# Nonlinear Elliptic Equations

## Michael E. Taylor

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## Introduction

Methods of the calculus of variations applied to problems in geometry and classical continuum mechanics often lead to elliptic PDE that are not linear. We discuss a number of examples and some of the developments that have arisen to treat such problems.

The simplest nonlinear elliptic problems are the semilinear ones, of the form $Lu = f(x, D^{m-1}u)$, where $L$ is a linear elliptic operator of order $m$ and the nonlinear term $f(x, D^{m-1}u)$ involves derivatives of $u$ of order $\leq m-1$. In §1 we look at semilinear equations of the form

$$\Delta u = f(x, u), \tag{0.1}$$

on a compact, Riemannian manifold $M$, with or without boundary. The Dirichlet problem for (0.1) is solvable provided $\partial_u f(x, u) \geq 0$ if each connected component of $M$ has a nonempty boundary. If $M$ has no boundary, this condition does not always imply the solvability of (0.1), but one can solve this equation if one requires $f(x, u)$ to be positive for $u > a_1$ and negative for $u < a_0$. We use three approaches to (0.1): a variational approach, minimizing a function defined on a certain function space, the "method of continuity," solving a one-parameter family of equations of the type (0.1), and a variant of the method of continuity that involves a Leray-Schauder fixed-point theorem. This fixed-point theorem is established in Appendix B, at the end of this chapter.

A particular example of (0.1) is

$$\Delta u = k(x) - K(x)e^{2u}, \tag{0.2}$$

which arises when one has a 2-manifold with Gauss curvature $k(x)$ and wants to multiply the metric tensor by the conformal factor $e^{2u}$ and obtain $K(x)$ as the Gauss curvature. The condition $\partial_u f(x, u) \geq 0$ implies that $K(x) \leq 0$ in (0.2).

In §2 we study (0.2) on a compact, Riemannian 2-fold without boundary, given $K(x) < 0$. The Gauss-Bonnet formula implies that $\chi(M) < 0$ is a necessary condition for solvability in this case; the main result of §2 is that this is also a sufficient condition. When you take $K \equiv -1$, this establishes the uniformization theorem for compact Riemann surfaces of negative Euler characteristic. When $\chi(M) = 0$, one takes $K = 0$ and (0.2) is linear. The remaining case of this uniformization theorem, $\chi(M) = 2$, is treated in Chapter 10, §9.

The next topic is local solvability of nonlinear elliptic PDE. We establish this via the inverse function theorem for $C^1$-maps on a Banach space. We treat underdetermined as well as determined elliptic equations. We obtain solutions in §3 with a high but finite degree of regularity. In some cases such solutions are actually $C^\infty$. In §4 we establish higher regularity for solutions to elliptic PDE that are already known to have a reasonably high degree of smoothness. This result suffices for applications made in §3, though PDE encountered further on will require much more powerful regularity results.

In §5 we establish the theorem of J. Nash, on isometric imbeddings of compact Riemannian manifolds in Euclidean space, largely following the ingenious simplification of M. Günther [Gu1], allowing one to apply the inverse function theorem for $C^1$-maps on a Banach space. Again, the regularity result of §4 applies, allowing one to obtain a $C^\infty$-isometric imbedding.

In §6 we introduce the venerable problem of describing minimal surfaces. We establish a number of classical results, in particular the solution to the Plateau problem, producing a (generalized) minimal surface, as the image of the unit disc under a harmonic and essentially conformal map, taking the boundary of the disc homeomorphically onto a given simple closed curve.

In §7 we begin to study the quasi-linear elliptic PDE satisfied by a function whose graph is a minimal surface. We use results of §6 to establish some results on the Dirichlet problem for the minimal surface equation, and we note several questions about this Dirichlet problem whose solutions are not simple consequences of the results of §6, such as boundary regularity. These questions serve as guides to the results of boundary problems for quasi-linear elliptic PDE derived in the next three sections.

In §8 we apply the paradifferential operator calculus developed in Chapter 13, §10, to obtain regularity results for nonlinear elliptic boundary problems. We concentrate on second-order PDE (possibly systems) on a compact manifold with boundary $\overline{M}$ and obtain higher regularity for a solution $u$, assumed a priori to belong to $C^{2+r}(\overline{M})$, for some $r > 0$, for a completely nonlinear elliptic PDE, or to $C^{1+r}(\overline{M})$, in the quasi-linear case. To check how much these results accomplish, we recall the minimal surface equation and note a gap between the regularity of a solution needed to apply the main result (Theorem 8.4) and the regularity a solution is known to possess as a consequence of results in §7.

Section 9 is devoted to filling that gap, in the scalar case, by the famous DeGiorgi-Nash-Moser theory. We follow mainly J. Moser [Mo2], together with complementary results of C. B. Morrey on nonhomogeneous equations. Morrey's results use spaces now known as Morrey spaces, which are discussed in Appendix A at the end of this chapter.

With the regularity results of §§8 and 9 under our belt, we resume the study of the Dirichlet problem for quasi-linear elliptic PDE in the scalar case, in §10, with particular attention to the minimal surface equation. We note that the Dirichlet problem for general boundary data is not solvable unless there is a restriction on the domain on which a solution $u$ is sought. This has to do with the fact that the minimal surface equation is not "uniformly elliptic." We give examples of some uniformly elliptic PDE, modeling stretched membranes, for which the Dirichlet problem has a solution for general smooth data, on a general, smooth, bounded domain. We do not treat the most general scalar, second-order, quasi-linear elliptic PDE, though our treatment does include cases of major importance. More material can be found in [GT] and [LU].

In §11 we return to the variational method, introduced in §1, and prove that a variety of functionals

$$I(u) = \int_\Omega F(x, u, \nabla u)\, dV(x) \tag{0.3}$$

possess minima in sets

$$V = \{u \in H^1(\Omega) : u = g \text{ on } \partial\Omega\}. \tag{0.4}$$

The analysis includes cases both of real-valued $u$ and of $u$ taking values in $\mathbb{R}^N$. The latter case gives rise to $N \times N$ elliptic systems, and some regularity results for quasi-linear elliptic systems are established in §12. Sometimes solutions are not smooth everywhere, but we can show that they are smooth on the complement of a closed set $\Sigma \subset \Omega$ of Hausdorff dimension $< n - 2$ ($n = \dim \Omega$). Results of this nature are called "partial regularity" results.

In §13 we establish results on linear elliptic equations in nondivergence form, due to N. Krylov and M. Safonov, which take the place of DeGiorgi-Nash-Moser estimates in the study of certain fully nonlinear equations, done in §14. In §15 we apply this to equations of the Monge-Ampere type.

In §16 we obtain some results for nonlinear elliptic equations for functions of two variables that are stronger than results available for functions of more variables.

One attack on second-order, scalar, nonlinear elliptic PDE that has been very active recently is the "viscosity method." We do not discuss this method here; one can consult the review article [CIL] for material on this.

## 1. A class of semilinear equations

In this section we look at equations of the form

$$\Delta u = f(x, u) \quad \text{on } M, \tag{1.1}$$

where $M$ is a Riemannian manifold, either compact or the interior of a compact manifold $\overline{M}$ with smooth boundary. We first consider the Dirichlet boundary condition

$$u\big|_{\partial M} = g, \tag{1.2}$$

where $\overline{M}$ is connected and has nonempty boundary. We suppose $f \in C^\infty(\overline{M} \times \mathbb{R})$. We will treat (1.1)–(1.2) under the hypothesis that

$$\frac{\partial f}{\partial u} \ge 0. \tag{1.3}$$

Other cases will be considered later in this section. Suppose $F(x, u) = \int_0^u f(x, s)\, ds$, so

$$f(x, u) = \partial_u F(x, u). \tag{1.4}$$

Then (1.3) is the hypothesis that $F(x, u)$ is a convex function of $u$. Let

$$I(u) = \frac{1}{2}\|du\|^2_{L^2(M)} + \int_M F\big(x, u(x)\big)\, dV(x). \tag{1.5}$$

We will see that a solution to (1.1)–(1.2) is a critical point of $I$ on the space of functions $u$ on $M$, equal to $g$ on $\partial M$.

We will make the following *temporary* restriction on $F$:

$$\text{For } |u| \ge K,\ \partial_u f(x, u) = 0, \tag{1.6}$$

so $F(x, u)$ is linear in $u$ for $u \ge K$ and for $u \le -K$. Thus, for some constant $L$,

$$|\partial_u\ F(x, u)| \le L \quad \text{on } \overline{M} \times \mathbb{R}. \tag{1.7}$$

Let

$$V = \{u \in H^1(M) : u = g \text{ on } \partial M\}. \tag{1.8}$$

**Lemma 1.1.** *Under the hypotheses (1.3)–(1.7), we have the following results about the functional $I : V \to \mathbb{R}$:*

$$I \text{ is strictly convex;} \tag{1.9}$$

$$I \text{ is Lipschitz continuous,} \tag{1.10}$$

*with the norm topology on $V$;*

$$I \text{ is bounded below;} \tag{1.11}$$

*and*

$$I(v) \to +\infty, \quad \text{as } \|v\|_{H^1} \to \infty. \tag{1.12}$$

**Proof.** (1.9) is trivial. (1.10) follows from

$$|F(x,u) - F(x,v)| \le L|u-v|, \tag{1.13}$$

which follows from (1.7). The convexity of $F(x,u)$ in $u$ implies

$$F(x,u) \ge -C_0|u| - C_1. \tag{1.14}$$

Hence

$$\begin{aligned} I(u) &\ge \frac{1}{2}\|du\|^2 - C_0\|u\|_{L^1} - C_1' \\ &\ge \frac{1}{4}\|du\|_{L^2}^2 + \frac{1}{2}B\|u\|_{L^2}^2 - C\|u\|_{L^2} - C', \end{aligned} \tag{1.15}$$

since

$$\frac{1}{2}\|du\|_{L^2}^2 \ge B\|u\|_{L^2}^2 - C'', \quad \text{for } u \in V. \tag{1.16}$$

The last line in (1.15) clearly implies (1.11) and (1.12).

**Proposition 1.2.** *Under the hypotheses (1.3)–(1.7), $I(u)$ has a unique minimum on $V$.*

**Proof.** Let $\alpha_0 = \inf_V I(u)$. By (1.11), $\alpha_0$ is finite. Pick $R$ such that $K = V \cap B_R(0) \ne \emptyset$, where $B_R(0)$ is the ball of radius $R$ centered at 0 in $H^1(M)$, and such that $\|u\|_{H^1} \ge R \Rightarrow I(u) \ge \alpha_0 + 1$, which is possible by (1.12). Note that $K$ is a closed, convex, bounded subset of $H^1(M)$. Let

$$K_\varepsilon = \{u \in K : \alpha_0 \le I(u) \le \alpha_0 + \varepsilon\}. \tag{1.17}$$

For each $\varepsilon > 0$, $K_\varepsilon$ is a closed, convex subset of $K$. It follows that $K_\varepsilon$ is weakly closed in $K$, which is weakly compact. Hence

$$\bigcap_{\varepsilon>0} K_\varepsilon = K_0 \neq \emptyset. \tag{1.18}$$

Now $\inf I(u)$ is assumed on $K_0$. By the strict convexity of $I(u)$, $K_0$ consists of a single point.

If $u$ is the unique point in $K_0$ and $v \in C_0^\infty(M)$, then $u + sv \in V$, for all $s \in \mathbb{R}$, and $I(u+sv)$ is a smooth function of $s$ which is minimal at $s = 0$, so

$$0 = \frac{d}{ds}\, I(u+sv)\big|_{s=0} = (-\Delta u, v) + \int_M f\big(x, u(x)\big) v(x)\, dV(x). \tag{1.19}$$

Hence (1.1) holds. We have the following regularity result:

**Proposition 1.3.** *For $k = 1, 2, 3, \dots$, if $g \in H^{k+1/2}(\partial M)$, then any solution $u \in V$ to (1.1)–(1.2) belongs to $H^{k+1}(M)$. Hence, if $g \in C^\infty(\partial M)$, then $u \in C^\infty(\overline{M})$.*

**Proof.** We start with $u \in H^1(M)$. Then the right side of (1.1) belongs to $H^1(M)$ if $f(x,u)$ satisfies (1.6). This gives $u \in H^2(M)$, provided $g \in H^{3/2}(\partial M)$. Additional regularity follows inductively.

We have uniqueness of the element $u \in V$ minimizing $I(u)$, under the hypotheses (1.3)–(1.7). In fact, under the hypothesis (1.3), there is uniqueness of solutions to (1.1)–(1.2) which are sufficiently smooth, as a consequence of the following application of the maximum principle.

**Proposition 1.4.** *Let $u$ and $v \in C^2(M) \cap C(\overline{M})$ satisfy (1.1), with $u = g$ and $v = h$ on $\partial M$. If (1.3) holds, then*

$$\sup_M\, (u - v) \le \sup_{\partial M}\, (g - h) \vee 0, \tag{1.20}$$

*where $a \vee b = \max(a, b)$ and*

$$\sup_M |u - v| \le \sup_{\partial M} |g - h|. \tag{1.21}$$

**Proof.** Let $w = u - v$. Then, by (1.3),

$$\Delta w = \lambda(x) w, \quad w\big|_{\partial M} = g - h, \tag{1.22}$$

where

$$\lambda(x) = \frac{f(x,u) - f(x,v)}{u - v} \ge 0.$$

If $\mathcal{O} = \{x \in M : w(x) \ge 0\}$, then $\Delta w \ge 0$ on $\mathcal{O}$, so the maximum principle applies on $\mathcal{O}$, yielding (1.20). Replacing $w$ by $-w$ gives (1.20) with the roles of $u$ and $v$, and of $g$ and $h$, reversed, and (1.21) follows.

One application will be the following first step toward relaxing the hypothesis (1.6).

**Corollary 1.5.** *Let $f(x,0) = \varphi(x) \in C^\infty(\overline{M})$. Take $g \in C^\infty(\partial M)$, and let $\Phi \in C^\infty(\overline{M})$ be the solution to*

$$\Delta\Phi = \varphi \text{ on } M, \quad \Phi = g \text{ on } \partial M. \tag{1.23}$$

*Then, under the hypothesis (1.3), a solution $u$ to (1.1)–(1.2) satisfies*

$$\sup_M \, u \le \sup_M \, \Phi + \bigl(\sup_M (-\Phi) \vee 0\bigr) \tag{1.24}$$

*and*

$$\sup_M \, |u| \le \sup_M \, 2|\Phi|. \tag{1.25}$$

**Proof.** We have

$$\Delta(u - \Phi) = f(x,u) - f(x,0) = \lambda(x)u, \tag{1.26}$$

with $\lambda(x) = [f(x,u) - f(x,0)]/u \ge 0$. Thus $\Delta(u - \Phi) \ge 0$ on $\mathcal{O} = \{x \in M : u(x) > 0\}$, so

$$\sup_{\mathcal{O}} (u - \Phi) = \sup_{\partial\mathcal{O}} (u - \Phi) \le \sup_M (-\Phi) \vee 0.$$

This gives (1.24). Also $\Delta(\Phi - u) \ge 0$ on $\mathcal{O}^- = \{x \in M : u(x) < 0\}$, so

$$\sup_{\mathcal{O}^-} (\Phi - u) = \sup_{\partial\mathcal{O}^-} (\Phi - u) \le \sup_M \, \Phi \vee 0,$$

which together with (1.24) gives (1.25).

We can now prove the following result on the solvability of (1.1)–(1.2).

**Theorem 1.6.** *Suppose $f(x,u)$ satisfies (1.3). Given $g \in C^\infty(\partial M)$, there is a unique solution $u \in C^\infty(\overline{M})$ to (1.1)–(1.2).*

**Proof.** Let $f_j(x,u)$ be smooth, satisfying

$$f_j(x,u) = f(x,u), \quad \text{for } |u| \le j, \tag{1.27}$$

and be such that (1.3)–(1.7) hold for each $f_j$, with $K = K_j$. We have solutions $u_j \in C^\infty(\overline{M})$ to

$$\Delta u_j = f_j(x,u_j), \quad u_j\big|_{\partial M} = g. \tag{1.28}$$

Now $f_j(x,0) = f(x,0) = \varphi(x)$, independent of $j$, and the estimate (1.25) holds for all $u_j$, so

$$\sup_M |u_j| \le \sup_M 2|\Phi|, \tag{1.29}$$

where $\Phi$ is defined by (1.23). Thus the sequence $(u_j)$ stabilizes for large $j$, and the proof is complete.

We next discuss a geometrical problem that can be solved using the results developed above. A more substantial variant of this problem will be tackled in the next section. The problem we consider here is the following. Let $\overline{M}$ be a connected, compact, two-dimensional manifold, with nonempty boundary. We suppose that we are given a Riemannian metric $g$ on $\overline{M}$, and we desire to construct a conformally related metric whose Gauss curvature $K(x)$ is a given function on $\overline{M}$. As shown in (3.46) of Appendix C, if $k(x)$ is the Gauss curvature of $g$ and if $g' = e^{2u}g$, then the Gauss curvature of $g'$ is given by

$$K(x) = \big(-\Delta u + k(x)\big)e^{-2u}, \tag{1.30}$$

where $\Delta$ is the Laplace operator for the metric $g$. Thus we want to solve the PDE

$$\Delta u = k(x) - K(x)e^{2u} = f(x,u), \tag{1.31}$$

for $u$. This is of the form (1.1). The hypothesis (1.3) is satisfied provided $K(x) \le 0$. Thus Theorem 1.6 yields the following.

**Proposition 1.7.** *If $\overline{M}$ is a connected, compact 2-manifold with nonempty boundary $\partial M$, $g$ a Riemannian metric on $\overline{M}$, and $K \in C^\infty(\overline{M})$ a given function satisfying*

$$K(x) \le 0 \text{ on } M, \tag{1.32}$$

*then there exists $u \in C^\infty(\overline{M})$ such that the metric $g' = e^{2u}g$ conformal to $g$ has Gauss curvature $K$. Given any $v \in C^\infty(\partial M)$, there is a unique such $u$ satisfying $u = v$ on $\partial M$.*

Results of this section do not apply if $K(x)$ is allowed to be positive somewhere; we refer to [KaW] and [Kaz] for results that do apply in that case.

If one desires to make $(M,g)$ conformally equivalent to a flat metric, that is, one with $K(x) = 0$, then (1.31) becomes the linear equation

$$\Delta u = k(x). \tag{1.33}$$

This can be solved whenever $M$ is connected with nonempty boundary, with $u$ prescribed on $\partial M$. As shown in Proposition 3.1 of Appendix C, when the curvature vanishes, one can choose local coordinates so that the metric tensor becomes $\delta_{jk}$. This could provide an alternative proof of the existence of local isothermal coordinates, which is established by a different argument in Chapter 5, §10. However, the following logical wrinkle should be pointed out. The derivation of the formula (1.30) in §3 of Appendix C made use of a reduction to the case $g_{jk} = e^{2v}\delta_{jk}$ and therefore relied on the existence of local isothermal coordinates. Now, one could grind out a direct proof of (1.30) without using this reduction, thus smoothing out this wrinkle.

We next tackle the equation (1.1) when $M$ is compact, without boundary. For now, we retain the hypothesis (1.3), $\partial f/\partial u \geq 0$. Without a boundary for $M$, we have a hard time bounding $u$, since (1.16) fails for constant functions on $M$. In fact, the equation (1.31) cannot be solved when $K(x) = -1$, $k(x) = 1$, and $M = S^2$, so some further hypotheses are necessary. We will make the following hypothesis: For some $a_j \in \mathbb{R}$,

$$u < a_0 \Rightarrow f(x,u) < 0, \quad u > a_1 \Rightarrow f(x,u) > 0. \tag{1.34}$$

If $\partial f/\partial u > 0$, this is equivalent to the existence of a function $u = \varphi(x)$ such that $f\big(x, \varphi(x)\big) = 0$. We see how this hypothesis controls the size of a solution.

**Proposition 1.8.** *If $u$ solves (1.1) and $M$ is compact, then*

$$a_0 \leq u(x) \leq a_1, \tag{1.35}$$

*provided (1.34) holds.*

**Proof.** If $u$ is maximal at $x_0$, then $\Delta u(x_0) \leq 0$, so $f\big(x_0, u(x_0)\big) \leq 0$, and so (1.34) implies $u \leq a_1$. The other inequality in (1.35) follows similarly.

To get an existence result out of this estimate, we use a technique known as the method of continuity. We show that, for each $\tau \in [0,1]$, there is a smooth solution to

$$\Delta u = (1-\tau)(u-b) + \tau f(x,u) = f_\tau(x,u), \tag{1.36}$$

where we pick $b = (a_0+a_1)/2$. Clearly, this equation is solvable when $\tau = 0$. Let $J$ be the largest interval in $[0,1]$, containing 0, with the property that (1.36) is solvable for all $\tau \in J$. We wish to show that $J = [0,1]$. First note that, for any $\tau \in [0,1]$,

$$u < a_0 \Rightarrow f_\tau(x,u) < 0, \quad u > a_1 \Rightarrow f_\tau(x,u) > 0, \tag{1.37}$$

so any solution $u = u_\tau$ to (1.36) must satisfy

$$a_0 \leq u_\tau(x) \leq a_1. \tag{1.38}$$

Using this, we can show that $J$ is *closed* in $[0,1]$. In fact, let $u_j = u_{\tau_j}$ solve (1.36) for $\tau_j \in J$, $\tau_j \nearrow \sigma$. We have $\|u_j\|_{L^\infty} \leq a < \infty$ by (1.38), so

$g_j(x) = f_{\tau_j}\big(x, u_j(x)\big)$ is bounded in $C(M)$. Thus elliptic regularity for the Laplace operator yields

$$\|u_j\|_{C^r(M)} \le b_r < \infty, \tag{1.39}$$

for any $r < 2$. This yields a $C^r$-bound for $g_j$, and hence (1.39) holds for any $r < 4$. Iterating, we get $u_j$ bounded in $C^\infty(M)$. Any limit point $u \in C^\infty(M)$ solves (1.36) with $\tau = \sigma$, so $J$ is closed.

We next show that $J$ is *open* in $[0,1]$. That is, if $\tau_0 \in J$, $\tau_0 < 1$, then, for some $\varepsilon > 0$, $[\tau_0, \tau_0 + \varepsilon) \subset J$. To do this, fix $k$ large and define

$$\Psi : [0,1] \times H^k(M) \longrightarrow H^{k-2}(M), \quad \Psi(\tau, u) = \Delta u - f_\tau(x, u). \tag{1.40}$$

This map is $C^1$, and its derivative with respect to the second argument is

$$D_2\Psi(\tau_0, u)v = Lv, \tag{1.41}$$

where

$$L : H^k(M) \longrightarrow H^{k-2}(M)$$

is given by

$$Lv = \Delta v - A(x)v, \quad A(x) = 1 - \tau_0 + \tau_0\, \partial_u f(x, u). \tag{1.42}$$

Now, if $f$ satisfies (1.3), then $A(x) \ge 1 - \tau_0$, which is $> 0$ if $\tau_0 < 1$. Thus $L$ is an invertible operator. The inverse function theorem implies that $\Psi(\tau, u) = 0$ is solvable for $|\tau - \tau_0| < \varepsilon$. We thus have the following:

**Proposition 1.9.** *If $M$ is a compact manifold without boundary and if $f(x,u)$ satisfies the conditions (1.3) and (1.34), then the PDE (1.1) has a smooth solution. If (1.3) is strengthened to $\partial_u f(x,u) > 0$, then the solution is unique.*

The only point left to establish is uniqueness. If $u$ and $v$ are two solutions, then, as in (1.22), we have for $w = u - v$ the equation

$$\Delta w = \lambda(x)w, \quad \lambda(x) = \big[f(x,u) - f(x,v)\big]/(u - v) \ge 0.$$

Thus

$$-\|\nabla w\|_{L^2}^2 = \int \lambda(x)|w(x)|^2\, dV,$$

which implies $w = 0$ if $\lambda(x) > 0$ on $M$.

Note that if we only have $\lambda(x) \ge 0$, then $w$ must be constant (if $M$ is connected), and that constant must be 0 if $\lambda(x) > 0$ on an open subset of $M$, so cases of nonuniqueness are rather restricted, under the hypotheses of Proposition 1.9. The reader can formulate further uniqueness results.

It is possible to obtain solutions to (1.1) without the hypothesis (1.3) if we retain the hypothesis (1.34). To do this, first alter $f(x,u)$ on $u \le a_0$ and on $u \ge a_1$ to a smooth $g(x,u)$ satisfying $g(x,u) = -\kappa_0 < 0$ for $u \le a_0 - \delta$

and $g(x, u) = \kappa_1 > 0$ for $u \geq a_1 + \delta$, where $\delta$ is some positive number. We want to show that, for each $\tau \in [0, 1]$, the equation

$$\Delta u = (1 - \tau)(u - b) + \tau g(x, u) = g_\tau(x, u) \tag{1.43}$$

is solvable, with solution satisfying (1.38). Convert (1.43) to

$$u = (\Delta - 1)^{-1}\big(g_\tau(x, u) - u\big) = \Phi_\tau(u). \tag{1.44}$$

Now each $\Phi_\tau$ is a continuous and compact map on the Banach space $C(M)$:

$$\Phi_\tau : C(M) \longrightarrow C(M), \tag{1.45}$$

with continuous dependence on $\tau$. For solvability we can use the Leray-Schauder fixed-point theorem, proved in Appendix B at the end of this chapter. Note that any solution to (1.44) is also a solution to (1.43) and hence satisfies (1.38). In particular,

$$u = \Phi_\tau(u) \Longrightarrow \|u\|_{C(M)} \leq A = \max\big(|a_0|, |a_1|\big). \tag{1.46}$$

Since $\Phi_0(u) = -(\Delta - 1)^{-1} b = b$, which is independent of $u$, it follows from Theorem B.5 that (1.44) is solvable for all $\tau \in [0, 1]$. We have the following improvement of Proposition 1.9.

