Limiting Absorption Principle
For Long Range Potentials
Lectures of S. Agmon

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Contents:

1. Introduction

We are going to study differential operators of the form

$$P = P(x, D) = P_0(D) + V(x, D)$$

where $P_0(D)$ has constant coefficients and $V(x, D)$ has coefficients which are assumed to decay slowly as $\langle x\rangle = (1 + |x|^2)^{1/2}$ tends to infinity. We will make the following assumptions.

(1.1) $\quad P_0$ and P are elliptic of order m, essentially self adjoint on $\mathcal{S}(\mathbb{R}^n)$.

(1.2) $$V = V^S + V^L;\ V(x, \xi) = \sum_{|\alpha|\le m} V_\alpha(x)\xi^\alpha \text{ with } V_\alpha(x) = V^S_\alpha(x) + V^L_\alpha(x).$$

(1.3) $$\begin{aligned} |V^L_\alpha(x)| &\le C\langle x\rangle^{-\varepsilon}, \text{ for some } \varepsilon > 0, \text{ and} \\ |D^\beta_x V^L_\alpha(x)| &\le C_\beta\langle x\rangle^{-1-\varepsilon}, \text{ for all } \beta > 0. \end{aligned}$$

(1.4) $$|V^S_\alpha(x)| \le C\langle x\rangle^{-1-\varepsilon}.$$

If the set of values taken on by $P_0(\xi)$, $\xi \in \mathbb{R}^n$, is $[a, \infty)$, then, as is well known, the spectrum of P is described by

$$\sigma(P) = [a, \infty) \cup \{\lambda_j\}$$

where $\lambda_j \to a$ are eigenvalues of finite multiplicity. Our goal is to show that, as z approaches $\lambda \in [a, \infty)$ from the upper (or lower) half plane in the complex domain, provided λ avoids a certain small exceptional set, $(z - P)^{-1}f$ converges, at least in some weak sense, if f is nice enough. Such a result is known as a limiting absorption principle.

The exceptional set consists of $\sigma_p(P)$, the point spectrum of P, and also the set of critical values of P_0 :

$$\Lambda_c = \Lambda_c(P_0) = \{P_0(\xi) \in \mathbb{R} : \nabla_\xi P_0(\xi) = 0\}.$$

It is easy to see that, by the ellipticity of the polynomial $P_0(\xi)$, the set $\{\xi \in \mathbb{R}^n : \nabla_\xi P_0(\xi) = 0\}$ is a compact real algebraic variety and that hence $\Lambda_c(P_0)$ is a finite set.

Now to examine $(P-z)^{-1}f$ as $z \to \lambda \in \sigma(P_0) \setminus [\sigma_p(P) \cup \Lambda_c]$, we will take f to belong to a certain space B and derive weak convergence in B^*. We now define the spaces B and B^*. Let $R_j = 2^{j-1}$ for $j \geq 1,\ R_0 = 0$.
Definition.

$$(1.5)\qquad u \in B \Longleftrightarrow \sum_{j=1}^{\infty}\Big(R_j \int\limits_{R_{j-1}<|x|<R_j} |u(x)|^2 dx\Big)^{1/2} < \infty.$$

$$(1.6)\qquad u \in B^* \Longleftrightarrow \sup_{R>1}\Big(R^{-1} \int\limits_{|x|<R} |u(x)|^2 dx\Big)^{1/2} < \infty.$$

$$(1.7)\qquad u \in B^{o*} \Longleftrightarrow R^{-1} \int\limits_{|x|<R} |u(x)|^2 dx \to 0 \text{ as } R \to \infty.$$

We use (1.5) and (1.6) to define the B-norm and B^*-norm, respectively. Note that B^* is the dual of B. However, B is not reflexive. We remark that the image of B under the Fourier transform is a Besov space, but that fact plays no role in our investigations.

The space B can be compared with some weighted L^2 spaces, defined as follows:

$$(1.8)\qquad f \in L^{2,s} \Longleftrightarrow \langle x\rangle^s f \in L^2(\mathbb{R}^n).$$

One has:

$$(1.9)\qquad L^{2,1/2+\varepsilon} \subset B \subset L^{2,1/2}.$$

Furthermore, the use of such weighted L^2 spaces facilitates the study of B and B^*. We mention the following interpolation result, which will be of use:

Proposition 1.A. *If $T : L^2 \longrightarrow L^2$ and also $T : L^{2,1} \longrightarrow L^{2,1}$, both bounded, then $T : B \longrightarrow B$. Consequently, if $T : L^2 \longrightarrow L^2$ and $T : L^{2,-1} \longrightarrow L^{2,-1}$, then $T : B^* \longrightarrow B^*$.*

Let us now state the main theorem, whose proof will be completed in section 5.

Theorem 1.B. *Let $\lambda \in [a,\infty)\setminus[\sigma_p(P)\cup\Lambda_c(P_0)]$. Then there exist bounded operators*

$$(1.10)\qquad T_\pm : B \longrightarrow B^*$$

such that

$$(1.11)\qquad \langle T_\pm f, g\rangle = \lim_{z\to\lambda,\ \pm Im\ z>0} \langle (z-P)^{-1}f, g\rangle \text{ for all } f,g \in B.$$

We will call this limit $T_\pm = R(\lambda \pm i0)$.

Remarks.
(1.12) In fact, there is not strong convergence. Although we will not prove this, one could get strong convergence in $\mathcal{L}(L^{2,s}, L^{2,-s})$, for $s > 1/2$.

(1.13) If $\lambda \in \sigma_p(P) \setminus \Lambda_c$, we would still get convergence in (1.11), provided that f and g are both chosen orthogonal to the λ-eigenspace of P. This would follow by minor modifications of the arguments we give, and we will not give the details.

(1.14) $\sigma_p(P) \setminus \Lambda_c(P_0)$ consists of isolated eigenvalues, of finite multiplicity (possibly accumulating at points of Λ_c). We will prove this in section 3; see Corollary 3.J.

