Spectral Asymptotics for a Spherical Sublaplacian

Michael Taylor

Abstract. We provide an elementary asymptotic analysis of the spectrum of the sublaplacian $X_1^2 + X_2^2 = \Delta - X_3^2$ on the 2D sphere S^2. Our result is consistent with the general asymptotic analysis of [MS], but here we have a sharper remainder estimate.

1. Introduction

Let S^2 be the unit sphere in $\mathbb{R}^3$, with its standard round metric. Let X_j denote the vector fields that generate period-2π rotation about the x_j-axis, for $j = 1, 2, 3$. Then the Laplace-Beltrami operator on S^2 is given by

$$\Delta = X_1^2 + X_2^2 + X_3^2. \tag{1.1}$$

We are interested in the sublaplacian

$$L = X_1^2 + X_2^2. \tag{1.2}$$

This operator is subelliptic with loss of one derivative, so

$$(1 - L)^{-1} : H^s(S^2) \longrightarrow H^{s+1}(S^2), \quad \forall\, s \in \mathbb{R}. \tag{1.3}$$

In particular, $(1 - L)^{-1}$ is a compact self-adjoint operator on $L^2(S^2)$, so $L^2(S^2)$ has an orthonormal basis $\{\varphi_k\}$ satisfying

$$L\varphi_k = -\lambda_k^2 \varphi_k, \quad \lambda_k \nearrow +\infty. \tag{1.4}$$

In fact, since Δ and X_3 commute, the functions φ_k can be taken to be eigenfunctions of Δ, so $\Delta\varphi_k = -\mu_k^2 \varphi_k$, but in the ordering for which (1.4) holds (μ_k) is not monotonically increasing.

Our goal is to analyze the spectrum of L. Our approach will be to consider the joint spectrum of the commuting pair (Δ, X_3). To begin, we recall that, if

$$\Lambda = \sqrt{-\Delta + \frac{1}{4}}, \tag{1.5}$$

then

$$\operatorname{Spec} \Lambda = \Big\{ k + \frac{1}{2} : k \in \mathbb{Z},\ k \geq 0 \Big\}. \tag{1.6}$$

We have the eigenspaces

$$V_k = \Big\{u \in L^2(S^2) : \Lambda u = \Big(k + \frac{1}{2}\Big)u\Big\}, \quad \dim V_k = 2k+1. \tag{1.7}$$

Also $X_3 : V_k \to V_k$, and

$$\operatorname{Spec} iX_3\Big|_{V_k} = \{-k, -k+1, \dots, k-1, k\}. \tag{1.8}$$

Cf. [T], Chapter 8, §4.

The operator L can be written

$$\begin{aligned} -L &= -\Delta + X_3^2 \\ &= \Lambda^2 + X_3^2 - \frac{1}{4}. \end{aligned} \tag{1.9}$$

Hence $L : V_k \to V_k$ for each k, and

$$\begin{aligned} \operatorname{Spec}(-L)\Big|_{V_k} &= -\frac{1}{4} + \Big\{\Big(k + \frac{1}{2}\Big)^2 - \ell^2 : -k \le \ell \le k\Big\} \\ &= k + \{k^2 - \ell^2 : -k \le \ell \le k\}. \end{aligned} \tag{1.10}$$

The counting function

$$\mathcal{N}_L(R) = \#\{k \in \mathbb{Z}^+ : \lambda_k^2 \in \operatorname{Spec}(-L),\ \lambda_k^2 \le R\} \tag{1.11}$$

can be identified as

$$\mathcal{N}_L(R) = \#\{(k,\ell) \in \mathbb{Z}^+ \times \mathbb{Z} : -k \le \ell \le k,\ k + k^2 - \ell^2 \le R\}. \tag{1.12}$$

Analyzing (1.12) directly is a lattice-counting problem, which we will tackle in §3. First, in §2, we will take another approach to the asymptotic analysis of $\mathcal{N}_L(R)$. We will establish the following.