**Theorem 1.10.** *If $M$ is a compact manifold without boundary and if the function $f(x, u)$ satisfies the condition (1.34), then the equation (1.1) has a smooth solution, satisfying $a_0 \leq u(x) \leq a_1$.*

The equation (1.31) for the conformal factor needed to adjust the curvature of a 2-manifold to a desired $K(x)$ satisfies the hypotheses of Theorem 1.10 (even those of Proposition 1.9) in the special case when $k(x) < 0$ and $K(x) < 0$, yielding a special case of a result to be proved in §2, where the assumption that $k(x) < 0$ is replaced by $\chi(M) < 0$. In some cases, Theorem 1.10 also applies to equations for such conformal factors in higher dimensions. When dim $M = n \geq 3$, we alter the metric by

$$g' = u^{4/(n-2)} g. \tag{1.47}$$

The scalar curvatures $\sigma$ and $S$ of the metrics $g$ and $g'$ are then related by

$$S = u^{-\alpha}(\sigma u - \gamma \Delta u), \quad \gamma = 4\frac{n-1}{n-2}, \quad \alpha = \frac{n+2}{n-2}, \tag{1.48}$$

where $\Delta$ is the Laplacian for the metric $g$. Hence, obtaining the scalar curvature $S$ for $g'$ is equivalent to solving

$$\gamma \Delta u = \sigma(x) u - S(x) u^\alpha, \tag{1.49}$$

for a smooth positive function $u$. Note that $\alpha > 1$ and $\gamma > 1$. For $n = 3$, we have $\gamma = 8$ and $\alpha = 5$.

Note that (1.34) holds, for some $a_j$ satisfying $0 < a_0 < a_1 < \infty$, provided both $\sigma(x)$ and $S(x)$ are negative on $M$. Thus we have the next result:

**Proposition 1.11.** *Let $M$ be a compact manifold of dimension $n \geq 2$. Let $g$ be a Riemannian metric on $M$ with scalar curvature $\sigma$. If both $\sigma$ and $S$ are negative functions in $C^\infty(M)$, then there exists a conformally equivalent metric $g'$ on $M$ with scalar curvature $S$.*

An important special case of Proposition 1.11 is that if $M$ has a metric with negative scalar curvature, then that metric can be conformally altered to one with *constant* negative scalar curvature. There is a very significant generalization of this result, first stated by H. Yamabe. Namely, for any compact manifold with a Riemannian metric $g$, there is a conformally equivalent metric with constant scalar curvature. This result, known as the solution to the Yamabe problem, was established by R. Schoen [Sch], following progress by N. Trudinger and T. Aubin.

Note that (1.3) also holds in the setting of Proposition 1.11; thus to prove this latter result, we could appeal as well to Proposition 1.9 as to Theorem 1.10. Here is a generalization of (1.49) to which Theorem 1.10 applies in some cases where Proposition 1.9 does not:

$$\gamma\Delta u = B(x)u^\beta + \sigma(x)u - A(x)u^\alpha, \quad \beta < 1 < \alpha. \tag{1.50}$$

It is possible that $\beta < 0$. Then we have (1.34), for some $a_j > 0$, and hence the solvability of (1.50), for some positive $u \in C^\infty(M)$, provided $A(x)$ and $B(x)$ are both negative on $M$, for any $\sigma \in C^\infty(M)$. If we assume $A < 0$ on $M$ but only $B \leq 0$ on $M$, we still have (1.34), and hence the solvability of (1.50), provided $\sigma(x) < 0$ on $\{x \in M : B(x) = 0\}$.

An equation of the form (1.50) arises in Chapter 18, in a discussion of results of J. York and N. O'Murchadha, describing permissible first and second fundamental forms for a compact, spacelike hypersurface of a Ricci-flat spacetime, in the case when the mean curvature is a given constant. See (9.28) of Chapter 18.

## Exercises

1. Assume $f(x,u)$ is smooth and satisfies (1.6). Define $F(x,u)$ and $I(u)$ as in (1.4) and (1.5). Show that $I$ has the strict convexity property (1.9) on the space $V$ given by (1.8), as long as

$$\frac{\partial}{\partial u} f(x,u) \geq -\lambda_0, \tag{1.51}$$

   where $\lambda_0$ is the smallest eigenvalue of $-\Delta$ on $M$, with Dirichlet conditions on $\partial M$. Extend Proposition 1.2 to cover this case, and deduce that the Dirichlet problem (1.1)–(1.2) has a unique solution $u \in C^\infty(\overline{M})$, for any $g \in C^\infty(\partial M)$, when $f(x,u)$ satisfies these conditions.
2. Extend Theorem 1.6 to the case where $f(x,u)$ satisfies (1.51) instead of (1.3). (*Hint*: To obtain sup norm estimates, use the variants of the maximum principle indicated in Exercises 5–7 of §2, Chapter 5.)

3. Let $\operatorname{spec}(-\Delta) = \{\lambda_j\}$, where $0 < \lambda_0 < \lambda_1 < \cdots$. Suppose there is a pair $\lambda_j < \lambda_{j+1}$ and $\varepsilon > 0$ such that

$$-\lambda_{j+1} + \varepsilon \le \frac{\partial}{\partial u} f(x,u) \le -\lambda_j - \varepsilon,$$

for all $x, u$. Show that, for $g \in C^\infty(\partial M)$, the boundary problem (1.1)–(1.2) has a unique solution $u \in C^\infty(\overline{M})$.
(*Hint*: With $\mu = (\lambda_j + \lambda_{j+1})/2,\ u = v + g,\ g \in C^\infty(\overline{M})$, rewrite (1.1)–(1.2) as

$$(\Delta + \mu)v = f(x, v+g) + \mu v - G, \quad v\Big|_{\partial M} = 0,$$

where $G = (\Delta + \mu)g$, or

$$v = (\Delta + \mu)^{-1}\Big[f(x, v+g) + \mu v\Big] - g = \Phi(v). \tag{1.52}$$

Apply the contraction mapping principle.)

4. In the context of Exercise 3, this time assume

$$-\lambda_{j+1} + \varepsilon \le \frac{\partial}{\partial u} f(x,u) \le -\lambda_{j-1} - \varepsilon,$$

so $\partial f/\partial u$ might assume the value $-\lambda_j$. Take $\mu = (\lambda_{j-1} + \lambda_{j+1})/2$, let $P_0$ be the orthogonal projection of $L^2(M)$ on the $\lambda_j$ eigenspace of $-\Delta$, and let $P_1 = I - P_0$. Writing

$$u - g = v = P_0 v + P_1 v = v_0 + v_1,$$

convert (1.1)–(1.2) to a system

$$\begin{aligned} v_1 &= (\Delta + \mu)^{-1} P_1 \Big[f(x, v_0 + v_1 + g) + \mu v_1\Big] - P_1 g, \\ v_0 &= (\mu - \lambda_j)^{-1} P_0 \Big[f(x, v_0 + v_1 + g) + \mu v_0\Big] - P_0 g. \end{aligned} \tag{1.53}$$

Given $v_0$, the first equation in (1.53) has a unique solution, $v_1 = \Xi(v_0)$, by the argument in Exercise 3. Thus the solvability of (1.1)–(1.2) is converted to the solvability of

$$v_0 = (\mu - \lambda_j)^{-1} P_0 \Big[f\Big(x, v_0 + \Xi(v_0) + g\Big) + \mu v_0\Big] - P_0 g = \Psi(v_0). \tag{1.54}$$

Here, $\Psi$ is a nonlinear operator on a finite-dimensional space. (Essentially, on the real line if $\lambda_j$ is a simple eigenvalue of $-\Delta$.) Examine various cases, where there will or will not be solutions, perhaps more than one in number.

5. Given a Riemanian manifold $M$ of dimension $n \ge 3$, with metric $g$ and Laplace operator $\Delta$, define the "conformal Laplacian" on functions:

$$Lf = \Delta f - \gamma_n^{-1}\sigma(x) f, \quad \gamma_n^{-1} = \frac{n-2}{4(n-1)}, \tag{1.55}$$

where $\sigma(x)$ is the scalar curvature of $(M, g)$. If $g' = u^{4/(n-2)} g$ as in (1.47), and $(M, g')$ has scalar curvature $S(x)$, set

$$\widetilde{L} f = \widetilde{\Delta} f - \gamma_n^{-1} S(x) f, \tag{1.56}$$

where $\widetilde{\Delta}$ is the Laplace operator for the metric $g'$. Show that

$$L(uf) = u^{4/(n-2)} u \widetilde{L} f. \tag{1.57}$$

(*Hint*: First show that $\Delta(uf) - uu^{4/(n-2)}\widetilde{\Delta} f = (\Delta u) f$. Then use the identity (1.49).)

6. Assume $M$ is compact and connected. Let $\lambda_0$ be the smallest eigenvalue of $-L = -\Delta + \gamma_n^{-1}\sigma(x)$. A $\lambda_0$-eigenfunction $v$ of $L$ is nowhere vanishing (by Proposition 2.9 of Chapter 8). Assume $v(x) > 0$ on $M$. Form the new metric $\widetilde{g} = v^{4/(n-2)} g$. Show that the scalar curvature $\widetilde{S}$ of $(M, \widetilde{g})$ is given by

$$\widetilde{S}(x) = \lambda_0 v^{-4/(n-2)}, \tag{1.58}$$

which is positive everywhere if $\lambda_0 > 0$, negative everywhere if $\lambda_0 < 0$, and zero if $\lambda_0 = 0$.

7. Establish existence for an $\ell \times \ell$ system

$$\Delta u = f(x, u),$$

where $M$ is a compact Riemannian manifold and $f : M \times \mathbb{R}^\ell \to \mathbb{R}^\ell$ satisfies the condition that, for some $A < \infty$,

$$|u| \ge A \Longrightarrow f(x, u) \cdot u > 0.$$

(*Hint*: Replace $f$ by $\tau f$, and let $0 \le \tau \le 1$. Show that any solution to such a system satisfies $|u(x)| < A$.)

8. Let $\overline{\Omega}$ be a compact, connected Riemannian manifold with nonempty boundary. Consider

$$\Delta u + f(x, u) = 0, \quad u\Big|_{\partial\Omega} = g, \tag{1.59}$$

for some real-valued $u$; assume $f \in C^\infty(\overline{\Omega} \times \mathbb{R}),\ g \in C^\infty(\partial\Omega)$. Assume there is an *upper solution* $\overline{u}$ and a *lower solution* $\underline{u}$, in $C^2(\Omega) \cap C(\overline{\Omega})$, satisfying

$$\Delta \overline{u} + f(x, \overline{u}) \le 0, \quad \overline{u}\Big|_{\partial\Omega} \ge g,$$
$$\Delta \underline{u} + f(x, \underline{u}) \ge 0, \quad \underline{u}\Big|_{\partial\Omega} \le g.$$

Also assume $\underline{u} \le \overline{u}$ on $\overline{\Omega}$.

Under these hypotheses, show that there exists a solution $u \in C^\infty(\overline{\Omega})$ to (1.59), such that $\underline{u} \le u \le \overline{u}$.

*One approach.* Let $K = \{v \in C(\overline{\Omega}) : \underline{u} \le v \le \overline{u}\}$, which is a closed, bounded, convex set in $C(\overline{\Omega})$. Pick $\lambda > 0$ so that $|\partial_u f(x, u)| \le \lambda$, for $\min\ \underline{u} \le u \le \max\ \overline{u}$. Let $\Phi(v) = w$ be the solution to

$$\Delta w - \lambda w = -\lambda v - f(x, v), \quad w\Big|_{\partial\Omega} = g.$$

Show that $\Phi : K \to K$ continuously and that $\Phi(K)$ is relatively compact in $K$. Deduce that $\Phi$ has a fixed point $u \in K$.

*Second approach.* If $u_0 = \underline{u}$ and $u_{j+1} = \Phi(u_j)$, show that

$$\underline{u} = u_0 \le u_1 \le \cdots \le u_j \le \cdots \le \overline{u}$$

and that $u_j \nearrow u$, solving (1.59).

## 2. Surfaces with negative curvature

In this section we examine the possibility of imposing a given Gauss curvature $K(x) < 0$ on a compact surface $M$ without boundary, by conformally altering a given metric $g$, whose Gauss curvature is $k(x)$. As noted in §1, if $g$ and $g'$ are conformally related,

$$g' = e^{2u} g, \tag{2.1}$$

then $K$ and $k$ are related by

$$K(x) = e^{-2u}(-\Delta u + k(x)), \tag{2.2}$$

where $\Delta$ is the Laplace operator for the original metric $g$, so we want to solve the PDE

$$\Delta u = k(x) - K(x)e^{2u}. \tag{2.3}$$

This is not possible if $M$ is diffeomorphic to the sphere $S^2$ or the torus $\mathbb{T}^2$, by virtue of the Gauss-Bonnet formula (proved in §5 of Appendix C):

$$\int_M k \, dV = \int_M K e^{2u} \, dV = 2\pi\chi(M), \tag{2.4}$$

where $dV$ is the area element on $M$, for the original metric $g$, and $\chi(M)$ is the Euler characteristic of $M$. We have

$$\chi(S^2) = 2, \quad \chi(\mathbb{T}^2) = 0. \tag{2.5}$$

For us to be able to arrange that $K < 0$ be the curvature of $M$, it is necessary for $\chi(M)$ to be negative. This is the only obstruction; following [Bgr], we will establish the following.

**Theorem 2.1.** *If $M$ is a compact surface satisfying $\chi(M) < 0$, with given Riemannian metric $g$, then for any negative $K \in C^\infty(M)$, the equation (2.3) has a solution, so $M$ has a metric, conformal to $g$, with Gauss curvature $K(x)$.*

We will produce the solution to (2.3) as an element where the function

$$F(u) = \int_M \Bigl(\frac{1}{2}|du|^2 + k(x)u\Bigr) \, dV \tag{2.6}$$

on the set

$$S = \Bigl\{u \in H^1(M) : \int_M K(x)e^{2u} \, dV = 2\pi\chi(M)\Bigr\} \tag{2.7}$$

achieves a minimum. Note that the Gauss-Bonnet formula is built into (2.7), since a metric $g' = e^{2u}g$ has volume element $e^{2u}dV$. While providing an obstruction to specifying $K(x)$, the Gauss-Bonnet formula also provides an aid in making a prescription of $K(x) < 0$ when it is possible to do so, as we will see below.

**Lemma 2.2.** *The set $S$ is a nonempty $C^1$-submanifold of $H^1(M)$ if $K < 0$ and $\chi(M) < 0$.*

**Proof.** Set

$$\Phi(u) = e^{2u}. \tag{2.8}$$

By Trudinger's inequality,

$$\Phi : H^1(M) \longrightarrow L^p(M), \tag{2.9}$$

for all $p < \infty$. Take $p = 1$. We see that $\Phi$ is differentiable at each $u \in H^1(M)$ and

$$D\Phi(u)v = 2e^{2u}v, \quad D\Phi(u) : H^1(M) \to L^1(M). \tag{2.10}$$

Furthermore,

$$\begin{aligned} \big\|\big(D\Phi(u) - D\Phi(w)\big)v\big\|_{L^1(M)} &\le 2\int_M |v| \cdot |e^{2u} - e^{2w}|\, dV \\ &\le 2\Big(\int |v|^4\, dV\Big)^{1/4}\Big(\int |u-w|^4\, dV\Big)^{1/4}\Big(\int e^{4|u|+4|w|}\, dV\Big)^{1/2} \\ &\le C\|v\|_{H^1} \cdot \|u - w\|_{H^1} \cdot\ \exp\Big[C\big(\|u\|_{H^1} + \|w\|_{H^1}\big)\Big], \end{aligned} \tag{2.11}$$

so the map $\Phi : H^1(M) \to L^1(M)$ is a $C^1$-map. Consequently,

$$J(u) = \int_M K e^{2u}\, dV \Longrightarrow J : H^1(M) \to \mathbb{R} \text{ is a } C^1\text{-map}. \tag{2.12}$$

Furthermore, $DJ(u) = 2K\, e^{2u}$, as an element of $H^{-1}(M) \approx \mathcal{L}(H^1(M), \mathbb{R})$, so $DJ(u) \neq 0$ on $S$. The implicit function theorem then implies that $S$ is a $C^1$-submanifold of $H^1(M)$. If $K < 0$ and $\chi(M) < 0$, it is clear that there is a *constant* function in $S$, so $S \neq \emptyset$.

**Lemma 2.3.** *Suppose $F : S \to \mathbb{R}$, defined by (2.6), assumes a minimum at $u \in S$. Then $u$ solves the PDE (2.3), provided the hypotheses of Theorem 2.1 hold.*

**Proof.** Clearly, $F : S \to \mathbb{R}$ is a $C^1$-map. If $\gamma(s)$ is any $C^1$-curve in $S$ with $\gamma(0) = u,\ \gamma'(0) = v$, we have

$$\begin{aligned} 0 = \frac{d}{ds}\, F(u + sv)\big|_{s=0} &= \int_M \big[(du, dv) + k(x)v\big]\, dV \\ &= \int_M \big(-\Delta u + k(x)\big)v\, dV. \end{aligned} \tag{2.13}$$

The condition that $v$ is tangent to $S$ at $u$ is

$$\int_M Ke^{2(u+sv)}\,dV = 2\pi\chi(M) + O(s^2), \tag{2.14}$$

which is equivalent to

$$\int_M vKe^{2u}\,dV = 0. \tag{2.15}$$

Thus, if $u \in S$ is a minimum for $F$, we have

$$v \in H^1(M),\ \int_M vKe^{2u}\,dV = 0 \Longrightarrow \int_M \big(-\Delta u + k(x)\big)v\,dV = 0,$$

and hence $-\Delta u + k(x)$ is parallel to $Ke^{2u}$ in $H^1(M)$; that is,

$$-\Delta u + k(x) = \beta Ke^{2u}, \tag{2.16}$$

for some constant $\beta$. Integrating and using the Gauss-Bonnet theorem yield $\beta = 1$ if $\chi(M) \neq 0$.

By Trudinger's estimate, the right side of (2.16) belongs to $L^2(M)$, so $u \in H^2(M)$. This implies $e^{2u} \in H^2(M)$, and an easy inductive argument gives $u \in C^\infty(M)$.

Our task is now to show that $F$ has a minimum on $S$, given $K < 0$ and $\chi(M) < 0$. Let us write, for any $u \in H^1(M)$,

$$u = u_0 + \alpha, \tag{2.17}$$

where $\alpha = (\text{Area } M)^{-1}\int_M u\,dV$ is the mean value of $u$, and

$$u_0 \in \overline{H}(M) = \Big\{v \in H^1(M) : \int_M v\,dV = 0\Big\}. \tag{2.18}$$

Then $u$ belongs to $S$ if and only if

$$e^{2\alpha}\int_M Ke^{2u_0}\,dV = 2\pi\chi(M),$$

or equivalently,

$$\alpha = \frac{1}{2}\log\Big[2\pi\chi(M)\Big/\int Ke^{2u_0}\,dV\Big]. \tag{2.19}$$

Thus, for $u \in S$,

$$\begin{aligned} F(u) = &\int_M \Big(\frac{1}{2}|du_0|^2 + ku_0\Big)\,dV \\ &+ \pi\chi(M)\left\{\log 2\pi|\chi(M)| - \log\Big|\int_M Ke^{2u_0}\,dV\Big|\right\}. \end{aligned} \tag{2.20}$$

**Lemma 2.4.** *If* $\chi(M) < 0$ *and* $K < 0$, *then* $\inf_S F(u) = a > -\infty$.

**Proof.** By (2.20), we need to estimate

$$-\chi(M)\ \log\ \Big|\int\limits_M Ke^{2u_0}\ dV\Big|$$

from below. Indeed, granted that $K(x) \le -\delta < 0$,

$$\int Ke^{2u_0}\ dV \le -\delta \int e^{2u_0}\ dV.$$

Since $e^x \ge 1 + x$, we have $\int e^{2u_0}\ dV \ge \int dV + \int 2u_0\ dV =$ area $M$, so

$$\int\limits_M Ke^{2u_0}\ dV \ge -\delta A \quad (A = \text{ Area } M),$$

and hence

$$-\chi(M)\ \log\ \Big|\int\limits_M Ke^{2u_0}\ dV\Big| \ge |\chi(M)|\ \log\ |\delta A| \ge b > -\infty. \tag{2.21}$$

Thus, for $u \in S$,

$$F(u) \ge \int\limits_M \Big(\frac{1}{2}|du_0|^2 + ku_0\Big)\, dV - C_2, \tag{2.22}$$

with $C_2$ independent of $u_0 \in H^1(M)$. Now, since $\|u_0\|_{L^2} \le C\|du_0\|_{L^2}$,

$$\Big|\int\limits_M ku_0\ dV\Big| \le C_3\varepsilon\|du_0\|^2_{L^2} + \frac{C_4}{\varepsilon}, \tag{2.23}$$

with $C_3$ and $C_4$ independent of $\varepsilon$. Taking $\varepsilon = 1/2C_3$, we get $F(u) \ge -C_3C_4 - C_2$, which proves the lemma.

We are now in a position to prove the main existence result.

**Proposition 2.5.** *If* $M$ *and* $K$ *are as in Theorem 2.1, then* $F$ *achieves a minimum at a point* $u \in S$, *which consequently solves (2.3).*

**Proof.** Pick $u_n \in S$ so that $a + 1 \ge F(u_n) \searrow a$. If we use (2.22) and (2.23), with $\varepsilon = 1/4C_3$, we have

$$a + 1 \ge \frac{1}{4}\|du_{n0}\|^2_{L^2} - C_5, \tag{2.24}$$

where $u_{n0} = u_n -$ mean value. But the mean value of $u_n$ is

$$\frac{1}{2}\ \log\ \Big[2\pi\chi(M)\Big/\int\limits_M Ke^{2u_{n0}}\ dV\Big],$$

which is bounded from above by the proof of Lemma 2.4. Hence

$$u_n \text{ is bounded in } H^1(M). \tag{2.25}$$

Passing to a subsequence, we have an element $u \in H^1(M)$ such that

$$u_n \longrightarrow u \quad \text{weakly in } H^1(M). \tag{2.26}$$

By Proposition 4.3 of Chapter 12, $e^{2u_n} \to e^{2u}$ in $L^1(M)$, in norm, so $u \in S$. Now (2.26) implies that $\int_M k(x)u_n\, dV \to \int_M k(x)u\, dV$ and that

$$\int_M |du|^2\, dV \le \liminf_{n\to\infty} \int_M |du_n|^2\, dV, \tag{2.27}$$

so $F(u) \le a = \int_S F(v)$, and the existence proof is completed.

The most important special case of Theorem 2.1 is the case $K = -1$. For any compact surface with $\chi(M) < 0$, given a Riemannian metric $g$, it is conformally equivalent to a metric for which $K = -1$. The universal covering surface

$$\widetilde{M} \longrightarrow M, \tag{2.28}$$

endowed with the lifted metric, also has curvature $-1$. A basic theorem of differential geometry is that any two complete, simply connected Riemannian manifolds, with the same constant curvature (and the same dimension), are isometric. See the exercises for dimension 2. For a proof in general, see [ChE]. One model surface of curvature $-1$ is the *Poincaré disk*,

$$\mathcal{D} = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} = \{z \in \mathbb{C} : |z| < 1\}, \tag{2.29}$$

with metric

$$ds^2 = 4(1 - x^2 - y^2)^{-2}\big(dx^2 + dy^2\big). \tag{2.30}$$

This was discussed in §5 of Chapter 8. Any compact surface $M$ with negative Euler characteristic is conformally equivalent to the quotient of $\mathcal{D}$ by a discrete group $\Gamma$ of isometries. If $M$ is orientable, all the elements of $\Gamma$ preserve orientation.

A group of orientation-preserving isometries of $\mathcal{D}$ is provided by the group $G$ of linear fractional transformations, where

$$T_g z = \frac{az+b}{cz+d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{2.31}$$

for

$$g \in G = \mathrm{SU}(1,1) = \Big\{ \begin{pmatrix} u & v \\ \overline{v} & \overline{u} \end{pmatrix} : u, v \in \mathbb{C},\ |u|^2 - |v|^2 = 1 \Big\}. \tag{2.32}$$

It is easy to see that $G$ acts transitively on $\mathcal{D}$; that is, for any $z_1, z_2 \in \mathcal{D}$, there exists $g \in G$ such that $T_g z_1 = z_2$. We claim $\{T_g : G \in G\}$ exhausts the group of orientation-preserving isometries of $\mathcal{D}$. In fact, let $T$ be such an isometry of $\mathcal{D}$; say $T(0) = z_0$. Pick $g \in G$ such that $T_g z_0 = 0$. Then $T_g \circ T$ is an orientation-preserving isometry of $\mathcal{D}$, fixing 0, and it is easy to deduce that $T_g \circ T$ must be a rotation, which is given by an element of $G$.