We now outline the method that will be developed in succeeding sections to prove the main theorem. For $\lambda \in [a, \infty) \setminus [\sigma_p(P) \cup \Lambda_c]$, let $\Delta(\lambda, \rho) = \{z \in \mathbb{C} : |z - \lambda| < \rho\}$ and take ρ small. We obtain in section 3 the basic a priori estimate that, if $u \in B^{o*} \cap H^m_{loc}$ and $(P - z)u = f \in B$ for some $z \in \Delta(\lambda, \rho)$, then there is a $\psi \in C_0^\infty(\mathbb{R}^n)$ such that

$$(1.15) \qquad \sum_{|\alpha| \le m} \|D^\alpha u\|_{B^*} \le C\|f\|_B + C\|\psi u\|_{L^2}; \ C \text{ independent of } z \in \Delta(\lambda, \rho).$$

To get such a regularity estimate, we will develop a calculus of pseudodifferential operators in section 2, which give good control on a function for large x. Indeed, it is large x, rather than large frequency, which is the most delicate point to handle. In frequency space, our difficulties are essentially confined to a small neighborhood of the compact set

$$(1.16) \qquad \mathcal{M}_\lambda = \{\xi \in \mathbb{R}^n : P_0(\xi) = \lambda\}.$$

The next step is to get rid of the second term on the right in (1.15). Indeed, as we will see in the proof, if this were not possible one could produce a nonzero $u \in B^{o*}$ such that $(P - \lambda)u = 0$. However, Theorem 3.I then yields $u \in L^2$, so $\lambda \in \sigma_p(P)$, contrary to assumption. This accomplished, we have the existence of weak limits in B^* of $(P - z)^{-1}f$ as $z \to \lambda$ from Im $z > 0$ and from Im $z < 0$. To check uniqueness of these respective limits, we need a radiation condition. This radiation condition is studied in section 4. It is given in terms of the 'radiation set' of an element $u \in B^*$, defined in section 2, a concept formally similar to the concept of wave front set except, again, the emphasis is on large $|x|$ rather than large frequency.

In section 6 we note that the absence of the singular continuous spectrum of P is a simple consequence of the main theorem.

The lectures of Agmon reported on here also were sketched in [1]. This material is also treated in detail in §30.2 of [2]. For Schrödinger operators with long range potentials, such results had been obtained in [3].

References.

[1] S.Agmon, Some new results in spectral and scattering theory of differential operators on $\mathbb{R}^n$. Sem. Goulaouic-Schwartz, 1978-79, Expose no.2, pp. 1-11.

[2] L.Hörmander, The analysis of Linear Partial Differential Operators IV, Springer-Verlag, 1985.

[3] T.Ikebe and Y.Saito, Limiting absorption method and absolute continuity for the Schrödinger operator, J.Math. Kyoto Univ. 12(1972), 513-542.

2. Pseudodifferential operators and the radiation set

As usual, a pseudodifferential operator with symbol $p(x,\xi)$ is defined by the formula

$$p(x,D)u = \int p(x,\xi)\hat{u}(\xi)e^{ix\cdot\xi}d\xi. \tag{2.1}$$

We will be interested in the following symbol classes. Let μ be a sequence:

$$\mu = \{\mu(0), \mu(1), \mu(2), \dots\}.$$

Assume $\mu(i)$ is monotonically decreasing, or at least non-increasing.
Definition. $p(x,\xi) \in S^m_\mu$ if and only if

$$|D^\beta_x D^\alpha_\xi p(x,\xi)| \le C\langle\xi\rangle^{m-|\alpha|}\langle x\rangle^{\mu(|\beta|)}. \tag{2.2}$$

We also define weighted Sobolev spaces. Namely,

$$u \in H_{m,s}(\mathbb{R}^n) \Longleftrightarrow \langle x\rangle^s(1-\Delta)^{m/2}u \in L^2(\mathbb{R}^n). \tag{2.3}$$

Our first result is elementary.

Theorem 2.A. *If $p(x,\xi) \in S^m_\mu$, then*

$$p(x,D) : H_{k,s} \longrightarrow H_{k-m,s-\mu(0)}.$$

Using the interpolation result, Proposition 1.A, we obtain the following.

Corollary 2.B. *If $p(x,\xi) \in S^0_{\{0,0,\dots\}}$, then*

$$p(x,D) : B \longrightarrow B,\ B^* \longrightarrow B^*,\ B^{o*} \longrightarrow B^{o*}.$$

The last conclusion follows from the second since $p(x,D) : \mathcal{S} \longrightarrow \mathcal{S}$ and the closure of $\mathcal{S}$ in B^* is B^{o*}.

The next few results record the basic behavior of operators in OPS^m_μ, i.e., operators defined by (2.1) with symbols in S^m_μ. We omit the proofs.

Theorem 3.C. *If $p_j \in S^{m_j}_{\mu_j}$ for $j = 1, 2$, then*

$$p(x,D) = p_1(x,D)p_2(x,D) \in OPS^{m_1+m_2}_{\mu_1(0)+\mu_2(0)} \tag{2.4}$$

and

$$p(x,\xi) = \sum_{j=0}^{N-1} q_j(x,\xi) + R_N(x,\xi)$$

with

$$q_j(x,\xi) \in S^{m_1+m_2-j}_{\mu_1(0)+\mu_2(j)} \text{ and } R_N(x,\xi) \in S^{m_1+m_2-N}_{\mu_1(0)+\mu_2(N)}. \tag{2.5}$$

Theorem 2.D. *If $p \in S^m_\mu$, then $p(x,D)^* \in OPS^m_\mu$, with symbol*

$$p^*(x,\xi) = \sum_{|\alpha|\le N} \frac{1}{\alpha!} D^\alpha_\xi D^\alpha_x \overline{p(x,\xi)} + R_N(x,\xi), \tag{2.6}$$

and

$$R_N(x,\xi) \in S^{m-N}_{\mu(N)}. \tag{2.7}$$

Corollary 2.E. *If $p(x,\xi) \in S^m_\mu$ is real, then*

$$p(x,D) - p(x,D)^* \in OPS^{m-1}_{\mu(1)}.$$

The next theorem is an important technical result which will be used in the proof of Theorem 2.J.

Theorem 2.F. *Suppose $p(x,\xi) \in S^0_{\{0,-\delta,\dots\}}$, $\delta > 0$. If $u \in B^*$, then $p(x,D)u \in B^*$ satisfies*

$$\limsup_{R\to\infty} R^{-1} \int\limits_{|x|<R} |p(x,D)u|^2 dx \le C_0^2 \ \limsup_{R\to\infty} R^{-1} \int\limits_{|x|<R} |u(x)|^2 dx \tag{2.8}$$

where

$$C_0 = \lim_{R\to\infty} \sup_{|x|>R, \xi\in\mathbb{R}^n} |p(x,\xi)|. \tag{2.9}$$

In particular, if $C_0 = 0$, then $p(x,D) : B^* \longrightarrow B^{o*}$, but that is not the only case of interest.

