Karamata's Tauberian Theorem

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1. Basics

Let μ be a positive Borel measure on $[0,\infty)$. Assume $e^{-s\lambda} \in L^1(\mathbb{R}^+,\mu)$ for each $s>0$, and assume

$$\int_0^\infty e^{-s\lambda}\, d\mu(\lambda) \sim A\varphi(s), \quad \text{as } s \searrow 0, \tag{1.1}$$

where $\varphi(s) \nearrow +\infty$ as $s \searrow 0$, and we say

$$\Phi(s) \sim A\varphi(s) \Longleftrightarrow \Phi(s) = A\varphi(s) + o(\varphi(s)), \quad \text{as } s \searrow 0. \tag{1.2}$$

Regarding $\varphi(s)$, we will assume that

$$\varphi(s) = \mathcal{L}\psi(s) = \int_0^\infty e^{-s\lambda}\psi(\lambda)\, d\lambda, \quad \psi > 0. \tag{1.3}$$

The classical examples are

$$\psi(\lambda) = \lambda^{\alpha-1}, \quad \varphi(s) = \Gamma(\alpha)s^{-\alpha}, \quad \alpha > 0. \tag{1.4}$$

Slightly more exotic examples are

$$\psi(\lambda) = \lambda^{\alpha-1}\log\lambda, \quad \varphi(s) = (\Gamma'(\alpha) - \Gamma(\alpha)\log s)s^{-\alpha}, \tag{1.5}$$

again for $\alpha > 0$. Actually, in this case $\psi(\lambda) < 0$ for $\lambda \in (0,1)$, so we would want to cut this off, obtaining

$$\psi(\lambda) = \lambda^{\alpha-1}(\log\lambda)_+, \quad \varphi(s) = \Gamma(\alpha)\Big(\log\frac{1}{s}\Big)s^{-\alpha} + O(s^{-\alpha}), \tag{1.6}$$

as $s \searrow 0$. See §2 for another example, in which $\varphi(s) \sim \log 1/s$.

Our goal is to establish (under some natural hypotheses on φ and ψ) that

$$\mu([0,R]) = A\int_0^R \psi(\lambda)\, d\lambda + o(\varphi(R^{-1})), \quad \text{as } R \nearrow +\infty. \tag{1.7}$$

In case φ and ψ are given by (1.4), this yields the implication

$$\begin{gathered}\int_0^\infty e^{-s\lambda}\, d\mu(\lambda) \sim As^{-\alpha}, \quad \text{as } s \searrow 0 \quad \Longrightarrow \\ \mu([0,R]) \sim \frac{A}{\Gamma(\alpha+1)}R^{\alpha}, \quad \text{as } R \nearrow +\infty,\end{gathered} \tag{1.8}$$

which is the most basic version of the Karamata Tauberian theorem.

We tackle the problem of establishing (1.7) in stages, examining when we can show that

$$\int_0^\infty f(s\lambda)\,d\mu(\lambda) = A\int_0^\infty f(s\lambda)\psi(\lambda)\,d\lambda + o(\varphi(s)), \tag{1.9}$$

for various classes of functions $f(\lambda)$, untimately including

$$\begin{aligned} \chi_I(\lambda) = 1 \quad &\text{for} \;\; 0 \le \lambda \le 1, \\ 0 \quad &\text{for} \;\; \lambda > 1. \end{aligned} \tag{1.10}$$

We start with the function space

$$\mathcal{E} = \Big\{\sum_{k=1}^{M} \gamma_k e^{-k\lambda} : \gamma_k \in \mathbb{R},\, M \in \mathbb{N}\Big\}. \tag{1.11}$$