Since each element of $G$ defines a holomorphic map of $\mathcal{D}$ to itself, we have the following result, a major chunk of the *uniformization theorem* for compact Riemann surfaces:

**Proposition 2.6.** *If $M$ is a compact Riemann surface, $\chi(M) < 0$, then there is a holomorphic covering map of $M$ by the unit disk $\mathcal{D}$.*

Let us take a brief look at the case $\chi(M) = 0$. We claim that any metric $g$ on such $M$ is conformally equivalent to a *flat* metric $g'$, that is, one for which $K = 0$. Note that the PDE (2.3) is linear in this case; we have

$$\Delta u = k(x). \tag{2.33}$$

This equation can be solved on $M$ if and only if

$$\int_M k(x)\, dV = 0, \tag{2.34}$$

which, by the Gauss-Bonnet formula (2.4) holds precisely when $\chi(M) = 0$. In this case, the universal covering surface $\widetilde{M}$ of $M$ inherits a flat metric, and it must be isometric to Euclidean space. Consequently, in analogy with Proposition 2.6, we have the following:

**Proposition 2.7.** *If $M$ is a compact Riemann surface, $\chi(M) = 0$, then $M$ is holomorphically equivalent to the quotient of $\mathbb{C}$ by a discrete group of translations.*

By the characterization

$$\chi(M) = \dim H^0(M) - \dim H^1(M) + \dim H^2(M) = 2 - \dim H^1(M),$$

if $M$ is a compact, connected Riemann surface, we must have $\chi(M) \le 2$. If $\chi(M) = 2$, it follows from the Riemann-Roch theorem that $M$ is conformally equivalent to the standard sphere $S^2$ (see §9 of Chapter 10). This implies the following.

**Proposition 2.8.** *If $M$ is a compact Riemannian manifold homeomorphic to $S^2$, with Riemannian metric tensor $g$, then $M$ has a metric tensor, conformal to $g$, with Gauss curvature $\equiv 1$.*

In other words, we can solve for $u \in C^\infty(M)$ the equation

$$\Delta u = k(x) - e^{2u}, \tag{2.35}$$

where $k(x)$ is the Gauss curvature of $g$. This result does not follow from Theorem 2.1. A PDE proof, involving a nonlinear parabolic equation, is given by [Chow], following work of [Ham3]. An elliptic PDE proof, under the hypothesis that $M$ has a metric with Gauss curvature $k(x) > 0$, has been given in Chapter 2 of [CK].

We end this section with a direct *linear* PDE proof of the following, which as noted above implies Proposition 2.8. This argument appeared in [MT].

**Proposition 2.9.** *If $M$ is a compact Riemannian manifold homeomorphic to $S^2$, there is a conformal diffeomorphism $F : M \to S^2$ onto the standard Riemann sphere.*

**Proof.** Pick a Riemannian metric on $M$, compatible with its conformal structure. Then pick $p \in M$, and pick $h \in \mathcal{D}'(M)$, supported at $p$, given in local coordinates as a first-order derivative of $\delta_p$ (plus perhaps a multiple of $\delta_p$), such that $\langle 1, h\rangle = 0$. Hence there exists a solution $u \in \mathcal{D}'(M)$ to

$$\Delta u = h. \tag{2.36}$$

Then $u \in C^\infty(M \setminus p)$, and $u$ is harmonic on $M \setminus p$ and has a $\text{dist}(x, p)^{-1}$ type of singularity. Now, if $M$ is homeomorphic to $S^2$, then $M \setminus p$ is simply connected, so $u$ has a single-valued harmonic conjugate on $M \setminus p$, given by $v(x) = \int_q^x *du$, where we pick $q \in M \setminus p$. We see that $v$ also has a $\text{dist}(x, p)^{-1}$ type singularity. Then $f = u + iv$ is holomorphic on $M \setminus p$ and has a simple pole at $p$. From here it is straightforward that $f$ provides a conformal diffeomorphism of $M$ onto the standard Riemann sphere.

Actually, the bulk of [MT] dealt with an attack on the curvature equation (2.3), with $M$ a planar domain and $K \equiv -1$, so the equation is

$$\Delta u = e^{2u} \quad \text{on} \quad \Omega \subset \mathbb{C}. \tag{2.37}$$

Here is one of the main results of [MT].

**Proposition 2.10.** *Assume $\Omega = \mathbb{C} \setminus S$, where $S$ is a closed subset of $\mathbb{C}$ with more than one point. Then there exists a solution to (2.37) on $\Omega$ such that $e^{2u}(dx^2 + dy^2)$ is a complete metric on $\Omega$ with curvature $\equiv -1$.*

As with Proposition 2.6, this has as a corollary the following special case of the general uniformization theorem.

**Corollary 2.11.** *If $\Omega \subset \mathbb{C}$ is as in Proposition 2.10, there exists a holomorphic covering of $\Omega$ by the unit disk $\mathcal{D}$.*

Techniques employed in the proof of Proposition 2.10 include maximal principle arguments and barrier constructions. We refer to [MT] for further details.

## Exercises

1. Let $M$ be a complete, simply connected 2-manifold, with Gauss curvature $K = -1$. Fix $p \in M$, and consider

$$\mathrm{Exp}_p : \mathbb{R}^2 \approx T_pM \longrightarrow M.$$

Show that this is a diffeomorphism.
(*Hint*: The map is onto by completeness. Negative curvature implies no Jacobi fields vanishing at 0 and another point, so $D\ \mathrm{Exp}_p$ is everywhere nonsingular. Use simple connectivity of $M$ to show that $\mathrm{Exp}_p$ must be one-to-one.)

2. For $M$ as in Exercise 1, take geodesic polar coordinates, so the metric is

$$ds^2 = dr^2 + G(r,\theta)\, d\theta^2.$$

Use formula (3.37) of Appendix C, for the Gauss curvature, to deduce that

$$\partial_r^2 \sqrt{G} = \sqrt{G}$$

if $K = -1$. Show that

$$\sqrt{G}(0,\theta) = 0, \quad \partial_r \sqrt{G}(0,\theta) = 1,$$

and deduce that $\sqrt{G}(r,\theta) = \varphi(r)$ is the unique solution to

$$\varphi''(r) - \varphi(r) = 0, \quad \varphi(0) = 0, \quad \varphi'(0) = 1.$$

Deduce that

$$G(r,\theta) = \sinh^2 r.$$

3. Using Exercise 2, deduce that any two complete, simply connected 2-manifolds with Gauss curvature $K = -1$ are isometric. Use (3.37) or (3.41) of Appendix C to show that the Poincaré disk (2.30) has this property.

## 3. Local solvability of nonlinear elliptic equations

We take a look at nonlinear PDE, of the form

$$f(x, D^m u) = g(x), \tag{3.1}$$

where, in the latter argument of $f$,

$$D^m u = \{D^\alpha u : |\alpha| \le m\}. \tag{3.2}$$

We suppose $f(x,\zeta)$ is smooth in its arguments, $x \in \Omega \subset \mathbb{R}^n$, and $\zeta = \{\zeta_\alpha : |\alpha| \le m\}$. The function $u$ might take values in some vector space $\mathbb{R}^k$. Set

$$F(u) = f(x, D^m u), \tag{3.3}$$

so $F : C^\infty(\Omega) \to C^\infty(\Omega)$; $F$ is the nonlinear differential operator. Let $u_0 \in C^m(\Omega)$. We say that the linearization of $F$ at $u_0$ is $DF(u_0)$, which is a linear map from $C^m(\Omega)$ to $C(\Omega)$. (Sometimes less smooth $u_0$ can be considered.) We have

$$DF(u_0)v = \frac{\partial}{\partial s}\, F(u_0 + sv)\big|_{s=0} = \sum_{|\beta|\le m} \frac{\partial f}{\partial \zeta_\beta}(x, D^m u_0)\; D^\beta v, \tag{3.4}$$

so $DF(u_0)$ is itself a linear differential operator of order $m$. We say the operator $F$ is elliptic at $u_0$ if its linearization $DF(u_o)$ is an elliptic, linear differential operator.

An operator of the form (3.3) with

$$f(x, D^m u) = \sum_{|\alpha|=m} a_\alpha(x, D^{m-1}u) D^\alpha u + f_1(x, D^{m-1}u) \tag{3.5}$$

is said to be *quasi-linear*. In that case, the linearization at $u_0$ is

$$DF(u_0) = \sum_{|\alpha|=m} a_\alpha(x, D^{m-1}u_0) D^\alpha v + Lv, \tag{3.6}$$

where $L$ is a linear differential operator of order $m-1$, with coefficients depending on $D^{m-1}u_0$. A nonlinear operator that is not quasi-linear is called *completely nonlinear*. The distinction is made because some aspects of the theory of quasi-linear operators are simpler than the general case.

An example of a completely nonlinear operator is the Monge-Ampere operator

$$F(u) = \det \begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix} = u_{xx}u_{yy} - u_{xy}^2, \tag{3.7}$$

with $(x, y) \in \Omega \subset \mathbb{R}^2$. In this case,

$$\begin{aligned} DF(u)v &= \mathrm{Tr}\left[ \begin{pmatrix} v_{xx} & v_{xy} \\ v_{xy} & v_{yy} \end{pmatrix} \begin{pmatrix} u_{yy} & -u_{xy} \\ -u_{xy} & u_{xx} \end{pmatrix} \right] \\ &= u_{yy}v_{xx} - 2u_{xy}v_{xy} + u_{xx}v_{yy}. \end{aligned} \tag{3.8}$$

Thus the linear operator $DF(u)$ acting on $v$ is elliptic provided the matrix

$$\begin{pmatrix} u_{yy} & -u_{xy} \\ -u_{xy} & u_{xx} \end{pmatrix} \tag{3.9}$$

is either positive-definite or negative-definite. Since, for $u$ real-valued, this is a real symmetric matrix, we see that this condition holds precisely when $F(u) > 0$.

More generally, for $\Omega \subset \mathbb{R}^n$, we consider the Monge-Ampere operator

$$F(u) = \det H(u), \tag{3.7a}$$

where $H(u) = (\partial_j \partial_k u)$ is the Hessian matrix of second-order derivatives. In this case, we have

$$DF(u)v = \mathrm{Tr}\,\big[\mathcal{C}(u)H(v)\big], \tag{3.8a}$$

where $H(v)$ is the Hessian matrix for $v$ and $\mathcal{C}(u)$ is the cofactor matrix of $H(u)$, satisfying

$$H(u)\mathcal{C}(u) = \big[\det H(u)\big] I.$$

In this setting we see that $DF(u)$ is a linear, second-order differential operator that is elliptic provided the matrix $\mathcal{C}(u)$ is either positive-definite or negative-definite, and this holds provided the Hessian matrix $H(u)$ is either positive-definite or negative-definite.

Having introduced the concepts above, we aim to establish the following local solvability result:

**Theorem 3.1.** *Let $g \in C^\infty(\Omega)$, and let $u_1 \in C^\infty(\Omega)$ satisfy*

$$F(u_1) = g(x), \quad \text{at } x = x_0, \tag{3.10}$$

*where $F(u)$ is of the form (3.3). Suppose that $F$ is elliptic at $u_1$. Then, for any $\ell$, there exists $u \in C^\ell(\Omega)$ such that*

$$F(u) = g \tag{3.11}$$

*on a neighborhood of $x_0$.*

We begin with a formal power-series construction to arrange that (3.11) hold to infinite order at $x_0$.

**Lemma 3.2.** *Under the hypotheses of Theorem 3.1, there exists $u_0 \in C^\infty(\Omega)$ such that*

$$F(u_0) - g(x) = O(|x - x_0|^\infty) \tag{3.12}$$

*and*

$$(u_0 - u_1)(x) = O(|x - x_0|^{m+1}). \tag{3.13}$$

**Proof.** Making a change of variable, we can suppose $x_0 = 0$. Denote coordinates near 0 in $\Omega$ by $(x, y) = (x_1, \dots, x_{n-1}, y)$. We write $u_0(x, y)$ as a formal power series in $y$:

$$u_0(x, y) = v_0(x) + v_1(x)y + \cdots + \frac{1}{k!} v_k(x) y^k + \cdots . \tag{3.14}$$

Set

$$v_0(x) = u_1(x, 0), \quad v_1(x) = \partial_y u_1(x, 0), \dots, v_{m-1}(x) = \partial_y^{m-1} u_1(x, 0). \tag{3.15}$$

Now the PDE $F(u) = g$ can be rewritten in the form

$$\frac{\partial^m u}{\partial y^m} = F^\#(x, y, D_x^m u, D_x^{m-1} D_y u, \dots, D_x^1 D_y^{m-1} u). \tag{3.16}$$

Then the equation for $v_m(x)$ becomes

$$v_m(x) = f^{\#}(x, 0, D_x^m v_0(x), \dots, D_x^1 v_{m-1}(x)). \tag{3.17}$$

Now, by (3.10), we have $v_m(0) = \partial_y^m u_1(0, 0)$, so (3.13) is satisfied. Taking $y$-derivatives of (3.16) yields inductively the other coefficients $v_j(x)$, $j \geq m + 1$, and the lemma follows from this construction.

Note that if $F$ is elliptic at $u_1$, then $F$ continues to be elliptic at $u_0$, at least on a neighborhood of $x_0$; shrink $\Omega$ appropriately.

To continue the proof of Theorem 3.1, for $k > m+1+n/2$, we have that

$$F : H^k(\Omega) \longrightarrow H^{k-m}(\Omega) \tag{3.18}$$

is a $C^1$-map. We have

$$\mathcal{L} = DF(u_0) : H^k(\Omega) \longrightarrow H^{k-m}(\Omega). \tag{3.19}$$

Now, $\mathcal{L}$ is an elliptic operator of order $m$. We know from Chapter 5 that the Dirichlet problem is a regular boundary problem for the strongly elliptic operator $\mathcal{L}\mathcal{L}^*$. Furthermore, if $\Omega$ is a sufficiently small neighborhood of $x_0$, the map

$$\mathcal{L}\mathcal{L}^* : H^{k+m}(\Omega) \cap H_0^m(\Omega) \longrightarrow H^{k-m}(\Omega) \tag{3.20}$$

is invertible. Hence the map (3.19) is surjetive, so we can apply the implicit function theorem. For any neighborhood $\mathcal{B}_k$ of $u_0$ in $H^k(\Omega)$, the image of $\mathcal{B}_k$ under the map $F$ contains a neighborhood $\mathcal{C}_k$ of $F(u_0)$ in $H^{k-m}(\Omega)$. Now if (3.12) holds, then any neighborhood of $r(x) = F(u_0)-g$ in $H^{k-m}(\Omega)$ contains functions that vanish on a neighborhood of $x_0$, so any neighborhood $\mathcal{C}_k$ of $F(u_0)$ contains functions equal to $g(x)$ on a neighborhood of $x_0$. This establishes the local solvability asserted in Theorem 3.1.

One would rather obtain a local solution $u \in C^\infty$ than just an $\ell$-fold differentiable solution. This can be achieved by using elliptic regularity results that will be established in the next section.

We now discuss a refinement of Theorem 3.1.

**Proposition 3.3.** *If $u_1, g \in C^\infty(\Omega)$ satisfy the hypotheses of Theorem 3.1 at $x = x_0$, with $F$ elliptic at $u_1$, then, for any $\ell$, there exists $u \in C^\ell(\Omega)$ such that, on a neighborhood of $x_0$,*

$$F(u) = g \tag{3.21}$$

*and, furthermore,*

$$(u - u_1)(x) = O(|x - x_0|^{m+1}). \tag{3.22}$$

In the literature, one frequently sees a result weaker than (3.22). The desirability of having this refinement was pointed out to the author by R. Bryant. As before, results of the next section will give $u \in C^\infty(\Omega)$.

To begin the proof, we invoke Lemma 3.2, as before, obtaining $u_0$. Now, for $k > m + 1 + n/2$, set

$$\begin{aligned} \mathcal{V}_k &= \bigl\{u \in H^k(\Omega) : (u - u_0)(x) = O(|x - x_0|^{m+1})\bigr\}, \\ \mathcal{G}_{k-m} &= \bigl\{h \in H^{k-m}(\Omega) : h(x_0) = g(x_0)\bigr\}. \end{aligned} \tag{3.23}$$

Then

$$F : \mathcal{V}_k \longrightarrow \mathcal{G}_{k-m} \tag{3.24}$$

is a $C^1$-map, and we want to show that $F$ maps a neighborhood of $u_0$ in $\mathcal{V}_k$ onto a neighborhood of $g_0 = F(u_0)$ in $\mathcal{G}_{k-m}$. We will again use the implicit function theorem. We want to show that the linear map

$$\mathcal{L} = DF(u_0) : \mathcal{V}_k^b \longrightarrow \mathcal{G}_{k-m}^b \tag{3.25}$$

is surjective, where

$$\begin{aligned} \mathcal{V}_k^b &= \{v \in H^k(\Omega) : D^\beta v(x_0) = 0 \text{ for } |\beta| \le m\}, \\ \mathcal{G}_{k-m}^b &= \{h \in H^{k-m}(\Omega) : h(x_0) = 0\} \end{aligned} \tag{3.26}$$

are the tangent spaces to $\mathcal{V}_j$ and $\mathcal{G}_{k-m}$, at $u_0$ and $g_0$, respectively.

By the previous argument involving (3.19) and (3.20), we know that, for any given $h \in \mathcal{G}_{k-m}^b$, we can find $v_1 \in H^k(\Omega)$ such that $\mathcal{L}v_1 = h$, perhaps after shrinking $\Omega$. To prove the surjectivity in (3.25), we need to find $v \in H^k(\Omega)$ such that $\mathcal{L}v = 0$ and such that $v - v_1 = O(|x - x_0|^{m+1})$, so that $v_1 - v \in \mathcal{V}_k^b$ and $\mathcal{L}(v_1 - v) = h$. We will actually produce $v \in C^\infty(\Omega)$. To work on this problem, we will find it convenient to use the notion of the $m$-jet $J_0^m(v)$ of a function $v \in C^\infty(\Omega)$, at $x_0$, being the Taylor polynomial of order $m$ for $v$ about $x_0$. Note that

$$J_0^m(v) = J_0^m(v^\#) \Longleftrightarrow (v - v^\#)(x) = O(|x - x_0|^{m+1}), \tag{3.27}$$

given that $v, v^\# \in C^\infty(\Omega)$. The existence of the function $v$ we seek here is guaranteed by the following assertion.

**Lemma 3.4.** *Given an elliptic operator $\mathcal{L}$ of order $m$, as above, let*

$$\mathcal{J} = \{J_0^m(v) : \mathcal{L}v(x_0) = 0\} \tag{3.28}$$

*and*

$$\mathcal{S} = \{J_0^m(v) : v \in C^\infty(\Omega),\ \mathcal{L}v = 0 \text{ on } \Omega\}. \tag{3.29}$$

*Clearly, $\mathcal{S} \subset \mathcal{J}$. If $\Omega$ is a sufficiently small neighborhood of $x_0$, then $\mathcal{S} = \mathcal{J}$.*

**Proof.** This result is a simple special case of our goal, Proposition 3.3; the beginning of the proof here just retraces arguments from the beginning of that proof. Namely, let $v_1 \in C^\infty(\Omega)$ have $m$-jet in $\mathcal{J}$, hence satisfying $\mathcal{L}v_1(x_0) = 0$. Then Lemma 3.2 applies, so there exists $v_0$ such that

$$J_0^m(v_0) = J_0^m(v_1) \text{ and } \mathcal{L}v_0 = O(|x - x_0|^\infty). \tag{3.30}$$

Set $h_0 = \mathcal{L}v_0$. Suppose $\Omega$ is shrunk so far that $\mathcal{L}\mathcal{L}^*$ in (3.20) is an isomorphism. Now, for any $\varepsilon > 0$, there exists $h_1 \in C^\infty(\overline{\Omega})$ such that

$$h_1 = h_0 \text{ near } x_0, \quad \|h_1\|_{H^\ell(\Omega)} < \varepsilon. \tag{3.31}$$

Then the Dirichlet problem

$$\mathcal{L}\mathcal{L}^*\tilde{w} = h_1 \text{ on } \Omega, \quad \tilde{w} \in H_0^m(\Omega)$$

has a unique solution $\tilde{w}$ satisfying estimates

$$\|\tilde{w}\|_{H^{\ell+2m}(\Omega)} \leq C_\ell \|h_1\|_{H^\ell(\Omega)}. \tag{3.32}$$

Fix $\ell > n/2$. By Sobolev's imbedding theorem, $w = \mathcal{L}^*\tilde{w}$ satisfies

$$\|w\|_{C^m(\Omega)} \leq C^\# \|w\|_{H^{\ell+m}(\Omega)}. \tag{3.33}$$

In light of this, we have

$$\|w\|_{C^m(\Omega)} \leq C_\ell^\# \varepsilon, \quad \mathcal{L}w = h_1 \text{ on } \Omega, \tag{3.34}$$

so $v = v_1 - w$ defines an element in $\mathcal{S}$, provided $\Omega$ is shrunk to $\Omega_1$, on which $h_1 = h_0$ in (3.31). Furthermore, $J_0^m(v)$ differs from $J_0^m(v_1)$ by $J_0^m(w)$, which is small (i.e., proportional to $\varepsilon$). Since $\mathcal{S}$ is a linear subspace of the finite-dimensional space $\mathcal{J}$, this approximability yields the identity $\mathcal{S} = \mathcal{J}$ and proves the lemma.

From the lemma, as we have seen, it follows that the map (3.25) is a surjective linear map between two Hilbert spaces, so the implicit function theorem therefore applies to the map $F$ in (3.24). In other words, $F$ maps a neighborhood of $u_0$ in $\mathcal{V}_k$ onto a neighborhood of $g_0 = F(u_0)$ in $\mathcal{G}_{k-m}$. As in the proof of Theorem 3.1, we see that any neighborhood of $r(x) = F(u_0) - g$ in $\mathcal{G}_{k-m}^b$ contains functions that vanish on a neighborhood of $x_0$, so any neighborhood of $F(u_0)$ in $\mathcal{G}_{k-m}$ contains functions equal to $g(x)$ on a neighborhood of $x_0$. This completes the proof of Proposition 3.3.

In some geometrical problems, it is useful to extend the notion of ellipticity. A differential operator of the form (3.3) is said to be *underdetermined elliptic* at $u_0$ provided $DF(u_0)$ has surjective symbol.

**Proposition 3.5.** *If $F(u_1)$ satisfies $F(u_1) = g$ at $x = x_0$, and if $F$ is underdetermined elliptic at $u_1$, then, for any $\ell$, there exists $u \in C^\ell(\Omega)$ such that $F(u) = g$ on a neighborhood of $x_0$ and such that $(u - u_1)(x) = O(|x - x_0|^{m+1})$.*

**Proof.** We produce $u$ in the form $u = u_1 + u_2$, where we want

$$F(u_1 + u_2) = g \text{ near } x_0, \quad u_2(x) = O(|x - x_0|^{m+1}). \tag{3.35}$$

We will find $u_2$ in the form $u_2 = \mathcal{L}^*w$, where $\mathcal{L} = DF(u_1)$. Thus we want to find $w \in C^{\ell+m}(\Omega)$ satisfying

$$\Phi(w) = F(u_1 + \mathcal{L}^*w) = g \text{ near } x_0, \quad w(x) = O(|x - x_0|^{2m+1}). \tag{3.36}$$

Now $\Phi(w)$ is strongly elliptic of order $2m$ at $w_1$ and $\Phi(w_1) = 0$ at $x_0$ if $w_1 = 0$. Thus the existence of $w$ satisfying (3.36) follows from Proposition 3.3, and the proof is finished.

We will apply the local existence theory to establish the following classical local isometric imbedding result.

**Proposition 3.6.** *Let $M$ be a 2-dimensonal Riemannian manifold. If $p_0 \in M$ and the Gauss curvature $K(p_0) > 0$, then there is a neighborhood $\mathcal{O}$ of $p_0$ in $M$ that can be smoothly isometrically imbedded in $\mathbb{R}^3$.*

The proof involves constructing a smooth, real-valued function $u$ on $\mathcal{O}$ such that $du(p_0) = 0$ and such that $g_1 = g - du^2$ is a flat metric on $\mathcal{O}$, where $g$ is the given metric tensor on $M$. Assuming this can be accomplished, then by the fundamental property of curvature (Proposition 3.1 of Appendix C), we can take coordinates $(x, y)$ on $\mathcal{O}$ (after possibly shrinking $\mathcal{O}$) such that $g_1 = dx^2 + dy^2$. Thus $g = dx^2 + dy^2 + du^2$, which implies that $(x, y, u) : \mathcal{O} \to \mathbb{R}^3$ provides the desired local isometric imbedding.

Thus our task is to find such a function $u$. We need a formula for the Gauss curvature $K_1$ of $\mathcal{O}$, with metric tensor $g_1 = g - du^2$. A lengthy but finite computation from the fundamental formulas given in §3 of Appendix C yields

$$(1 - |\nabla u|^2)^2 K_1 = (1 - |\nabla u|^2) K - \det H_g(u). \tag{3.37}$$

Here, $|\nabla u|^2 = g^{jk} u_{;j} u_{;k}$, and $H_g(u)$ is the Hessian of $u$ relative to the Levi-Civita connection of $g$:

$$H_g(u) = \big(u^{;j}{}_{;k}\big). \tag{3.38}$$

This is the tensor field of type (1,1) associated to the tensor field $\nabla^2 u$ of type (0,2), such as defined by (2.3)–(2.4) of Appendix C, or equivalently by (3.27) of Chapter 2. In normal coordinates centered at $p \in M$, we have $H_g(u) = (\partial_j \partial_k u)$, at $p$.

