Harmonic Forms on Exterior Domains in $\mathbb{R}^n$

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1. Introduction

Let $\mathcal{O}$ be a smoothly bounded open set in $\mathbb{R}^n$ (it can have several connected components), and let $\Omega = \mathbb{R}^n \setminus \overline{\mathcal{O}}$. We study harmonic k-forms on Ω that belong to $L^p(\Omega)$ and satisfy absolute or relative boundary conditions on $\partial\Omega$. That is to say, we study

$$\begin{aligned} \mathcal{H}^p_A(\Omega, \Lambda^k) &= \{u \in L^p(\Omega, \Lambda^k) : du = 0 = d * u \text{ on } \Omega,\ u \rfloor \nu = 0 \text{ on } \partial\Omega\}, \\ \mathcal{H}^p_R(\Omega, \Lambda^k) &= \{u \in L^p(\Omega, \Lambda^k) : du = 0 = d * u \text{ on } \Omega,\ \nu \wedge u = 0 \text{ on } \partial\Omega\}. \end{aligned} \tag{1.1}$$

Here, ν is the unit conormal to $\partial\Omega$.

Regarding the existence of boundary values, we can take a collar neighborhood $I \times \partial\Omega$ if $\partial\Omega$ in $\overline{\Omega}$, with local coordinates (y, y'), $y = 0$ defining $\partial\Omega$, and write the elliptic system $du = d * u = 0$ on functions with values in $\Lambda^* = \oplus\Lambda^k$ as

$$\frac{\partial u}{\partial y} = K(y, y', D_{y'})u, \tag{1.2}$$

yielding, for $u \in L^p(\Omega)$ satisfying this system, the behavior

$$\frac{\partial u}{\partial y} \in L^p(I, H^{-1,p}(\partial\Omega)), \tag{1.3}$$

hence

$$u \in C(I, H^{-1,p}(\partial\Omega)), \tag{1.4}$$

and consequently $u|_{\partial\Omega} \in H^{-1,p}(\partial\Omega)$. This is not sharp. As shown in [Se], one actually has the Besov space trace result

$$u\big|_{\partial\Omega} \in B^{-1/p}_{p,p}(\partial\Omega). \tag{1.5}$$

The result (1.4) allows one to apply methods of Chapter 10 of [H] to obtain C^∞ regularity results, and estimates, such as

$$\|u\|_{C^1(\overline{\Omega}\cap B_K)} \le C_K \|u\|_{L^p(\Omega\cap B_{2K})}, \tag{1.6}$$

for all $u \in \mathcal{H}^p_B(\Omega, \Lambda^k)$, with $B = A$ or R, where we pick $K \in (0,\infty)$ such that

$$\overline{\mathcal{O}} \subset B_K = \{x \in \mathbb{R}^n : |x| < K\}. \tag{1.7}$$

The spaces $\mathcal{H}^p_B(\Omega, \Lambda^k)$, for $k = 1$ and $n = 2$ or 3, are of interest in the study of Euler equations for incompressible fluids, and results on these spaces have been treated in several papers, notably [HY1]–[HY2]. These papers stimulate one to investigate more general cases of n and k.

In §2 we establish that, for all $n \ge 2$ and $k \in \{1,\dots,n\}$, and all $p \in (1,\infty)$,

$$\dim \mathcal{H}^p_B(\Omega, \Lambda^k) < \infty, \tag{1.8}$$

with $B = A$ or R. These results were established in the papers cited above for $k = 1$ and $n = 2, 3$.

In §3, we specialize to $n = 2, k = 1$, and give a proof that

$$\begin{aligned} \dim \mathcal{H}^p_A(\Omega, \Lambda^1) = \dim \mathcal{H}^p_R(\Omega, \Lambda^1) = L, &\quad \text{if } 2 < p < \infty, \\ L-1, &\quad \text{if } 1 < p \le 2, \end{aligned} \tag{1.9}$$

where L is the number of connected components of $\partial\Omega$. This is one of the main results of [HY2], but the details of the proof given here differ from those of that paper.

