda} h\Big(\frac{\mu}{\lambda}\Big) d_{\lambda\mu} \\ = \int_I h(y)\,d\gamma(y) = \int_I h(y)\Psi(y)\,dy,\end{aligned} \tag{5.27}$$

for $h\in C^\infty(I)$. We want to extend this to a broader class of functions h. The extension to $h\in C(I)$ is easy enough, but we want to go further. To do this, suppose $h: I\to\mathbb{R}$ is bounded and that

$$h_-\le h\le h_+, \tag{5.28}$$

where $h_\pm$ belong to a class $\mathcal{C}(I)$ of bounded, Borel functions on I for which (5.27) is known to hold (with h replaced by h_+, h_-). We deduce that

$$\begin{aligned} \limsup_{R\to\infty} \frac{1}{N_\varphi(R)} \sum_\lambda \varphi(\lambda - R) \sum_{\mu\in\mathcal{A}_\lambda} h\Bigl(\frac{\mu}{\lambda}\Bigr) d_{\lambda\mu} &\le \int_I h_+\, d\gamma, \\ \liminf_{R\to\infty} \frac{1}{N_\varphi(R)} \sum_\lambda \varphi(\lambda - R) \sum_{\mu\in\mathcal{A}_\lambda} h\Bigl(\frac{\mu}{\lambda}\Bigr) d_{\lambda\mu} &\ge \int_I h_-\, d\gamma. \end{aligned} \tag{5.29}$$

This in turn yields the following.

Lemma 5.5 *Let $h : I \to \mathbb{R}$ be a bounded function. Assume that, for each $\varepsilon > 0$, there exist $h_\pm \in \mathcal{C}(I)$ such that* (5.28) *holds and*

$$\int_I (h_+ - h_-)\, d\gamma < \varepsilon. \tag{5.30}$$

then (5.27) *holds for h.*

If $h : I \to \mathbb{R}$ is bounded and, for each $\varepsilon > 0$, there exist $h_\pm \in C(I)$ such that (5.30) holds (in which case there exist $h_\pm \in C^\infty(I)$ such that (5.30) holds), we say h is Riemann integrable on the measured metric space (I, γ), and write

$$h \in \mathcal{R}(I, \gamma). \tag{5.31}$$

The content of Lemma 5.5 is that

$$(5.27) \text{ holds for } \ h \in \mathcal{R}(I, \gamma).$$

See Appendix A for a brief treatment of Riemann integrable functions on a compact measured metric space. The standard example, of course, is $\mathcal{R}(I)$, the space of Riemann integrable functions on a compact interval $I \subset \mathbb{R}$, in case γ is Lebesgue measure. Our next goal is to establish:

Proposition 5.6 *For M, Λ, X as in Proposition 5.2, $f \in C^\infty(M)$, and φ as in* (5.11)*, we have* (5.27) *for all $h \in \mathcal{R}(I)$.*

Proof. It remains only to prove that

$$\mathcal{R}(I) \subset \mathcal{R}(I, \gamma), \tag{5.32}$$

when γ is absolutely continuous with respect to Lebesgue measure. To do this, we use the fact that, for any finite Borel measure γ on I, a bounded function $h : I \to \mathbb{R}$ belongs to $\mathcal{R}(I, \gamma)$ if and only if the set of points in I at which h is discontinuous has γ-measure 0. See Proposition A.1. Since this characterization applies both to $\mathcal{R}(I, \gamma)$ and to $\mathcal{R}(I)$ (where it is classical), we have (5.32). $\square$

We next compute the functions Ψ that arise in (5.24) and (5.27), in the two basic cases emphasized in the introduction.

Example 1. $M = \mathbb{T}^2$.
In this case, one has $S^*_x M$ canonically equivalent to the unit circle $S^1 \subset \mathbb{R}^2$, for each $x \in \mathbb{T}^2$, and the push-forward of arc length on $S^*_x M$ under σ_x is $2/\sqrt{1-y^2}\, dy$. Hence, normalizing, we obtain

$$\Psi(y) = \frac{1}{\pi} \frac{1}{\sqrt{1-y^2}}, \quad y \in [-1, 1]. \tag{5.33}$$

Example 2. $M = S^2$.
In this case we have $S^*M = \{(x, \xi) \in S^2 \times S^2 : \xi \perp x\}$. This is naturally diffeomorphic to

$$SO(3) = \{X = (x, \xi, \eta) \in M(3, \mathbb{R}) : (x, \xi) \in S^*M,\, \eta = x \times \xi\}. \tag{5.34}$$

The action of $SO(2)$ in S^*M is given by

$$(g, X) \mapsto gX, \quad X \in SO(3), \quad g = \begin{pmatrix} \cos\theta & -\sin\theta & \\ \sin\theta & \cos\theta & \\ & & 1 \end{pmatrix} \in SO(2). \tag{5.35}$$

We have a smooth map

$$\Xi : S^*M \longrightarrow S^2, \tag{5.36}$$

or equivalently

$$\Xi : SO(3) \longrightarrow S^2, \ \text{ given by } \ \Xi(X) = X^t e_3, \tag{5.37}$$

where e_3 is the third standard basis vector of $\mathbb{R}^3$. Note that

$$g \in SO(2) \Longrightarrow \Xi(gX) = X^t g^t e_3 = \Xi(X), \tag{5.38}$$

so Ξ induces a diffeomorphism

$$SO(2) \setminus SO(3) \longrightarrow S^2. \tag{5.39}$$

The explicit formula for Ξ in (5.36) is

$$\Xi(x,\xi) = \begin{pmatrix} x^t \\ \xi^t \\ \eta^t \end{pmatrix} e_3 = \begin{pmatrix} x \cdot e_3 \\ \xi \cdot e_3 \\ \eta \cdot e_3 \end{pmatrix}, \tag{5.40}$$

where, recall, $\eta = x \times \xi$.