We now define a couple of further classes of symbols:

$$\tilde{\Sigma}^0_0 \subset \Sigma^0_0 \subset S^0_{\{0,-1,-2,\dots\}};\ \Sigma^m_\ell \subset S^m_{\{\ell,\ell-1,\ell-2,\dots\}}$$

as follows.

Definition. $p(x,\xi) \in \Sigma^m_\ell$ if and only if $p(x,\xi) \in S^m_{1,0}$ and is homogeneous in x of degree ℓ for $|x|$ large.

Next, let $\tilde{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$. We say $f(\xi)$ is C^∞ near ∞ provided $g(\xi) = f(\xi/|\xi|^2)$ is C^∞ near 0. Similarly, $C^\infty(S^{n-1} \times \tilde{\mathbb{R}}^n)$ is defined. Note that, if $\mathcal{Z}(x)$ is C^∞ with $\mathcal{Z}(x) = 0$ for $|x| \le \frac{1}{2}$, $\mathcal{Z}(x) = 1$ for $|x| \ge 1$, and if $p_0(\omega,\xi) \in C^\infty(S^{n-1} \times \tilde{\mathbb{R}}^n)$, then

$$\mathcal{Z}(x) p_0(x/|x|, \xi) \tag{2.10}$$

belongs to Σ_0^0.

Definition. $p(x,\xi) \in \tilde{\Sigma}_0^0$ if and only if $p(x,\xi)$ is of the form (2.10) for some $p_0(\omega,\xi) \in C^\infty(S^{n-1} \times \tilde{\mathbb{R}}^n)$.

We now define the concept of the radiation set of an element $u \in L^\# = \cup_s L^{2,s}$. We consider a more general concept than is absolutely necessary. Thus, let $\cap_{k,s} H_{k,s} \subset \mathcal{L} \subset \cup_s L^{2,s}$ and assume

$$OPS^0_{\{0,\dots\}} : \mathcal{L} \longrightarrow \mathcal{L}.$$

The example of greatest interest is $\mathcal{L} = B^{o*}$.

Definition. Let $u \in L^\#$. Then $\mathcal{L}/RS(u)$, called the $\mathcal{L}$-radiation set of u, is defined as follows.

$$\mathcal{L}/RS(u) \subset S^{n-1} \times \tilde{\mathbb{R}}^n$$

and $(\omega_0,\xi_0) \notin \mathcal{L}/RS(u)$ if and only if there exist $h(\omega) \in C^\infty(S^{n-1})$ with $h(\omega_0) \neq 0$ and $\varphi(\xi) \in C^\infty(\tilde{\mathbb{R}}^n)$ with $\varphi(\xi_0) \neq 0$ such that

$$\mathcal{Z}(x)h(x/|x|)\varphi(D)u \in \mathcal{L}. \tag{2.11}$$

The following results are elementary consequences of the previous theorems.

Theorem 2.G. *$\mathcal{L}/RS(u) = \emptyset$ if and only if $u \in \mathcal{L}$.*

Theorem 2.H. *Let $p(x,D) \in OPS^0_\mu$. Assume $\mu(0) = 0,\ \mu(i) \to -\infty$. Then*

$$\mathcal{L}/RS(p(x,D)u) \subset \mathcal{L}/RS(u). \tag{2.12}$$

Furthermore, if $p_0(\omega,\xi) = 0$ on a neighborhood of $\mathcal{L}/RS(u)$, $p_0(\omega,\xi) \in C^\infty(S^{n-1} \times \tilde{\mathbb{R}}^n)$, then $\mathcal{L}/RS(\mathcal{Z}(x)p_0(\frac{x}{|x|},D)u) = \emptyset$.

Using the first part of Theorem 2.H, one sees that the following result is a generalization of the second part.

Theorem 2.I. *Take $p(x,D) \in OPS^0_\mu$ as in 2.H Theorem. Let $\mathcal{O} \subset S^{n-1} \times \tilde{\mathbb{R}}^n$ be open, and assume*

$$|D_x^\beta D_\xi^\alpha p(x,\xi)| \leq C_N \langle x\rangle^{-N} \langle\xi\rangle^{-|\alpha|},\ (x/|x|,\xi) \in \mathcal{O}.$$

Then $\mathcal{O} \cap \mathcal{L}/RS(p(x,D)u) = \emptyset$.

We emphasize that the case of special interest for us is

$$B^{o*}/RS(u)$$

which we shall merely call the radiation set of u. In this case we have the following technically important strengthening of the second half of Theorem 2.H. See the proof of Proposition 4.B.

Theorem 2.J. *Let* $u \in B^*$. *Take* $p(x,\xi) \in \tilde{\Sigma}_0^0$, *of the form* (2.11), *such that*

$$p_0(\omega,\xi) = 0 \text{ on } B^{o*}/RS(u).$$

Then $p(x,D)u \in B^{o*}$.

Proof. Write $p(x,\xi) = p_1(x,\xi) + p_2(x,\xi)$ with $p_j = \mathcal{Z} p_{j0}(x/|x|,\xi)$. Suppose $p_{10}(\omega,\xi) = 0$ on a neighborhood of $B^{o*}/RS(u)$ and $|p_{20}(\omega,\xi)| \le \delta$. Then $p_1(x,D)u \in B^{o*}$ by the second half of Theorem 2.H. Meanwhile, Theorem 2.F implies that

$$\limsup_{R\to\infty} R^{-1} \int_{|x|<R} |p(x,D)u|^2 dx \le \delta^2 \|u\|_{B^*}^2 .$$

Taking $\delta \to 0$ completes the proof.

3. The basic a priori estimate

This section is devoted to the proof and a few consequences of the following estimate. Fix $\lambda \in [a,\infty) \setminus [\sigma_p(P) \cup \Lambda_c(P_0)]$ and pick ρ so small that $\Delta(\lambda,\rho) = \{z \in \mathbb{C} : |z-\lambda| < \rho\}$ is bounded away from $\sigma_p(P) \cup \Lambda_c(P_0)$.