One useful tool in the analysis is that

$$du = 0 = d * u \Longrightarrow \Delta u = 0 \text{ on } \Omega, \tag{1.10}$$

where Δ is the Laplace operator, acting on k-forms, which for a k-form u on $\Omega \subset \mathbb{R}^n$ just means the standard Laplace operator $\Delta = \partial_1^2 + \cdots + \partial_n^2$, acting componentwise on u.

2. Finite dimensionality results

As in §1, let B denote either A or R, and assume $p \in (1, \infty)$. Our analysis of $\mathcal{H}^p(\Omega, \Lambda^k)$ starts with the following.

Proposition 2.1. *If $u \in \mathcal{H}^p_B(\Omega, \Lambda^k)$, then*

$$\lim_{|x| \to \infty} |u(x)| = 0. \tag{2.1}$$

To establish this, tile a neighborhood of $\Omega \setminus B_K$ in Ω with cubes Q_ν, of a fixed size, such that, for all ν, $2Q_\nu \subset\subset \Omega$. Then, thanks to (1.10) and local regularity for harmonic functions, we have

$$\begin{aligned} \sum_\nu \|u\|^p_{L^\infty(Q_\nu)} &\le C_1 \sum_\nu \|u\|^p_{L^p(2Q_\nu)} \\ &\le C_2 \|u\|^p_{L^p(\Omega)}, \end{aligned} \tag{2.2}$$

for all $u \in \mathcal{H}^p_B(\Omega, \Lambda^k)$. This implies (2.1).

Another corollary of (2.2), in concert with (1.6), is that

$$\|u\|_{L^\infty(\Omega)} \le C \|u\|_{L^p(\Omega)}, \quad \forall\, u \in \mathcal{H}^p_B(\Omega, \Lambda^k), \tag{2.3}$$

given $p \in (1, \infty)$. This leads to the following.

Proposition 2.2. *If $1 < p < q < \infty$, then*

$$\mathcal{H}^p_B(\Omega, \Lambda^k) \subset \mathcal{H}^q_B(\Omega, \Lambda^k). \tag{2.4}$$

Proposition 2.1 allows us to prove the following uniform estimate.

Proposition 2.3. *Let $\Delta u = 0$ on Ω and assume (2.1) holds. Pick K such that (1.7) holds. Then, if $n \ge 3$, we have, for $|x| \ge K$,*

$$|u(x)| \le \Big(\sup_{|y|=K} |u(y)| \Big) \Big| \frac{x}{K} \Big|^{-(n-2)}. \tag{2.5}$$

If $n = 2$, we have, for $|x| \ge K$,

$$|u(x)| \le C_K \Big(\sup_{|y|=K} |u(y)| \Big) \Big| \frac{x}{K} \Big|^{-1}. \tag{2.6}$$

Proof. It suffices to get such estimates for real valued harmonic functions. Since $|x|^{-(n-2)}$ is harmonic on $\mathbb{R}^n \setminus 0$, the estimate (2.8) follows from the maximum principle. When $n = 2$, we argue as follows. Set

$$v(x) = u(|x|^{-2}x), \quad |x| \le \frac{1}{K},\ x \neq 0. \tag{2.7}$$

Then, by invariance of the class of harmonic functions under conformal maps, $v \in C^\infty(\overline{B}_{1/K} \setminus 0)$ is harmonic and tends to 0 as $x \to 0$. Hence 0 is a removable singularity, and v extends to be harmonic on $B_{1/K}$, with $v(0) = 0$. Hence

$$\begin{aligned} &|v(x)| \le A(v)|x|, \quad \text{for } |x| \le \frac{1}{2K}, \\ &A(v) = \sup_{|x|\le 1/2K} |\nabla v(x)| \le C\Big(\sup_{|y|=1/K} |v(y)|\Big), \end{aligned} \tag{2.8}$$

the last estimate thanks to interior elliptic regularity. This yields (2.6).

In concert with (1.6), Proposition 2.3 yields the following uniform bounds.