We seek a formula for $\sigma_X : S^*M \to \mathbb{R}$, given by $\sigma_X(x,\xi) = \langle Y(x), \xi \rangle$, that factors through (5.40). Note that, in this case, $M = S^2$, we have

$$Y(x) = e_3 \times x, \tag{5.41}$$

hence

$$\sigma_X(x,\xi) = (e_3 \times x) \cdot \xi = (x \times \xi) \cdot e_3 = \eta \cdot e_3, \tag{5.42}$$

so

$$\sigma_X(x,\xi) = \Xi(x,\xi) \cdot e_3. \tag{5.43}$$

Now Ξ in (5.37) is measure-preserving, up to scaling. Furthermore, a classical area computation implies that the map

$$p_3 : S^2 \longrightarrow \mathbb{R}, \quad p_3(x) = x \cdot e_3 \tag{5.44}$$

pushes the standard area measure on S^2 onto Lebesgur measure on $[-1,1]$, up to scaling. We deduce that

$$\Psi(y) = \frac{1}{2}, \quad y \in [-1,1]. \tag{5.45}$$

Remark. To make contact with formulas below, we find it convenient to permute variables in (5.40), and define

$$\widetilde{\Xi} : S^*M \longrightarrow S^2, \quad \widetilde{\Xi}(x,\xi) = \begin{pmatrix} \eta \cdot e_3 \\ \xi \cdot e_3 \\ x \cdot e_3 \end{pmatrix}, \tag{5.46}$$

so

$$\sigma_X(x,\xi) = \widetilde{\Xi}(x,\xi) \cdot e_1. \tag{5.47}$$

Having derived the formulas for the factor $\Psi(y)$ in Examples 1 and 2, we consider their significance in the formula (5.27) for joint spectral asymptotics of (Λ, X) in these two cases. To begin, the formulas (5.33) and (5.45) are strikingly different. The first is singular at the endpoints of $[-1,1]$, while the second is simply constant along this whole interval. The singularities of $\Psi(y)$ in (5.33) provide a quantitative description of the clustering of points of $\operatorname{Spec}(\Lambda, X)$ at the edges $\mu = \pm\lambda$, in case $M = \mathbb{T}^2$, depicted in Figure 1.1. On the other hand, the flat graph of $\Psi(y)$ in (5.45) reflects the even distribution within $\{(\lambda,\mu) : |\mu| \le \lambda\}$ of points of $\operatorname{Spec}(\Lambda, X)$ for $M = S^2$, depicted in Figure 1.3.

6 Further examples, illustrating spectral clustering, concentration, and shadowing

In §5 we illustrated the spectral asymptotic result given in Proposition 5.6 by two examples, $M = \mathbb{T}^2$ and $M = S^2$. We turn to some further examples of types of surfaces of revolution in $\mathbb{R}^3$, giving rise to joint spectral clustering factors $\Psi(y)$ that exhibit various behaviors, ranging from smoothness on the entire interval $I = [-A_1, A_1]$ to cases with jumps and blowups, particularly log blowups, which are less severe than the blowup in (5.27). We will also make some comments on shadow regions and concentration regions for certain classes of joint eigenfunctions.

Example 3. More general convex surfaces of revolution.
Let C be a simple, closed, smooth curve in the (x_1, x_3)-plane, symmetric about the x_3-axis and with positive curvature everywhere. Let M be the surface of revolution in $\mathbb{R}^3$ obtained by rotating C about the x_3-axis. See the left side of Figure 1.7 for an illustration. The surface M has two poles. Translating and scaling, we will assume that they are at e_3 and $-e_3$. Under these hypotheses, there is a diffeomorphism $\psi : M \to S^2$, taking level curves of $x_3|_M$ to level curves of $x_3|_{S^2}$, commuting with the $SO(2)$ action of rotation about the x_3-axis. This gives rise to a diffeomorphism $\psi : TM \to TS^2$, linear on each fiber, hence (via the Riemannian metrics on M and S^2) to a diffeomorphism $\psi : T^*M \to T^*S^2$, yielding in turn a diffeomorphism $\psi : S^*M \to S^*S^2$, commuting with the $SO(2)$-action (but in general not preserving Liouville measure). We can follow this with the diffeomorphism $SO(2) \setminus S^*S^2 \to S^2$, discussed in Example 2, with the goal of analyzing $\sigma_X : S^*M \to \mathbb{R}$.

Actually, for this, it is more direct to proceed via the following observation. Let C_L denote the left side of C (where $x_1 < 0$). Pick $a = (a_1, 0, a_3)^t \in C_L$, and consider $\xi \in S^*_a M$. (Use the inner product on $T_a M$ to identify $S_a M$ and $S^*_a M$.) We map the circle $S^*_a M$ onto the circle $\{x \in M : x_3 = a_3\}$ so that $Y(a)/|Y(a)| \in S_a M \approx S^*_a M$ maps to $|Y(a)|e_1 + a_3 e_3$, and so that the map intertwines counterclockwise rotation in $T^*_a M$ with the $SO(2)$-action of rotation about the x_3 axis specified in (5.35). Call this map

$$\tilde{\Xi} : S^*M\big|_{C_L} \longrightarrow M. \tag{6.1}$$

Using the diffeomorphism ψ described above and the analysis in Example 2, we have a smooth extension to

$$\tilde{\Xi} : S^*M \longrightarrow M, \tag{6.2}$$

invariant under the $SO(2)$ action on S^*M, yielding a diffeomorphism $SO(2)\backslash S^*M \to M$, and, extending (5.47),

$$\sigma_X(x,\xi) = \widetilde{\widetilde{\Xi}}(x,\xi)\cdot e_1. \tag{6.3}$$

In the special case $M = S^2$, $\widetilde{\widetilde{\Xi}}$ pushes Liouville measure on S^*M onto area measure on M (up to scaling). Generally, this might not hold, but the pushforward will be a smooth, positive multiple of area measure on M. We are hence in a position to use (6.3) to describe the behavior of the factor $\Psi(y)$. Indeed, the function $\Xi_0 : M \to \mathbb{R}$ given by $\Xi_0(x) = x\cdot e_1$ is a Morse function, having two critical points, a nondegenerate minumum and a nondegenerate maximum, with critical values $-r_1$ and r_1, where

$$r_1 = \max\, x_1\big|_C. \tag{6.4}$$

We deduce that

$$\Psi \in C^\infty([-r_1, r_1]), \quad \Psi > 0. \tag{6.5}$$

This is a mild variation of the behavior in (5.45), and it leads to somewhat regular density of $\mathrm{Spec}(\Lambda, X)$ in $\{(\lambda,\mu) : |\mu| \le r_1\lambda\}$, without the sort of clustering arising for $\mathbb{T}^2$, as in Figure 1.1.