Proposition 1.1. *We have*

$$\operatorname{Tr} e^{tL} \sim \frac{A}{t} \log \frac{1}{t}, \quad \text{as } t \searrow 0, \tag{1.13}$$

where

$$A = \frac{1}{2}. \tag{1.14}$$

Given this result, we can apply Karamata's Tauberian theorem (cf. [T2]) to deduce that

$$\mathcal{N}_L(R) \sim AR\big(\log R\big), \quad R \nearrow +\infty. \tag{1.15}$$

By contrast, we recall that standard heat asymptotics yields

$$\operatorname{Tr} e^{t\Delta} \sim \frac{1}{t}, \quad \text{as} \ \ t \searrow 0, \tag{1.16}$$

hence

$$\mathcal{N}_\Delta(R) \sim R, \quad \text{as} \ \ R \nearrow +\infty, \tag{1.17}$$

consistent with (1.6)–(1.7).

Since L is not elliptic, construction of a parametrix for e^{tL} is harder than for $e^{t\Delta}$. However, results of [MS] apply to this, and they yield (1.13)–(1.15). In §2 we will give another demonstration of this asymptotic behavior. Our approach to the analysis of $\operatorname{Tr} e^{tL}$ will proceed directly from (1.10), which gives

$$\operatorname{Tr} e^{tL} = \sum_{k=0}^{\infty} \sum_{\ell=-k}^{k} e^{-tk} e^{-t(k^2-\ell^2)}. \tag{1.18}$$

By contrast, note that (1.6)–(1.7) implies

$$\operatorname{Tr} e^{t(\Delta-1/4)} = \sum_{k=0}^{\infty} (2k+1) e^{-t(k+1/2)^2}, \tag{1.19}$$

hence

$$\operatorname{Tr} e^{t\Delta} = \sum_{k=0}^{\infty} (2k+1) e^{-tk} e^{-tk^2}. \tag{1.20}$$

In §3 we tackle the lattice point problem directly, and use it to prove the following refinement of (1.15):

Proposition 1.2. *We have*

$$\mathcal{N}_L(R) = \frac{1}{2} R(\log R) + O(R), \quad R \nearrow +\infty. \tag{1.21}$$

2. Asymptotic analysis of $\mathrm{Tr}\, e^{tL}$

To begin the asymptotic analysis of (1.15) as $t \searrow 0$, let us set

$$\mu = k + \ell, \quad \nu = k - \ell, \tag{2.1}$$

and write (1.18) as

$$\Phi(t) = \sum_{\mu,\nu \ge 0,\, \mu-\nu \text{ even}} e^{-t(\mu+\nu)/2} e^{-t\mu\nu}. \tag{2.2}$$

We split this sum into 2 pieces, over

$$\begin{aligned} &\mu,\nu \ge 0 \text{ both even, say } \mu = 2j,\ \nu = 2k,\ \ j,k \ge 0, \\ &\mu,\nu \ge 1 \text{ both odd, say } \mu = 2j+1,\ \nu = 2k+1,\ \ j,k \ge 0. \end{aligned} \tag{2.3}$$

Thus

$$\begin{aligned} \Phi(t) &= \sum_{j,k\ge 0} e^{-t(j+k)} e^{-4tjk} + \sum_{j,k \ge 0} e^{-t(j+k+1)} e^{-t(4jk+2j+2k+1)} \\ &= \sum_{j,k\ge 0} e^{-t(j+k)} e^{-4tjk} + e^{-2t} \sum_{j,k\ge 0} e^{-3t(j+k)} e^{-4tjk}. \end{aligned} \tag{2.4}$$

Thus our task becomes to analyze the behavior as $t \searrow 0$ of

$$\Phi_a(t) = \sum_{j,k\ge 0} e^{-at(j+k)} e^{-4tjk}, \tag{2.5}$$

for $a = 1, 3$. We first sum over j:

$$\begin{aligned} \sum_{j=0}^{\infty} e^{-atj} e^{-4tkj} &= \sum_{j=0}^{\infty} e^{-t(a+4k)j} \\ &= \frac{1}{1 - e^{-t(a+4k)}}. \end{aligned} \tag{2.6}$$