Proposition 2.4. *For all* $u \in \mathcal{H}^p_B(\Omega, \Lambda^k),\ x \in \overline{\Omega}$,

$$|u(x)| \le C\|u\|_{L^p(\Omega)}(1 + |x|)^{-\alpha_n}, \tag{2.9}$$

where $\alpha_n = n - 2$ *for* $n \ge 3$, *and* $\alpha_2 = 1$.

Note that

$$\int_{\mathbb{R}^n} (1 + |x|)^{-\alpha_n q}\, dx \le C + C\int_1^\infty r^{-\alpha_n q} r^{n-1}\, dr, \tag{2.10}$$

which is finite if and only if

$$\begin{aligned} q > \frac{n}{\alpha_n} = \ &\frac{n}{n-2}, \quad \text{if } \ n \ge 3, \\ &2 \qquad \text{if } \ n = 2. \end{aligned} \tag{2.11}$$

We hence complement (2.4) with the following.

Proposition 2.5. *In the setting of (2.9),*

$$p > \frac{n}{\alpha_n} \Longrightarrow \mathcal{H}^p_B(\Omega, \Lambda^k) \subset \mathcal{H}^q_B(\Omega, \Lambda^k), \quad \forall\, q > \frac{n}{\alpha_n}. \tag{2.12}$$

Consequently,

$$\mathcal{H}^p_B(\Omega, \Lambda^k) = \mathcal{H}^q_B(\Omega, \Lambda^k), \quad \forall\, p, q \in (n/\alpha_n, \infty). \tag{2.13}$$

We are now amply prepared for the following.

Theorem 2.6. *Given* $p \in (1,\infty),\ n \ge 2,\ k \in \{1,\dots,n\}$*, and* $B = A$ *or* R*,*

$$\dim \mathcal{H}^p_B(\Omega,\Lambda^k) < \infty. \tag{2.14}$$

Proof. By (2.4), it suffices to prove (2.14) for $p > n/\alpha_n$. Fix such p, and suppose $u_\nu \in \mathcal{H}^p_B(\Omega,\Lambda^k)$ satisfies $\|u_\nu\|_{L^p(\Omega)} \le 1$. By Proposition 2.4, we have a uniform estimate

$$|u_\nu(x)| \le C(1+|x|)^{-\alpha_n}, \quad \forall\, \nu. \tag{2.15}$$

Also, using (1.6), applied to a sequence $K_\mu \to \infty$, and using a diagonal argument, we can pass to a subsequence (which we still denote (u_ν)) and an element $u \in \mathcal{H}^p_B(\Omega,\Lambda^k)$, such that

$$\sup_{\Omega\cap B_K} |u - u_\nu| \longrightarrow 0, \quad \forall\, K < \infty. \tag{2.16}$$

In particular,

$$u_\nu(x) \longrightarrow u(x), \quad \forall\, x \in \Omega. \tag{2.17}$$

Then the uniform upper bound (2.15) allows us to apply the Lebesgue dominated convergence theorem, to deduce that

$$\int_\Omega |u_\nu - u|^p\, dx \longrightarrow 0. \tag{2.18}$$

Thus the closed unit ball in $\mathcal{H}^p_B(\Omega,\Lambda^k)$ is compact, so (2.14) holds.

While Propositions 2.4–2.5 are adequate for the proof of Theorem 2.6, it is nevertheless of interest to record the following improvement.

Proposition 2.7. *The results (2.9)–(2.13) hold with*

$$\alpha_n = n - 1. \tag{2.19}$$

Proof. We have this for $n = 2$, so we focus on $n \ge 3$. In this case, given $u \in \mathcal{H}^p_B(\Omega,\Lambda^k)$, arguments above, involving Proposition 2.1, imply that, for $|x| \ge K$, $u(x)$ has a convergent expansion

$$u(x) = A_0|x|^{-(n-2)} + \sum_{\ell\ge 1} A_\ell\Bigl(\frac{x}{|x|}\Bigr)|x|^{-(n-2)-\ell}, \tag{2.20}$$

where

$$A_0 = \sum_{j_1<\cdots<j_k} a_{j_1\cdots j_k}\, dx_{j_1}\wedge\cdots\wedge dx_{j_k}, \tag{2.21}$$