Example 4. Symmetric dumbbell.
Here we examine an example of a nonconvex surface of revolution, in which there is clustering of $\mathrm{Spec}(\Lambda, X)$. As in Example 3, we start with a simple, closed smooth curve C in the (x_1, x_3)-plane, symmetric about the x_3-axis, and rotate it about the x_3-axis in $\mathbb{R}^3$ to produce a surface of revolution M. The difference here is that C will not have curvature that is everywhere positive. We take C as illustrated in Figure 6.1. We assume

$$\begin{aligned}&\text{the curvature of } C \text{ is nonzero}\\ &\text{at } x_1 = \pm r_1 \text{ and at } x_1 = \pm r_2.\end{aligned}$$

We assume the left side C_L of C is the graph of $x_1 = \beta(x_3)$, $x_3 \in [-1,1]$. In the example we consider here, $\beta(x_3) = \beta(-x_3)$, so M has not only $SO(2)$ symmetry, but also the symmetry $\iota : M \to M$ given by

$$\iota(x_1,x_2,x_3) = (x_1,x_2,-x_3). \tag{6.6}$$

Arguments from Example 3 extend, to produce a smooth map $\widetilde{\widetilde{\Xi}}$, as in (6.2), invariant under the $SO(2)$ action on S^*M, yielding a diffeomorphism

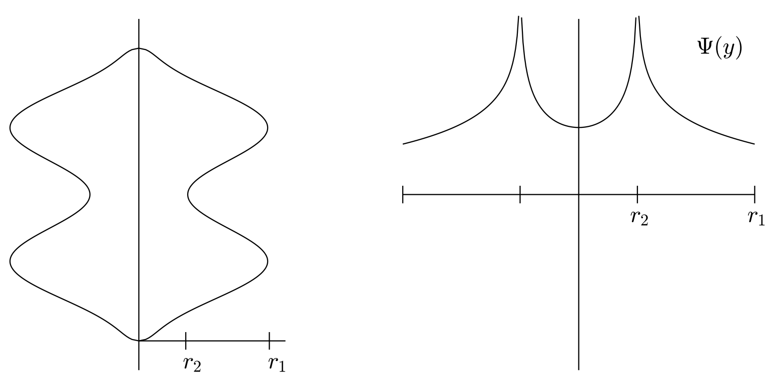

Figure 6.1: Dumbbell figure with singular factor $\Psi(y)$ (Example 4)

$SO(2) \setminus S^*M \to M$, for which (6.3) holds. Again the qualitative features of the factor $\Psi(y)$ can be read off from how

$$\Xi_0 : M \longrightarrow \mathbb{R}, \quad \Xi_0(x) = x \cdot e_1 \tag{6.7}$$

pushes forward the area element on M to a measure on $[-r_1, r_1]$, with r_1 as in (6.4). The difference in this case is that Ξ_0 has more critical values. As before, we have $\pm r_1$ as maxima and minima of Ξ_0. This time, there are a pair of points in M at which $\Xi_0 = r_1$ and a pair of points at which $\Xi_0 = -r_1$. These four points are nondegenerate critical points for Ξ_0. In addition, there are two points in M at which Ξ_0 has a saddle, with critical values $\pm r_2$, as illustrated in Figure 6.1. One sees that the inverse images under Ξ_0 of $(\pm r_2, \pm r_2 + \varepsilon)$ and $(\pm r_2 - \varepsilon, \pm r_2)$ have area $\sim C\varepsilon \log \varepsilon$, so

$$\Psi(y) \sim C \log \frac{1}{|y \mp r_2|} \tag{6.8}$$

for y near $\pm r_2$. See the right side of Figure 6.1 for a depiction of the graph of $\Psi(y)$ in such a case. As a consequence

There is clustering of $\operatorname{Spec}(\Lambda, X)$ along the rays $\mu = \pm r_2 \lambda$, as $\lambda \to \infty$, (6.9)

though the clustering along these rays is less dense than it is along the rays $\mu = \pm\lambda$ for $M = \mathbb{T}^2$, as illustrated in Figure 1.1.

Remark. The result (6.8) assumes the curvature of C is nonzero at $x_1 = \pm r_2$. If the curvature vanished at these points, $\Psi(y)$ would be have a stronger singularity at $y = \pm r_2$. Similarly, if the curvature of C vanished at $x_1 = \pm r_1$, $\Psi(y)$ would have a singularity at $y = \pm r_1$.

Continuing with Example 4, we now look at the part

$$\Sigma(A) = \Big\{ (\lambda, \mu) \in \operatorname{Spec}(\Lambda, X) : \Big| \frac{\mu}{\lambda} \Big| \geq A \Big\} \tag{6.10}$$

of the joint spectrum, for a given $A \in (r_2, r_1)$, and see that there is a bit of "spectral pairing" in this region of $\operatorname{Spec}(\Lambda, X)$. This is related to the eigenfunction concentration result of Proposition 4.3. To recall it, let us set

$$\begin{aligned} \Omega_A &= \{ x \in M : |Y(x)| \leq A \} \\ &= \{ x \in M : r(x) \leq A \}, \end{aligned} \tag{6.11}$$

where $r(x) = (x_1^2 + x_2^2)^{1/2}$, and take

$$f \in C^\infty(M), \quad \operatorname{supp} f \subset \Omega_A. \tag{6.12}$$

Proposition 4.3 implies the following result.