Note that this is an algebra of functions that separates the points of $[0,\infty)$, hence, by the Stone-Weierstrass theorem, it is dense in

$$C_0([0,\infty)) = \{f \in C([0,\infty]) : f(\infty) = 0\}. \tag{1.12}$$

Now, if $f \in \mathcal{E}$, say

$$f(\lambda) = \sum_{k=1}^{M} \gamma_k e^{-k\lambda}, \tag{1.13}$$

then the hypothesis (1.1) implies

$$\begin{aligned} \int_0^\infty f(s\lambda)\,d\mu(\lambda) &= \sum_{k=1}^{M} \gamma_k \int_0^\infty e^{-sk\lambda}\,d\mu(\lambda) \\ &= A\sum_{k=1}^{M} \gamma_k \varphi(ks) + o\Big(\sum_{k=1}^{M} \varphi(ks)\Big) \\ &= A\int_0^\infty f(s\lambda)\psi(\lambda)\,d\lambda + o(\varphi(s)), \end{aligned} \tag{1.14}$$

since

$$\varphi(ks) \le \varphi(s), \quad \text{for} \;\; k \ge 1. \tag{1.15}$$

Hence (1.9) holds for all $f \in \mathcal{E}$. The following is the next key result.

Lemma 1.1. *Given (1.1), the result (1.9) holds for all*

$$f \in C_0([0,\infty)) \quad \textit{such that} \quad e^{\lambda} f \in C_0([0,\infty)). \tag{1.16}$$

Proof. Given such f, and given $\varepsilon > 0$, take $h \in \mathcal{E}$ such that $\sup |h(\lambda) - e^{\lambda} f(\lambda)| \le \varepsilon$, and set $g = e^{-\lambda} h$, so

$$g \in \mathcal{E}, \quad |f(\lambda) - g(\lambda)| \le \varepsilon e^{-\lambda}. \tag{1.17}$$

This implies

$$\int_0^\infty |f(s\lambda) - g(s\lambda)| \, d\mu(\lambda) \le \varepsilon \int_0^\infty e^{-s\lambda} \, d\mu(\lambda) \tag{1.18}$$

and

$$\int_0^\infty |f(s\lambda) - g(s\lambda)| \psi(\lambda) \, d\lambda \le \varepsilon \int_0^\infty e^{-s\lambda} \psi(\lambda) \, d\lambda. \tag{1.19}$$

The fact that the right sides of (1.18) and (1.19) are both $\le C\varepsilon\varphi(s)$, for $s \in (0,1]$, follows from (1.1) and (1.3), respectively. But we know that (1.9) holds with g in place of f. Hence

$$\Bigl| \int_0^\infty f(s\lambda) \, d\mu(\lambda) - A \int_0^\infty f(s\lambda) \psi(\lambda) \, d\lambda \Bigr| \le 2C\varepsilon\varphi(s) + o(\varphi(s)), \tag{1.20}$$

for each $\varepsilon > 0$. Taking $\varepsilon \searrow 0$ yields the lemma.

We now tackle (1.9) for $f = \chi_I$, given by (1.10). For each $\delta \in (0, 1/2]$, take $f_\delta, g_\delta \in C_0([0,\infty))$ such that

$$0 \le f_\delta \le \chi_I \le g_\delta \le 1, \tag{1.21}$$

with

$$\begin{aligned} f_\delta(\lambda) &= 1 \quad \text{for} \quad 0 \le \lambda \le 1 - \delta, \\ &\, 0 \quad \text{for} \quad \lambda \ge 1, \end{aligned} \tag{1.22}$$

and

$$\begin{aligned} g_\delta(\lambda) &= 1 \quad \text{for} \quad 0 \le \lambda \le 1, \\ &\, 0 \quad \text{for} \quad \lambda \ge 1 + \delta. \end{aligned} \tag{1.23}$$

Note that Lemma 1.1 is applicable to each f_δ and g_δ. Hence

$$\begin{aligned} \int_0^\infty \chi_I(s\lambda) \, d\mu(\lambda) &\le \int_0^\infty g_\delta(s\lambda) \, d\mu(\lambda) \\ &= A \int_0^\infty g_\delta(s\lambda) \psi(\lambda) \, d\lambda + o(\varphi(s)), \end{aligned} \tag{1.24}$$

and

$$\begin{aligned}\int_0^\infty \chi_I(s\lambda)\,d\mu(\lambda) &\ge \int_0^\infty f_\delta(s\lambda)\,d\mu(\lambda)\\ &= A\int_0^\infty f_\delta(s\lambda)\psi(\lambda)\,d\lambda + o(\varphi(s)).\end{aligned} \tag{1.25}$$