and, for $\ell \geq 1$, $A_\ell(y)$ is a finite linear combination of spherical harmonics, with coefficients in $\Lambda^k\mathbb{R}^n$ (i.e., of a form like (2.21)). It suffices to show that

$$du = d * u = 0 \Longrightarrow A_0 = 0. \tag{2.22}$$

Note that (2.20) says $u(x) = \sum_{\ell\geq 0} u_\ell(x)$, with u_ℓ homogeneous in x of degree $-(n-2)-\ell$. Hence, for each ℓ, du_ℓ and $d * u_\ell$ are homogeneous in x of degree $-(n-2)-\ell-1$. Thus the hypothesis in (2.22) implies $du_\ell = d * u_\ell = 0$ for each ℓ. Now

$$du_0(x) = -\frac{n-2}{|x|^n}\sum_\ell x_\ell\, dx_\ell \wedge A_0, \tag{2.23}$$

so

$$\begin{aligned} du_0 = 0 &\Rightarrow dx_\ell \wedge A_0 = 0, \quad \forall\, \ell \in \{1,\dots,n\} \\ &\Rightarrow k = n. \end{aligned} \tag{2.24}$$

Similarly $d * u_0 = 0 \Rightarrow k = 0$. This proves (2.22), and yields (2.19).

3. Dimension computation for $n = 2$

As advertised in §1, the purpose of this section is to prove the following.

Proposition 3.1. *Let $\mathcal{O} \subset \mathbb{R}^2$ be a smoothly bounded open set, $\Omega = \mathbb{R}^2 \setminus \overline{\mathcal{O}}$. Assume $\partial\Omega$ has L connected components. Then*

$$\begin{aligned} \dim \mathcal{H}^p_A(\Omega, \Lambda^1) = \dim \mathcal{H}^p_R(\Omega, \Lambda^1) = L, &\quad \text{if } 2 < p < \infty, \\ L - 1, &\quad \text{if } 1 < p \le 2. \end{aligned} \tag{3.1}$$

In general (for $\Omega \subset \mathbb{R}^n$), the Hodge $*$ operator provides isomorphisms

$$* : \mathcal{H}^p_A(\Omega, \Lambda^k) \xrightarrow{\approx} \mathcal{H}^p_R(\Omega, \Lambda^{n-k}). \tag{3.2}$$

In particular, when $n = 2$,

$$* : \mathcal{H}^p_A(\Omega, \Lambda^1) \xrightarrow{\approx} \mathcal{H}^p_R(\Omega, \Lambda^1), \tag{3.3}$$

so it suffices to establish (3.1) for $\mathcal{H}^p_R(\Omega, \Lambda^1)$.

We start with a lower bound on $\dim \mathcal{H}^p_R(\Omega, \Lambda^1)$. The following result is useful.

Lemma 3.2. *Let $f \in C^\infty(\overline{\Omega})$ be a real valued harmonic function. Assume f is constant on each connected component Γ_ℓ of $\partial\Omega$:*

$$f\big|_{\Gamma_\ell} = c_\ell. \tag{3.4}$$

Then

$$u = df \Longrightarrow u \wedge \nu = 0 \text{ on } \partial\Omega. \tag{3.5}$$

Hence, for such u,

$$u \in L^p(\Omega) \Longrightarrow u \in \mathcal{H}^p_R(\Omega, \Lambda^1). \tag{3.6}$$

Proof. The result (3.5) is clear. Also $u = df \Rightarrow du = d^2 f = 0$. Finally,

$$u = df \Longrightarrow *d * u = \Delta f, \tag{3.7}$$

so we have (3.6).