Theorem 3.A. *If $u \in B^{o*} \cap H^m_{loc}$ and, for some $z \in \Delta(\lambda,\rho)$,*

$$(P-z)u = f \in B,$$

then there is a $\psi \in C_0^\infty(\mathbb{R}^n)$ and a $C < \infty$, independent of $z \in \Delta(\lambda,\rho)$, such that

$$\sum_{|\alpha|\le m} \|D^\alpha u\|_{B^*} \le C\|f\|_B + C\|\psi u\|_{L^2}. \tag{3.1}$$

We begin the proof, isolating the hard part as 3.B Proposition . Rewrite $V(x,\xi)$ as $\tilde V^L + \tilde V^S$ with $\tilde V^L \in S^m_{\{-\varepsilon,-1-\varepsilon,\dots\}}$ such that all the roots of $P_0(\xi)+\tilde V^L(x,\xi)-\lambda = 0$ are in $\mathcal{O}_\lambda$, a small neighborhood of

$$\mathcal{M}_\lambda = \{\xi \in \mathbb{R}^n : P_0(\xi) = \lambda\}.$$

Let $\tilde P^L(x,\xi) = P_0(\xi) + \tilde V^L(x,\xi)$. Take $\omega(\xi)$ such that $\omega(\xi) = 0$ on $\mathcal{O}_\lambda$, $\omega(\xi) = \langle\xi\rangle^m$ for $|\xi|$ large. Let

$$K(x,\xi,z) = \omega(\xi)(\tilde P^L(x,\xi) - z)^{-1}. \tag{3.2}$$

Then $K \in OPS^0_\mu$ with $\{\mu(i)\} = \{0, -1-\varepsilon, \dots\}$. Thus

$$K(x,D,z)(\tilde P^L(x,D) - z) - \omega(D) = L(x,D,z) \in OPS^{m-1}_{\tilde\mu};\ \tilde\mu(i) = \mu(i+1). \tag{3.3}$$

Thus, for u satisfying the hypotheses of Theorem 3.A,

$$\begin{aligned} \omega(D)u &= K(\tilde P^L - z)u - Lu \\ &= Kf - K(\tilde V^S u) - Lu \end{aligned} \tag{3.4}$$

so

$$\|\omega(D)u\|_{B^*} \le C\|f\|_{B^*} + C\|u\|_{m,-\frac12-\frac{\varepsilon}{2}}. \tag{3.5}$$

Now, setting $\chi(D) = I - \omega(D)$, we claim:

Proposition 3.B. *$\chi(D)u$ satisfies the estimate*

$$\sum_{|\alpha|\le m} \|D^\alpha \chi(D)u\|_{B^*} \le C\|f\|_B + C\|u\|_{m,-\frac{1}{2}-\frac{\varepsilon}{2}}.$$

Granted this result, we have the following, from (3.5):

$$\sum_{|\alpha|\le m} \|D^\alpha u\|_{B^*} \le C\|f\|_B + C\|u\|_{m,-\frac{1}{2}-\frac{\varepsilon}{2}}. \tag{3.6}$$

Now, if $\psi(x)$ is a cut-off, equal to 1 on a large set,

$$\|(1-\psi)u\|_{m,-1/2-\varepsilon/2} < \frac{1}{2}\sum_{|\alpha|\le m} \|D^\alpha u\|_{B^*},$$

so we get

$$\sum_{|\alpha|\le m} \|D^\alpha u\|_{B^*} \le C\|f\|_B + C'\|\psi u\|_m. \tag{3.7}$$

Now we can use $(P-z)(\psi u) = \psi f + [P,\psi]u$, noting that the coefficients of $[P,\psi]$ are compactly supported, to get $\|\psi u\|_m \le C\|f\| + C\|\psi_1 u\|_{m-1}$, with some $\psi_1 \in C_0^\infty(\mathbb{R}^n)$. Relabelling ψ, calling it ψ, we have

$$\sum_{|\alpha|\le m} \|D^\alpha u\|_{B^*} \le C\|f\|_B + C\|\psi u\|_{m-1}. \tag{3.8}$$

The estimate (3.1) follows from (3.8) by Poincare's inequality. Thus we have given the proof of Theorem 3.A, modulo the proof of Proposition 3.B.

To prove Proposition 3.B, we cover the zero set of $P_0(\xi) - \tilde{V}^L(x,\xi) - \lambda$ in $\mathbb{R}^n \times \mathbb{R}^n$ (which is close to $\mathcal{M}_\lambda$) with little balls and write $\chi(D) = \sum \chi_j(D)$ where each $\chi_j(\xi)$ is supported on such a ball. We take the balls so small that, for each j, there is a k such that $\partial P_0/\partial \xi_k$ is non-vanishing on supp χ_j. For simplicity we will assume that $k = 1$, which can be arranged for each fixed j by a coordinate rotation, and that

$$\partial P_0/\partial \xi_1 > 0 \text{ on supp } \chi_j. \tag{3.9}$$

The situation $\partial P_0/\partial \xi_1 < 0$ is handled by a similar argument. Now, on supp χ_j, write (with $\xi = (\xi_1, \xi')$)

$$P^L(x,\xi_1,\xi') - z = (\xi_1 - a(x,\xi',z))F(x,\xi,z) \tag{3.10}$$

with $F(x,\xi,z)$ non-vanishing. It is clear that if $z = \lambda \in \mathbb{R}$ then Im $a = 0$. More generally, we have the following fact, which will be technically useful.