Proposition 6.1 *In the current setting (Example 4), if f satisfies* (6.12), *with $A \in (r_2, r_1)$, then*

$$\begin{aligned} &u \in V_{\lambda\mu},\ (\lambda,\mu) \in \Sigma(A'),\ A' > A \\ &\Longrightarrow \|fu\|_{C^m(M)} \le \frac{C}{\lambda^m}\|u\|_{L^2}, \quad C = C_m(A,A'). \end{aligned} \tag{6.13}$$

Let us now take f satisfying (6.12) and also

$$Yf = 0, \quad \iota^* f = f, \quad f \equiv 1 \text{ on } \Omega_B, \text{ for some } B \in (r_2, A). \tag{6.14}$$

We take u as in (6.13), and set

$$v = (1-f)u, \tag{6.15}$$

so $v = 0$ on a neighborhood of the "neck," $\{x \in M : r(x) = r_2\}$. It follows that

$$(-\Delta - \lambda^2)v = w, \quad \|w\|_{L^2} \le \frac{c_m}{\lambda^m}\|u\|_{L^2}, \tag{6.16}$$

while

$$Xv = \mu v. \tag{6.17}$$

We can rewrite (6.16) as

$$(\Lambda - \lambda)v = w_0 = (\Lambda + \lambda)^{-1}w, \quad \|w_0\|_{L^2} \le \frac{c_m}{\lambda^{m+1}}\|u\|_{L^2}. \tag{6.18}$$

The function v decouples into two pieces with disjoint support:

$$v = v_+ + v_-, \quad \operatorname{supp} v_\pm \subset \{x \in M : \pm x_3 > 0\}, \tag{6.19}$$

and v_+ and v_- separately satisfy conditions of the form (6.16)–(6.18). Since M has a pole, we have from Proposition 2.4 that

$$(\lambda,\mu) \in \operatorname{Spec}(\Lambda, X) \Longrightarrow \dim V_{\lambda\mu} = 1, \tag{6.20}$$

so

$$u \in V_{\lambda\mu} \Longrightarrow \iota^* u = u \text{ or } \iota^* u = -u. \tag{6.21}$$

Whatever parity u has under ι^*, v in (6.15) has the same parity, so

$$u^\# = v_+ - v_- \tag{6.22}$$

has the opposite parity. This also satisfies conditions like (6.16)–(6.18), i.e.,

$$Xu^\# = \mu u^\#, \quad (\Lambda - \lambda)u^\# = w_0^\#, \quad \|w_0^\#\|_{L^2} \le \frac{c_m}{\lambda^{m+1}}\|u^\#\|_{L^2}. \tag{6.23}$$

A function that satisfies (6.18) is called a λ-quasimode of Λ. If it also satisfies (6.17), we say it is a joint (λ,μ)-quasimode of (Λ, X). What is established above is that if $u \in V_{\lambda\mu}$, then the functions v_+, v_-, and $u^{\#}$ produced in (6.19) and (6.22) are all joint (λ,μ)-quasimodes of (Λ, X). As noted above, every joint eigenfunction is either even or odd with respect to ι^*, so it is clear that neither of the quasimodes v_+ or v_- is close to an actual joint eigenfunction.

To proceed, it is convenient to set up some notation. Set

$$\begin{aligned} L^2_\alpha(M) &= \{u \in L^2(M) : \iota^* u = \alpha u\}, \quad \alpha \in \{1,-1\}, \\ \mathrm{Spec}_\alpha(\Lambda, X) &= \{(\lambda,\mu) \in \mathrm{Spec}(\Lambda, X) : V_{\lambda\mu} \subset L^2_\alpha(M)\}, \\ \Sigma_\alpha(M) &= \Sigma(A) \cap \mathrm{Spec}_\alpha(\Lambda, X). \end{aligned} \tag{6.24}$$

Note that $\Sigma_\alpha(A) \cap \Sigma_{-\alpha}(A) = \emptyset$. We will establish the following.

Proposition 6.2 *In the current setting (Example 4), given $A \in (r_2, r_1)$, $m \in \mathbb{N}$, there exist L, C such that*

$$\begin{aligned} &\textit{for } \ (\lambda,\mu) \in \Sigma_\alpha(X), \ \lambda \ge L, \ \alpha \in \{1,-1\}, \\ &\mathrm{dist}\Big((\lambda,\mu), \mathrm{Spec}_{-\alpha}(\Lambda, X)\Big) \le \frac{C}{\lambda^m}. \end{aligned} \tag{6.25}$$

Proof. Let us denote by Λ_α the restriction of Λ to $H^1(M) \cap L^2_\alpha(M)$, an unbounded self-adjoint operator on the Hilbert space $L^2_\alpha(M)$, as is X. Their joint spectrum is $\mathrm{Spec}_\alpha(\Lambda, X)$. Given that $u \in V_{\lambda\mu} \subset L^2_\alpha(M)$, the construction above produces $u^{\#} \in \mathcal{D}(\Lambda_{-\alpha})$ such that (6.23) holds. Hence

$$\|(\Lambda_{-\alpha} - \lambda)^{-1}\| \ge \frac{1}{c_m} \lambda^{m+1}. \tag{6.26}$$

Now the spectral theorem implies

$$\mathrm{dist}(\lambda, \mathrm{Spec}\, \Lambda_{-\alpha}) = \|(\Lambda_{-\alpha} - \lambda)^{-1}\|^{-1}, \tag{6.27}$$

so we have (6.25). □

Example 5. Top-heavy dumbbell.
This example is similar to Example 4, except that now the surface does not have the symmetry (6.6). One starts with a curve C in the (x_1, x_3)-plane, symmetric about the x_3-axis, as before, and assumes the left half C_L is the

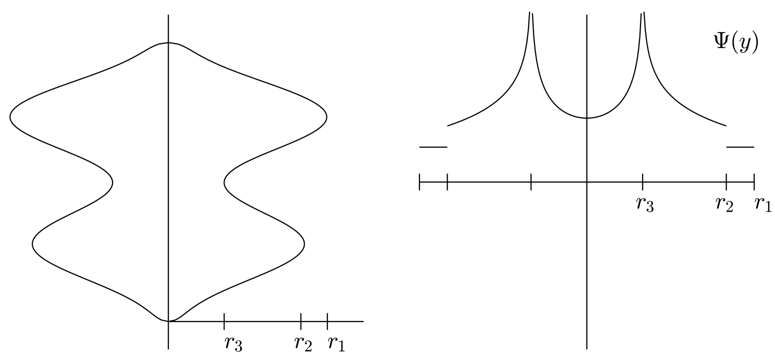

Figure 6.2: Top-heavy dumbbell figure and its factor $\Psi(y)$ (Example 5)

graph of $x_1 = \beta(x_3)$, for $x_3 \in [-1, 1]$, but we do not have $\beta(x_3) = \beta(-x_3)$. Rather, this time, the appearance is as illustrated in Figure 6.2. We assume

$$\text{the curvature of } C \text{ is nonzero} \\ \text{at } x_1 = \pm r_1,\ \pm r_2 \text{ and } \pm r_3.$$