To proceed, it is convenient to identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ and bring in the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, and set

$$\overline{\Omega}^* = \overline{\Omega} \cup \{\infty\} \subset \widehat{\mathbb{C}}, \tag{3.8}$$

so $\overline{\Omega}^*$ is a smoothly bounded, compact subset of the compact Riemann surface $\widehat{\mathbb{C}}$, on which we can solve the Dirichlet problem, producing a map

$$\mathrm{PI} : C^\infty(\partial\Omega) \longrightarrow C^\infty(\overline{\Omega}^*). \tag{3.9}$$

Then, given $g \in C^\infty(\partial\Omega)$, $f = \mathrm{PI}\, g$ can be restricted to $\overline{\Omega}$, yielding

$$f \in C^\infty(\overline{\Omega}), \quad \Delta f = 0 \text{ on } \Omega, \quad f\big|_{\partial\Omega} = g, \quad f(z) \to f(\infty) \text{ as } |z| \to \infty. \tag{3.10}$$

Let $\mathcal{C}(\partial\Omega)$ denote the space of real valued functions on $\partial\Omega$ that are constant on each connected component. We have a map

$$\begin{aligned} &\mathfrak{h} : \mathcal{C}(\partial\Omega) \longrightarrow \{u \in C^\infty(\overline{\Omega}, \Lambda^1) : du = d * u = 0 \text{ on } \Omega,\ u \wedge \nu = 0 \text{ on } \partial\Omega\}, \\ &\mathfrak{h} g = d\ \mathrm{PI}\, g. \end{aligned} \tag{3.11}$$

Expanding $\mathrm{PI}\, g(re^{i\theta})$ in a Fourier series in θ, we have, for $|z| \geq K$ (K as in (1.7)),

$$\mathrm{PI}\, g(z) = a_0 + \sum_{\ell \geq 1} (a_\ell \cos \ell\theta + b_\ell \sin \ell\theta) r^{-\ell}, \tag{3.12}$$

with $a_0, a_\ell, b_\ell \in \mathbb{R}$. The constant term a_0 is annihilated by d, and the images of $a_\ell (\cos \ell\theta) r^{-\ell}$ and $b_\ell (\sin \ell\theta) r^{-\ell}$ are homogeneous of degree $-\ell - 1$. We hence have

$$|\mathfrak{h} g(z)| = O(|z|^{-2}), \quad \forall\, g \in \mathcal{C}(\partial\Omega), \tag{3.13}$$

so

$$\mathfrak{h} : \mathcal{C}(\partial\Omega) \longrightarrow \mathcal{H}^p_R(\Omega, \Lambda^1), \quad \forall\, p > 1. \tag{3.14}$$

Now $\mathcal{C}(\partial\Omega)$ is a real vector space of dimension L, and the null space of $\mathfrak{h}$ is the one-dimensional subspace of functions assuming the same constant value on each connected component of $\partial\Omega$. Hence the range if $\mathfrak{h}$ in (3.14) is a real vector space of dimension $L - 1$. We have

$$\dim \mathcal{H}^p_R(\Omega, \Lambda^1) \geq L - 1, \quad \forall\, p > 1. \tag{3.15}$$

We next produce another element of $\mathcal{H}^p_R(\Omega, \Lambda^1)$, when $p \in (2, \infty)$, in the form

$$u_L = d(\log |x| - \mathrm{PI}\, g), \tag{3.16}$$

where we arrange (by translating coordinates if necessary) that $0 \in \mathcal{O}$, and set

$$g(x) = \log |x|, \quad \text{for } \ x \in \partial\Omega, \tag{3.17}$$

so $g \in C^\infty(\partial\Omega)$. We define $\mathrm{PI}\, g$ as in (3.9)–(3.10). Then $\mathrm{PI}\, g$ again has the form (3.12), so $|d\ \mathrm{PI}\, g(x)| = O(|x|^{-2})$. Hence

$$u_L(x) = |x|^{-2} \sum_\ell x_\ell \, dx_\ell + O(|x|^{-2}). \tag{3.18}$$

Note that $\log|x| - \mathrm{PI}\, g$ vanishes on $\partial\Omega$. Hence

$$u_L \in \mathcal{H}^p_R(\Omega, \Lambda^1) \Longleftrightarrow 2 < p < \infty. \tag{3.19}$$