Next,

$$\begin{aligned}&\int_0^\infty [g_\delta(s\lambda) - f_\delta(s\lambda)]\psi(\lambda)\,d\lambda\\ &\le \int_{(1-\delta)/s}^{(1+\delta)/s} \psi(\lambda)\,d\lambda\\ &\le \frac{2\delta}{s}\,\max\Big\{|\psi(\lambda)| : \Big|\lambda - \frac{1}{s}\Big| \le \frac{\delta}{s}\Big\}.\end{aligned} \tag{1.26}$$

We now make the hypothesis that, for some $\varepsilon > 0,\ b > 0,\ B < \infty$,

$$\max\Big\{|\psi(\lambda)| : \Big|\lambda - \frac{1}{s}\Big| \le \frac{\varepsilon}{s}\Big\} \le Bs\varphi(s), \quad \text{for } \ 0 < s \le b. \tag{1.27}$$

Note that such a condition holds in cases (1.4) and (1.6). When such an estimate holds, (1.26) yields

$$\int_0^\infty [g_\delta(s\lambda) - f_\delta(s\lambda)]\psi(\lambda)\,d\lambda \le 2B\delta\,\varphi(s), \quad \text{for } \ \delta \le \varepsilon,\ s \le b. \tag{1.28}$$

It then follows from (1.24)–(1.26) that

$$\begin{aligned}&\lim_{s\searrow 0}\ \varphi(s)^{-1}\Big|\int_0^\infty \chi_I(s\lambda)\,d\mu(\lambda) - A\int_0^\infty \chi_I(s\lambda)\psi(\lambda)\,d\lambda\Big|\\ &\le \inf_{\delta\le\varepsilon}\ 2B\delta = 0.\end{aligned} \tag{1.29}$$

We have the following conclusion.

Proposition 1.2. *Let μ be a positive measure on $[0,\infty)$, and assume (1.1)–(1.3) hold, with $\psi \notin L^1(\mathbb{R}^+)$, and that (1.27) holds. Then μ satisfies (1.7).*

The special case (1.8) has already been mentioned. We turn to the case (1.6), for which (1.1) leads to (1.7) with

$$\begin{aligned}\int_0^R \psi(\lambda)\,d\lambda &= \int_1^R \lambda^{\alpha-1}(\log\lambda)\,d\lambda\\ &= \frac{1}{\alpha}\int_1^R \Big(\frac{d}{d\lambda}\lambda^\alpha\Big)(\log\lambda)\,d\lambda\\ &= \frac{1}{\alpha}\,R^\alpha(\log R) + O(R^\alpha).\end{aligned} \tag{1.30}$$

This leads to the following.

Corollary 1.3. *Let μ be a positive measure on $[0,\infty)$. Assume*

$$\int_0^\infty e^{-s\lambda}\, d\mu(\lambda) \sim A\Bigl(\log \frac{1}{s}\Bigr) s^{-\alpha}, \quad s \searrow 0, \tag{1.31}$$

with $\alpha > 0$. then

$$\mu([0,R]) \sim \frac{A}{\Gamma(\alpha+1)}\, R^\alpha (\log R), \quad R \nearrow +\infty. \tag{1.32}$$

We mention some other results of Karamata. To state them, let us set

$$\Psi(R) = \int_0^R \psi(\lambda)\, d\lambda. \tag{1.33}$$

Here is Karamata's Abelian theorem.