Hence

$$\Phi_a(t) = \sum_{k=0}^{\infty} \frac{e^{-atk}}{1 - e^{-t(a+4k)}}. \tag{2.7}$$

To analyze this, set

$$F(x) = \frac{x}{1 - e^{-x}}, \tag{2.8}$$

satisfying

$$F \in C^\infty([0,\infty)), \quad F(0) = 1. \tag{2.9}$$

We have

$$\begin{aligned} \Phi_a(t) &= \frac{1}{t}\sum_{k=0}^{\infty} \frac{e^{-atk}}{a+4k}\, F(ta+4tk) \\ &= \frac{1}{t}\sum_{k=0}^{\infty} \frac{e^{-atk}}{a+4k} + R_a(t), \end{aligned} \tag{2.10}$$

where

$$R_a(t) = \frac{1}{t}\sum_{k=0}^{\infty} \frac{e^{-atk}}{a+4k} \Big[F(ta+4tk) - 1\Big]. \tag{2.11}$$

We see that

$$|F(x) - 1| \le Cx, \quad \text{for} \ \ x \ge 0, \tag{2.12}$$

so

$$|R_a(t)| \le C\sum_{k=0}^{\infty} e^{-atk} = \frac{C}{1-e^{-at}} = \frac{C}{at}\, F(at). \tag{2.13}$$

Thus we are left with the task of analyzing

$$\Psi_a(t) = \frac{1}{t}\sum_{k=0}^{\infty} \frac{e^{-atk}}{a+4k} = \frac{1}{t} G_a(e^{-at}), \tag{2.14}$$

where

$$G_a(z) = \sum_{k=0}^{\infty} \frac{z^k}{a+4k}, \quad \text{for} \ \ |z| < 1. \tag{2.15}$$

Hence, for $|z| < 1$,

$$z^a G_a(z^4) = H_a(z) = \sum_{k=0}^{\infty} \frac{z^{4k+a}}{4k+a}. \tag{2.16}$$

Now

$$H_a'(z) = \sum_{k=0}^{\infty} z^{4k+a-1} = z^{a-1} \sum_{k=0}^{\infty} z^{4k} = \frac{z^{a-1}}{1-z^4} = \frac{1}{1-z}\,\frac{z^{a-1}}{1+z+z^2+z^3}. \tag{2.17}$$

Integrating, we get

$$H_a(z) \sim \frac{1}{4} \log \frac{1}{1-z}, \quad \text{as} \quad z \nearrow 1. \tag{2.18}$$

This implies

$$G_a(z) \sim \frac{1}{4} \log \frac{1}{1-z^{1/4}} \sim \frac{1}{4} \log \frac{1}{1-z}. \tag{2.19}$$

Hence, as $t \searrow 0$,

$$\Phi_a(t) \sim \frac{1}{t} G_a(e^{-at}) \sim \frac{1}{4t} \log \frac{1}{1-e^{-at}} \sim \frac{1}{4t} \log \frac{1}{at}. \tag{2.20}$$

Therefore,

$$\operatorname{Tr} e^{tL} = \Phi(t) = \Phi_1(t) + e^{-2t}\Phi_3(t) \sim \frac{1}{2t} \log \frac{1}{t}. \tag{2.21}$$

This gives (1.13)–(1.14), and completes the proof of Proposition 1.1.

Remark. We could probably examine (2.2)–(2.11) more carefully, and produce further terms in the asymptotic expansion of $\operatorname{Tr} e^{tL}$.

3. A lattice counting problem

As seen in (1.10)–(1.12), the counting function $\mathcal{N}_L(R)$ for the spectrum of L is given by

$$\mathcal{N}_L(R) = \#\{(k,\ell) \in \mathbb{Z}^+ \times \mathbb{Z} : (k,\ell) \in \Omega_R\}, \tag{3.1}$$

where

$$\Omega_R = \{(x,y) \in \mathbb{R}^+ \times \mathbb{R} : |y| \le x,\ x + x^2 - y^2 \le R\}. \tag{3.2}$$

Our goal is to establish the following.