Therefore, $g_1$ is a flat metric if and only if $u$ satisfies the PDE

$$\det H_g(u) = (1 - |\nabla u|^2) K. \tag{3.39}$$

By the sort of analysis done in (3.7)–(3.9), we see that this equation is elliptic, provided $K > 0$ and $|\nabla u| < 1$. Thus Proposition 3.3 applies, to yield a local solution $u \in C^\ell(\mathcal{O})$, for arbitrarily large $\ell$, provided the metric tensor $g$ is smooth. As mentioned above, results of §4 will imply that $u \in C^\infty(\mathcal{O})$.

If $K(p_0) < 0$, then (3.39) will be hyperbolic near $p_0$, and results of Chapter 16 will apply, to produce an analogue of Proposition 3.6 in that case. No matter what the value of $K(p_0)$, if the metric tensor $g$ is real analytic, then the nonlinear Cauchy-Kowalewsky theorem, proved in §4 of

Chapter 16, will apply, yielding in that case a real analytic, local isometric imbedding of $M$ into $\mathbb{R}^3$.

If $M$ is compact (diffeomorphic to $S^2$) and has a metric with $K > 0$ everywhere, then in fact $M$ can be *globally* isometrically imbedded in $\mathbb{R}^3$. This result is established in [Ni2] and [Po]. Of course, it is not true that a given compact Riemannian 2-manifold $M$ can be globally isometrically imbedded in $\mathbb{R}^3$ (for example, if $K < 0$), but it can always be isometrically imbedded in $\mathbb{R}^N$ for sufficiently large $N$. In fact, this is true no matter what the dimension of $M$. This important result of J. Nash will be proved in §5 of this chapter.

## Exercises

1. Given the formula (3.8a) for the linearization of $F(u) = \det H(u)$, show that the symbol of $DF(u)$ is given by

$$\sigma_{DF(u)}(x, \xi) = -\mathcal{C}(u)\xi \cdot \xi. \tag{3.40}$$

2. Let a surface $M \subset \mathbb{R}^3$ be given by $x_3 = u(x_1, x_2)$. Given $K(x_1, x_2)$, to construct $u$ such that the Gauss curvature of $M$ at $(x_1, x_2, u(x_1, x_2))$ is equal to $K(x_1, x_2)$ is to solve

$$\det H(u) = \Bigl(1 + |\nabla u|^2\Bigr)^2 K. \tag{3.41}$$

   See (4.29) of Appendix C. If one is given a smooth $K(x_1, x_2) > 0$, then this PDE is elliptic. Applying Proposition 3.3, what geometrical properties of $M$ can you prescribe at a given point and guarantee a local solution?
3. Verify (3.37). Compare with formula (**) on p. 210 of [Spi], Vol. 5.
4. Show that, in local coordinates on a 2-dimensional Riemannian manifold, the left side of (3.39) is given by

$$\det\Bigl(u^{;j}{}_{;k}\Bigr) = g^{-1} \det(\partial_j \partial_k u) + A^{jk}(x, \nabla u)\, \partial_j \partial_k u + Q(\nabla u, \nabla u),$$

   where $g = \det(g_{jk})$,

$$A^{jk}(x, \nabla u) = \pm g^{jk} \sigma^{j'\ell}{}_{k'} \partial_\ell u,$$

   with "+" if $j = k$, "−" if $j \neq k$, $j'$ and $k'$ the indices complementary to $j$ and $k$, and

$$\sigma^{j\ell}{}_k = \partial_k g^{j\ell} + \Gamma^j{}_{mk} g^{m\ell},$$

   and

$$Q(\nabla u, \nabla u) = \det(\tau^j{}_k), \quad \tau^j{}_k = \sigma^{j\ell}{}_k \partial_\ell u.$$

# 4. Elliptic regularity I (interior estimates)

Here we will discuss two methods of establishing regularity of solutions to nonlinear elliptic PDE. The first is to consider regularity for a linear elliptic differential operator of order $m$

$$A(x, D) = \sum_{|\alpha| \le m} a_\alpha(x) \; D^\alpha, \tag{4.1}$$

whose coefficients have limited regularity. The second method will involve use of paradifferential operators. For both methods, we will make use of the Hölder spaces $C^s(\mathbb{R}^n)$ and Zygmund spaces $C^s_*(\mathbb{R}^n)$, discussed in §8 of Chapter 13. Material in this section largely follows the exposition in [T].

Let us suppose $a_\alpha(x) \in C^s(\mathbb{R}^n), \; s \in (0, \infty) \setminus \mathbb{Z}$. Then $A(x, \xi)$ belongs to the symbol space $C^s_* S^m_{1,0}$, as defined in §9 of Chapter 13. Recall that $p(x, \xi) \in C^s_* S^m_{1,\delta}$, provided

$$|D^\alpha_\xi p(x, \xi)| \le C_\alpha \langle \xi \rangle^{m - |\alpha|} \tag{4.2}$$

and

$$\|D^\alpha_\xi p(\cdot, \xi)\|_{C^s_*(\mathbb{R}^n)} \le C^\alpha \langle \xi \rangle^{m - |\alpha| + \delta s}. \tag{4.3}$$

We would like to establish regularity results for elliptic $A(x, \xi) \in C^s_* S^m_{1,0}$, by pseudodifferential operator techniques. It is not so convenient to work with an operator with symbol $A(x, \xi)^{-1}$. Rather, we will decompose $A(x, \xi)$ into a sum

$$A(x, \xi) = A^\#(x, \xi) + A^b(x, \xi), \tag{4.4}$$

in such a way that a good parametrix can be constructed for $A^\#(x, D)$, while $A^b(x, D)$ is regarded as a remainder term to be estimated. Pick $\delta \in (0, 1)$. As shown in Proposition 9.9 of Chapter 13, any $A(x, \xi) \in C^s_* S^m_{1,0}$ can be written in the form (4.4), with

$$A^\#(x, \xi) \in S^m_{1,\delta}, \quad A^b(x, \xi) \in C^s_* S^{m - \delta s}_{1,\delta}. \tag{4.5}$$

To $A^b(x, D)$ we apply Proposition 9.10 of Chapter 13, which, we recall, states that

$$p(x, \xi) \in C^s_* S^\mu_{1,\delta} \Longrightarrow p(x, D) : C^{\mu + r}_* \longrightarrow C^r_*, \quad -(1 - \delta)s < r < s. \tag{4.6}$$

Consequently,

$$A^b(x, D) : C^{m + r - \delta s}_* \longrightarrow C^r_*, \quad -(1 - \delta)s < r < s. \tag{4.7}$$

Now let $p(x, D) \in OPS^{-m}_{1,\delta}$ be a parametrix for $A^\#(x, D)$, which is elliptic. Hence, mod $C^\infty$,

$$p(x, D) A(x, D) u = u + p(x, D) A^b(x, D) u, \tag{4.8}$$

so if

$$A(x, D) u = f, \tag{4.9}$$

then, mod $C^\infty$,

$$u = p(x, D) f - p(x, D) A^b(x, D) u. \tag{4.10}$$

In view of (4.7), we see that when (4.10) is satisfied,

$$u \in C_*^{m+r-\delta s},\ f \in C_*^r \Longrightarrow u \in C_*^{m+r}. \tag{4.11}$$

We then have the following.

**Proposition 4.1.** *Let $A(x,\xi) \in C_*^s S_{1,0}^m$ be elliptic, and suppose $u$ solves (4.9). Assuming*

$$s > 0, \quad 0 < \delta < 1 \quad \text{and} \quad -(1-\delta)s < r < s, \tag{4.12}$$

*we have*

$$u \in C^{m+r-\delta s},\ f \in C_*^r \Longrightarrow u \in C_*^{m+r}. \tag{4.13}$$

Note that, for $|\alpha| = m$, $D^\alpha u \in C_*^{r-\delta s}$, and $r - \delta s$ could be negative. However, $a_\alpha(x) D^\alpha u$ will still be well defined for $a_\alpha \in C^s$. Indeed, if (4.6) is applied to the special case of a multiplication operator, we have

$$a \in C^s,\ u \in C_*^\sigma \Longrightarrow au \in C_*^\sigma, \quad \text{for } -s < \sigma < s. \tag{4.14}$$

Note that the range of $r$ in (4.12) can be rewritten as $-s < r - \delta s < (1-\delta)s$. If we set $r - \delta s = -s + \varepsilon$, this means $0 < \varepsilon < (2-\delta)s$, so we can rewrite (4.13) as

$$u \in C^{m-s+\varepsilon},\ f \in C_*^r \Longrightarrow u \in C_*^{m+r}, \quad \text{provided } \varepsilon > 0,\ r < s, \tag{4.15}$$

as long as the relation $r = -(1-\delta)s + \varepsilon$ holds. Letting $\delta$ range over $(0,1)$, we see that this will hold for any $r \in (-s+\varepsilon, \varepsilon)$. However, if $r \in [\varepsilon, s)$, we can first obtain from the hypothesis (4.15) that $u \in C_*^{m+\rho}$, for any $\rho < \varepsilon$. This improves the a priori regularity of $u$ by almost $s$ units. This argument can be iterated repeatedly, to yield:

**Theorem 4.2.** *If $A(x,\xi) \in C^s S_{1,0}^m$ is elliptic and $u$ solves (4.9), then (assuming $s > 0$)*

$$\begin{aligned} &u \in C^{m-s+\varepsilon},\ f \in C_*^r \Longrightarrow u \in C_*^{m+r}, \\ &\text{provided } \varepsilon > 0 \text{ and } -s < r < s. \end{aligned} \tag{4.16}$$

We can sharpen this up to obtain the following Schauder regularity result:

**Theorem 4.3.** *Under the hypotheses above,*

$$u \in C^{m-s+\varepsilon},\ f \in C_*^s \Longrightarrow u \in C_*^{m+s}. \tag{4.17}$$

**Proof.** Applying (4.16), we can assume $u \in C_*^{m+r}$ with $s - r > 0$ arbitrarily small. Now if we invoke Proposition 9.7 of Chapter 13, which says

$$p(x,\xi) \in C^r S^m_{1,1} \Longrightarrow p(x,D) : C^{m+r+\varepsilon}_* \longrightarrow C^r_*, \tag{4.18}$$

for all $\varepsilon > 0$, we can supplement (4.7) with

$$A^b(x,D) : C^{m+s-\delta s+\varepsilon}_* \longrightarrow C^s_*, \quad \varepsilon > 0. \tag{4.19}$$

If $\delta > 0$, and if $\varepsilon > 0$ is picked small enough, we can write $m+s-\delta s+\varepsilon = m+r$ with $r < s$, so, under the hypotheses of (4.17), the right side of (4.8) belongs to $C^{m+s}_*$, proving the theorem. We note that a similar argument also produces the regularity result:

$$u \in H^{m-s+\varepsilon,p}, \ f \in C^s_* \Longrightarrow u \in C^{m+s}_*. \tag{4.20}$$

We now apply these results to solutions to the *quasi-linear* elliptic PDE

$$\sum_{|\alpha|\le m} a_\alpha(x, D^{m-1}u)\ D^\alpha u = f. \tag{4.21}$$

As long as $u \in C^{m-1+s}$, $a_\alpha(x, D^{m-1}u) \in C^s$. If also $u \in C^{m-s+\varepsilon}$, we obtain (4.16) and (4.17). If $r > s$, using the conclusion $u \in C^{m+s}_*$, we obtain $a_\alpha(x, D^{m-1}u) \in C^{s+1}$, so we can reapply (4.16) and (4.17) for further regularity, obtaining the following:

**Theorem 4.4.** *If $u$ solves the quasi-linear elliptic PDE (4.21), then*

$$u \in C^{m-1+s} \cap C^{m-s+\varepsilon}, \ f \in C^r_* \Longrightarrow u \in C^{m+r}_*, \tag{4.22}$$

*provided $s > 0$, $\varepsilon > 0$, and $-s < r$. Thus*

$$u \in C^{m-1+s}, \ f \in C^r_* \Longrightarrow u \in C^{m+r}_*, \tag{4.23}$$

*provided*

$$s > \frac{1}{2}, \quad r > s-1. \tag{4.24}$$

We can sharpen Theorem 4.4 a bit as follows. Replace the hypothesis in (4.22) by

$$u \in C^{m-1+s} \cap H^{m-1+\sigma,p}, \tag{4.25}$$

with $p \in (1,\infty)$. Recall that Proposition 9.10 of Chapter 13 gives both (4.6) and, for $p \in (1,\infty)$,

$$\begin{gathered} p(x,\xi) \in C^s_* S^m_{1,\delta} \Longrightarrow p(x,D) : H^{r+m,p} \longrightarrow H^{r,p}, \\ -(1-\delta)s < r < s. \end{gathered} \tag{4.26}$$

Parallel to (4.14), we have

$$a \in C^s, \ u \in H^{\sigma,p} \Longrightarrow au \in H^{\sigma,p}, \quad \text{for } -s < \sigma < s, \tag{4.27}$$

as a consequence of (4.26), so we see that the left side of (4.21) is well defined provided $s+\sigma > 1$. We have (4.8) and, by (4.26),

$$A^b(x,D) : H^{m+r-\delta s,p} \longrightarrow H^{r,p}, \quad \text{for } -(1-\delta)s < r < s, \tag{4.28}$$

parallel to (4.7). Thus, if (4.25) holds, we obtain

$$p(x,D)A^b(x,D)u \in H^{m-1+\sigma+\delta s,p}, \tag{4.29}$$

provided $-(1-\delta)s < \delta s - 1 + \sigma < s$, i.e., provided

$$s+\sigma > 1 \quad \text{and} \quad -1+\sigma+\delta s < s. \tag{4.30}$$

Thus, if $f \in H^{\rho,p}$ with $\rho > \sigma - 1$, we manage to improve the regularity of $u$ over the hypothesized (4.25). One way to record this gain is to use the Sobolev imbedding theorem:

$$H^{m-1+\sigma+\delta s,p} \subset H^{m-1+\sigma,p_1}, \quad p_1 = \frac{pn}{n-\delta s} > p\Bigl(1+\frac{\delta s p}{n}\Bigr). \tag{4.31}$$

If we assume $f \in C^r_*$ with $r > \sigma - 1$, we can iterate this argument sufficiently often to obtain $u \in C^{m-1+\sigma-\varepsilon}$, for arbitrary $\varepsilon > 0$. Now we can arrange $s+\sigma > 1+\varepsilon$, so we are now in a position to apply Theorem 4.4. This proves the following:

**Theorem 4.5.** *If $u$ solves the quasi-linear elliptic PDE* (4.21)*, then*

$$u \in C^{m-1+s} \cap H^{m-1+\sigma,p}, \ f \in C^r_* \Longrightarrow u \in C^{m+r}_*, \tag{4.32}$$

*provided* $1 < p < \infty$ *and*

$$s > 0, \quad s+\sigma > 1, \quad r > \sigma - 1. \tag{4.33}$$

Note that if $u \in H^{m,p}$ for some $p > n$, then $u \in C^{m-1+s}$ for $s = 1-n/p > 0$, and then (4.32) applies, with $\sigma = 1$, or even with $\sigma = n/p + \varepsilon$.

We next obtain a result regarding the regularity of solutions to a completely nonlinear elliptic system

$$F(x, D^m u) = f. \tag{4.34}$$

We could apply Theorems 4.2 and 4.3 to the equation for $u_j = \partial u/\partial x_j$:

$$\sum_{|\alpha|\le m} \frac{\partial F}{\partial \zeta_\alpha}(x, D^m u) D^\alpha u_j = -F_{x_j}(x, D^m u) + \frac{\partial f}{\partial x_j} = f_j. \tag{4.35}$$

Suppose $u \in C^{m+s}$, $s > 0$, so the coefficients $a_\alpha(x) = (\partial F/\partial \zeta_\alpha)(x, D^m u) \in C^s$. If $f \in C^r_*$, then $f_j \in C^s + C^{r-1}_*$. We can apply Theorems 4.2 and 4.3 to $u_j$ provided $u \in C^{m+1-s+\varepsilon}$, to conclude that $u \in C^{m+s+1}_* \cup C^{m+r}_*$. This implication can be iterated as long as $s+1 < r$, until we obtain $u \in C^{m+r}_*$.

This argument has the drawback of requiring too much regularity of $u$, namely that $u \in C^{m+1-s+\varepsilon}$ as well as $u \in C^{m+s}$. We can fix this up by considering difference quotients rather than derivatives $\partial_j u$. Thus, for $y \in \mathbb{R}^n$, $|y|$ small, set

$$v_y(x) = |y|^{-1}\big[u(x+y) - u(x)\big];$$

$v_y$ satisfies the PDE

$$\sum_{|\alpha|\le m} \Phi_{\alpha y}(x) D^\alpha v_y(x) = G_y(x, D^m u), \tag{4.36}$$

where

$$\Phi_{\alpha y}(x) = \int_0^1 (\partial F/\partial \zeta_\alpha)\big(x, tD^m u(x) + (1-t)D^m u(x+y)\big)\, dt \tag{4.37}$$

and $G_y$ is an appropriate analogue of the right side of (4.35). Thus $\Phi_{\alpha y}$ is in $C^s$, uniformly as $|y| \to 0$, if $u \in C^{m+s}$, while this hypothesis also gives a uniform bound on the $C^{m-1+s}$-norm of $v_y$. Now, for each $y$, Theorems 4.2 and 4.3 apply to $v_y$, and one can get an *estimate* on $\|v_y\|_{C^{m+\rho}}, \rho = \min(s, r-1)$, *uniform* as $|y| \to 0$. Therefore, we have the following.

**Theorem 4.6.** *If $u$ solves the elliptic PDE* (4.34), *then*

$$u \in C^{m+s},\ f \in C^r_* \Longrightarrow u \in C^{m+r}_*, \tag{4.38}$$

*provided*

$$0 < s < r. \tag{4.39}$$

We shall now give a second approach to regularity results for nonlinear elliptic PDE, making use of the paradifferential operator calculus developed in §10 of Chapter 13. In addition to giving another perspective on interior estimates, this will also serve as a warm-up for the work on boundary estimates in §8.

If $F$ is smooth in its arguments, then, as shown in (10.53)–(10.55) of Chapter 13,

$$F(x, D^m u) = \sum_{|\alpha|\le m} M_\alpha(x, D) D^\alpha u + F(x, D^m \Psi_0(D) u), \tag{4.40}$$

where $F\big(x, D^m \Psi_0(D) u\big) \in C^\infty$ and

$$M_\alpha(x,\xi) = \sum_k m_k^\alpha(x) \psi_{k+1}(\xi), \tag{4.41}$$

with

$$m_k^\alpha(x) = \int_0^1 \frac{\partial F}{\partial \zeta_\alpha}\big(\Psi_k(D) D^m u + t\psi_{k+1}(D) D^m u\big)\, dt. \tag{4.42}$$

As shown in Proposition 10.7 of Chapter 13, we have, for $r \ge 0$,

$$u \in C^{m+r} \Longrightarrow M_\alpha(x,\xi) \in \mathcal{A}_0^r S^0_{1,1} \subset S^0_{1,1} \cap C^r S^0_{1,0}. \tag{4.43}$$

We recall from (10.31) of Chapter 13 that

$$p(x,\xi) \in \mathcal{A}_0^r S^m_{1,\delta} \Longleftrightarrow \|D_\xi^\alpha p(\cdot,\xi)\|_{C^{r+s}} \le C_{\alpha s}\ \langle\xi\rangle^{m-|\alpha|+\delta s}, \quad s \ge 0. \tag{4.44}$$

Consequently, if we set

$$M(u;x,D) = \sum_{|\alpha|\le m} M_\alpha(x,D) D^\alpha, \tag{4.45}$$

we obtain

**Proposition 4.7.** *If $u \in C^{m+r}$, $r \ge 0$, then*

$$F(x, D^m u) = M(u;x,D)u + R, \tag{4.46}$$

*with $R \in C^\infty$ and*

$$M(u;x,\xi) \in \mathcal{A}_0^r S^m_{1,1} \subset S^m_{1,1} \cap C^r S^m_{1,0}. \tag{4.47}$$

Decomposing each $M_\alpha(x,\xi)$, we have, by (10.60)–(10.61) of Chapter 13,

$$M(u;x,\xi) = M^\#(x,\xi) + M^b(x,\xi), \tag{4.48}$$

with

$$M^\#(x,\xi) \in \mathcal{A}_0^r S^m_{1,\delta} \subset S^m_{1,\delta} \tag{4.49}$$

and

$$M^b(x,\xi) \in C^r S^{m-\delta r}_{1,\delta} \cap \mathcal{A}_0^r S^m_{1,1} \subset S^{m-r\delta}_{1,1}. \tag{4.50}$$

Let us explicitly recall that (4.49) implies

$$D_x^\beta M^\#(x,\xi) \in \begin{array}{ll} S^m_{1,\delta}, & |\beta| \le r, \\ S^{m+\delta(|\beta|-r)}_{1,\delta}, & |\beta| \ge r. \end{array} \tag{4.51}$$

Note that the linearization of $F(x, D^m u)$ at $u$ is given by

$$Lv = \sum_{|\alpha|\le m} \tilde{M}_\alpha(x) D^\alpha v, \tag{4.52}$$

where

$$\tilde{M}_\alpha(x) = \frac{\partial F}{\partial \zeta_\alpha}(x, D^m u). \tag{4.53}$$

Comparison with (4.40)–(4.42) gives (for $u \in C^{m+r}$)

$$M(u;x,\xi) - L(x,\xi) \in C^r S^{m-r}_{1,1}, \tag{4.54}$$

by the same analysis as in the proof of the $\delta = 1$ case of (9.35) of Chapter 13. More generally, the difference in (4.54) belongs to $C^r S^{m-r\delta}_{1,\delta}$, $0 \le \delta \le 1$. Thus $L(x,\xi)$ and $M(u;x,\xi)$ have many qualitative properties in common.

Consequently, given $u \in C^{m+r}$, the operator $M^\#(x,D) \in OPS^m_{1,\delta}$ is microlocally elliptic in any direction $(x_0,\xi_0) \in T^*\mathbb{R}^n \setminus 0$ that is noncharacteristic for $F(x,D^m u)$, which by definition means noncharacteristic for $L$. In particular, $M^\#(x,D)$ is elliptic if $F(x,D^m u)$ is. Now if

$$F(x,D^m u) = f \tag{4.55}$$

is elliptic and $Q \in OPS^{-m}_{1,\delta}$ is a parametrix for $M^\#(x,D)$, we have

$$u = Q(f - M^b(x,D)u), \quad \text{mod } C^\infty. \tag{4.56}$$

By (4.50) we have

$$QM^b(x,D) : H^{m-r\delta+s,p} \longrightarrow H^{m+s,p}, \quad s > 0. \tag{4.57}$$

(In fact $s > -(1-\delta)r$ suffices.) We deduce that

$$u \in H^{m-\delta r+s,p},\ f \in H^{s,p} \Longrightarrow u \in H^{m+s,p}, \tag{4.58}$$

granted $r > 0$, $s > 0$, and $p \in (1,\infty)$. There is a similar implication, with Sobolev spaces replaced by Hölder (or Zygmund) spaces. This sort of implication can be iterated, leading to a second proof of Theorem 4.6. We restate the result, including Sobolev estimates, which could also have been obtained by the first method used to prove Theorem 4.6.

**Theorem 4.8.** *Suppose, given $r > 0$, that $u \in C^{m+r}$ satisfies* (4.55) *and this PDE is elliptic. Then, for each $s > 0$, $p \in (1,\infty)$,*

$$f \in H^{s,p} \Longrightarrow u \in H^{m+s,p} \ \text{ and } \ f \in C^s_* \Longrightarrow u \in C^{m+s}_*. \tag{4.59}$$

By way of further comparison with the methods used earlier in this section, we now rederive Theorem 4.5, on regularity for solutions to a quasi-linear elliptic PDE. Note that, in the quasi-linear case,

$$F(x,D^m u) = \sum_{|\alpha|\le m} a_\alpha(x, D^{m-1}u) D^\alpha u = f, \tag{4.60}$$

the construction above gives $F(x,D^m u) = M(u;x,D)u + R_0(u)$ with the property that, for $r \ge 0$,

$$u \in C^{m+r} \Longrightarrow M(u;x,\xi) \in C^{r+1}S^m_{1,0} \cap S^m_{1,1} + C^r S^{m-1}_{1,0} \cap S^{m-1}_{1,1}. \tag{4.61}$$

Of more interest to us now is that, for $0 < r < 1$,

$$u \in C^{m-1+r} \Longrightarrow M(u;x,\xi) \in C^r S^m_{1,0} \cap S^m_{1,1} + S^{m-r}_{1,1}, \tag{4.62}$$

which follows from (10.23) of Chapter 13. Thus we can decompose the term in $C^r S^m_{1,0} \cap S^m_{1,1}$ via symbol smoothing, as in (10.60)–(10.61) of Chapter 13, and throw the term in $S^{m-r}_{1,1}$ into the remainder, to get

$$M(u;x,\xi) = M^\#(x,\xi) + M^b(x,\xi), \tag{4.63}$$

with

$$M^{\#}(x,\xi) \in S^m_{1,\delta}, \quad M^b(x,\xi) \in S^{m-r\delta}_{1,1}. \tag{4.64}$$

If $P(x,D) \in OPS^{-m}_{1,\delta}$ is a parametrix for the elliptic operator $M^{\#}(x,D)$, then whenever $u \in C^{m-1+r} \cap H^{m-1+\rho,p}$ is a solution to (4.60), we have, mod $C^\infty$,

$$u = P(x,D)f - P(x,D)M^b(x,D)u. \tag{4.65}$$

Now

$$P(x,D)M^b(x,D) : H^{m-1+\rho,p} \longrightarrow H^{m-1+\rho+r\delta,p} \quad \text{if } r+\rho > 1, \tag{4.66}$$

by the last part of (4.64). As long as this holds, we can iterate this argument and obtain Theorem 4.5, with a shorter proof than the one given before.