Arguments used in Example 4 show that the qualitative features of the factor $\Psi(y)$ can be read off from how

$$\Xi_0 : M \longrightarrow \mathbb{R}, \quad \Xi_0(x) = x \cdot e_1 \tag{6.28}$$

pushes forward the area element of M to a measure on $[-r_1, r_1]$. Again, Ξ_0 has $\pm r_1$ as maxima and minima. This time, $\pm r_2$ are local maxima and minima, and there are two points in M at which Ξ_0 has a saddle, with critical values $\pm r_3$. All 6 critical points are nondegenerate. The local max and min yield jumps in $\Psi(y)$ at $y = \pm r_2$, and the saddles yield log blowups of $\Psi(y)$ at $y = \pm r_3$, as illustrated in Figure 6.2.

We next examine concentration of eigenfunctions, using the sets

$$\Sigma(A) \subset \operatorname{Spec}(\Lambda, X), \quad \Omega_A \subset M,$$

defined in (6.10)–(6.11). Arguing as in Example 4, we have again the conclusion of Proposition 6.1, which we restate:

Proposition 6.3 *In the current setting (Example 5), given $A \in (r_2, r_1)$, we have*

$$\begin{aligned} &u \in V_{\lambda\mu},\ (\lambda,\mu) \in \Sigma(A'),\ A' > A \\ &\Longrightarrow \|u\|_{C^m(\Omega_A)} \le \frac{C}{\lambda^m}\|u\|_{L^2}, \quad C = C_m(A, A'). \end{aligned} \tag{6.29}$$

.

Note that, for A close to r_1, the concentration set $M \setminus \Omega_A$ is a small strip about the curve

$$\{x \in M : r(x) = r_1\}, \tag{6.30}$$

which is a closed, elliptic geodesic. By contrast, in Example 4 the concentration set (6.30) consisted of a pair of closed, elliptic geodesics. We saw in that case that eigenfunctions could not concentrate on just one of these geodesics. Now, in Example 5, there is a family of eigenfunctions concentrating on the one closed geodesic given by (6.30). On the other hand, in Example 5, the set

$$\{x \in M : r(x) = r_2\} \tag{6.31}$$

is also a closed, eliptic geodesic. It follows that there is a sequence of quasimodes that concentrate on this set (cf. [1]). Actually, in this case, one can cut and paste quasimodes that arise in Example 4. In light of other results holding for Example 4, we speculate that there is not a sequence of actual joint eigenfunctions that concentrate on the set (6.31). This might motivate further study.

In Example 4, the set described by (6.31) was a hyperbolic closed geodesic. Here in Example 5, the set

$$\{x \in M : r(x) = r_3\} \tag{6.32}$$

is our hyperbolic closed geodesic.

Example 6. Surface with inflective invariant geodesic.
Here one starts with a smooth curve C in the (x_1, x_3)-plane, symmetric about the x_3-axis, whose curvature vanishes simply, at two points, say $(\pm r_2, z_0)$, such that $\pm r_2$ are critical values of the x_1-coordinate on C. See

the left half of Figure 6.3. As in the previous examples, the singularities in the factor $\Psi(y)$ can be read off from how the function $\Xi_0 : M \to \mathbb{R}$ given by $\Xi_0(x) = x \cdot e_1$ pushes forward the area measure of M to a measure on $[-r_1, r_1]$, where r_1 is the maximum value of Ξ_0, again a nondegenerate critical value. Again $\Psi(y)$ has a graph that looks somewhat like the graph on the right half of Figure 6.1, but this time the singularities of Ψ at $y = \pm r_2$ are stronger. In fact, the area of the inverse image $\Xi_0^{-1}([r_2, r_2 + \varepsilon))$ behaves essentially like $A_+(\varepsilon)$, where

$$A_\pm(\varepsilon) = \pm \int_0^1 \Big\{(x^2 \pm \varepsilon)^{1/3} - x^{2/3}\Big\}\, dx. \tag{6.33}$$

One has

$$A'_+(\varepsilon) = \frac{1}{3} \int_0^1 (x^2 + \varepsilon)^{-2/3}\, dx \sim C\varepsilon^{-1/6}, \tag{6.34}$$

and similarly for $A'_-(\varepsilon)$, and hence

$$\Psi(y) \sim C|y \mp r_2|^{-1/6}, \tag{6.35}$$

for y near $\pm r_2$. Again we have a conclusion similar to (6.9), namely clustering of $\mathrm{Spec}(\Lambda, X)$ along the rays $\mu = \pm r_2 \lambda$, as $\lambda \to \infty$. This clustering is stronger than in Example 4, but not as pronounced as that along the rays $\mu = \pm\lambda$ in Example 1 (where $M = \mathbb{T}^2$).

Example 7. Inner tube.
The previous examples in this section all arose by taking a smooth closed curve C in the (x_1, x_3)-plane that was symmetric about the x_3-axis, and rotating it about this axis in $\mathbb{R}^3$. By contrast, this example takes C to be a smooth closed curve contained in the half-plane $\{x_1 > 0\}$, namely $C = C_{ab}$, the circle

$$C_{ab} = \{(x_1, x_3) : (x_1 - a)^2 + x_3^2 = b^2\}, \quad 0 < b < a. \tag{6.36}$$

Rotating this about the x_3-axis in $\mathbb{R}^3$ produces an "inner tube," somewhat like that pictured in the right half of Figure 1.7. We take

$$r_1 = a + b, \quad r_2 = a - b.$$

Again, looking at $\Xi_0 : M \to \mathbb{R}$ defined as above, we see that $\Psi \in L^1([-r_1, r_1])$ is smooth except for logarithmic singularities at $y = \pm r_2$, as in (6.8). Thus we see how clustering of $\mathrm{Spec}(\Lambda, X)$ in this case occurs, but differs from that for the flat torus $M = \mathbb{T}^2$ described in Example 1.