Proposition 3.1. *We have*

$$\mathcal{N}_L(R) = \frac{1}{2} R \log R + O(R), \quad R \to +\infty. \tag{3.3}$$

Note that this is sharper than the result (1.15), established via the results of §2. Our first step is to show that

$$\operatorname{Area} \Omega_R = \frac{1}{2} R \log R + O(R) \quad \text{as} \quad R \to +\infty, \tag{3.4}$$

To get (3.4), we apply the transformation

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{3.5}$$

to get

$$\operatorname{Area} \Omega_R = \frac{1}{2} \operatorname{Area} \widetilde{\Omega}_R, \tag{3.6}$$

where

$$\begin{aligned} \widetilde{\Omega}_R &= \Big\{(x,y) \in \mathbb{R}^+ \times \mathbb{R}^+ : \frac{x+y}{2} + xy \le R\Big\} \\ &= \Big\{(x,y) \in \mathbb{R}^2 : 0 \le x \le 2R,\ 0 \le y \le \frac{2R - x}{2x+1}\Big\}. \end{aligned} \tag{3.7}$$

We have

$$\begin{aligned} \operatorname{Area} \widetilde{\Omega}_R &= \int_0^{2R} \frac{2R - x}{2x+1}\, dx \\ &= R \int_0^{2R} \frac{dx}{x + 1/2} - \frac{1}{2} \int_0^{2R} \frac{2x}{2x+1}\, dx \\ &= R \log\Big(x + \frac{1}{2}\Big)\Big|_0^{2R} - R + \frac{1}{4} \log\Big(x + \frac{1}{2}\Big)\Big|_0^{2R} \\ &= \Big(R + \frac{1}{4}\Big)\Big[\log\Big(2R + \frac{1}{2}\Big) + \log 2\Big] - R. \end{aligned} \tag{3.8}$$

This gives (3.4).

The next step is to deduce (3.3) from (3.4). This involves a general, elementary argument, which goes as follows. For $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{Z}^n$, let

$$Q_\nu = \Big\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n : \nu_j - \frac{1}{2} \le x_j < \nu_j + \frac{1}{2},\ \forall\, j \Big\}, \tag{3.9}$$

which are basically unit cubes centered at $\nu \in \mathbb{Z}^n$. These cubes partition $\mathbb{R}^n$ into mutually disjoint sets. Let

$$\Omega \subset \mathbb{R}^n \quad \text{be open}, \tag{3.10}$$

and let

$$\begin{aligned} S_\Omega &= \{\nu \in \mathbb{Z}^n : Q_\nu \subset \Omega\}, \\ S_{\partial\Omega} &= \{\nu \in \mathbb{Z}^n : Q_\nu \cap \partial\Omega \neq \emptyset\}. \end{aligned} \tag{3.11}$$

Note that

$$\bigcup_{\nu \in S_\Omega} Q_\nu \subset \Omega \subset \overline{\Omega} \subset \bigcup_{\nu \in S_\Omega \cup S_{\partial\Omega}} Q_\nu. \tag{3.12}$$

Hence

$$\# S_\Omega \le \operatorname{Vol} \Omega \le \# S_\Omega + \# S_{\partial\Omega}, \tag{3.13}$$

where $\# S$ denotes the number of elements of the set S. Furthermore,

$$S_\Omega \subset \Omega \cap \mathbb{Z}^n \subset \overline{\Omega} \cap \mathbb{Z}^n \subset S_\Omega \cup S_{\partial\Omega}, \tag{3.14}$$

so

$$\# S_\Omega \le \#(\Omega \cap \mathbb{Z}^n) \le \#(\overline{\Omega} \cap \mathbb{Z}^n) \le \# S_\Omega + \# S_{\partial\Omega}. \tag{3.15}$$

Comparing (3.13) and (3.15), we see that

$$\begin{aligned} \big| \operatorname{Vol} \Omega - \#(\Omega \cap \mathbb{Z}^n) \big| &\le \# S_{\partial\Omega}, \\ \big| \operatorname{Vol} \Omega - \#(\overline{\Omega} \cap \mathbb{Z}^n) \big| &\le \# S_{\partial\Omega}. \end{aligned} \tag{3.16}$$

We apply the general estimate (3.16) to the family $\Omega_R \subset \mathbb{Z}^2$ described in (3.3). One readily verifies that

$$\# S_{\partial\Omega_R} \le CR. \tag{3.17}$$

Hence (3.16) implies for $\mathcal{N}_L(R)$ in (3.1) that

$$\big| \mathcal{N}_L(R) - \operatorname{Area} \Omega_R \big| \le CR. \tag{3.18}$$

This completes the derivation of (3.3) from (3.4).