Lemma 3.C. *For certain real valued $b(x,\xi',z)$ which, on supp χ_j, satisfies*

$$|D^{\alpha}_{\xi'} b| \leq C_{\alpha}\langle \xi\rangle^{-|\alpha|} \tag{3.11}$$

and

$$D^{\beta}_{x'} D^{\alpha}_{\xi'} b(x_1,x',\xi',z)| \leq C_{\alpha\beta}\langle x_1\rangle^{-1-\varepsilon}\langle\xi\rangle^{-|\alpha|} \text{ for } |\beta|\geq 1 \tag{3.12}$$

one has, on supp χ_j,

$$Im\ a(x,\xi',z) = (Im\ z) b(x,\xi',z)^2. \tag{3.13}$$

Proof. Since P^L is a small perturbation of P_0, we can drop the extra parameters. Let $p(\xi_1)$ be some polynomial and let $\rho(z)$ be the unique zero of $p-z$ near $\rho_0 = \rho(\lambda_0)$, assuming $p(\rho_0)-\lambda_0 = 0$, λ_0 real, and $p'(\rho_0) > 0$. The implicit function theorem gives

$$\frac{d}{d\lambda}\rho(\lambda_0) > 0. \tag{3.14}$$

We want Im $\rho(z) = (\text{Im } z)\sigma(z)^2$, $\sigma(z)$ real and smooth. Indeed, $\rho(z)-\rho_0 = \int\limits_{\lambda_0}^{z} \rho'(\zeta)d\zeta$, so

$$\begin{aligned} \text{Im } \rho(z) &= \int\limits_{\lambda_0}^{z} \text{Im } [\rho'(\zeta)d\zeta] \\ &= \int\limits_{0}^{Im\ z} \text{Re } \rho'(\lambda+is)ds \\ &= (\text{Im } z)\int\limits_{0}^{1} \text{Re } \rho'(\lambda + i\tau \text{ Im } z)d\tau \\ &= (\text{Im } z)\sigma(z)^2, \end{aligned}$$

by (3.14). This proves (3.13), and the proofs of (3.11) and (3.12) are straightforward.

To return to (3.10), this factorization implies that

$$\begin{aligned} \Big(-i\frac{\partial}{\partial x_1} - a(x,D_{x'},z)\Big)(\chi_j(D)u) &= G(x,D_x,z)(P^L-z)+Ru \\ &= Gf - GV^S u + Ru. \end{aligned} \tag{3.15}$$

Here $G \in OPS_{\mu}^{-\infty}$ and $R \in OPS_{\tilde{\mu}}^{-\infty}$, with $\{\mu(j)\} = \{0, -1-\varepsilon, \dots\}$, $\tilde{\mu}(j) = \mu(j+1)$. We claim that, from (3.15), it follows that

$$\sum_{|\alpha| \le m} \|D^{\alpha} \chi_j(D) u\|_{B^*} \le C\|f\|_B + C\|GV^S u\|_B + C\|Ru\|_B. \tag{3.16}$$

We briefly potspone the proof of (3.16) and show how this estimate leads to the proof of 3.B Proposition.

Indeed, the short range hypothesis on V^S and the operator properties of G and R give

$$\sum_{|\alpha| \le m} \|D^{\alpha} \chi_j(D) u\|_{B^*} \le C\|f\|_B + C\|u\|_{m, -\frac{1}{2} - \frac{\varepsilon}{2}}, \tag{3.17}$$

and summing over j gives Proposition 3.B.

Thus, to complete the proof of Theorem 3.A, it remains only to establish that (3.16) follows from (3.15). From the left hand side of (3.l5), we see that we are considering an operator-valued ODE of the form

$$\frac{du}{dy} - A(y)u(y) = g(y). \tag{3.18}$$

Here it is convenient to change notation, replacing x_1 by y and replacing x' by x. So we denote $a(x, D_{x'}, z)$ by $a(y, x, D_x, z) = A(y)$ (with z as a parameter). We look for a priori estimates of solutions $u(y)$, taking values in some Hilbert space H, to (3.18), assuming $f \in L^1((0, T], H)$ and $A(y) \in C((-\infty, T], \mathcal{L}(H))$. As one can imagine, using standard energy estimates and Gronwall inequality arguments, one can get estimates on solutions $u(y)$ to (3.18) provided $A(y)$ satisfies the semi-boundedness condition:

$$A(y) + A(y)^* \le B(y);\ B(y) \text{ symmetric and } \int_{-\infty}^{T} \|B(y)\| dy < \infty. \tag{3.19}$$

For example, the following result is elementary.

Proposition 3.D. *Let u solve* (3.18) *and suppose* $\liminf_{t \to -\infty} \|u(t)\| = 0$. *Then*

$$\|u(t)\| \le \int_{-\infty}^{t} \|f(s)\| ds \cdot \ exp\ \Big(\int_{-\infty}^{t} \|B(s)\| ds \Big).$$

One can also get weighted estimates, such as the following:

Proposition 3.E. *Suppose in addition that* $\|B(y)\| \le \langle y\rangle^{-1-\varepsilon}$. *Then, for* $k > 1$,

$$\int_{-\infty}^{T} (1+|y|)^{k-2}\|u(y)\|^2 dy \le C_{k,T} \int_{-\infty}^{T} (1+|y|)^{k}\|f(y)\|^2 dy,\ T<\infty.$$

In our situation, $A(y) = ia(y,x,D_x,z)$, with (3.13) holding. Let us consider the case Im $z \le 0$. Otherwise, do the same argument on $[T,\infty)$. Thus, with $(-\text{Im } z)^{\frac{1}{2}}b$ relabeled as b, we have

$$\text{Im } a = -b^2.$$

It follows from (3.11), (3.12), and the pseudodifferential operator calculus, that

$$A(y) + A(y)^* = C(y) - 2b^*b \le C(y) \text{ and } \|C(y)\| \le C\langle y\rangle^{-1-\delta}. \tag{3.20}$$

Thus (3.19) is satisfied in our case.

Remark. It might be amusing to prove some sort of 'sharp Gårding inequality,' which would make Lemma 3.C and the computation (3.20) unnecessary. However, we will not pursue this.

The crucial abstract result, which allows us to deduce (3.16) from (3.15), is the following. Let

$$\Omega_T^- = \{(y,x) : y < T\}.$$

Proposition 3.F. *Suppose* $u,\ \partial u/\partial y$ *belong to* $L^{2,-N}(\Omega_T^-)$ *for some* $N<0$. *Consider* $u,\ \partial u/\partial y$ *as functions of* y *with values in* $L^{2,-N}(\mathbb{R}^{n-1})$. *Let* $a(y,x,D_x)$ *be as above. Assume that*

$$\liminf_{R\to\infty} R^{-1} \int_{\Omega_T^-\cap\{y^2+|x|^2<R^2\}} |u(y,x)|^2 dy\ dx = 0. \tag{3.21}$$

Suppose $-i\partial u/\partial y - a(y,x,D_x)u = f(y,x)$ *and*

$$\int_{-\infty}^{T} \|f(y,\cdot)\|_{L^2(\mathbb{R}^{n-1})} dy < \infty.$$

Then

$$\|u(y,\cdot)\|_{L^2(\mathbb{R}^{n-1})} \le K \int_{-\infty}^{t} \|f(s)\|_{L^2(\mathbb{R}^{n-1})} ds. \tag{3.22}$$

Note. Given the crucial nature of this result, a complete proof should be written down.