Next we look at one example of a quasi-linear elliptic system in divergence form, with a couple of special features. One is that we will be able to assume less regularity a priori on $u$ than in results above. The other is that the lower-order terms have a more significant impact on the analysis than above. After analyzing the following system, we will show how it arises in the study of the Ricci tensor.

We consider second-order elliptic systems of the form

$$\sum \partial_j a_{jk}(x,u)\partial_k u + B(x,u,\nabla u) = f. \tag{4.67}$$

We assume that $a_{jk}(x,u)$ and $B(x,u,p)$ are smooth in their arguments and that

$$|B(x,u,p)| \le C\langle p\rangle^2. \tag{4.68}$$

**Proposition 4.9.** *Assume that a solution $u$ to (4.67) satisfies*

$$\nabla u \in L^q, \quad \textit{for some } q > n, \quad \textit{hence } u \in C^r, \tag{4.69}$$

*for some $r \in (0,1)$. Then, if $p \in (q,\infty)$ and $s \ge -1$, we have*

$$f \in H^{s,p} \Longrightarrow u \in H^{s+2,p}. \tag{4.70}$$

To begin the proof of Proposition 4.9, we write

$$\sum_k a_{jk}(x,u)\,\partial_k u = A_j(u;x,D)u \tag{4.71}$$

mod $C^\infty$, with

$$u \in C^r \Longrightarrow A_j(u;x,\xi) \in C^r S^1_{1,0} \cap S^1_{1,1} + S^{1-r}_{1,1}, \tag{4.72}$$

as established in Chapter 13. Hence, given $\delta \in (0,1)$,

$$\begin{aligned} A_j(u;x,\xi) &= A^{\#}_j(x,\xi) + A^b_j(x,\xi), \\ A^{\#}_j(x,\xi) \in S^1_{1,\delta}, &\quad A^b_j(x,\xi) \in S^{1-r\delta}_{1,1}. \end{aligned} \tag{4.73}$$

It follows that we can write

$$\sum \partial_j a_{jk}(x,u)\,\partial_k u = P^{\#}u + P^b u, \tag{4.74}$$

with

$$P^{\#} = \sum \partial_j A_j^{\#}(x,D) \in OPS^2_{1,\delta}, \quad \text{elliptic}, \tag{4.75}$$

and

$$P^b = \sum \partial_j A_j^b(x,D). \tag{4.76}$$

By Theorem 9.1 of Chapter 13, we have

$$A_j^b(x,D) : H^{1-r\delta+\mu,p'} \longrightarrow H^{\mu,p'}, \quad \text{for } \mu > 0,\ 1 < p' < \infty. \tag{4.77}$$

In particular (taking $\mu = r\delta,\ p' = q$),

$$\nabla u \in L^q \Longrightarrow P^b u \in H^{-1+r\delta,q}. \tag{4.78}$$

Now, if

$$E^{\#} \in OPS^{-2}_{1,\delta} \tag{4.79}$$

denotes a parametrix of $P^{\#}$, we have, mod $C^\infty$,

$$u = E^{\#}f - E^{\#}B(x,u,\nabla u) - E^{\#}P^b u, \tag{4.80}$$

and we see that under the hypothesis (4.69), we have some control over the last term:

$$E^{\#}P^b u \in H^{1+r\delta,q} \subset H^{1,\tilde{q}}, \quad \frac{1}{\tilde{q}} = \frac{1}{q} - \frac{r\delta}{n}. \tag{4.81}$$

Note also that under our hypothesis on $B(x,u,p)$,

$$\nabla u \in L^q \Longrightarrow B(x,u,\nabla u) \in L^{q/2}. \tag{4.82}$$

Now, by Sobolev's imbedding theorem,

$$E^{\#}B(x,u,\nabla u) \in H^{1,\tilde{p}}, \tag{4.83}$$

with $\tilde{p} = q/(2-q/n)$ if $q < 2n$ and for all $\tilde{p} < \infty$ if $q \ge 2n$. Note that $\tilde{p} > q(1+a/n)$ if $q = n+a$. This treats the middle term on the right side of (4.80). Of course, the hypothesis on $f$ yields

$$E^{\#}f \in H^{s+2,p}, \quad s+2 \ge 1, \tag{4.84}$$

which is just where we want to place $u$.

Having thus analyzed the three terms on the right side of (4.80), we have

$$u \in H^{1,q^{\#}}, \quad q^{\#} = \min(\tilde{p}, p, \tilde{q}). \tag{4.85}$$

Iterating this argument a finite number of times, we get

$$u \in H^{1,p}. \tag{4.86}$$

If $s = -1$ in (4.70), our work is done.

If $s > -1$ in (4.70), we proceed as follows. We already have $u \in H^{1,p}$, so $\nabla u \in L^p$. Thus, on the next pass through estimates of the form (4.78)–(4.83), we obtain

$$\begin{aligned} E^{\#} P^b u &\in H^{1+r\delta,p}, \\ E^{\#} B(x,u,\nabla u) &\in H^{2,p/2} \subset H^{2-n/p,p}, \end{aligned} \tag{4.87}$$

and hence

$$u \in H^{1+\sigma,p}, \quad \sigma = \min\big(r\delta, 1-\frac{n}{p}, 1+s\big). \tag{4.88}$$

We can iterate this sort of argument a finite number of times until the conclusion in (4.70) is reached.

Further results on elliptic systems of the form (4.67) will be given in §12B. We now apply apply Proposition 4.9 to estimates involving the Ricci tensor. Consider a Riemannian metric $g_{jk}$ defined on the unit ball $B_1 \subset \mathbb{R}^n$. We will work under the following hypotheses:

*(i) For some constants* $a_j \in (0,\infty)$, *there are estimates*

$$0 < a_0 I \le \big(g_{jk}(x)\big) \le a_1 I. \tag{4.89}$$

*(ii) The coordinates* $x_1, \dots, x_n$ *are harmonic, namely*

$$\Delta x_\ell = 0. \tag{4.90}$$

Here, $\Delta$ is the Laplace operator determined by the metric $g_{jk}$. In general,

$$\Delta v = g^{jk} \partial_j \partial_k v - \lambda^\ell \partial_\ell v, \quad \lambda^\ell = g^{jk} \Gamma^\ell{}_{jk}. \tag{4.91}$$

Note that $\Delta x_\ell = -\lambda^\ell$, so the coordinates are harmonic if and only if $\lambda^\ell = 0$. Thus, in harmonic coordinates,

$$\Delta v = g^{jk}\, \partial_j \partial_k v. \tag{4.92}$$

We will also assume some bounds on the Ricci tensor, and we desire to see how this influences the regularity of $g_{jk}$ in these coordinates. Generally, as can be derived from formulas in §3 of Appendix C, the Ricci tensor is given by

$$\begin{aligned} \mathrm{Ric}_{jk} &= \frac{1}{2} g^{\ell m} \big[ -\partial_\ell \partial_m g_{jk} - \partial_j \partial_k g_{\ell m} \\ &\qquad + \partial_k \partial_m g_{\ell j} + \partial_\ell \partial_j g_{km} \big] + M_{jk}(g, \nabla g) \\ &= -\frac{1}{2} g^{\ell m} \partial_\ell \partial_m g_{jk} + \frac{1}{2} g_{j\ell} \partial_k \lambda^\ell + \frac{1}{2} g_{k\ell} \partial_j \lambda^\ell + H_{jk}(g, \nabla g), \end{aligned} \tag{4.93}$$

with $\lambda^\ell$ as in (4.91). In harmonic coordinates, we obtain

$$-\frac{1}{2} \sum \partial_j g^{jk}(x)\, \partial_k g_{\ell m} + Q_{\ell m}(g, \nabla g) = \mathrm{Ric}_{\ell m}, \tag{4.94}$$

and $Q_{\ell m}(g, \nabla g)$ is a quadratic form in $\nabla g$, with coefficients that are smooth functions of $g$, as long as (4.89) holds. Also, when (4.89) holds, the equation

(4.94) is elliptic, of the form (4.67). Thus Proposition 4.9 implies the following.

**Proposition 4.10.** *Assume the metric tensor satisfies hypotheses (i) and (ii). Also assume that, on $B_1$,*

$$\nabla g_{jk} \in L^q, \quad \text{for some } q > n, \tag{4.95}$$

*and*

$$Ric_{\ell m} \in H^{s,p}, \tag{4.96}$$

*for some $p \in (q, \infty)$, $s \geq -1$. Then, on the ball $B_{9/10}$,*

$$g_{jk} \in H^{s+2,p}. \tag{4.97}$$

In [DK] it was shown that if $g_{jk} \in C^2$, in harmonic coordinates, then, for $k \in \mathbb{Z}^+$, $\alpha \in (0,1)$, $\mathrm{Ric}_{\ell m} \in C^{k+\alpha} \Rightarrow g_{jk} \in C^{k+2+\alpha}$. Such results also follow by the methods used to prove Proposition 4.10. A result stronger than Proposition 4.10, using Morrey spaces, is proved in [T2].

## Exercises

1. Consider the system $F(x, D^m u) = f$ when
$$F(x, D^m u) = \sum_{|\alpha| \leq m} a_\alpha(x, D^j u)\ D^\alpha u,$$
for some $j$ such that $0 \leq j < m$. Assume this quasi-linear system is elliptic. Given $p, q \in (1, \infty)$, $r > 0$, assume
$$u \in C^{j+r} \cap H^{m-1+\rho,p}, \quad r + \rho > 1.$$
Show that
$$f \in H^{s,q} \Longrightarrow u \in H^{s+m,q}.$$

# 5. Isometric imbedding of Riemannian manifolds

In this section we will establish the following result.

**Theorem 5.1.** *If $M$ is a compact Riemannian manifold, there exists a $C^\infty$-map*

$$\Phi : M \longrightarrow \mathbb{R}^N, \tag{5.1}$$

*which is an isometric imbedding.*

This was first proved by J. Nash [Na1], but the proof was vastly simplified by M. Günther [Gu1]–[Gu3]. These works also deal with noncompact

Riemannian manifolds and derive good bounds for $N$, but to keep the exposition simple we will not cover these results.

To prove Theorem 5.1, we can suppose without loss of generality that $M$ is a torus $\mathbb{T}^k$. In fact, imbed $M$ smoothly in some Euclidean space $\mathbb{R}^k$; $M$ will sit inside some box; identify opposite faces to have $M \subset \mathbb{T}^k$. Then smoothly extend the Riemannian metric on $M$ to one on $\mathbb{T}^k$.

If $\mathcal{R}$ denotes the set of smooth Riemannian metrics on $\mathbb{T}^k$ and $\mathcal{E}$ is the set of such metrics arising from smooth imbeddings of $\mathbb{T}^k$ into some Euclidean space, our goal is to prove

$$\mathcal{E} = \mathcal{R}. \tag{5.2}$$

Now $\mathcal{R}$ is clearly an open convex cone in the Fréchet space

$$V = C^\infty(\mathbb{T}^k, S^2T^*)$$

of smooth, second-order, symmetric, covariant tensor fields. As a preliminary to demonstrating (5.2), we show that the subset $\mathcal{E}$ shares some of these properties.

**Lemma 5.2.** *$\mathcal{E}$ is a convex cone in $V$.*

**Proof.** If $g_0 \in \mathcal{E}$, it is obvious from scaling the imbedding producing $g_0$ that $\alpha g_0 \in \mathcal{E}$, for any $\alpha \in (0, \infty)$. Suppose also that $g_1 \in \mathcal{E}$. If these metrics $g_j$ arise from imbeddings $\varphi_j : \mathbb{T}^k \to \mathbb{R}^{\nu_j}$, then $g_0 + g_1$ is a metric arising from the imbedding $\varphi_0 \oplus \varphi_1 : \mathbb{T}^k \to \mathbb{R}^{\nu_0+\nu_1}$. This proves the lemma.

Using Lemma 5.2 plus some functional analysis, we will proceed to establish that any Riemannian metric on $\mathbb{T}^k$ can be *approximated* by one in $\mathcal{E}$. First, we define some more useful objects. If $u : \mathbb{T}^k \to \mathbb{R}^m$ is any smooth map, let $\gamma_u$ denote the symmetric tensor field on $\mathbb{T}^k$ obtained by pulling back the Euclidean metric on $\mathbb{R}^m$. In a natural local coordinate system on $\mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$, arising from standard coordinates $(x_1, \dots, x_k)$ on $\mathbb{R}^k$,

$$\gamma_u = \sum_{i,j,\ell} \frac{\partial u_\ell}{\partial x_i} \frac{\partial u_\ell}{\partial x_j} \, dx_i \otimes dx_j. \tag{5.3}$$

Whenever $u$ is an immersion, $\gamma_u$ is a Riemannian metric; and if $u$ is an imbedding, then $\gamma_u$ is of course an element of $\mathcal{E}$. Denote by $\mathcal{C}$ the set of tensor fields on $\mathbb{T}^k$ of the form $\gamma_u$. By the same reasoning as in Lemma 5.2, $\mathcal{C}$ is a convex cone in $V$.

**Lemma 5.3.** *$\mathcal{E}$ is a dense subset of $\mathcal{R}$.*

**Proof.** If not, take $g \in \mathcal{R}$ such that $g \notin \overline{\mathcal{E}}$, the closure of $\mathcal{E}$ in $V$. Now $\overline{\mathcal{E}}$ is a closed, convex subset of $V$, so the Hahn-Banach theorem implies that there is a continuous linear functional $\ell : V \to \mathbb{R}$ such that $\ell(\overline{\mathcal{E}}) \le 0$ while $\ell(g) = a > 0$.

Let us note that $\mathcal{C} \subset \overline{\mathcal{E}}$ (and hence $\overline{\mathcal{C}} = \overline{\mathcal{E}}$). In fact, if $u : \mathbb{T}^k \to \mathbb{R}^m$ is any smooth map and $\varphi : \mathbb{T}^k \to \mathbb{R}^n$ is an imbedding, then, for any $\varepsilon > 0$, $\varepsilon\varphi \oplus u : \mathbb{T}^k \to \mathbb{R}^{n+m}$ is an imbedding, and $\gamma_{\varepsilon\varphi\oplus u} = \varepsilon^2\gamma_\varphi + \gamma_u \in \mathcal{E}$. Taking $\varepsilon \searrow 0$, we have $\gamma_u \in \overline{\mathcal{E}}$.

Consequently, the linear functional $\ell$ produced above has the property $\ell(\overline{\mathcal{C}}) \le 0$. Now we can represent $\ell$ as a $k \times k$ symmetric matrix of distributions $\ell_{ij}$ on $\mathbb{T}^k$, and we deduce that

$$\sum_{i,j} \langle \partial_i f\, \partial_j f, \ell_{ij} \rangle \le 0, \quad \forall f \in C^\infty(\mathbb{T}^k). \tag{5.4}$$

If we apply a Friedrichs mollifier $J_\varepsilon$, in the form of a convolution operator on $\mathbb{T}^k$, it follows easily that (5.4) holds with $\ell_{ij} \in \mathcal{D}'(\mathbb{T}^k)$ replaced by $\lambda_{ij} = J_\varepsilon \ell_{ij} \in C^\infty(\mathbb{T}^k)$. Now it is an exercise to show that if $\lambda_{ij} \in C^\infty(\mathbb{T}^k)$ satisfies both $\lambda_{ij} = \lambda_{ji}$ and the analogue of (5.4), then $\Lambda = (\lambda_{ij})$ is a negative-semidefinite, matrix-valued function on $\mathbb{T}^k$, and hence, for any positive-definite $G = (g_{ij}) \in C^\infty(\mathbb{T}^k, S^2T^*)$,

$$\sum_{i,j} \langle g_{ij}, \lambda_{ij} \rangle \le 0. \tag{5.5}$$

Taking $\lambda_{ij} = J_\varepsilon \ell_{ij}$ and passing to the limit $\varepsilon \to 0$, we have

$$\sum_{i,j} \langle g_{ij}, \ell_{ij} \rangle \le 0, \tag{5.6}$$

for any Riemannian metric tensor $(g_{ij})$ on $\mathbb{T}^k$. This contradicts the hypothesis that we can take $g \notin \overline{\mathcal{E}}$, so Lemma 5.3 is proved.

The following result, to the effect that $\mathcal{E}$ has nonempty interior, is the analytical heart of the proof of Theorem 5.1.

**Lemma 5.4.** *There exist a Riemannian metric $g_0 \in \mathcal{E}$ and a neighborhood $U$ of $0$ in $V$ such that $g_0 + h \in \mathcal{E}$ whenever $h \in U$.*

We now prove (5.2), hence Theorem 5.1, granted this result. Let $g \in \mathcal{R}$, and take $g_0 \in \mathcal{E}$, given by Lemma 5.4. Then set $g_1 = g + \alpha(g - g_0)$, where $\alpha > 0$ is picked sufficiently small that $g_1 \in \mathcal{R}$. It follows that $g$ is a convex combination of $g_0$ and $g_1$; that is, $g = ag_0 + (1-a)g_1$ for some $a \in (0,1)$. By Lemma 5.4, we have an open set $U \subset V$ such that $g_0 + h \in \mathcal{E}$ whenever $h \in U$. But by Lemma 5.3, there exists $h \in U$ such that $g_1 - bh \in \mathcal{E}$, $b = a/(1-a)$. Thus $g = a(g_0 + h) + (1-a)(g_1 - bh)$ is a convex combination of elements of $\mathcal{E}$, so by Lemma 5.1, $g \in \mathcal{E}$, as desired.

We turn now to a proof of Lemma 5.4. The metric $g_0$ will be one arising from a *free* imbedding

$$u : \mathbb{T}^k \longrightarrow \mathbb{R}^\mu, \tag{5.7}$$

defined as follows.

**Definition.** *An imbedding as in (5.7) is free provided that the* $k + k(k+1)/2$ *vectors*

$$\partial_j u(x), \quad \partial_j \partial_k u(x) \tag{5.8}$$

*are linearly independent in* $\mathbb{R}^\mu$*, for each* $x \in \mathbb{T}^k$.

Here, we regard $\mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$, so $u : \mathbb{R}^k \to \mathbb{R}^\mu$, invariant under the translation action of $\mathbb{Z}^k$ on $\mathbb{R}^k$, and $(x_1, \dots x_k)$ are the standard coordinates on $\mathbb{R}^k$. It is not hard to establish the existence of free imbeddings; see the exercises.

Now, given that $u$ is a free imbedding and that $(h_{ij})$ is a smooth, symmetric tensor field that is small in some norm (stronger than the $C^2$-norm), we want to find $v \in C^\infty(\mathbb{T}^k, \mathbb{R}^\mu)$, small in a norm at least as strong as the $C^1$-norm, such that, with $g_0 = \gamma_u$,

$$\sum_\ell \partial_i(u_\ell + v_\ell)\partial_j(u_\ell + v_\ell) = g_{0ij} + h_{ij}, \tag{5.9}$$

or equivalently, using the dot product on $\mathbb{R}^\mu$,

$$\partial_i u \cdot \partial_j v + \partial_j u \cdot \partial_i v + \partial_i v \cdot \partial_j v = h_{ij}. \tag{5.10}$$

We want to solve for $v$. Now, such a system turns out to be highly underdetermined, and the key to success is to append convenient side conditions. Following [Gu3], we apply $\Delta - 1$ to (5.10), where $\Delta = \sum \partial_j^2$, obtaining

$$\begin{aligned} \partial_i\Big\{(\Delta - 1)(\partial_j u \cdot v) + \Delta v \cdot \partial_j v\Big\} + \partial_j\Big\{(\Delta - 1)(\partial_i u \cdot v) + \Delta v \cdot \partial_i v\Big\} \\ -2\Big\{(\Delta - 1)(\partial_i\partial_j u \cdot v) + \frac{1}{2}\partial_i v \cdot \partial_j v - \partial_i\partial_\ell v \cdot \partial_j\partial_\ell v \\ +\Delta v \cdot \partial_i\partial_j v + \frac{1}{2}(\Delta - 1)h_{ij}\Big\} = 0, \end{aligned} \tag{5.11}$$

where we sum over $\ell$. Thus (5.10) will hold whenever $v$ satisfies the new system

$$\begin{aligned} (\Delta - 1)\big(\zeta_i(x) \cdot v\big) &= -\Delta v \cdot \partial_i v, \\ (\Delta - 1)\big(\zeta_{ij}(x) \cdot v\big) &= -\frac{1}{2}(\Delta - 1)h_{ij} \\ &\quad + \Big(\partial_i\partial_\ell v \cdot \partial_j\partial_\ell v - \Delta v \cdot \partial_i\partial_j v - \frac{1}{2}\partial_i v \cdot \partial_j v\Big). \end{aligned} \tag{5.12}$$

Here we have set $\zeta_i(x) = \partial_i u(x)$, $\zeta_{ij}(x) = \partial_i\partial_j u(x)$, smooth $\mathbb{R}^\mu$-valued functions on $\mathbb{T}^k$.

Now (5.12) is a system of $k(k+3)/2 = \kappa$ equations in $\mu$ unknowns, and it has the form

$$(\Delta - 1)\big(\xi(x)v\big) + Q(D^2 v, D^2 v) = H = \Big(0, -\frac{1}{2}(\Delta - 1)h_{ij}\Big), \tag{5.13}$$

where $\xi(x) : \mathbb{R}^\mu \to \mathbb{R}^\kappa$ is *surjective* for each $x$, by the linear independence hypothesis on (5.8), and $Q$ is a bilinear function of its arguments $D^2 v = \{D^\alpha v : |\alpha| \le 2\}$. This is hence an underdetermined system for $v$. We can obtain a determined system for a function $w$ on $\mathbb{T}^k$ with values in $\mathbb{R}^\kappa$, by setting

$$v = \xi(x)^t w, \tag{5.14}$$

namely

$$(\Delta - 1)\big(A(x)w\big) + \widetilde{Q}(D^2 w, D^2 w) = H, \tag{5.15}$$

where, for each $x \in \mathbb{T}^k$,

$$A(x) = \xi(x)\xi(x)^t \in \ \mathrm{End}(\mathbb{R}^\kappa) \text{ is invertible.} \tag{5.16}$$

If we denote the left side of (5.15) by $F(w)$, the operator $F$ is a nonlinear differential operator of order 2, and we have

$$DF(w)f = (\Delta - 1)\big(A(x)f\big) + B(D^2 w, D^2 f), \tag{5.17}$$

where $B$ is a bilinear function of its arguments. In particular,

$$DF(0)f = (\Delta - 1)\big(A(x)f\big). \tag{5.18}$$

We thus see that, for any $r \in \mathbb{R}^+ \setminus \mathbb{Z}^+$,

$$DF(0) : C^{r+2}(\mathbb{T}^k, \mathbb{R}^\kappa) \longrightarrow C^r(\mathbb{T}^k, \mathbb{R}^\kappa) \text{ is invertible.} \tag{5.19}$$

Consequently, if we fix $r \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, and if $H \in C^r(\mathbb{T}^k, \mathbb{R}^\kappa)$ has sufficiently small norm (i.e., if $(h_{ij}) \in C^{r+2}(\mathbb{T}^k, S^2 T^*)$ has sufficiently small norm), then (5.15) has a unique solution $w \in C^{r+2}(\mathbb{T}^k, \mathbb{R}^\kappa)$ with small norm, and via (5.14) we get a solution $v \in C^{r+2}(\mathbb{T}^k, \mathbb{R}^\mu)$, with small norm, to (5.13). If the norm of $v$ is small enough, then of course $u + v$ is also an imbedding.

Furthermore, if the $C^{r+2}$-norm of $w$ is small enough, then (5.15) is an elliptic system for $w$. By the regularity result of Theorem 4.6, we can deduce that $w$ is $C^\infty$ (hence $v$ is $C^\infty$) if $h$ is $C^\infty$. This concludes the proof of Lemma 5.4, hence of Nash's imbedding theorem.

## Exercises

In Exercises 1–3, let $B$ be the unit ball in $\mathbb{R}^k$, centered at 0. Let $(\lambda_{ij})$ be a smooth, symmetric, matrix-valued function on $B$ such that

$$\sum_{i,j} \int (\partial_i f)(x)\ (\partial_j f)(x)\ \lambda_{ij}(x)\ dx \le 0, \quad \forall f \in C_0^\infty(B). \tag{5.20}$$

1. Taking $f_\varepsilon \in C_0^\infty(B)$ of the form
$$f_\varepsilon(x) = f(\varepsilon^{-2} x_1, \varepsilon^{-1} x'), \quad 0 < \varepsilon < 1,$$
examine the behavior as $\varepsilon \searrow 0$ of (5.20), with $f$ replaced by $f_\varepsilon$. Establish that $\lambda_{11}(0) \le 0$.