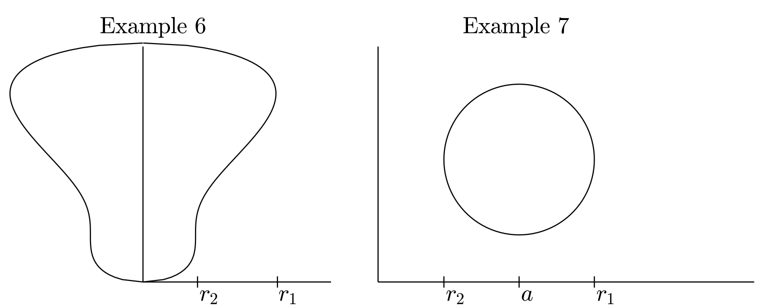

Figure 6.3: Generating curves for the surfaces in Examples 6 and 7

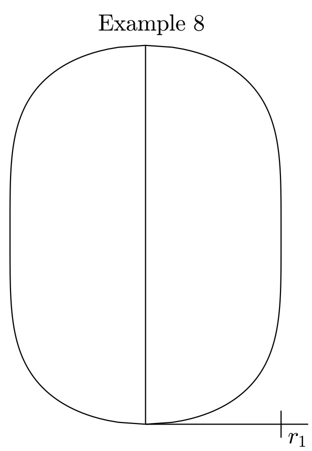

Figure 6.4: Generating curves for the surfaces in Examples 8 and 9

Example 8. Surface with flattened equator.
Here the curve C in the (x_1, x_3)-plane that we rotate about the x_3-axis is given by

$$x_1^4 + \frac{x_3^2}{2} = 1. \tag{6.37}$$

See the left half of Figure 6.4. The Gauss curvature of this surface is > 0 everywhere except at the equator ($x_3 = 0$), where it vanishes to second order. A calculation similar to that done for Example 6 yields $\Psi \in C^\infty((-1,1))$, blowing up at the endpoints,

$$\Psi(y) \sim C(1-y^2)^{-1/4}, \quad y \in (-1,1). \tag{6.38}$$

This blow-up is less severe than that arising for flat $\mathbb{T}^2$ in Example 1 (cf. (5.33)), but stronger than the blow-up at $y = \pm r_2$ in Example 6 (cf. (6.35)).

For $(\lambda, \mu) \in \operatorname{Spec}(\Lambda, X)$, the joint eigenspace $V_{\lambda\mu}$ is one-dimensional, since M has a pole. Also, in this case M is invariant under the involution $x_3 \mapsto -x_3$, so a joint eigenfunction in $V_{\lambda\mu}$ is either even or odd with respect to this involution.

We have the shadow phenomenon for elements of $V_{\lambda\mu}$, given $|\mu/\lambda| \geq A$, $A \in (0,1)$, as described in Proposition 4.3. As $A \nearrow 1$, the joint eigenfunctions concentrate on the equator.

Example 9. Capped cylinder.
Here the curve C, illustrated in the right half of Figure 6.4, contains the line segments

$$x_1 = \pm 1, \quad x_3 \in [-1,1]. \tag{6.39}$$

The resulting surface of revolution M consists of a circular cylinder with two caps. The Gauss curvature of M is positive on the caps, and zero on the cylinder. In this case we have $\Psi \in C^\infty((-1,1))$, blowing up at the endpoints at the same rate as in Example 1,

$$\Psi(y) \sim C(1-y^2)^{-1/2}, \quad y \in (-1,1). \tag{6.40}$$

Then $\operatorname{Spec}(\Lambda, X)$ exhibits clustering near $\mu = \pm\lambda$ in a fashion similar to that illustrated in Figure 1.1, though of course, unlike in that case, it cannot be expected to be arranged along hyperbolic arcs.

As in Example 7, if $(\lambda, \mu) \in \operatorname{Spec}(\Lambda, X)$, then $V_{\lambda\mu}$ is one dimensional. Again we arrange that M be invariant under the involution $x_3 \mapsto -x_3$, so eigenfunctions in such $V_{\lambda\mu}$ are either even or odd under this involution. Let us write

$$M = M_+ \cup M_0 \cup M_-, \tag{6.41}$$

where $M_\pm$ are the top and bottom caps, and

$$M_0, \text{ isometric to } [-1,1] \times \mathbb{R}/2\pi\mathbb{Z}, \tag{6.42}$$

is the cylindrical part of M. Note that

$$\text{on } M_0, \quad \Delta = X^2 + \partial_{x_3}^2, \tag{6.43}$$

so, given $(\lambda,\mu) \in \operatorname{Spec}(\Lambda, X)$,

$$\begin{aligned} u \in V_{\lambda\mu} &\Rightarrow (X^2 + \partial_{x_3}^2)u = -\lambda^2 u \\ &\Rightarrow \partial_{x_3}^2 u = -(\lambda^2 - \mu^2)u, \text{ on } M_0. \end{aligned} \tag{6.44}$$

Hence

$$u \in V_{\lambda\mu} \Longrightarrow u = e^{i\mu\psi} v(x_3), \quad \text{on } M_0, \tag{6.45}$$

where ψ is the angular coordinate on M_0, and

$$v''(x_3) = -(\lambda^2 - \mu^2) v(x_3), \quad |x_3| \le 1. \tag{6.46}$$

As we have seen, v is either even or odd, so, for some $\alpha \in \mathbb{C}$,

$$\begin{aligned} v(x_3) &= \alpha \cos \sqrt{\lambda^2 - \mu^2}\, x_3, \quad \text{or} \\ v(x_3) &= \alpha \sin \sqrt{\lambda^2 - \mu^2}\, x_3, \end{aligned} \tag{6.47}$$

for $x_3 \in [-1,1]$.