Remarks.

(3.23) This is better than Proposition 3.D, since we are only assuming $u(y,\cdot) \in L^{2,-N}$.

(3.24) If the symbol a is real, we get such a result in $\mathbb{R}^n = \Omega_0^- \cup \Omega_0^+$.

(3.25) We also can prove weighted estimates.

Corollary 3.G. *Under the hypotheses of the Theorem above, one has*

$$\|u\|_{B^*(\Omega_T^-)} \leq C\|f\|_{B(\Omega_T^-)}, \; C \textit{ independent of } T. \tag{3.26}$$

Thus we can take $T = +\infty$; $\Omega_T^- = \mathbb{R}^n$.

Proof. Use

$$\|u(y,\cdot)\|_{L^2(\mathbb{R}^{n-1})} \leq C \int_{-\infty}^{t} \|f(y,\cdot)\|_{L^2} dy \leq C'\|f\|_{B(\Omega_t^-)}$$

and

$$\|u\|_{B^*(\Omega_T^-)} \leq C \sup_{s \leq T} \|u(s,\cdot)\|_{L^2}.$$

Corollary 3.H. *If* a *is real and* $u \in B^{o*}$, *then* $u(y,\cdot) \in L^2(\mathbb{R}^{n-1})$ *for almost all* y, *and, for* $k > 1$,

$$\int_{-\infty}^{\infty} \langle y\rangle^{k-2}\|u(y,\cdot)\|_{L^2}^2 dy \leq C \int_{-\infty}^{\infty} \langle y\rangle^{k}\|f(y,\cdot)\|^2 dy, \tag{3.27}$$

Now the estimate (3.26), for $T = +\infty$, directly shows that (3.l6) follows from (3.15). At this point, the proof of the basic a priori estimate, Theorem 3.A, is complete. In the same fashion, via 3.H Corollary, one obtains the folowing.

Theorem 3.I. *Let* $\lambda \in \mathbb{R}$, $f \in L^{2,s}$, $s > \frac{1}{2}$. *Then, if the principal part of* P *is real,*

$$(P - \lambda)u = f, \; u \in B^{o*} \Longrightarrow u \in L^{2,s-1}. \tag{3.28}$$

Corollary 2.J. *All eigenvalues of* P *different from critical values (elements of* $\Lambda_c(P_0)$*) are isolated and have finite multiplicity.*

Proof. Given $[a,b] \subset\subset \sigma(P) \setminus \Lambda_c(P_0)$, 3.I Theorem yields, for $\lambda \in [a,b]$,

$$(P - \lambda)u = f, \; u \in L^2 \subset B^{o*} \Longrightarrow \|u\|_{m,s-1} \leq C_s(\|f\|_{0,s} + \|\psi u\|_{L^2}).$$

Taking $s = 2$, $f = 0$,

$$\|u\|_{m,1} \leq C\|\psi u\|_{L^2} \leq C'\|u\|_{L^2}.$$

The Rellich theorem implies the finite dimensionality of the sum of the eigenspaces of P with eigenvalues in $[a,b]$.

Note that all such eigenfunctions associated with eigenvalues $\lambda \in \sigma_p(P) \setminus \Lambda_c(P_0)$ belong to $L^{2,s}$ for all $s < \infty$.

4. The radiation condition

Let $u \in B^*$ and suppose $D^\alpha u \in B^*$ for $|\alpha| \le m$. Suppose $\lambda \in \sigma(P) \setminus [\sigma_p(P) \cup \Lambda_c(P_0)]$, and

$$(P - \lambda)u = f \in B.$$

We say u is λ-*outgoing* if there exist $z_j \to \lambda$, Im $z_j > 0$, such that

$$(P - z_j)^{-1} f \to u \text{ weakly in } B^*, \text{ with all derivatives of order } \le m.$$

Similarly we say u is λ-*incoming* if such a limit holds with $z_j \to \lambda$, Im $z_j < 0$. Once the limiting absorption principle is proved, in the next section, we can say that u is λ-outgoing (resp., λ-incoming) if and only if, for some $f \in B$, $u = R(\lambda + i0)f$ (resp., $u = R(\lambda - i0)f$). Here, as an aid in proving our main theorem, we give a characterization of λ-incoming and λ-outgoing functions u in terms of the radiation set.

Theorem 4.A. *Assume $(P - \lambda)u = f \in B$ for some $\lambda \in \sigma(P) \setminus [\sigma_p(P) \cup \Lambda_c]$ and suppose $D^\alpha u \in B^*$ for $|\alpha| \le m$. Then u is λ-outgoing if and only if, for $|\alpha| \le m$,*

$$B^{o*}/RS(D^\alpha u) \subset \left\{ (\omega, \xi) \in S^{n-1} \times \tilde{\mathbb{R}^n} : \xi \in \mathcal{M}_\lambda \text{ and } \omega = \frac{\nabla P_0(\xi)}{|\nabla P_0(\xi)|} \right\} = Y_\lambda^+. \tag{4.1}$$

There is a similar characterization for u λ-incoming, with Y_λ^+ replaced by Y_λ^-, characterized by

$$\xi \in \mathcal{M}_\lambda, \ \omega = -\frac{\nabla P_0(\xi)}{|\nabla P_0(\xi)|}.$$

Proof. As in the proof of Theorem 3.A, we need only consider the behavior of $\chi_j(D)u$. If u is λ-outgoing then Corollary 3.G implies that u restricted to Ω_0^- belongs to $B^{o*}(\Omega_0^-)$. Now we can re-define the ξ_1-axis so that it points in any direction non-tangent to $\mathcal{M}_\lambda$ on supp χ_j and hence conclude that $\chi_j(D)u$ is in B^{o*} on the complement of an arbitrarily small cone about the normal to $\mathcal{M}_\lambda$ at a point in supp χ_j, if this support is taken small enough, by taking a sufficiently fine covering of $\mathcal{M}_\lambda$. From this, (4.1) follows. A similar argument works if u is assumed λ-incoming.