We see by comparing (3.13) with (3.18) that u_L does not belong to the range of $\mathfrak{h}$ in (3.14). Therefore, we have

$$\dim \mathcal{H}^p_R(\Omega, \Lambda^1) \ge L, \quad \forall\, p \in (2, \infty). \tag{3.20}$$

We next seek to account for the jump in dimensions described in (3.1). Pick $q \in (2, \infty)$ and suppose $u \in \mathcal{H}^q_R(\Omega, \Lambda^1)$. Picking K as in (3.12) and expanding $u(re^{i\theta})$ in a Fourier series, we have (parallel to (2.20))

$$u(x) = \sum_{\ell \ge 1} u_\ell = \sum_{\ell \ge 1} (A_\ell \cos \ell\theta + B_\ell \sin \ell\theta) r^{-\ell}, \tag{3.21}$$

with $A_\ell, B_\ell \in \Lambda^1\mathbb{R}^2$. In particular,

$$u_1(x) = |x|^{-2} \sum_{i,j} \alpha_{ij} x_i \, dx_j, \quad \alpha_{ij} \in \mathbb{R}. \tag{3.22}$$

Lemms 3.3. *In the setting of (3.21), with $u \in \mathcal{H}^q_R(\Omega, \Lambda^1)$, there exists $\alpha \in \mathbb{R}$ such that*

$$u_1(x) = \alpha \, d \log|x|. \tag{3.23}$$

Proof. The set of 1-forms given by the right side of (3.22) is a 4-dimensional real vector space (call it $\mathcal{V}$), on which $*$ acts, satisfying $** = -I$. The element u_1 belongs to $\mathcal{V}$, and satisfies the constraint that $du_1 = d * u_1 = 0$. A 2-dimensional linear subspace of $\mathcal{V}$ satisfying this constraint is spanned by

$$\beta_1 = d \log r, \quad \beta_2 = *\beta_1 = d\theta. \tag{3.24}$$

If $\mathcal{V}$ contained a linearly independent β_3 satisfying $d\beta_3 = d * \beta_3 = 0$, then $\beta_4 = *\beta_3$ would also satisfy this condition. The span of $\beta_1, \beta_2, \beta_3, \beta_4$ is invariant under $*$, so it cannot be 3 dimensional; it must be all of $\mathcal{V}$. But $\tilde\beta = |x|^{-2} x_1 \, dx_1 \in \mathcal{V}$, and $d\tilde\beta \ne 0$. It follows that there exist $\alpha, \alpha' \in \mathbb{R}$ such that

$$u_1 = \alpha \, d \log r + \alpha' d\theta. \tag{3.25}$$

It remains to show that $\alpha' = 0$. To see this, note that

$$\int_{\partial\Omega} u = 0, \tag{3.26}$$

so, for each K such that $\overline{\mathcal{O}} \subset\subset B_K$, we have

$$\int_{\partial B_K} u = 0, \tag{3.27}$$

by Stokes' theorem. On the other hand, we see from (3.21) that $\int_{\partial B_K} u_1$ is independent of K, while, for $\ell \geq 2$, $\int_{\partial B_K} u_\ell \to 0$ as $K \to \infty$. This forces

$$\int_{\partial B_K} u_1 = 0, \tag{3.28}$$

which in turn forces $\alpha' = 0$ in (3.25).

This leads to the following result.

Proposition 3.4. *For $\Omega \subset \mathbb{R}^2$ as in Proposition 3.1, and u_L as in (3.16)–(3.18), if $1 < p \leq 2 < q < \infty$, then*

$$\mathcal{H}^q_R(\Omega, \Lambda^1) = \mathcal{H}^p_R(\Omega, \Lambda^1) \oplus \mathrm{Span}(u_L). \tag{3.29}$$

With this in hand, we deduce Proposition 3.1 from the following.