Again we have the shadow phenomenon for elements of $V_{\lambda\mu}$, given $|\mu/\lambda| \ge A,\ A \in (0,1)$, as described in Proposition 4.3. If $\mathcal{O}$ is a fixed neighborhood if the cylinder M_0, then there exists $A < 1$ with the property that such eigenfunctions vanish rapidly on $M \setminus \mathcal{O}$, as $\lambda \to \infty$.

A Riemann integrable functions on a compact measured metric space

Let X be a compact metric space, equipped with a finite Borel measure μ. Let $f : X \to \mathbb{R}$ be bounded. We define

$$\begin{aligned} \overline{I}(f) &= \inf\Big\{ \int_X v\, d\mu : v \ge f,\ v \in C(X) \Big\}, \\ \underline{I}(f) &= \sup\Big\{ \int_X u\, d\mu : u \le f,\ u \in C(X) \Big\}, \end{aligned} \tag{A.1}$$

where $C(X)$ denotes the space of continuous, real-valued functions on X. Clearly $\underline{I}(f) \le \overline{I}(f)$. We say

$$f \in \mathcal{R}(X, \mu) \Longleftrightarrow \overline{I}(f) = \underline{I}(f). \tag{A.2}$$

In case X is a product of n closed, bounded intervals in $\mathbb{R}^n$ and μ is Lebesgue measure, it is easy to show that (A.2) is equivalent to the standard definition of Riemann integrability, involving taking partitions; cf. [12], Proposition 3.1.11. The following is a generalization of Lebesgue's theorem characterizing Riemann integrability.

Proposition A.1 *Given $f : X \to \mathbb{R}$ bounded, set*

$$\mathcal{D}_f = \{x \in X : f \text{ not continuous at } x\}. \tag{A.3}$$

Then $\mathcal{D}_f$ is a Borel subset of X, and

$$f \in \mathcal{R}(X, \mu) \Longleftrightarrow \mu(\mathcal{D}_f) = 0. \tag{A.4}$$

To start the proof, we note that $C(X)$ is a separable Banach space, and let $\mathcal{E} \subset C(X)$ be a countable dense subset. Set

$$\begin{aligned} \mathcal{C}_f^+ &= \{v \in C(X) : v \ge f\}, \quad \mathcal{E}_f^+ = \{v \in \mathcal{E} : v \ge f\}, \\ \mathcal{C}_f^- &= \{u \in C(X) : u \le f\}, \quad \mathcal{E}_f^- = \{u \in \mathcal{E} : u \le f\}. \end{aligned} \tag{A.5}$$

Note that $\mathcal{E}_f^+$ is dense in $\mathcal{C}_f^+$ and $\mathcal{E}_f^-$ is dense in $\mathcal{C}_f^-$. Then set

$$\begin{aligned} \psi &= \inf\{v : v \in \mathcal{C}_f^+\} = \inf\{v : v \in \mathcal{E}_f^+\}, \\ \varphi &= \sup\{u : u \in \mathcal{C}_f^-\} = \sup\{u : u \in \mathcal{E}_f^-\}. \end{aligned} \tag{A.6}$$

Note that $\varphi \le f \le \psi$. Using countability, write

$$\mathcal{E}_f^+ = \{v_j : j \in \mathbb{N}\}, \quad \mathcal{E}_f^- = \{u_j : j \in \mathbb{N}\}. \tag{A.7}$$

Let

$$\psi_j = \min_{k \le j} v_k, \quad \varphi_j = \max_{k \le j} u_k. \tag{A.8}$$

Then

$$\psi_j \in \mathcal{C}_f^+, \quad \varphi_j \in \mathcal{C}_f^-, \tag{A.9}$$

and these are bounded monotone sequences, so they converge at each point of X,

$$\psi_j \searrow \psi, \quad \varphi_j \nearrow \varphi, \tag{A.10}$$

with ψ and φ as in (A.6). Thus ψ and φ are Borel functions, and the monotone convergence theorem implies

$$\int_X \psi_j \, d\mu \searrow \int_X \psi \, d\mu, \quad \int_X \varphi_j \, d\mu \nearrow \int_X \varphi \, d\mu. \tag{A.11}$$

We see that if $v \in \mathcal{E}_f^+$ then $\psi_j \le v$ for j sufficiently large, and if $u \in \mathcal{E}_f^-$ then $\varphi_j \ge u$ for j sufficiently large. Also

$$\begin{aligned} \overline{I}(f) &= \inf\Big\{\int_X v \, d\mu : v \in \mathcal{C}_f^+\Big\} = \inf\Big\{\int_X v \, d\mu : v \in \mathcal{E}_f^+\Big\}, \\ \underline{I}(f) &= \sup\Big\{\int_X u \, d\mu : u \in \mathcal{C}_f^-\Big\} = \sup\Big\{\int_X u \, d\mu : u \in \mathcal{E}_f^-\Big\}. \end{aligned} \tag{A.12}$$

Thus, by (A.11),

$$\overline{I}(f) = \int_X \psi \, d\mu, \quad \underline{I}(f) = \int_X \varphi \, d\mu. \tag{A.13}$$

It follows that

$$\overline{I}(f) - \underline{I}(f) = \int_X (\psi - \varphi) \, d\mu. \tag{A.14}$$

The next lemma brings in $\mathcal{D}_f$.

Lemma A.2 *Given $x \in X$, the function f is continuous at x if and only if $\varphi(x) = \psi(x)$.*

Proof. It is equivalent to say f is continuous at x if and only if

$$\begin{aligned}&\text{for each } \varepsilon > 0,\ \exists \text{ continuous } u_\varepsilon \le f \text{ and continuous } v_\varepsilon \ge f\\ &\text{such that } v_\varepsilon(x) - u_\varepsilon(x) < \varepsilon.\end{aligned} \tag{A.15}$$

Indeed if (A.15) holds, then $x_j \to x$ implies

$$\limsup_{j\to\infty} f(x_j) \le v_\varepsilon(x), \quad \liminf_{j\to\infty} f(x_j) \ge u_\varepsilon(x), \tag{A.16}$$

so these differ by at most ε, for each $\varepsilon > 0$, hence f is continuous at x.