To prove the converse, we establish the folowing result, of independent interest.

Proposition 4.B. *Assume $D^\alpha u \in B^*$, for $|\alpha| \le m$, and $(P - \lambda)u = f \in B$, $\lambda \in \sigma(P) \setminus \Lambda_c$. Suppose that, with one choice of sign, $|\alpha| \le m$,*

$$B^{o*}/RS(D^\alpha u) \subset Y_\lambda^\pm. \tag{4.2}$$

Take $\psi(\xi) \in C^\infty(\tilde{\mathbb{R}}^n)$ *with* $\psi(\xi)^2 = |\nabla P_0(\xi)|$ *on* $\mathcal{M}_\lambda$. *Then*

$$\lim_{R\to\infty} R^{-1} \int\limits_{|x|<R} |\psi(D)u|^2 dx = \pm 2 \ Im \ \langle u, f\rangle. \tag{4.3}$$

To see how this result completes the proof of Theorem 4.A, assume u satisfies (4.1). In the next section, as a preliminary step toward proving the main theorem (see Lemma 5.B) it will be shown that there exists a weak limit $v = w - lim \ (P - z_j)^{-1} f$ in B^*, $z_j \to \lambda$, Im $z_j > 0$; this result will be established without the use of the 'only if' part of Theorem 4.A. Granted this, the part of Theorem 4.A proved so far yields $B^{o*}/RS(v) \subset Y_\lambda^+$. Consider $w = u - v$. We have $(P - \lambda)w = 0$ and $B^{o*}/RS(w) \subset Y_\lambda^+$. By Proposition 4.B, we conclude that

$$\psi(D)w \in B^{o*}.$$

On the other hand, certainly $\omega(D)w \in B^{o*}$ since $B^{o*}/RS(w) \subset Y_\lambda^+$. This implies that

$$w \in B^{o*}.$$

But now Theorem 3.I yields $w \in L^{2,s}$ for all $s < \infty$. In particular either w is an eigenfunction of P, contrary to the assumption that $\lambda \notin \sigma_p(P)$, or $w = 0$, so $u = v$, which shows that if u satisfies (4.1), then u is λ-outgoing. The same argument shows that (4.1) with Y_λ^+ replaced by Y_λ^- implies u is λ-incoming. Thus, granted Proposition 4.B and Lemma 5.B, the proof of Theorem 4.A is complete.

We turn now to the proof of Proposition 4.B. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be of the form $\varphi(x) = \varphi_1(|x|)$ with $\varphi_1 \in C_0^\infty(\mathbb{R})$; assume $\varphi_1(s) = 1$ for $s \le 1$. Let $\varphi_R(x) = \varphi(x/R)$. Note that

$$\begin{aligned} 2i \ \text{Im} \ \langle u, f\rangle &= \lim_{R\to\infty} [\langle \varphi_R u, (P-\lambda)u\rangle - \langle (P-\lambda)u, \varphi_R u\rangle] \\ &= \lim_{R\to\infty} \langle [P-\lambda, \varphi_R]u, u\rangle. \end{aligned} \tag{4.4}$$

Now

$$[P, \varphi_R] = -\frac{i}{R}\sum_{j=1}^n (\nabla_x \varphi)(\frac{x}{R}) \cdot (\nabla_\xi P)(x, D) + R^{-2} Q_R(x, D)$$

and the coefficients of $Q_R(x, D)$ are uniformly bounded in x and R. Thus

$$\lim_{R\to\infty} R^{-2} |\langle Q_R(x, D)u, u\rangle| = 0 \text{ if } D^\alpha u \in B^* \text{ for } |\alpha| \le m.$$

Also $\nabla_\xi P(x, \xi) = \nabla_\xi P_0(\xi) + \nabla_\xi V(x, \xi)$, so our hypotheses on V give

$$\lim_{R\to\infty} |R^{-1} \langle \nabla\varphi(x/R) \cdot (\nabla_\xi V)(x, D)u, u\rangle| = 0.$$

Thus

$$
(4.5)\qquad \begin{aligned} 2i \text{ Im } \langle u, f\rangle &= \lim_{R\to\infty} \langle [P_0, \varphi_R]u, u\rangle \\ &= -i \lim_{R\to\infty} R^{-1}\langle (\nabla_x\varphi)(x/R)\cdot(\nabla_\xi P_0)(x,D)u, u\rangle. \end{aligned}
$$

Now let

$$
(4.6)\qquad Q_\pm(x,\xi) = \frac{x}{|x|}\cdot\nabla_\xi P_0(\xi) \mp |\psi(\xi)|^2.
$$

Thus (4.5) becomes

$$
(4.7)\qquad \begin{aligned} -2 \text{ Im } \langle u, f\rangle = \lim_{R\to\infty} R^{-1}\langle \varphi_1'(\frac{|x|}{R})Q_\pm(x,D)u, u\rangle \\ \mp \lim_{R\to\infty} R^{-1}\langle \varphi_1'(\frac{|x|}{R})\psi(D)^2u, u\rangle. \end{aligned}
$$

On the other hand, we see that

$$
(4.8)\qquad Q_\pm\Big(\pm\frac{\nabla P_0(\xi)}{|\nabla P_0(\xi)|}, \xi\Big) = \pm|P_0(\xi)| \mp |\psi(\xi)|^2 = 0 \text{ on } \mathcal{M}_\lambda.
$$

Thus $Q_\pm(x,\xi)$ vanishes, by hypothesis, on $B^{o*}/RS(D^\alpha u)$, $|\alpha|\le m$. Consequently, Theorem 2.J implies $Q_\pm(x,D)u \in B^{o*}$, so (4.7) becomes

$$
(4.9)\qquad 2 \text{ Im } \langle u, f\rangle = \pm \lim_{R\to\infty} R^{-1}\langle \varphi_1'(\frac{|x|}{R})\psi(D)^2u, u\rangle.
$$

If we take a sequence of $\varphi_{1j}(\xi)$ approaching in the limit $\varphi_0(s) = 1-s$ for $0\le s\le 1$, 0 for $s\ge 1$, then in the limit (4.9) becomes (4.3), and the proof is complete.