Proposition 3.5. *For $\Omega \subset \mathbb{R}^2$ as in Proposition 3.1,*

$$\dim \mathcal{H}^p_R(\Omega, \Lambda^1) = L - 1, \quad \text{if} \ \ 1 < p \leq 2. \tag{3.30}$$

Proof. Take $u \in \mathcal{H}^p_R(\Omega, \Lambda^1)$. The analysis of (3.21) gives, for $|x| \geq K$,

$$u(x) = \sum_{\ell \geq 2} (A_\ell \cos \ell\theta + B_\ell \sin \ell\theta) r^{-\ell}, \tag{3.31}$$

with $A_\ell, B_\ell \in \Lambda^1\mathbb{R}^2$. Such u defines a 1-form on $\overline{\Omega}^* \setminus \{\infty\} \subset \widehat{\mathbb{C}}$. Note that

$$w = \frac{1}{z} \Longrightarrow dw = -\frac{dz}{z^2} \text{ and } d\overline{w} = -\frac{d\overline{z}}{\overline{z}^2}. \tag{3.32}$$

Thanks to the fact that the Hodge star operator is conformally invariant on 1-forms on a 2-dimensional Riemannian manifold, we see that the pull-back of u to a 1-form on $\{w \in \mathbb{C} : 0 < |w| < 1/K\}$ is a harmonic 1-form, with coefficients that are bounded on $B_{1/K} \setminus \{0\}$. Hence 0 is a removable singularity, so in fact such u defines a harmonic 1-form

$$u \in \mathcal{H}_R(\overline{\Omega}^*, \Lambda^1) = \{u \in C^\infty(\overline{\Omega}^*, \Lambda^1) : du = d * u = 0,\ \nu \wedge u = 0 \text{ on } \partial\Omega\}. \tag{3.33}$$

As shown in [Mor], or [T], Chapter 5, Proposition 9.9, one has

$$\mathcal{H}_R(\overline{\Omega}^*, \Lambda^1) \approx H^1(\overline{\Omega}^*, \partial\Omega), \tag{3.34}$$

the relative singular cohomology group, hence

$$\dim \mathcal{H}_R(\overline{\Omega}^*, \Lambda^1) = L - 1. \tag{3.35}$$

Thus the natural injection

$$\mathcal{H}^p_R(\Omega, \Lambda^1) \longrightarrow \mathcal{H}_R(\overline{\Omega}^*, \Lambda^1) \tag{3.36}$$

described above implies

$$\dim \mathcal{H}^p_R(\Omega, \Lambda^1) \le L - 1. \tag{3.37}$$

In concert with (3.15), we have the asserted conclusion (3.30). In addition, we see that (3.36) is an isomorphism, for $p \in (1, 2]$.

4. The dimension of $\mathcal{H}^p_R(\Omega, \Lambda^1)$, for $n \geq 3$

Our goal in this section is to prove the following.

Proposition 4.1. *Assume $n \geq 3$, and let $\mathcal{O} \subset \mathbb{R}^n$ be a smoothly bounded open set, $\Omega = \mathbb{R}^n \setminus \overline{\mathcal{O}}$. Assume $\partial\Omega$ has L connected components. Then*

$$\begin{aligned} \dim \mathcal{H}^p_R(\Omega, \Lambda^1) = L, &\quad \text{if } \ \beta_n < p < \infty, \\ L - 1, &\quad \text{if } \ 1 < p \leq \beta_n, \end{aligned} \tag{4.1}$$

where

$$\beta_n = \frac{n}{n-1}. \tag{4.2}$$

For $n = 3$, this is one of the main results on [HY1].

To begin our analysis, as in earlier sections we pick $K \in (0, \infty)$ such that (1.7) holds. The proof of Proposition 2.7 implies that, for $u \in \mathcal{H}^p_R(\Omega, \Lambda^1)$, $|x| \geq K$, there is a convergent expansion

$$u(x) = \sum_{\ell \geq 1} u_\ell(x) = \sum_{\ell \geq 1} A_\ell\Bigl(\frac{x}{|x|}\Bigr)|x|^{-n+2-\ell}, \tag{4.3}$$

where each $A_\ell(y)$ is a finite linear combination of spherical harmonics (harmonic polynomials in y, homogeneous of degree ℓ), with coefficients in $\Lambda^1\mathbb{R}^n$. Note that

$$u_\ell \in L^p(\mathbb{R}^n \setminus B_K) \Longleftrightarrow \int_1^\infty r^{-(n+\ell-2)p} r^{n-1}\, dr < \infty, \tag{4.4}$$

hence

$$u_1 \in L^p(\mathbb{R}^n \setminus B_K) \Longleftrightarrow p > \frac{n}{n-1}, \tag{4.5}$$

while

$$u_\ell \in L^p(\mathbb{R}^n \setminus B_K), \quad \forall\, \ell \geq 2,\ p > 1. \tag{4.6}$$