2. Show that the condition (5.20) is invariant under rotations of $\mathbb{R}^k$, and deduce that $\big(\lambda_{ij}(0)\big)$ is a negative-semidefinite matrix.
3. Deduce that $\big(\lambda_{ij}(x)\big)$ is negative-semidefinite for all $x \in B$.
4. Using the results above, demonstrate the implication (5.4) $\Rightarrow$ (5.5), used in the proof of Lemma 5.3.
5. Suppose we have a $C^\infty$-imbedding $\varphi : \mathbb{T}^k \to \mathbb{R}^n$. Define a map

$$\psi : \mathbb{T}^k \longrightarrow \mathbb{R}^n \oplus S^2\mathbb{R}^n \approx \mathbb{R}^\mu, \quad \mu = n + \frac{1}{2}n(n+1),$$

to have components

$$\varphi_j(x),\ 1 \le j \le n, \quad \varphi_i(x)\varphi_j(x),\ 1 \le i \le j \le n.$$

Show that $\psi$ is a free imbedding.
6. Using Leibniz' rule to expand derivatives of products, verify that (5.10) and (5.11) are equivalent, for $v \in C^\infty(\mathbb{T}^k, \mathbb{R}^\mu)$.
7. In [Na1] the system (5.10) was augmented with $\partial_i u \cdot v = 0$, yielding, instead of (5.12), the system

$$\begin{aligned} \zeta_i(x) \cdot v &= 0, \\ \zeta_{ij}(x) \cdot v &= \frac{1}{2}\Big(\partial_i v \cdot \partial_j v - h_{ij}\Big). \end{aligned} \tag{5.21}$$

What makes this system more difficult to solve than (5.12)?

## 6. Minimal surfaces

A *minimal surface* is one that is critical for the area functional. To begin, we consider a $k$-dimensional manifold $M$ (generally with boundary) in $\mathbb{R}^n$. Let $\xi$ be a compactly supported normal field to $M$, and consider the one-parameter family of manifolds $M_s \subset \mathbb{R}^n$, images of $M$ under the maps

$$\varphi_s(x) = x + s\xi(x), \quad x \in M. \tag{6.1}$$

We want a formula for the derivative of the $k$-dimensional area of $M_s$, at $s = 0$. Let us suppose $\xi$ is supported on a single coordinate chart, and write

$$A(s) = \int_\Omega \big\|\partial_1 X \wedge \cdots \wedge \partial_k X\big\| \, du_1 \cdots du_k, \tag{6.2}$$

where $\Omega \subset \mathbb{R}^k$ parameterizes $M_s$ by $X(s,u) = X_0(u) + s\xi(u)$. We can also suppose this chart is chosen so that $\|\partial_1 X_0 \wedge \cdots \wedge \partial_k X_0\| = 1$. Then we have

$$A'(0) = \sum_{j=1}^{k} \int \langle \partial_1 X_0 \wedge \cdots \wedge \partial_j \xi \wedge \cdots \wedge \partial_k X_0, \partial_1 X_0 \wedge \cdots \wedge \partial_k X_0 \rangle \, du_1 \cdots du_k. \tag{6.3}$$

By the Weingarten formula (see (4.9) of Appendix C), we can replace $\partial_j \xi$ by $-A_\xi E_j$, where $E_j = \partial_j X_0$. Without loss of generality, for any fixed $x \in M$, we can assume that $E_1, \dots, E_k$ is an orthonormal basis of $T_x M$. Then

$$\langle E_1 \wedge \cdots \wedge A_\xi E_j \wedge \cdots \wedge E_k, E_1 \wedge \cdots \wedge E_k \rangle = \langle A_\xi E_j, E_j \rangle, \tag{6.4}$$

at $x$. Summing over $j$ yields Tr $A_\xi(x)$, which is invariantly defined, so we have

$$A'(0) = -\int_M \mathrm{Tr}\ A_\xi(x)\ dA(x), \tag{6.5}$$

where $A_\xi(x) \in \mathrm{End}(T_x M)$ is the Weingarten map of $M$ and $dA(x)$ the Riemannian $k$-dimensional area element. We say $M$ is a minimal submanifold of $\mathbb{R}^n$ provided $A'(0) = 0$ for all variations of the form (6.1), for which the normal field $\xi$ vanishes on $\partial M$.

If we specialize to the case where $k = n - 1$ and $M$ is an oriented hypersurface of $\mathbb{R}^n$, letting $N$ be the "outward" unit normal to $M$, for a variation $M_s$ of $M$ given by

$$\varphi_s(x) = x + sf(x)N(x), \quad x \in M, \tag{6.6}$$

we hence have

$$A'(0) = -\int_M \mathrm{Tr}\ A_N(x)\ f(x)\ dA(x). \tag{6.7}$$

The criterion for a hypersurface $M$ of $\mathbb{R}^n$ to be minimal is hence that Tr $A_N = 0$ on $M$.

Recall from §4 of Appendix C that $A_N(x)$ is a symmetric operator on $T_x M$. Its eigenvalues, which are all real, are called the principal curvatures of $M$ at $x$. Various symmetric polynomials in these principal curvatures furnish quantities of interest. The mean curvature $H(x)$ of $M$ at $x$ is defined to be the mean value of these principal curvatures, that is,

$$H(x) = \frac{1}{k}\ \mathrm{Tr}\ A_N(x). \tag{6.8}$$

Thus a hypersurface $M \subset \mathbb{R}^n$ is a minimal submanifold of $\mathbb{R}^n$ precisely when $H = 0$ on $M$.

Note that changing the sign of $N$ changes the sign of $A_N$, hence of $H$. Under such a sign change, the mean curvature vector

$$\mathfrak{H}(x) = H(x)N(x) \tag{6.9}$$

is invariant. In particular, this is well defined whether or not $M$ is orientable, and its vanishing is the condition for $M$ to be a minimal submanifold. There is the following useful formula for the mean curvature of a hypersurface $M \subset \mathbb{R}^n$. Let $X : M \hookrightarrow \mathbb{R}^n$ be the isometric imbedding. We claim that

$$\mathfrak{H}(x) = \frac{1}{k}\Delta X, \tag{6.10}$$

with $k = n-1$, where $\Delta$ is the Laplace operator on the Riemannian manifold $M$, acting componentwise on $X$. This is easy to see at a point $p \in M$ if we translate and rotate $\mathbb{R}^n$ to make $p = 0$ and represent $M$ as the image of $\mathbb{R}^k = \mathbb{R}^{n-1}$ under

$$Y(x') = \big(x', f(x')\big), \quad x' = (x_1, \dots, x_k), \ \nabla f(0) = 0. \tag{6.11}$$

Then one verifies that

$$\Delta X(p) = \partial_1^2 Y(0) + \cdots + \partial_k^2 Y(0) = \big(0, \dots, 0, \partial_1^2 f(0) + \cdots + \partial_k^2 f(0)\big),$$

and (6.10) follows from the formula

$$\langle A_N(0)X, Y\rangle = \sum_{i,j=1}^{k} \partial_i \partial_j f(0)\ X_i Y_j \tag{6.12}$$

for the second fundamental form of $M$ at $p$, derived in (4.19) of Appendix C.

More generally, if $M \subset \mathbb{R}^n$ has dimension $k \le n-1$, we can define the mean curvature vector $\mathfrak{H}(x)$ by

$$\langle \mathfrak{H}(x), \xi\rangle = \frac{1}{k}\,\mathrm{Tr}\, A_\xi(x), \quad \mathfrak{H}(x) \perp T_x M, \tag{6.13}$$

so the criterion for $M$ to be a minimal submanifold is that $\mathfrak{H} = 0$. Furthermore, (6.10) continues to hold. This can be seen by the same type of argument used above; represent $M$ as the image of $\mathbb{R}^k$ under (6.11), where now $f(x') = (x_{k+1}, \dots, x_n)$. Then (6.12) generalizes to

$$\langle A_\xi(0)X, Y\rangle = \sum_{i,j=1}^{k} \langle \xi, \partial_i \partial_j f(0)\rangle\ X_i Y_j, \tag{6.14}$$

which yields (6.10). We record this observation.

**Proposition 6.1.** *Let $X : M \to \mathbb{R}^n$ be an isometric immersion of a Riemannian manifold into $\mathbb{R}^n$. Then $M$ is a minimal submanifold of $\mathbb{R}^n$ if and only if the coordinate functions $x_1, \dots, x_n$ are harmonic functions on $M$.*

A two-dimensional minimal submanifold of $\mathbb{R}^n$ is called a minimal surface. The theory is most developed in this case, and we will concentrate on the two-dimensional case in the material below.

When dim $M = 2$, we can extend Proposition 6.1 to cases where $X : M \to \mathbb{R}^n$ is not an isometric map. This occurs because, in such a case, the class of harmonic functions on $M$ is invariant under conformal changes of metric. In fact, if $\Delta$ is the Laplace operator for a Riemannian metric $g_{ij}$ on $M$ and $\Delta_1$ that for $g_{1ij} = e^{2u} g_{ij}$, then, since $\Delta f = g^{-1/2}\,\partial_i(g^{ij} g^{1/2}\,\partial_j f)$ and $g_1^{ij} = e^{-2u} g^{ij}$, while $g_1^{1/2} = e^{ku} g^{1/2}$ (if dim $M = k$), we have

$$\Delta_1 f = e^{-2u}\ \Delta f + e^{-ku} \langle df, de^{(k-2)u}\rangle = e^{-2u} \Delta f \quad \text{if } k = 2. \tag{6.15}$$

Hence ker $\Delta =$ ker $\Delta_1$ if $k = 2$. We hence have the following:

**Proposition 6.2.** *If $\Omega$ is a Riemannian manifold of dimension 2 and $X : \Omega \to \mathbb{R}^n$ a smooth immersion, with image $M$, then $M$ is a minimal surface provided $X$ is harmonic and $X : \Omega \to M$ is conformal.*

In fact, granted that $X : \Omega \to M$ is conformal, $M$ is minimal if and only if $X$ is harmonic on $\Omega$.

We can use this result to produce lots of examples of minimal surfaces, by the following classical device. Take $\Omega$ to be an open set in $\mathbb{R}^2 = \mathbb{C}$, with coordinates $(u_1, u_2)$. Given a map $X : \Omega \to \mathbb{R}^n$, with components $x_j : \Omega \to \mathbb{R}$, form the complex-valued functions

$$\psi_j(\zeta) = \frac{\partial x_j}{\partial u_1} - i\frac{\partial x_j}{\partial u_2} = 2\frac{\partial}{\partial \zeta} x_j, \quad \zeta = u_1 + iu_2. \tag{6.16}$$

Clearly, $\psi_j$ is holomorphic if and only if $x_j$ is harmonic (for the standard flat metric on $\Omega$), since $\Delta = 4(\partial/\partial\overline{\zeta})(\partial/\partial\zeta)$. Furthermore, a short calculation gives

$$\sum_{j=1}^n \psi_j(\zeta)^2 = \left|\partial_1 X\right|^2 - \left|\partial_2 X\right|^2 - 2i\ \partial_1 X \cdot \partial_2 X. \tag{6.17}$$

Granted that $X : \Omega \to \mathbb{R}^n$ is an immersion, the criterion that it be conformal is precisely that this quantity vanish. We have the following result.

**Proposition 6.3.** *If $\psi_1, \dots, \psi_n$ are holomorphic functions on $\Omega \subset \mathbb{C}$ such that*

$$\sum_{j=1}^n \psi_j(\zeta)^2 = 0 \quad \text{on } \Omega, \tag{6.18}$$

*while $\sum |\psi_j(\zeta)|^2 \neq 0$ on $\Omega$, then setting*

$$x_j(u) = \ Re \int \psi_j(\zeta)\ d\zeta \tag{6.19}$$

*defines an immersion $X : \Omega \to \mathbb{R}^n$ whose image is a minimal surface.*

If $\Omega$ is not simply connected, the domain of $X$ is actually the universal covering surface of $\Omega$.

We mention some particularly famous minimal surfaces in $\mathbb{R}^3$ that arise in such a fashion. Surely the premier candidate for (6.18) is

$$\sin^2\zeta + \cos^2\zeta - 1 = 0. \tag{6.20}$$

Here, take $\psi_1(\zeta) = \sin\zeta$, $\psi_2(\zeta) = -\cos\zeta$, and $\psi_3(\zeta) = -i$. Then (6.19) yields

$$x_1 = (\cos u_1)(\cosh u_2), \quad x_2 = (\sin u_1)(\cosh u_2), \quad x_3 = u_2. \tag{6.21}$$

The surface obtained in $\mathbb{R}^3$ is called the *catenoid.* It is the surface of revolution about the $x_3$-axis of the curve $x_1 = \cosh x_3$ in the $(x_1 - x_3)$-plane. Whenever $\psi_j(\zeta)$ are holomorphic functions satisfying (6.18), so are $e^{i\theta}\psi_j(\zeta)$, for any $\theta \in \mathbb{R}$. The resulting immersions $X_\theta : \Omega \to \mathbb{R}^n$ give rise to a family of minimal surfaces $M_\theta \subset \mathbb{R}^n$, which are said to be *associated.* In particular, $M_{\pi/2}$ is said to be *conjugate* to $M = M_0$. When $M_0$ is the catenoid, defined by (6.21), the conjugate minimal surface arises from $\psi_1(\zeta) = i\ \sin\zeta$, $\psi_2(\zeta) = -i\ \cos\zeta$, and $\psi_3(\zeta) = 1$ and is given by

$$x_1 = (\sin u_1)(\sinh u_2), \quad x_2 = (\cos u_1)(\sinh u_2), \quad x_3 = u_1. \tag{6.22}$$

This surface is called the *helicoid.* We mention that associated minimal surfaces are locally isometric but generally not congruent; that is, the isometry between the surfaces does not extend to a rigid motion of the ambient Euclidean space.

The catenoid and helicoid were given as examples of minimal surfaces by Meusnier, in 1776.

One systematic way to produce triples of holomorphic functions $\psi_j(\zeta)$ satisfying (6.18) is to take

$$\psi_1 = \frac{1}{2}f(1-g^2), \quad \psi_2 = \frac{i}{2}f(1+g^2), \quad \psi_3 = fg, \tag{6.23}$$

for arbitrary holomorphic functions $f$ and $g$ on $\Omega$. More generally, $g$ can be meromorphic on $\Omega$ as long as $f$ has a zero of order $2m$ at each point where $g$ has a pole of order $m$. The resulting map $X : \Omega \to M \subset \mathbb{R}^3$ is called the Weierstrass-Enneper representation of the minimal surface $M$. It has an interesting connection with the Gauss map of $M$, which will be sketched in the exercises. The example arising from $f = 1, g = \zeta$ produces "Enneper's surface." This surface is immersed in $\mathbb{R}^3$ but not imbedded.

For a long time the only known examples of complete imbedded minimal surfaces in $\mathbb{R}^3$ of finite topological type were the plane, the catenoid, and the helicoid, but in the 1980s it was proved by [HM1] that the surface obtained by taking $g = \zeta$ and $f(\zeta) = \wp(\zeta)$ (the Weierstrass $\wp$-function) is another example. Further examples have been found; computer graphics have been a valuable aid in this search; see [HM2].

A natural question is how general is the class of minimal surfaces arising from the construction in Proposition 6.3. In fact, it is easy to see that every minimal $M \subset \mathbb{R}^n$ is at least locally representable in such a fashion, using the existence of local isothermal coordinates, established in §10 of Chapter 5. Thus any $p \in M$ has a neighborhood $\mathcal{O}$ such that there is a conformal diffeomorphism $X : \Omega \to \mathcal{O}$, for some open set $\Omega \subset \mathbb{R}^2$. By Proposition 6.2 and the remark following it, if $M$ is minimal, then $X$ must be harmonic, so (6.16) furnishes the functions $\psi_j(\zeta)$ used in Proposition 6.3. Incidentally, this shows that any minimal surface in $\mathbb{R}^n$ is real analytic.

As for the question of whether the construction of Proposition 6.3 globally represents every minimal surface, the answer here is also "yes." A proof uses the fact that every noncompact Riemann surface (without boundary) is covered by either $\mathbb{C}$ or the unit disk in $\mathbb{C}$. This is a more complete version of the uniformization theorem than the one we established in §2 of this chapter. A positive answer, for simply connected, compact minimal surfaces, with smooth boundary, is implied by the following result, which will also be useful for an attack on the Plateau problem.

**Proposition 6.4.** *If $\overline{M}$ is a compact, connected, simply connected Riemannian manifold of dimension 2, with nonempty, smooth boundary, then there exists a conformal diffeomorphism*

$$\Phi : \overline{M} \longrightarrow \overline{D}, \tag{6.24}$$

*where* $\overline{D} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$.

This is a slight generalization of the Riemann mapping theorem, established in §4 of Chapter 5, and it has a proof along the lines of the argument given there. Thus, fix $p \in M$, and let $G \in \mathcal{D}'(M) \cap C^\infty(\overline{M} \setminus p)$ be the unique solution to

$$\Delta G = 2\pi\delta, \quad G = 0 \text{ on } \partial M. \tag{6.25}$$

Since $M$ is simply connected, it is orientable, so we can pick a Hodge star operator, and $*dG = \beta$ is a smooth closed 1-form on $\overline{M} \setminus p$. If $\gamma$ is a curve in $M$ of degree 1 about $p$, then $\int_\gamma \beta$ can be calculated by deforming $\gamma$ to be a small curve about $p$. The parametrix construction for the solution to (6.25), in normal coordinates centered at $p$, gives $G(x) \sim \log\ \text{dist}(x, p)$, and one establishes that $\int_\gamma \beta = 2\pi$. Thus we can write $\beta = dH$, where $H$ is a smooth function on $\overline{M} \setminus p$, well defined mod $2\pi\mathbb{Z}$. Hence $\Phi(x) = e^{G+iH}$ is a single-valued function, tending to 0 as $x \to p$, which one verifies to be the desired conformal diffeomorphism (6.24), by the same reasoning as used to complete the proof of Theorem 4.1 in Chapter 5.

An immediate corollary is that the argument given above for the local representation of a minimal surface in the form (6.19) extends to a global representation of a compact, simply connected minimal surface, with smooth boundary

So far we have dealt with smooth surfaces, at least immersed in $\mathbb{R}^n$. The theorem of J. Douglas and T. Rado that we now tackle deals with "generalized" surfaces, which we will simply define to be the images of two-dimensional manifolds under smooth maps into $\mathbb{R}^n$ (or some other manifold). The theorem, a partial answer to the "Plateau problem," asserts the existence of an area-minimizing generalized surface whose boundary is a given simple, closed curve in $\mathbb{R}^n$.

To be precise, let $\gamma$ be a smooth, simple, closed curve in $\mathbb{R}^n$, that is, a diffeomorphic image of $S^1$. Let

$$\begin{aligned} \mathfrak{X}_\gamma = \{\varphi \in C(\overline{D}, \mathbb{R}^n) \cap C^\infty(D, \mathbb{R}^n) :\\ \varphi : S^1 \to \gamma \text{ monotone, and } \alpha(\varphi) < \infty\}, \end{aligned} \tag{6.26}$$

where $\alpha$ is the area functional:

$$\alpha(\varphi) = \int_D |\partial_1\varphi \wedge \partial_2\varphi| \; dx_1 dx_2. \tag{6.27}$$

Then let

$$\mathcal{A}_\gamma = \inf\{\alpha(\varphi) : \varphi \in \mathfrak{X}_\gamma\}. \tag{6.28}$$

The existence theorem of Douglas and Rado is the following:

**Theorem 6.5.** *There is a map $\varphi \in \mathfrak{X}_\gamma$ such that $\alpha(\varphi) = \mathcal{A}_\gamma$.*

We can choose $\varphi_\nu \in \mathfrak{X}_\gamma$ such that $\alpha(\varphi_\nu) \searrow \mathcal{A}_\gamma$, but $\{\varphi_\nu\}$ could hardly be expected to have a convergent subsequence unless some structure is imposed on the maps $\varphi_\nu$. The reason is that $\alpha(\varphi) = \alpha(\varphi \circ \psi)$ for any $C^\infty$-diffeomorphism $\psi : \overline{D} \to \overline{D}$. We say $\varphi \circ \psi$ is a *reparameterization* of $\varphi$. The key to success is to take $\varphi_\nu$, which approximately minimize not only the area functional $\alpha(\varphi)$ but also the energy functional

$$\vartheta(\varphi) = \int_D |\nabla\varphi(x)|^2 \; dx_1 dx_2, \tag{6.29}$$

so that we will also have $\vartheta(\varphi_\nu) \searrow d_\gamma$, where

$$d_\gamma = \inf\{\vartheta(\varphi) : \varphi \in \mathfrak{X}_\gamma\}. \tag{6.30}$$

To relate these, we compare (6.29) and the area functional (6.27).

To compare integrands, we have

$$|\nabla\varphi|^2 = |\partial_1\varphi|^2 + |\partial_2\varphi|^2, \tag{6.31}$$

while the square of the integrand in (6.27) is equal to

$$\begin{aligned} |\partial_1\varphi \wedge \partial_2\varphi|^2 &= |\partial_1\varphi|^2|\partial_2\varphi|^2 - \langle\partial_1\varphi, \partial_2\varphi\rangle \\ &\le |\partial_1\varphi|^2|\partial_2\varphi|^2 \\ &\le \frac{1}{4}\big(|\partial_1\varphi|^2 + |\partial_2\varphi|^2\big)^2, \end{aligned} \tag{6.32}$$

where equality holds if and only if

$$|\partial_1\varphi| = |\partial_2\varphi| \quad \text{and} \quad \langle \partial_1\varphi, \partial_2\varphi\rangle = 0. \tag{6.33}$$

Whenever $\nabla\varphi \neq 0$, this is the condition that $\varphi$ be conformal. More generally, if (6.33) holds, but we allow $\nabla\varphi(x) = 0$, we say that $\varphi$ is essentially conformal. Thus, we have seen that, for each $\varphi \in \mathfrak{X}_\gamma$,

$$\alpha(\varphi) \leq \frac{1}{2}\vartheta(\varphi), \tag{6.34}$$

with equality if and only if $\varphi$ is essentially conformal. The following result allows us to transform the problem of minimizing $\alpha(\varphi)$ over $\mathfrak{X}_\gamma$ into that of minimizing $\vartheta(\varphi)$ over $\mathfrak{X}_\gamma$, which will be an important tool in the proof of Theorem 6.5. Set

$$\mathfrak{X}_\gamma^\infty = \{\varphi \in C^\infty(\overline{D}, \mathbb{R}^n) : \varphi : S^1 \to \gamma \text{ diffeo.}\}. \tag{6.35}$$

**Proposition 6.6.** *Given $\varepsilon > 0$, any $\varphi \in \mathfrak{X}_\gamma^\infty$ has a reparameterization $\varphi \circ \psi$ such that*

$$\frac{1}{2}\vartheta(\varphi \circ \psi) \leq \alpha(\varphi) + \varepsilon. \tag{6.36}$$

**Proof.** We will obtain this from Proposition 6.4, but that result may not apply to $\varphi(\overline{D})$, so we do the following. Take $\delta > 0$ and define $\Phi_\delta : \overline{D} \to \mathbb{R}^{n+2}$ by $\Phi_\delta(x) = \big(\varphi(x), \delta x\big)$. For any $\delta > 0$, $\Phi_\delta$ is a diffeomorphism of $\overline{D}$ onto its image, and if $\delta$ is very small, area $\Phi_\delta(\overline{D})$ is only a little larger than area $\varphi(D)$. Now, by Proposition 6.4, there is a conformal diffeomorphism $\Psi : \Phi_\delta(\overline{D}) \to \overline{D}$. Set $\psi = \psi_\delta = \big(\Psi \circ \Phi_\delta\big)^{-1} : \overline{D} \to \overline{D}$. Then $\Phi_\delta \circ \psi = \Psi^{-1}$ and, as established above, $(1/2)\vartheta(\Psi^{-1}) =$ Area$(\Psi^{-1}(\overline{D}))$, i.e.,

$$\tfrac{1}{2}\vartheta(\Phi_\delta \circ \psi) = \text{Area}\big(\Phi_\delta(\overline{D})\big). \tag{6.37}$$

Since $\vartheta(\varphi \circ \psi) \leq \vartheta(\Phi_\delta \circ \psi)$, the result (6.34) follows if $\delta$ is taken small enough.

One can show that

$$\mathcal{A}_\gamma = \inf\{\alpha(\varphi) : \varphi \in \mathfrak{X}_\gamma^\infty\}, \quad d_\gamma = \inf\{\vartheta(\varphi) : \varphi \in \mathfrak{X}_\gamma^\infty\}. \tag{6.38}$$

It then follows from Proposition 6.6 that $\mathcal{A}_\gamma = (1/2)d_\gamma$, and if $\varphi_\nu \in \mathfrak{X}_\gamma^\infty$ is chosen so that $\vartheta(\varphi_\nu) \to d_\gamma$, then a fortiori $\alpha(\varphi_\nu) \to \mathcal{A}_\gamma$.