5. Proof of the main theorem

Let $\lambda \in \sigma(P) \setminus [\sigma_p(P) \cup \Lambda_c(P_0)]$. As stated in the Introduction, our goal is to prove:

Theorem 5.A. *There exist bounded operators*

$$T_\pm : B \longrightarrow B^* \tag{5.1}$$

such that

$$\langle T_\pm f, g\rangle = \lim_{z\to\lambda, \pm Im\ z>0} \langle (z-P)^{-1} f, g\rangle \text{ for all } f, g \in B. \tag{5.2}$$

Denoting $(z-P)^{-1}f$ by $R(z)f = u$, we have established in Section 3 the basic a priori estimate

$$\sum_{|\alpha|\le m} \|D^\alpha u\|_{B^*} \le C\|f\|_B + C\|\psi u\|_{L^2}, \quad z \in \Delta(\lambda, \rho). \tag{5.3}$$

Our next goal is to improve this estimate by omitting the term $\|\psi u\|_{L^2}$. (Obviously, for any fixed $z \notin \mathbb{R}$, one could omit this term.) Thus we try to show that $\|\psi u\|_{L^2} \le C\|f\|_B$. We argue by contradiction. We consider only the case Im $z > 0$. Thus, suppose there exists a sequence $z_j \to \lambda \in [a,\infty) \setminus [\sigma_p(P) \cup \Lambda_c]$, Im $z_j > 0$, such that one can find $f_j \in B$ with

$$\|f_j\|_B \to 0 \text{ and } \|\psi R(z_j) f_j\|_{L^2} = 1. \tag{5.4}$$

The estimate (5.3) then implies, with $u_j = R(z_j)f_j$,

$$\sum_{|\alpha|\le m} \|D^\alpha u_j\|_{B^*} \le C\|f_j\|_B + C\|\psi u_j\|_{L^2} \le C'. \tag{5.5}$$

Thus there is a weak limit, $u_j \to u$ weakly in H^m_{loc}. We also have $D^\alpha u \in B^*$, $|\alpha| \le m$. Note that, by Rellich's theorem, $\psi u_j \to \psi u$ in the L^2 norm. Hence $\|\psi u\|_{L^2} = 1$. In particular, u is not identically zero. On the other hand,

$$(P-\lambda)u = 0. \tag{5.6}$$

Now this element u satisfies the λ-outgoing condition introduced in Section 4. Thus the direct part of Theorem 4.A (which has been completely proved) implies that

$$B^{o*}/RS(D^\alpha u) \subset Y^+_\lambda, \quad |\alpha| \le m. \tag{5.7}$$

Consequently the hypotheses of Proposition 4.B are satisfied and, by (5.6), we have

$$\lim_{R\to\infty} R^{-1} \int\limits_{|x|<R} |\psi(D)u|^2 dx = 0$$

or

$$\psi(D)u \in B^{o*}. \tag{5.8}$$

But this implies $B^{o*}/RS(D^\alpha u) \cap Y_\lambda^+ = \emptyset$, and together with (5.7) this gives

$$D^\alpha u \in B^{o*}, \quad |\alpha| \leq m. \tag{5.9}$$

Now Theorem 3.I gives $u \in L^{2,s}$ for all $s < \infty$. In particular, by (5.6), u must be an eigenfunction of P, contradicting the hypothesis $\lambda \notin \sigma_p(P)$. Thus we have the following sharpening of (5.3):

$$\sum_{|\alpha|\leq m} \|D^\alpha u\|_{B^*} \leq C\|f\|_B, \quad z \in \Delta(\lambda, \rho). \tag{5.10}$$

Since B^* is the dual of B, an immediate consequence is:

Lemma 5.B. *For $f \in B$, as $z \to \lambda \in [a, \infty) \setminus [\sigma_p(P) \cup \Lambda_c]$ with $\pm Im\ z > 0$, then there exists a limit point $u_\pm \in B^*$,*

$$R(z)f \to u_\pm \text{ weakly in } B^*. \tag{5.11}$$

We are now almost through with the proof of Theorem 5.A. It remains only to show that the limits in (5.11) are unique. So suppose $z_j \to \lambda$, Im $z_j > 0$, and suppose there exists another limit point $v \in B^*$. It follows that

$$(P - \lambda)(u_+ - v) = 0.$$

It also follows that $u_+ - v \in B^*$ satisfies the λ-outgoing condition. Hence, by Theorem 4.A and Proposition 4.B, by an argument we have seen before, we conclude that $u_+ - v \in B^{o*}$. Again, Theorem 3.I implies $u_+ - v$ must be an eigenfunction of P unless $u_+ - v = 0$, so since we assume $\lambda \notin \sigma_p(P)$, we conclude $u_+ = v$. The main theorem is now proved. For notational convenience, we set

$$R(\lambda \pm i0) = T_\pm.$$

6. The absence of singular continuous spectrum

Here we deduce from the limiting absorption principle:

Theorem 6.A. *The singular continuous spectrum of P is empty.*

Proof. We want to show that, for any compact interval $[\alpha, \beta]$ in $[a, \infty) \setminus \sigma_p(P)$, the spectrum of P is absolutely continuous. With E_λ denoting the spectral resolution of P, use the formula

$$((E_\beta - E_\alpha)f, f) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_\alpha^\beta ([R(\lambda + i\varepsilon) - R(\lambda - i\varepsilon)]f, f) d\lambda,$$

which holds for any $f \in L^2(\mathbb{R}^n)$. Now suppose $f \in B$. It follows from 5.A Theorem that

$$((E_\beta - E_\alpha)f, f) = \frac{1}{2\pi i} \int_\alpha^\beta ([R(\lambda + i0) - R(\lambda - i0)]f, f) d\lambda.$$

It follows that $(E_\lambda f, f)$ is C^1 on $[\alpha, \beta]$ and

$$(d/d\lambda)(E_\lambda f, f) = \frac{1}{2\pi i}([R(\lambda + i0) - R(\lambda - i0)]f, f), \quad \lambda \in [\alpha, \beta],$$

for any $f \in B$. Now the set of functions $f \in L^2(\mathbb{R}^n)$ for which $(E_\lambda f, f)$ is an absolutely continuous function on $[\alpha, \beta]$ is known to be closed. Since we have shown it is dense, it must be all L^2. The proof is complete.

Remark. This proof is well known for the short range case.