We now produce a variant of the construction in §2 of a map

$$\mathfrak{h} : \mathcal{C}(\partial\Omega) \longrightarrow \mathcal{H}^p_R(\Omega, \Lambda^1), \tag{4.7}$$

where $\mathcal{C}(\partial\Omega)$ denotes the L-dimensional space of real valued functions on $\partial\Omega$ that are constant on each connected component. This starts with the following variant of (3.9):

$$\text{PI}: C^\infty(\partial\Omega) \longrightarrow \{f \in C^\infty(\overline{\Omega}) : \Delta f = 0,\ |f(x)| \le C(1+|x|)^{-(n-2)}\}, \tag{4.8}$$

whose existence follows from taking into account the function $|x|^{-(n-2)}$, harmonic on $\mathbb{R}^n \setminus \{0\}$, together with the maximum principle. We set

$$\mathfrak{h} g = d\ \text{PI}\, g, \tag{4.9}$$

for $g \in \mathcal{C}(\partial\Omega)$. For all such g,

$$|d\ \text{PI}\, g(x)| \le C(1+|x|)^{-(n-1)}. \tag{4.10}$$

Noting that Lemma 3.2 holds for $\Omega \subset \mathbb{R}^n$, we see that

$$\mathfrak{h} : \mathcal{C}(\partial\Omega) \longrightarrow \mathcal{H}^p_R(\Omega, \Lambda^1), \quad \forall\, p > \frac{n}{n-1}. \tag{4.11}$$

As opposed to the situation for $\mathfrak{h}$ in (3.14), the map $\mathfrak{h}$ in (4.11) is *injective*. We deduce that

$$\dim \mathcal{H}^p_R(\Omega, \Lambda^1) \ge L, \quad \forall\, p > \frac{n}{n-1}. \tag{4.12}$$

Let us also note that, for $g \in C^\infty(\partial\Omega)$, the function $\text{PI}\, g(x)$ has (for $|x| \ge K$) the convergent expansion

$$\text{PI}\, g(x) = \sum_{\ell \ge 0} f_\ell(x) = B_0 |x|^{-(n-2)} + \sum_{\ell \ge 1} B_\ell\Big(\frac{x}{|x|}\Big)|x|^{-(n-2)-\ell}, \tag{4.13}$$

where $B_0 \in \mathbb{R}$ is constant and $B_\ell(y)$ is a linear combination of spherical harmonics, with real coefficients. Compare Proposition 2.7. These coefficients depend linearly on g, for example, $B_0 : C^\infty(\partial\Omega) \longrightarrow \mathbb{R}$, and in particular

$$B_0 : \mathcal{C}(\partial\Omega) \longrightarrow \mathbb{R}. \tag{4.14}$$

It is readily verified (via the maximum principle) that $B_0(1) > 0$, so the null space $\mathcal{N}(B_0)$ in (4.14) satisfies

$$\mathcal{N}(B_0) \subset \mathcal{C}(\partial\Omega), \quad \dim \mathcal{N}(B_0) = L - 1. \tag{4.15}$$

An examination of (4.13) for $g \in \mathcal{N}(B_0)$ shhows that

$$\mathfrak{h} : \mathcal{N}(B_0) \longrightarrow \mathcal{H}^p_R(\Omega, \Lambda^1), \quad \forall\, p > 1. \tag{4.16}$$

Since $\mathfrak{h}$ in (4.11) is injective, so is $\mathfrak{h}$ here. We deduce that

$$\dim \mathcal{H}^p(\Omega, \Lambda^1) \ge L - 1, \quad \forall\, p > 1. \tag{4.17}$$

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