There is still an obstacle to obtaining a convergent subsequence of such $\{\varphi_\nu\}$. Namely, the energy integral (6.29) is invariant under reparameterizations $\varphi \mapsto \varphi \circ \psi$ for which $\psi : \overline{D} \to \overline{D}$ is a conformal diffeomorphism. We can put a clamp on this by noting that, given any two triples of (distinct) points $\{p_1, p_2, p_2\}$ and $\{q_1, q_2, q_3\}$ in $S^1 = \partial D$, there is a unique conformal diffeomorphism $\psi : \overline{D} \to \overline{D}$ such that $\psi(p_j) = q_j, 1 \leq j \leq 3$. Let us now make one choice of $\{p_j\}$ on $S^1$—for example, the three cube roots of 1—and make one choice of a triple $\{q_j\}$ of distinct points in $\gamma$. The following key compactness result will enable us to prove Theorem 6.5.

**Proposition 6.7.** *For any* $d \in (d_\gamma, \infty)$*, the set*

$$\Sigma_d = \big\{\varphi \in \mathfrak{X}_\gamma^\infty : \varphi \text{ harmonic}, \varphi(p_j) = q_j, \text{ and } \vartheta(\varphi) \le d\big\} \tag{6.39}$$

*is relatively compact in* $C(\overline{D}, \mathbb{R}^n)$.

In view of the mapping properties of the Poisson integral, this result is equivalent to the relative compactness in $C(\partial D, \gamma)$ of

$$\mathcal{S}_K = \{u \in C^\infty(S^1, \gamma) \text{ diffeo.} : u(p_j) = q_j, \text{ and } \|u\|_{H^{1/2}(S^1)} \le K\}, \tag{6.40}$$

for any given $K < \infty$. For $u \in \mathcal{S}_K$, we have $\|u\|_{H^{1/2}(S^1)} \approx \|\mathrm{PI}\, u\|_{H^1(D)}$. To demonstrate this compactness, there is no loss of generality in taking $\gamma = S^1 \subset \mathbb{R}^2$ and $p_j = q_j$.

We will show that the oscillation of $u$ over any arc $I \subset S^1$ of length $2\delta$ is $\le CK\big/\sqrt{\log(1/\delta)}$. This modulus of continuity will imply the compactness, by Ascoli's theorem.

Pick a point $z \in S^1$. Let $C_r$ denote the portion of the circle of radius $r$ and center $z$ which lies in $\overline{D}$. Thus $C_r$ is an arc, of length $\le \pi r$. Let $\delta \in (0,1)$. As $r$ varies from $\delta$ to $\sqrt{\delta}$, $C_r$ sweeps out part of an annulus, as illustrated in Fig. 6.1.

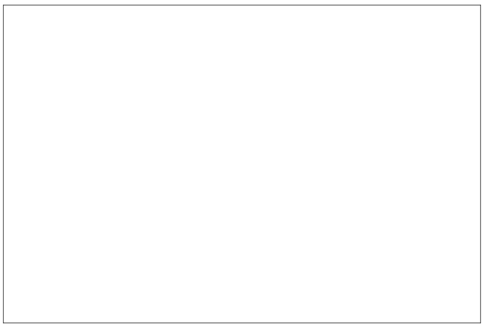

FIGURE 6.1

We claim there exists $\rho \in [\delta, \sqrt{\delta}]$ such that

$$\int_{C_\rho} |\nabla\varphi|\, ds \le K \sqrt{\frac{2\pi}{\log \frac{1}{\delta}}} \tag{6.41}$$

if $K = \|\nabla\varphi\|_{L^2(D)}$, $\varphi = \mathrm{PI}\, u$. To establish this, let

$$\omega(r) = r \int_{C_r} |\nabla\varphi|^2\, ds.$$

Then

$$\int_{\delta}^{\sqrt{\delta}} \omega(r)\ \frac{dr}{r} = \int_{\delta}^{\sqrt{\delta}} \int\limits_{C_r} |\nabla \varphi|^2\ ds\ dr = I \le K^2.$$

By the mean-value theorem, there exists $\rho \in [\delta, \sqrt{\delta}]$ such that

$$I = \omega(\rho) \int_{\delta}^{\sqrt{\delta}} \frac{dr}{r} = \frac{\omega(\rho)}{2}\ \log \frac{1}{\delta}.$$

For this value of $\rho$, we have

$$\rho \int\limits_{C_\rho} |\nabla \varphi|^2\ ds = \frac{2I}{\log \frac{1}{\delta}} \le \frac{2K^2}{\log \frac{1}{\delta}}. \tag{6.42}$$

Then Cauchy's inequality yields (6.41), since $\text{length}(C_\rho) \le \pi \rho$.

FIGURE 6.2

This almost gives the desired modulus of continuity. The arc $C_\rho$ is mapped by $\varphi$ into a curve of length $\le K\sqrt{2\pi/\log(1/\delta)}$, whose endpoints divide $\gamma$ into two segments, one rather short (if $\delta$ is small) and one not so short. There are two possibilities: $\varphi(z)$ is contained in either the short segment (as in Fig. 6.2) or the long segment (as in Fig. 6.3). However, as long as $\varphi(p_j) = p_j$ for three points $p_j$, this latter possibility cannot occur. We see that

$$|u(a) - u(b)| \le K\sqrt{\frac{2\pi}{\log \frac{1}{\delta}}},$$

if $a$ and $b$ are the points where $C_\rho$ intersects $S^1$. Now the monotonicity of $u$ along $S^1$ guarantees that the total variation of $u$ on the (small) arc from $a$ to $b$ in $S^1$ is $\le K\sqrt{2\pi/\log(1/\delta)}$. This establishes the modulus of continuity and concludes the proof.

Now that we have Proposition 6.7, we proceed as follows. Pick a sequence $\varphi_\nu$ in $\mathfrak{X}_\gamma^\infty$ such that $\vartheta(\varphi_\nu) \to d_\gamma$, so also $\alpha(\varphi_\nu) \to \mathcal{A}_\gamma$. Now we do

Figure 6.3

not increase $\vartheta(\varphi_\nu)$ if we replace $\varphi_\nu$ by the Poisson integral of $\varphi_\nu\big|_{\partial D}$, and we do not alter this energy integral if we reparameterize via a conformal diffeomorphism to take $\{p_j\}$ to $\{q_j\}$. Thus we may as well suppose that $\varphi_\nu \in \Sigma_d$. Using Proposition 6.7 and passing to a subsequence, we can assume

$$\varphi_\nu \longrightarrow \varphi \quad \text{in } C(\overline{D}, \mathbb{R}^n), \tag{6.43}$$

and we can furthermore arrange

$$\varphi_\nu \longrightarrow \varphi \quad \text{weakly in } H^1(D, \mathbb{R}^n). \tag{6.44}$$

Of course, by interior estimates for harmonic functions, we have

$$\varphi_\nu \longrightarrow \varphi \quad \text{in } C^\infty(D, \mathbb{R}^n). \tag{6.45}$$

The limit function $\varphi$ is certainly harmonic on $D$. By (6.44), we of course have

$$\vartheta(\varphi) \le \underline{\lim}_{\nu\to\infty}\ \vartheta(\varphi_\nu) = d_\gamma. \tag{6.46}$$

Now (6.34) applies to $\varphi$, so we have

$$\alpha(\varphi) \le \frac{1}{2}\vartheta(\varphi) \le \frac{1}{2}d_\gamma = \mathcal{A}_\gamma. \tag{6.47}$$

On the other hand, (6.43) implies that $\varphi : \partial D \to \gamma$ is monotone. Thus $\varphi$ belongs to $\mathfrak{X}_\gamma$. Hence we have

$$\alpha(\varphi) = \mathcal{A}_\gamma. \tag{6.48}$$

This proves Theorem 6.5 and most of the following more precise result.

**Theorem 6.8.** *If $\gamma$ is a smooth, simple, closed curve in $\mathbb{R}^n$, there exists a continuous map $\varphi : \overline{D} \to \mathbb{R}^n$ such that*

$$\vartheta(\varphi) = d_\gamma \ \text{ and } \ \alpha(\varphi) = \mathcal{A}_\gamma, \tag{6.49}$$

$$\varphi : D \longrightarrow \mathbb{R}^n \text{ is harmonic and essentially conformal,} \tag{6.50}$$

$$\varphi : S^1 \longrightarrow \gamma, \quad \textit{homeomorphically.} \tag{6.51}$$

**Proof.** We have (6.49) from (6.46)–(6.48). By the argument involving (6.31) and (6.32), this forces $\varphi$ to be essentially conformal. It remains to demonstrate (6.51).

We know that $\varphi : S^1 \to \gamma$, monotonically. If it fails to be a homeomorphism, there must be an interval $I \subset S^1$ on which $\varphi$ is constant. Using a linear fractional transformation to map $D$ conformally onto the upper half-plane $\Omega^+ \subset \mathbb{C}$, we can regard $\varphi$ as a harmonic and essentially conformal map of $\Omega^+ \to \mathbb{R}^n$, constant on an interval $I$ on the real axis $\mathbb{R}$. Via the Schwartz reflection principle, we can extend $\varphi$ to a harmonic function

$$\varphi : \mathbb{C} \setminus (\mathbb{R} \setminus I) \longrightarrow \mathbb{R}^n.$$

Now consider the holomorphic function $\psi : \mathbb{C} \setminus (\mathbb{R} \setminus I) \to \mathbb{C}^n$, given by $\psi(\zeta) = \partial\varphi/\partial\zeta$. As in the calculations leading to Proposition 6.3, the identities

$$|\partial_1\varphi|^2 - |\partial_2\varphi|^2 = 0, \quad \partial_1\varphi \cdot \partial_2\varphi = 0, \tag{6.52}$$

which hold on $\Omega^+$, imply $\sum_{j=1}^n \psi_j(\zeta)^2 = 0$ on $\Omega^+$; hence this holds on $\mathbb{C} \setminus (\mathbb{R} \setminus I)$, and so does (6.52). But since $\partial_1\varphi = 0$ on $I$, we deduce that $\partial_2\varphi = 0$ on $I$, hence $\psi = 0$ on $I$, hence $\psi \equiv 0$. This implies that $\varphi$, being both $\mathbb{R}^n$-valued and antiholomorphic, must be constant, which is impossible. This contradiction establishes (6.51).

Theorem 6.8 furnishes a generalized minimal surface whose boundary is a given smooth, closed curve in $\mathbb{R}^n$. We know that $\varphi$ is smooth on $D$. It has been shown by [Hild] that $\varphi$ is $C^\infty$ on $\overline{D}$ when the curve $\gamma$ is $C^\infty$, as we have assumed here. It should be mentioned that Douglas and others treated the Plateau problem for simple, closed curves $\gamma$ that were not smooth. We have restricted attention to smooth $\gamma$ for simplicity. A treatment of the general case can be found in [Nit1]; see also [Nit2].

There remains the question of the smoothness of the image surface $M = \varphi(D)$. The map $\varphi : D \to \mathbb{R}^n$ would fail to be an immersion at a point $z \in D$ where $\nabla\varphi(z) = 0$. At such a point, the $\mathbb{C}^n$-valued holomorphic function $\psi = \partial\varphi/\partial\zeta$ must vanish; that is, each of its components must vanish. Since a holomorphic function on $D \subset \mathbb{C}$ that is not identically zero can vanish only on a discrete set, we have the following:

**Proposition 6.9.** *The map $\varphi : D \to \mathbb{R}^n$ parameterizing the generalized minimal surface in Theorem 6.8 has injective derivative except at a discrete set of points in $D$.*

If $\nabla\varphi(z) = 0$, then $\varphi(z) \in M = \varphi(D)$ is said to be a *branch point* of the generalized minimal surface $M$; we say $M$ is a branched surface. If

$n \geq 4$, there are indeed generalized minimal surfaces with branch points that arise via Theorem 6.8. Results of Osserman [Oss2], complemented by [Gul], show that if $n = 3$, the construction of Theorem 6.8 yields a smooth minimal surface, immersed in $\mathbb{R}^3$. Such a minimal surface need not be imbedded; for example, if $\gamma$ is a knot in $\mathbb{R}^3$, such a surface with boundary equal to $\gamma$ is certainly not imbedded. If $\gamma$ is analytic, it is known that there cannot be branch points on the boundary, though this is open for merely smooth $\gamma$. An extensive discussion of boundary regularity is given in Vol. 2 of [DHKW].

The following result of Rado yields one simple criterion for a generalized minimal surface to have no branch points.

**Proposition 6.10.** *Let $\gamma$ be a smooth, closed curve in $\mathbb{R}^n$. If a minimal surface with boundary $\gamma$ produced by Theorem 6.8 has any branch points, then $\gamma$ has the property that*

$$\begin{gathered}\text{for some } p \in \mathbb{R}^n, \text{ every hyperplane through } p \\ \text{intersects } \gamma \text{ in at least four points.}\end{gathered} \tag{6.53}$$

**Proof.** Suppose $z_0 \in D$ and $\nabla\varphi(z_0) = 0$, so $\psi = \partial\varphi/\partial\zeta$ vanishes at $z_0$. Let $L(x) = \alpha \cdot x + c = 0$ be the equation of an arbitrary hyperplane through $p = \varphi(z_0)$. Then $h(x) = L\big(\varphi(x)\big)$ is a (real-valued) harmonic function on $D$, satisfying

$$\Delta h = 0 \text{ on } D, \quad \nabla h(z_0) = 0. \tag{6.54}$$

The proposition is then proved, by the following:

**Lemma 6.11.** *Any real-valued $h \in C^\infty(D) \cap C(\overline{D})$ having the property (6.54) must assume the value $h(z_0)$ on at least four points on $\partial D$.*

We leave the proof as an exercise for the reader.

The following result gives a condition under which a minimal surface constructed by Theorem 6.8 is the graph of a function.

**Proposition 6.12.** *Let $\mathcal{O}$ be a bounded convex domain in $\mathbb{R}^2$ with smooth boundary. Let $g : \partial\mathcal{O} \to \mathbb{R}^{n-2}$ be smooth. Then there exists a function*

$$f \in C^\infty(\mathcal{O}, \mathbb{R}^{n-2}) \cap C(\overline{\mathcal{O}}, \mathbb{R}^{n-2}), \tag{6.55}$$

*whose graph is a minimal surface, and whose boundary is the curve $\gamma \subset \mathbb{R}^n$ that is the graph of $g$, so*

$$f = g \quad \text{on } \partial\mathcal{O}. \tag{6.56}$$

**Proof.** Let $\varphi : \overline{D} \to \mathbb{R}^n$ be the function constructed in Theorem 6.8. Set $F(x) = \big(\varphi_1(x), \varphi_2(x)\big)$. Then $F : \overline{D} \to \mathbb{R}^2$ is harmonic on $D$ and $F$ maps

$S^1 = \partial D$ homeomorphically onto $\partial \mathcal{O}$. It follows from the convexity of $\mathcal{O}$ and the maximum principle for harmonic functions that $F : \overline{D} \to \overline{\mathcal{O}}$.

We claim that $DF(x)$ is invertible for each $x \in D$. Indeed, if $x_0 \in D$ and $DF(x_0)$ is singular, we can choose nonzero $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ such that, at $x = x_0$,

$$\alpha_1 \frac{\partial \varphi_1}{\partial x_j} + \alpha_2 \frac{\partial \varphi_2}{\partial x_j} = 0, \quad j = 1, 2.$$

Then the function $h(x) = \alpha_1 \varphi_1(x) + \alpha_2 \varphi_2(x)$ has the property (6.54), so $h(x)$ must take the value $h(x_0)$ at four distinct points of $\partial D$. Since $F : \partial D \to \partial \mathcal{O}$ is a homeomorphism, this forces the *linear* function $\alpha_1 x_1 + \alpha_2 x_2$ to take the same value at four distinct points of $\partial \mathcal{O}$, which contradicts the convexity of $\mathcal{O}$.

Thus $F : D \to \mathcal{O}$ is a local diffeomorphism. Since $F$ gives a homeomorphism of the boundaries of these regions, degree theory implies that $F$ is a diffeomorphism of $D$ onto $\mathcal{O}$ and a homeomorphism of $\overline{D}$ onto $\overline{\mathcal{O}}$. Consequently, the desired function in (6.55) is $f = \widetilde{\varphi} \circ F^{-1}$, where $\widetilde{\varphi}(x) = \big(\varphi_2(x), \dots, \varphi_n(x)\big)$.

Functions whose graphs are minimal surfaces satisfy a certain nonlinear PDE, called the *minimal surface equation*, which we will derive and study in §7.

Let us mention that while one ingredient in the solution to the Plateau problem presented above is a version of the Riemann mapping theorem, Proposition 6.4, there are presentations for which the Riemann mapping theorem is a consequence of the argument, rather than an ingredient (see, e.g., [Nit2]).

It is also of interest to consider the analogue of the Plateau problem when, instead of immersing the disk in $\mathbb{R}^n$ as a minimal surface with given boundary, one takes a surface of higher genus, and perhaps several boundary components. An extra complication is that Proposition 6.4 must be replaced by something more elaborate, since two compact surfaces with boundary which are diffeomorphic to each other but not to the disk may not be conformally equivalent. One needs to consider spaces of "moduli" of such surfaces; Theorem 4.2 of Chapter 5 deals with the easiest case after the disk. This problem was tackled by Douglas [Dou2] and by Courant [Cou2], but their work has been criticized by [ToT] and [Jos], who present alternative solutions. The paper [Jos] also treats the Plateau problem for surfaces in Riemannian manifolds, extending results of [Mor1].

There have been successful attacks on problems in the theory of minimal submanifolds, particularly in higher dimension, using very different techniques, involving geometric measure theory, currents, and varifolds. Material on these important developments can be found in [Alm], [Fed], and [Morg].

So far in this section, we have devoted all our attention to minimal submanifolds of Euclidean space. It is also interesting to consider minimal submanifolds of other Riemannian manifolds. We make a few brief comments on this topic. A great deal more can be found in [Cher], [Law], [Law2], [Mor1], and [Pi], and in survey articles in [Bom].

Let $Y$ be a smooth, compact Riemannian manifold. Assume $Y$ is isometrically imbedded in $\mathbb{R}^n$, which can always be arranged, by Nash's theorem. Let $M$ be a compact, $k$-dimensional submanifold of $Y$. We say $M$ is a minimal submanifold of $Y$ if its $k$-dimensional volume is a critical point with respect to small variations of $M$, within $Y$. The computations in (6.1)–(6.13) extend to this case. We need to take $X = X(s,u)$ with $\partial_s X(s,u) = \xi(s,u)$, tangent to $Y$, rather than $X(s,u) = X_0(u) + s\xi(u)$. Then these computations show that $M$ is a minimal submanifold of $Y$ if and only if, for each $x \in M$,

$$\mathfrak{H}(x) \perp T_x Y, \tag{6.57}$$

where $\mathfrak{H}(x)$ is the mean curvature vector of $M$ (as a submanifold of $\mathbb{R}^n$), defined by (6.13).

There is also a well-defined mean curvature vector $\mathfrak{H}_Y(x) \in T_x Y$, orthogonal to $T_x M$, obtained from the second fundamental form of $M$ as a submanifold of $Y$. One sees that $\mathfrak{H}_Y(x)$ is the orthogonal projection of $\mathfrak{H}(x)$ onto $T_x Y$, so the condition that $M$ be a minimal submanifold of $Y$ is that $\mathfrak{H}_Y = 0$ on $M$.

The formula (6.10) continues to hold for the isometric imbedding $X : M \to \mathbb{R}^n$. Thus $M$ is a minimal submanifold of $Y$ if and only if, for each $x \in M$,

$$\Delta X(x) \perp T_x Y. \tag{6.58}$$

If dim $M = 2$, the formula (6.15) holds, so if $M$ is given a new metric, conformally scaled by a factor $e^{2u}$, the new Laplace operator $\Delta_1$ has the property that $\Delta_1 X = e^{-2u}\Delta X$, hence is parallel to $\Delta X$. Thus the property (6.58) is unaffected by such a conformal change of metric; we have the following extension of Proposition 6.2:

**Proposition 6.13.** *If $M$ is a Riemannian manifold of dimension 2 and $X : M \to \mathbb{R}^n$ is a smooth imbedding, with image $M_1 \subset Y$, then $M_1$ is a minimal submanifold of $Y$ provided $X : M \to M_1$ is conformal and, for each $x \in M$,*

$$\Delta X(x) \perp T_{X(x)} Y. \tag{6.59}$$

We note that (6.59) alone specifies that $X$ is a *harmonic* map from $M$ into $Y$. Harmonic maps will be considered further in §§11 and 12B; they will also be studied, via parabolic PDE, in Chapter 15, §2.

## Exercises

1. Consider the Gauss map $N : M \to S^2$, for a smooth, oriented surface $M \subset \mathbb{R}^3$. Show that $N$ is *antiholomorphic* if and only if $M$ is a minimal surface. (*Hint*: If $N(p) = q$, $DN(p) : T_pM \to T_qS^2 \approx T_pM$ is identified with $-A_N$. Compare (4.67) in Appendix C. Check when $A_N J = -JA_N$, where $J$ is counterclockwise rotation by $90^\circ$, on $T_pM$.) Thus, if we define the antipodal Gauss map $\widetilde{N} : M \to S^2$ by $\widetilde{N}(p) = -N(p)$, this map is holomorphic precisely when $M$ is a minimal surface.
2. If $x \in S^2 \subset \mathbb{R}^3$, pick $v \in T_xS^2 \subset \mathbb{R}^3$, set $w = Jv \in T_xS^2 \subset \mathbb{R}^3$, and take $\xi = v + iw \in \mathbb{C}^3$. Show that the one-dimensional, complex span of $\xi$ is independent of the choice of $v$, and that we hence have a holomorphic map
$$\Xi : S^2 \longrightarrow \mathbb{CP}^3.$$
Show that the image $\Xi(S^2) \subset \mathbb{CP}^3$ is contained in the image of $\{\zeta \in \mathbb{C}^3 \setminus 0 : \zeta_1^2 + \zeta_2^2 + \zeta_3^3 = 0\}$ under the natural map $\mathbb{C}^3 \setminus 0 \to \mathbb{CP}^3$.
3. Suppose that $M \subset \mathbb{R}^3$ is a minimal surface constructed by the method of Proposition 6.3, via $X : \Omega \to M \subset \mathbb{R}^3$. Define $\Psi : \Omega \to \mathbb{C}^3 \setminus 0$ by $\Psi = (\psi_1, \psi_2, \psi_3)$, and define $\mathfrak{X} : \Omega \to \mathbb{CP}^3$ by composing $\Psi$ with the natural map $\mathbb{C}^3 \setminus 0 \to \mathbb{CP}^3$. Show that, for $u \in \Omega$,
$$\mathfrak{X}(u) = \Xi \circ \widetilde{N}\Big(X(u)\Big).$$
For the relation between $\psi_j$ and the Gauss map for minimal surfaces in $\mathbb{R}^n$, $n > 3$, see [Law].
4. Give a detailed demonstration of (6.38).
5. In analogy with Proposition 6.4, extend Theorem 4.3 of Chapter 5 to the following result:

   **Proposition.** *If $\overline{M}$ is a compact Riemannian manifold of dimension 2 which is homeomorphic to an annulus, then there exists a conformal diffeomorphism*
$$\Psi : \overline{M} \longrightarrow \overline{\mathfrak{A}}_\rho,$$
   *for a unique $\rho \in (0,1)$, where $\overline{\mathfrak{A}}_\rho = \{z \in \mathbb{C} : \rho \le |z| \le 1\}$.*

6. If $\widetilde{II}$ is the second fundamental form of a minimal hypersurface $M \subset \mathbb{R}^n$, show that $\widetilde{II}$ has divergence zero. As in Chapter 2, §3, we define the divergence of a second-order tensor field $T$ by $T^{jk}{}_{;k}$. (*Hint*: Use the Codazzi equation (cf. Appendix C, §4, especially (4.18)) plus the zero trace condition.)
7. Similarly, if $\widetilde{II}$ is the second fundamental form of a minimal submanifold $M$ of codimension 1 in $S^n$ (with its standard metric), show that $\widetilde{II}$ has divergence zero.
(*Hint*: The Codazzi equation, from (4.16) of Appendix C, is
$$(\nabla_Y \widetilde{II})(X, Z) - (\nabla_Y \widetilde{II})(Y, Z) = \langle R(X, Y)Z, N\rangle,$$
where $\nabla$ is the Levi-Civita connection on $M$; $X, Y, Z$ are tangent to $M$; $Z$ is normal to $M$ (but tangent to $S^n$); and $R$ is the curvature tensor of $S^n$. In such a case, the right side vanishes. (See Exercise 6 in §4 of Appendix C.) Thus the argument needed for Exercise 6 above extends.)

8. Extend the result of Exercises 6–7 to the case where $M$ is a codimension-1 minimal submanifold in any Riemannian manifold $\Omega$ with constant sectional curvature.
9. Let $M$ be a two-dimensional minimal submanifold of $S^3$, with its standard metric. Assume $M$ is diffeomorphic to $S^2$. Show that $M$ must be a "great sphere" in $S^3$.
(*Hint*: By Exercise 7, $\widetilde{II}$ is a symmetric trace free tensor of divergence zero; that is, $\widetilde{II}$ belongs to
$$\mathcal{V} = \{u \in C^\infty(M, S_0^2 T^*) : \text{div } u = 0\},$$
a space introduced in (10.47) of Chapter 10. As noted there, when $M$ is a Riemann surface, $\mathcal{V} \approx \mathcal{O}(\kappa \otimes \kappa)$. By Corollary 9.4 of Chapter 10, $\mathcal{O}(\kappa \otimes \kappa) = 0$ when $M$ has genus $g = 0$.)
10. Prove Lemma 6.11.