We dwell a little further on lattice point counts. Note that, by (1.5)–(1.7),

$$\mathcal{N}_\Delta(R) = \#\{(k,\ell) \in \mathbb{Z}^+ \times \mathbb{Z} : -k \le \ell \le k,\ k^2+k \le R\}. \tag{3.19}$$

Also, for $k \ge 0$,

$$k^2 + k \le R \Longleftrightarrow k \le \sqrt{R+\frac{1}{4}} - \frac{1}{2} = \rho, \tag{3.20}$$

so

$$\mathcal{N}_\Delta(R) = \#(\mathcal{T}_\rho \cap \mathbb{Z}^2), \tag{3.21}$$

where

$$\mathcal{T}_\rho = \rho\mathcal{T}_1, \quad \mathcal{T}_1 = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1,\ |y| \le x\}. \tag{3.22}$$

Since

$$\operatorname{Area}\mathcal{T}_\rho = \rho^2, \quad \text{and} \quad \#S_{\partial\mathcal{T}_\rho} \le C\rho, \tag{3.23}$$

we deduce from (3.16) and (3.21) that

$$\mathcal{N}_\Delta(R) = R + O(R^{1/2}). \tag{3.24}$$

In this case, $\mathcal{N}_\Delta(R)$ jumps by $2k+1$ as ρ crosses k, so the remainder estimate in (3.24) is sharp.

Note that the remainder in (3.24) is much smaller, compared to the leading term, than is the remainder term in (3.3) for $\mathcal{N}_L(R)$.

The application of (3.16) to (3.21), leading to (3.24), is a special case of the following corollary of (3.16).

Proposition 3.2. *Let $\mathcal{O} \subset \mathbb{R}^2$ be a bounded open set with piecewise smooth boundary, and set $\mathcal{O}_\rho = \rho\mathcal{O} = \{\rho x : x \in \mathcal{O}\}$. Then*

$$\#(\overline{\mathcal{O}}_\rho \cap \mathbb{Z}^2) = (\operatorname{Area}\mathcal{O})\rho^2 + O(\rho), \tag{3.25}$$

as $\rho \to \infty$.

While the remainder estimate here is sharp in some cases, such as $\mathcal{O} = \mathcal{T}_1$ in (3.22), there are many important cases where it is far from sharp. One much-studied, classical case is the disk

$$D = \{x \in \mathbb{R}^2 : |x| < 1\}. \tag{3.26}$$

A classical estimate here is

$$\#(\overline{D}_\rho \cap \mathbb{Z}^2) = \pi\rho^2 + O(\rho^{2/3}). \tag{3.27}$$

This is directly applicable to the spectral counting function for the Laplace operator on the flat torus

$$\mathbb{T}^2 = \mathbb{T}^1 \times \mathbb{T}^1, \quad \mathbb{T}^1 = \mathbb{R}/(2\pi\mathbb{Z}). \tag{3.28}$$

Methods of Fourier analysis are used to prove (3.27). These methods extend to other situations, including analysis of numerical integration (cf. [RT], especially §3). On the other hand, it is known that the remainder estimate in (3.27) is not optimal. In fact, finding an optimal remainder estimate for this lattice point problem is considered a major open problem. See §4 of [H] for a discussion and further references.

Returning to our principal interest, we close with the following.

Question. Can the asymptotic analysis of $\mathcal{N}_L(R)$ in (3.3) be improved?

References

[H] D. Hejhal, The Selberg trace formula and the Riemann zeta function, Duke Math. J. 43 (1976), 441–482.

[MS] A. Menikoff and J. Sjöstrand, On the eigenvalues of a class of hypoelliptic operators, Math Ann. 235 (1978), 55–85.

[RT] J. Rauch and M. Taylor, Quadrature estimates for multidimensional integrals, Houston J. Math. 36 (2010), 727–749.

[T] M. Taylor, Partial Differential Equations, Vols. 1–3, Springer, New York 1996 (2nd ed., 2011).

[T2] M. Taylor, Karamata's Tauberian theorem, Preprint, 2018.