For the converse, if $|f| \le M$ on X, and f is continuous at x, then, given $\varepsilon > 0$, there exists a ball $B_\delta(x)$ such that

$$y \in B_\delta(x) \Longrightarrow |f(x) - f(y)| < \frac{\varepsilon}{4}. \tag{A.17}$$

Then, assuming $\varepsilon < M$, we can define $v_\varepsilon \in C(X)$ by

$$\begin{aligned} v_\varepsilon(y) = f(x) + \frac{\varepsilon}{3} + \frac{4M}{\delta} d(y,x), &\quad y \in B_\delta(x),\\ f(x) + \frac{\varepsilon}{3} + 4M, &\quad y \in X \setminus B_\delta(x),\end{aligned} \tag{A.18}$$

and similarly define $u_\varepsilon \in C(X)$ so that (A.15) holds. □

Having Lemma A.2, we see that

$$\mathcal{D}_f = \{x \in X : \varphi(x) < \psi(x)\}. \tag{A.19}$$

This guarantees that $\mathcal{D}_f$ is a Borel subset of X, and (A.14) yields

$$\overline{I}(f) - \underline{I}(f) = \int_{\mathcal{D}_f} (\psi - \varphi)\, d\mu. \tag{A.20}$$

This proves Proposition A.1.

Furthermore, we see that if $f \in \mathcal{R}(X,\mu)$, then f is equal to each of the Borel functions φ and ψ, on the complement of a set of μ-measure 0. Hence f is $\overline{\mu}$-measurable, where $\overline{\mu}$ is the completion of μ. We deduce that

$$f \in \mathcal{R}(X,\mu) \Longrightarrow f \in L^1(X,\overline{\mu}) \ \text{ and } \int_X f\, d\overline{\mu} = \overline{I}(f) = \underline{I}(f). \tag{A.21}$$

B Finite symmetry group actions on eigenspaces

Let M be a compact, connected, n-dimensional Riemannian manifold, with Laplace operator Δ, $\Lambda = \sqrt{-\Delta}$. Suppose $B \in OPS^0(M)$ is self adjoint and commutes with Λ. We also assume K is a finite group of isometries of M. Let $\widehat{K}$ denote a complete set of irreducible unitary representations of K, and, for $\rho \in \widehat{K}$, consider $P_\rho \in \mathcal{L}(L^2(M))$, given by

$$P_\rho u(x) = \frac{d_\rho}{\#(K)} \sum_{g\in K} \overline{\chi_\rho(g)} u(g^{-1}x), \tag{B.1}$$

where $\chi_\rho(g) = \operatorname{Tr} \rho(g),\ d_\rho = \chi_\rho(I)$, and $\#(K)$ is the number of elements of K. Then P_ρ is the orthogonal projection of $L^2(M)$ onto the subspace of K on which K acts like copies of ρ.

We are interested in the behavior of

$$P_\rho B\varphi(\Lambda - R), \tag{B.2}$$

and its trace, as $R \to \infty$. As in §5, we assume

$$\varphi \in \mathcal{S}(\mathbb{R}), \quad \varphi \geq 0, \quad \operatorname{supp} \hat{\varphi} \subset (-\tau, \tau), \quad \tau < \operatorname{Inj} M. \tag{B.3}$$

To tackle (B.2), we define the integral kernel $\Phi_{B,R}(x,y)$ of $B\varphi(\Lambda - R)$:

$$B\varphi(\Lambda - R)u(x) = \int_M \Phi_{B,R}(x,y)u(y)\, dV(y). \tag{B.4}$$

Then

$$P_\rho B\varphi(\Lambda - R)u(x) = \frac{d_\rho}{\#(K)} \sum_{g\in K} \overline{\chi_\rho(g)} \int_M \Phi_{B,R}(g^{-1}x,y)u(y)\, dV(y). \tag{B.5}$$

Consequently,

$$\begin{aligned} \operatorname{Tr} P_\rho B\varphi(\Lambda - R) = {} & \frac{d_\rho^2}{\#(K)} \operatorname{Tr} B\varphi(\Lambda - R) \\ & + \frac{d_\rho}{\#(K)} \sum_{g\neq I} \overline{\chi_\rho(g)} \int_M \Phi_{B,R}(g^{-1}x,x)\, dV(x). \end{aligned} \tag{B.6}$$

In §5 we recalled that analysis in [6] of

$$B\varphi(\Lambda - R) = \int_{-\infty}^{\infty} Be^{it\Lambda}e^{-itR}\hat{\varphi}(t)\, dt \tag{B.7}$$

yields

$$\operatorname{Tr} \varphi(\Lambda - R) = N_\varphi(R) \sim C(\varphi, M) R^{n-1}, \quad R \to \infty, \tag{B.8}$$

and

$$\lim_{R\to\infty} \frac{1}{N_\varphi(R)} \operatorname{Tr} B\varphi(\Lambda - R) = \int\limits_{S^*M} \sigma_B(x,\xi)\, dS(x,\xi), \tag{B.9}$$

where $dS(x,\xi)$ is Liouville measure on S^*M, normalized to have total mass 1. Furthermore, an analysis of the Schwartz kernel $W_t(x,y)$ of $e^{it\Lambda}$, for $|t| < \operatorname{Inj} M$, gives, for $g \neq I$,

$$\int\limits_M \Phi_{B,R}(g^{-1}x, x)\, dV(x) = o(R^{n-1}), \quad R \to \infty. \tag{B.10}$$

We have the following conclusion.

Proposition B.1 *In the setting described above,*

$$\lim_{R\to\infty} \frac{1}{N_\varphi(R)} \operatorname{Tr} P_\rho B\varphi(\Lambda - R) = \frac{d_\rho^2}{\#(K)} \int\limits_{S^*M} \sigma_B(x,\xi)\, dS(x,\xi). \tag{B.11}$$

In connection with this result, we mention the classical result that

$$\sum_{\rho\in\widehat{K}} d_\rho^2 = \#(K), \tag{B.12}$$

and $d_\rho^2/\#(K)$ is the fraction of the finite dimensional Hilbert space $\ell^2(K)$ on which the regular representation of K acts like copies of ρ.

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