Proposition 1.4. *Let $\psi > 0$. Assume that Ψ, given by (1.33), has the form*

$$\Psi(R) = R^\alpha F(R), \quad \text{with} \ \ \alpha > 0, \tag{1.34}$$

where F is slowly varying at ∞, in the sense that

$$\lim_{R\to\infty} \frac{F(tR)}{F(R)} = 1, \tag{1.35}$$

uniformly in t in compact subsets of $(0,\infty)$. Then

$$\begin{aligned} \mathcal{L}\psi(s) &\sim \Gamma(\alpha+1) s^{-\alpha} F(s^{-1}) \\ &= \Gamma(\alpha+1)\Psi(s^{-1}), \end{aligned} \tag{1.36}$$

as $s \searrow 0$.

Note that (1.4) and (1.6) provide special cases of the functions ψ, considered here, with $F(R) = 1$ and $F(R) = (\log R)_+$, respectively.

The following result is Karamata's Tauberian theorem.

Proposition 1.5. *Take $\psi(\lambda)$ and $\Psi(R)$ as in Proposition 1.4. In particular, assume that (1.34)–(1.35) hold. Let μ be a positive measure on $[0,\infty)$, and assume*

$$\int_0^\infty e^{-s\lambda}\, d\mu(\lambda) \sim A\varphi(s), \tag{1.37}$$

as $s \searrow 0$, with

$$\begin{aligned} \varphi(s) &= \Gamma(\alpha+1)\Psi(s^{-1}) \\ &= \Gamma(\alpha+1) s^{-\alpha} F(s^{-1}). \end{aligned} \tag{1.38}$$

Then

$$\mu([0,R]) \sim A\Psi(R), \quad \text{as} \ \ R \nearrow +\infty. \tag{1.39}$$

2. Another special case

We want to extend the treatment of (1.4) to $\alpha = 0$. Since $\psi(\lambda) = \lambda^{-1}$ is not integrable on $(0,1]$, we instead take

$$\psi(\lambda) = \begin{cases} \lambda^{-1}, & \text{for } \lambda \geq 1, \\ 0, & \text{for } 0 < \lambda < 1. \end{cases} \tag{2.1}$$

Then

$$\begin{aligned} \varphi(s) = \int_0^\infty e^{-s\lambda} \psi(\lambda)\, d\lambda &= \int_s^\infty e^{-y} \frac{dy}{y} \\ &= \int_s^1 e^{-y} \frac{dy}{y} + \int_1^\infty e^{-y} \frac{dy}{y} \\ &= \log \frac{1}{s} - \int_s^1 (1 - e^{-y}) \frac{dy}{y} + c \\ &\sim \log \frac{1}{s}, \end{aligned} \tag{2.2}$$

as $s \searrow 0$.

Let μ be a positive measure on $[0, \infty)$, and assume (1.1) holds, with $\varphi(s)$ given by (2.1)–(2.2). We want to check that Proposition 1.2 holds. Certainly $\psi \notin L^1(\mathbb{R})$. It remains to check (1.27). We look at

$$\begin{aligned} \max \left\{ \frac{1}{\lambda} : \left| \lambda - \frac{1}{s} \right| \leq \frac{\varepsilon}{s} \right\} &= \max \left\{ \frac{1}{\lambda} : \lambda \geq \frac{1-\varepsilon}{s} \right\} \\ &= \frac{s}{1-\varepsilon}, \end{aligned} \tag{2.3}$$

while, for some $b > 0$,

$$s\varphi(s) \geq \frac{s}{2} \log \frac{1}{s}, \quad \text{for } 0 < s \leq b. \tag{2.4}$$

Hence (1.27) holds, and we deduce from Proposition 1.2 the following.

Corollary 2.1. *Let μ be a positive measure on $[0, \infty)$. Assume*

$$\int_0^\infty e^{-s\lambda}\, d\mu(\lambda) \sim A \log \frac{1}{s}, \quad s \searrow 0. \tag{2.5}$$

Then

$$\mu([0, R]) \sim A \int_1^R \frac{1}{\lambda}\, d\lambda = A \log R, \quad R \nearrow +\infty. \tag{2.6}$$