## 6B. Second variation of area

In this appendix to §6, we take up a computation of the second variation of the area integral, and some implications, for a family of manifolds of dimension $k$, immersed in a Riemannian manifold $Y$. First, we take $Y = \mathbb{R}^n$ and suppose the family is given by $X(s,u) = X_0(u) + s\xi(u)$, as in (6.1)–(6.5).

Suppose, as in the computation (6.2)–(6.5), that $\|\partial_1 X_0 \wedge \cdots \wedge \partial_k X_0\| = 1$ on $M$, while $E_j = \partial_j X_0$ form an orthonormal basis of $T_x M$, for a given point $x \in M$. Then, extending (6.3), we have

$$A'(s) = \sum_{j=1}^k \int \frac{\langle \partial_1 X \wedge \cdots \wedge \partial_j \xi \wedge \cdots \wedge \partial_k X, \partial_1 X \wedge \cdots \wedge \partial_k X \rangle}{\|\partial_1 X \wedge \cdots \wedge \partial_k X\|} \, du_1 \cdots du_k. \tag{6b.1}$$

Consequently, $A''(0)$ will be the integral with respect to $du_1 \cdots du_k$ of a sum of three terms:

$$\begin{aligned} &-\sum_{i,j} \langle \partial_1 X_0 \wedge \cdots \wedge \partial_i \xi \wedge \cdots \wedge \partial_k X_0, \partial_1 X_0 \wedge \cdots \wedge \partial_k X_0 \rangle \\ &\qquad \times \langle \partial_1 X_0 \wedge \cdots \wedge \partial_j \xi \wedge \cdots \wedge \partial_k X_0, \partial_1 X_0 \wedge \cdots \wedge \partial_k X_0 \rangle \\ &+ 2\sum_{i<j} \langle \partial_1 X_0 \wedge \cdots \wedge \partial_i \xi \wedge \cdots \wedge \partial_j \xi \wedge \cdots \wedge \partial_k X_0, \partial_1 X_0 \wedge \cdots \wedge \partial_k X_0 \rangle \\ &+ \sum_{i,j} \langle \partial_1 X_0 \wedge \cdots \wedge \partial_j \xi \wedge \cdots \wedge \partial_k X_0, \partial_1 X_0 \wedge \cdots \wedge \partial_i \xi \wedge \cdots \wedge \partial_k X_0 \rangle. \end{aligned} \tag{6b.2}$$

Let us write

$$A_\xi E_i = \sum_\ell a_\xi^{i\ell} E_\ell, \tag{6b.3}$$

with $E_j = \partial_j X_0$ as before. Then, as in (6.4), the first sum in (6.b2) is equal to

$$-\sum_{i,j} a_\xi^{ii} a_\xi^{jj}. \tag{6b.4}$$

Let us move to the last sum in (6b.2). We use the Weingarten formula $\partial_j \xi = \nabla_j^1 \xi - A_\xi E_j$, to write this sum as

$$\sum_{i,j} a_\xi^{jj} a_\xi^{ii} + \sum_{i,j} \langle \nabla_j^1 \xi, \nabla_i^1 \xi \rangle, \tag{6b.5}$$

at $x$. Note that the first sum in (6b.5) cancels (6b.4), while the last sum in (6b.5) can be written as $\|\nabla^1 \xi\|^2$. Here, $\nabla^1$ is the connection induced on the normal bundle of $M$.

Now we look at the middle term in (6b.2), namely,

$$2 \sum_{i<j} \sum_{\ell,m} a_\xi^{i\ell} a_\xi^{jm} \langle E_1 \wedge \cdots \wedge E_\ell \wedge \cdots \wedge E_m \wedge \cdots \wedge E_k, E_1 \wedge \cdots \wedge E_k \rangle, \tag{6b.6}$$

at $x$, where $E_\ell$ appears in the $i$th slot and $E_m$ appears in the $j$th slot in the $k$-fold wedge product. This is equal to

$$2 \sum_{i<j} \bigl( a_\xi^{ii} a_\xi^{jj} - a_\xi^{ij} a_\xi^{ji} \bigr) = 2 \ \mathrm{Tr}\ \Lambda^2 A_\xi, \tag{6b.7}$$

at $x$. Thus we have

$$A''(0) = \int_M \Bigl[ \|\nabla^1 \xi\|^2 + 2 \ \mathrm{Tr}\ \Lambda^2 A_\xi \Bigr] \, dA(x). \tag{6b.8}$$

If $M$ is a hypersurface of $\mathbb{R}^n$, and we take $\xi = fN$, where $N$ is a unit normal field, then $\|\nabla^1 \xi\|^2 = \|\nabla f\|^2$ and (6b.7) is equal to

$$2 \sum_{i<j} \langle R(E_j, E_i) E_i, E_j \rangle f^2 = S f^2, \tag{6b.9}$$

by the *Theorema Egregium*, where $S$ is the scalar curvature of $M$. Consequently, if $M \subset \mathbb{R}^n$ is a hypersurface (with boundary), and the hypersurfaces $M_s$ are given by (6.6), with area integral (6.2), then

$$A''(0) = \int_M \Bigl[ \|\nabla f\|^2 + S(x) f^2 \Bigr] \, dA(x). \tag{6b.10}$$

Recall that when dim $M = 2$, so $M \subset \mathbb{R}^3$,

$$S = 2K, \tag{6b.11}$$

where $K$ is the Gauss curvature, which is $\le 0$ whenever $M$ is a minimal surface in $\mathbb{R}^3$.

If $M$ has general codimension in $\mathbb{R}^n$, we can rewrite (6b.8) using the identity

$$2 \text{ Tr } \Lambda^2 A_\xi = (\text{Tr } A_\xi)^2 - \|A_\xi\|^2, \tag{6b.12}$$

where $\|A_\xi\|$ denotes the Hilbert-Schmidt norm of $A_\xi$, that is,

$$\|A_\xi\|^2 = \text{ Tr}(A_\xi^* A_\xi).$$

Recalling (6.13), if $k = \dim M$, we get

$$A''(0) = \int_M \Big[ \|\nabla^1 \xi\|^2 - \|A_\xi\|^2 + k^2 \langle \mathfrak{H}(x), \xi\rangle^2 \Big] \, dA(x). \tag{6b.13}$$

Of course, the last term in the integrand vanishes for all compactly supported fields $\xi$ normal to $M$ when $M$ is a minimal submanifold of $\mathbb{R}^n$.

We next suppose the family of manifolds $M_s$ is contained in a manifold $Y \subset \mathbb{R}^n$. Hence, as before, instead of $X(s,u) = X_0(u) + s\xi(u)$, we require $\partial_s X(s,u) = \xi(s,u)$ to be tangent to $Y$. We take $X(0,u) = X_0(u)$. Then (6b.1) holds, and we need to add to (6b.2) the following term, in order to compute $A''(0)$:

$$\begin{aligned} \Phi &= \sum_{j=1}^k \big\langle \partial_1 X_0 \wedge \cdots \wedge \partial_j \kappa \wedge \cdots \wedge \partial_k X_0, \partial_1 X_0 \wedge \cdots \wedge \partial_k X_0 \big\rangle, \\ \kappa &= \partial_s \xi = \partial_s^2 X. \end{aligned} \tag{6b.14}$$

If, as before, $\partial_j X_0 = E_j$ form an orthonormal basis of $T_x M$, for a given $x \in M$, then

$$\Phi = \sum_{j=1}^k \langle \partial_j \kappa, E_j \rangle, \quad \text{at } x. \tag{6b.15}$$

Now, given the compactly supported field $\xi(0,u)$, tangent to $Y$ and normal to $M$, let us suppose that, for each $u$, $\gamma_u(s) = X(s,u)$ is a constant-speed geodesic in $Y$, such that $\gamma_u'(0) = \xi(0,u)$. Thus $\kappa = \gamma_u''(0)$ is normal to $Y$, and, by the Weingarten formula for $M \subset \mathbb{R}^n$,

$$\partial_j \kappa = \nabla^1_{E_j} \kappa - A_\kappa E_j, \tag{6b.16}$$

at $x$, where $\nabla^1$ is the connection on the normal bundle to $M \subset \mathbb{R}^n$ and $A$ is as before the Weingarten map for $M \subset \mathbb{R}^n$. Thus

$$\Phi = -\sum_j \langle A_\kappa E_j, E_j \rangle = -\text{Tr } A_\kappa = -k \langle \mathfrak{H}(x), \kappa \rangle, \tag{6b.17}$$

where $k = \dim M$.

If we suppose $M$ is a minimal submanifold of $Y$, then $\mathfrak{H}(x)$ is normal to $Y$, so, for any compactly supported field $\xi$, normal to $M$ and tangent to

$Y$, the computation (6b.13) supplemented by (6b.14)–(6b.17) gives

$$A''(0) = \int_M \Big[ \|\nabla^1 \xi\|^2 - \|A_\xi\|^2 - k\langle \mathfrak{H}(x), \kappa \rangle \Big] \, dA(x). \tag{6b.18}$$

Recall that $A_\xi$ is the Weingarten map of $M \subset \mathbb{R}^n$.

We prefer to use $B_\xi$, the Weingarten map of $M \subset Y$. It is readily verified that

$$A_\xi = B_\xi \in \text{ End } T_x M \tag{6b.19}$$

if $\xi \in T_x Y$ and $\xi \perp T_x M$; see Exercise 13 in §4 of Appendix C. Thus in (6b.18) we can simply replace $\|A_\xi\|^2$ by $\|B_\xi\|^2$. Also recall that $\nabla^1$ in (6b.18) is the connection on the normal bundle to $M \subset \mathbb{R}^n$. We prefer to use the connection on the normal bundle to $M \subset Y$, which we denote by $\nabla^\#$. To relate these two objects, we use the identities

$$\begin{gathered} \partial_j \xi = \nabla^1_j \xi - A_\xi E_j, \quad \partial_j \xi = \widetilde{\nabla}_j \xi + II^Y(E_j, \xi), \\ \widetilde{\nabla}_j \xi = \nabla^\#_j \xi - B_\xi E_j, \end{gathered} \tag{6b.20}$$

where $\widetilde{\nabla}$ denotes the covariant derivative on $Y$, and $II^Y$ is the second fundamental form of $Y \subset \mathbb{R}^n$. In view of (6b.19), we obtain

$$\nabla^1_j \xi = \nabla^\#_j \xi + II^Y(E_j, \xi), \tag{6b.21}$$

a sum of terms tangent to $Y$ and normal to $Y$, respectively. Hence

$$\|\nabla^1 \xi\|^2 = \|\nabla^\# \xi\|^2 + \sum_j \|II^Y(E_j, \xi)\|^2. \tag{6b.22}$$

Thus we can rewrite (6b.18) as

$$A''(0) = \int_M \Big[ \|\nabla^\# \xi\|^2 - \|B_\xi\|^2 + \sum_j \|II^Y(E_j, \xi)\|^2 - \text{ Tr } A_\kappa \Big] \, dA(x). \tag{6b.23}$$

We want to replace the last two terms in this integrand by a quantity defined intrinsically by $M_s \subset Y$, not by the way $Y$ is imbedded in $\mathbb{R}^n$. Now Tr $A_\kappa = \sum \langle II^M(E_j, E_j), \kappa \rangle$, where $II^M$ is the second fundamental form of $M \subset \mathbb{R}^n$. On the other hand, it is easily verified that

$$\kappa = \gamma''_u(0) = II^Y(\xi, \xi). \tag{6b.24}$$

Thus the last two terms in the integrand sum to

$$\Psi = \sum_j \Big[ \|II^Y(E_j, \xi)\|^2 - \langle II^Y(\xi, \xi), II^M(E_j, E_j) \rangle \Big]. \tag{6b.25}$$

We can replace $II^M(E_j, E_j)$ by $II^Y(E_j, E_j)$ here, since these two objects have the same component normal to $Y$. Then Gauss' formula implies

$$\Psi = \sum_j \langle R^Y(\xi, E_j)\xi, E_j \rangle, \tag{6b.26}$$

where $R^Y$ is the Riemann curvature tensor of $Y$. We define $\overline{\mathfrak{R}} \in \operatorname{End} N_x M$, where $N(M)$ is the normal bundle of $N \subset Y$, by

$$\langle \overline{\mathfrak{R}}(\xi), \eta \rangle = \sum_j \langle R^Y(\xi, E_j)\eta, E_j \rangle, \tag{6b.27}$$

at $x$, where $\{E_j\}$ is an orthonormal basis of $T_x M$. It follows easily that this is independent of the choice of such an orthonormal basis.

Our calculation of $A''(0)$ becomes

$$A''(0) = \int_M \Big[ \|\nabla^\# \xi\|^2 - \|B_\xi\|^2 + \big\langle \overline{\mathfrak{R}}(\xi), \xi \big\rangle \Big] \, dA(x) \tag{6b.28}$$

when $M$ is a minimal submanifold of $Y$, where $\nabla^\#$ is the connection on the normal bundle to $M \subset Y$, $B$ is the Weingarten map for $M \subset Y$, and $\overline{\mathfrak{R}}$ is defined by (6b.27). If we define a second-order differential operator $\mathfrak{L}_0$ and a zero-order operator $\mathfrak{B}$ on $C_0^\infty\big(M, N(M)\big)$ by

$$\mathfrak{L}_0 \xi = (\nabla^\#)^* \nabla^\# \xi, \quad \langle \mathfrak{B}(\xi), \eta \rangle = \operatorname{Tr}(B_\eta^* B_\xi), \tag{6b.29}$$

respectively, we can write this as

$$A''(0) = (\mathfrak{L}\xi, \xi)_{L^2(M)}, \quad \mathfrak{L}\xi = \mathfrak{L}_0 \xi - \mathfrak{B}(\xi) + \overline{\mathfrak{R}}(\xi). \tag{6b.30}$$

We emphasize that these formulas, and the ones below, for $A''(0)$ are valid for immersed minimal submanifolds of $Y$ as well as for imbedded submanifolds.

Suppose that $M$ has codimension 1 in $Y$ and that $Y$ and $M$ are orientable. Complete the basis $\{E_j\}$ of $T_x M$ to an orthonormal basis

$$\{E_j : 1 \le j \le k+1\}$$

of $T_x Y$. In this case, $E_{k+1}(x)$ and $\xi(x)$ are parallel, so

$$\langle R^Y(\xi, E_{k+1})\eta, E_{k+1} \rangle = 0.$$

Thus (6b.27) becomes

$$\overline{\mathfrak{R}}(\xi) = -\operatorname{Ric}^Y \xi \quad \text{if } \dim Y = \dim M + 1, \tag{6b.31}$$

where $\operatorname{Ric}^Y$ denotes the Ricci tensor of $Y$. In such a case, taking $\xi = f E_{k+1} = f\nu$, where $\nu$ is a unit normal field to $M$, tangent to $Y$, we obtain

$$\begin{aligned} A''(0) &= \int_M \Big[ \|\nabla f\|^2 - \big( \|B_\nu\|^2 + \langle \operatorname{Ric}^Y \nu, \nu \rangle \big) |f|^2 \Big] \, dA(x) \\ &= (Lf, f)_{L^2(M)}, \end{aligned} \tag{6b.32}$$

where

$$Lf = -\Delta f + \varphi f, \quad \varphi = -\|B_\nu\|^2 - \langle \operatorname{Ric}^Y \nu, \nu \rangle. \tag{6b.33}$$

We can express $\varphi$ in a different form, noting that

$$\langle \operatorname{Ric}^Y \nu, \nu \rangle = S^Y - \sum_{j=1}^{k} \langle \operatorname{Ric}^Y E_j, E_j \rangle, \tag{6b.34}$$

where $S^Y$ is the scalar curvature of $Y$. From Gauss' formula we readily obtain, for general $M \subset Y$ of any codimension,

$$\begin{aligned} \langle \operatorname{Ric}^Y E_j, E_j \rangle = \langle R^Y(E_j, \nu)\nu, E_j \rangle + \langle \operatorname{Ric}^M E_j, E_j \rangle \\ + \sum_{\ell} \| II(E_j, E_\ell) \|^2 - k \big\langle \mathfrak{H}_Y, II(E_j, E_j) \big\rangle, \end{aligned} \tag{6b.35}$$

where $II$ denotes the second fundamental form of $M \subset Y$. Summing over $1 \le j \le k$, when $M$ has codimension 1 in $Y$, and $\nu$ is a unit normal to $M$, we get

$$2\langle \operatorname{Ric}^Y \nu, \nu \rangle = S^Y - S^M - \|B_\nu\|^2 + \|\mathfrak{H}_Y\|^2. \tag{6b.36}$$

If $M$ is a minimal submanifold of $Y$ of codimension 1, this implies that

$$\begin{aligned} \varphi &= \frac{1}{2}(S^M - S^Y) - \frac{1}{2}\|B_\nu\|^2 \\ &= \frac{1}{2}(S^M - S^Y) + \operatorname{Tr} \Lambda^2 B_\nu. \end{aligned} \tag{6b.37}$$

We also note that when dim $M = 2$ and dim $Y = 3$, then, for $x \in M$,

$$\operatorname{Tr} \Lambda^2 B_\nu(x) = K^M(x) - K^Y(T_x M), \tag{6b.38}$$

where $K^M = (1/2)S^M$ is the Gauss curvature of $M$ and $K^Y(T_xM)$ is the sectional curvature of $Y$, along the plane $T_xM$.

We consider another special case, where dim $M = 1$. We have $\langle \overline{\mathfrak{R}}(\xi), \xi \rangle = -|\xi|^2 K^Y(\Pi_{M\xi})$, where $K^Y(\Pi_{M\xi})$ is the sectional curvature of $Y$ along the plane in $T_xY$ spanned by $T_xM$ and $\xi$. In this case, to say $M$ is minimal is to say it is a geodesic; hence $B_\xi = 0$ and $\nabla^\# \xi = \widetilde{\nabla}_T \xi$, where $\widetilde{\nabla}$ is the covariant derivative on $Y$, and $T$ is a unit tangent vector to $M$. Thus (6b.28) becomes the familiar formula for the second variation of arc length for a geodesic:

$$\ell''(0) = \int_\gamma \Big[ \|\widetilde{\nabla}_T \xi\|^2 - |\xi|^2 K^Y(\Pi_{\gamma\xi}) \Big] \, ds, \tag{6b.39}$$

where we have used $\gamma$ instead of $M$ to denote the curve, and also $\ell$ instead of $A$ and $ds$ instead of $dA$, to denote arc length.

The operators $\mathfrak{L}$ and $L$ are second-order elliptic operators that are self-adjoint, with domain $H^2(M)$, if $M$ is compact and without boundary, and with domain $H^2(M) \cap H_0^1(M)$, if $\overline{M}$ is compact with boundary. In such cases, the spectra of these operators consist of eigenvalues $\lambda_j \nearrow +\infty$. If $M$ is not compact, but $B$ and $\overline{\mathfrak{R}}$ are bounded, we can use the Friedrichs method to define self-adjoint extensions $\mathfrak{L}$ and $L$, which might have continuous spectrum.

We say a minimal submanifold $M \subset Y$ is *stable* if $A''(0) \geq 0$ for all smooth, compactly supported variations $\xi$, normal to $M$ (and vanishing on $\partial M$). Thus the condition that $M$ be stable is that the spectrum of $\mathfrak{L}$ (equivalently, of $L$, if codim $M = 1$) be contained in $[0, \infty)$. In particular, if $M$ is actually area minimizing with respect to small perturbations, leaving $\partial M$ fixed (which we will just call "area minimizing"), then it must be stable, so

$$M \text{ area minimizing} \implies \operatorname{spec} \mathfrak{L} \subset [0, \infty). \tag{6b.40}$$

The second variational formulas above provide necessary conditions for a minimal immersed submanifold to be stable. For example, suppose $M$ is a boundaryless, codimension-1 minimal submanifold of $Y$, and both are orientable. Then we can take $f = 1$ in (6b.32), to get

$$M \text{ stable} \implies \int_M \Big( \|B_\nu\|^2 + \langle \operatorname{Ric}^Y \nu, \nu \rangle \Big)\, dA \leq 0. \tag{6b.41}$$

If dim $M = 2$ and dim $Y = 3$, then, by (6b.37), we have

$$M \text{ stable} \implies \int_M \Big( \|B_\nu\|^2 + S^Y - 2K^M \Big)\, dA \leq 0. \tag{6b.42}$$

In this case, if $M$ has genus $g$, the Gauss-Bonnet theorem implies that $\int K^M \, dA = 4\pi(1-g)$, so

$$M \text{ stable} \implies \int_M \Big( \|B_\nu\|^2 + S^Y \Big)\, dA \leq 8\pi(1-g). \tag{6b.43}$$

This implies some nonexistence results.

**Proposition 6B.1.** *Assume that $Y$ is a compact, oriented Riemannian manifold and that $Y$ and $M$ have no boundary.*

*If the Ricci tensor $Ric^Y$ is positive-definite, then $Y$ cannot contain any compact, oriented, area-minimizing immersed hypersurface $M$. If $Ric^Y$ is positive-semidefinite, then any such $M$ would have to be totally geodesic in $Y$.*

*Now assume dim $Y = 3$. If $Y$ has scalar curvature $S^Y > 0$ everywhere, then $Y$ cannot contain any compact, oriented, area-minimizing immersed surface $M$ of genus $g \geq 1$.*

*More generally, if $S^Y \geq 0$ everywhere, and if $M$ is a compact, oriented, immersed hypersurface of genus $g \geq 1$, then for $M$ to be area minimizing it is necessary that $g = 1$ and that $M$ be totally geodesic in $Y$.*

R. Schoen and S.-T. Yau [SY] obtained topological consequences for a compact, oriented 3-manifold $Y$ from this together with the following existence theorem. Suppose $M$ is a compact, oriented surface of genus $g \geq 1$, and suppose the fundamental group $\pi_1(Y)$ contains a subgroup isomorphic to $\pi_1(M)$. Then, given any Riemannian metric on $Y$, there is a smooth immersion of $M$ into $Y$ which is area minimizing with respect to small perturbations, as shown in [SY]. It follows that if $Y$ is a compact, oriented Riemannian 3-manifold, whose scalar curvature $S^Y$ is everywhere positive, then $\pi_1(Y)$ cannot have a subgroup isomorphic to $\pi_1(M)$, for any compact Riemann surface $M$ of genus $g \geq 1$.

We will not prove the result of [SY] on the existence of such minimal immersions. Instead, we demonstrate a topological result, due to Synge, of a similar flavor but simpler to prove. It makes use of the second variational formula (6b.39) for arc length.

**Proposition 6B.2.** *If $Y$ is a compact, oriented Riemannian manifold of even dimension, with positive sectional curvature everywhere, then $Y$ is simply connected.*

**Proof.** It is a simple consequence of Ascoli's theorem that there is a length-minimizing, closed geodesic in each homotopy class of maps from $S^1$ to $Y$. Thus, if $\pi_1(Y) \neq 0$, there is a nontrivial stable geodesic, $\gamma$. Pick $p \in \gamma$, $\xi_p$ normal to $\gamma$ at $p$ (i.e., $\xi_p \in N_p(\gamma)$), and parallel translate $\xi$ about $\gamma$, obtaining $\overline{\xi}_p \in N_p(\gamma)$ after one circuit. This defines an orientation-preserving, orthogonal, linear transformation $\tau : N_p\gamma \to N_p\gamma$. If $Y$ has dimension $2k$, then $N_p\gamma$ has dimension $2k-1$, so $\tau \in \mathrm{SO}(2k-1)$. It follows that $\tau$ must have an eigenvector in $N_p\gamma$, with eigenvalue 1. Thus we get a nontrivial, smooth section $\xi$ of $N(\gamma)$ which is parallel over $\gamma$, so (6b.39) implies

$$\int_\gamma K^Y(\Pi_{\gamma\xi})\, ds \leq 0. \tag{6b.44}$$

If $K^Y(\Pi) > 0$ everywhere, this is impossible.

One might compare these results with Proposition 4.7 of Chapter 10, which states that if $Y$ is a compact Riemannian manifold and $\mathrm{Ric}^Y > 0$, then the first cohomology group $\mathcal{H}^1(Y) = 0$.

## 7. The minimal surface equation

We now study a nonlinear PDE for functions whose graphs are minimal surfaces. We begin with a formula for the mean curvature of a hypersurface $M \subset \mathbb{R}^{n+1}$ defined by $u(x) = c$, where $\nabla u \neq 0$ on $M$. If $N = \nabla u/|\nabla u|$, we have the formula

$$\langle A_N X, Y\rangle = -|\nabla u|^{-1}(D^2 u)(X, Y), \tag{7.1}$$

for $X, Y \in T_x M$, as shown in (4.26) of Appendix C. To take the trace of the restriction of $D^2 u$ to $T_x M$, we merely take $\mathrm{Tr}(D^2 u) - D^2 u(N, N)$. Of course, $\mathrm{Tr}(D^2 u) = \Delta u$. Thus, for $x \in M$,

$$\text{(7.2)} \qquad \mathrm{Tr}\, A_N(x) = -|\nabla u(x)|^{-1} \Big[\Delta u - |\nabla u|^{-2} D^2 u(\nabla u, \nabla u)\Big].$$

Suppose now that $M$ is given by the equation

$$x_{n+1} = f(x'), \quad x' = (x_1, \dots, x_n).$$

Thus we take $u(x) = x_{n+1} - f(x')$, with $\nabla u = (-\nabla f, 1)$. We obtain for the mean curvature the formula

$$\text{(7.3)} \qquad nH(x) = -\frac{1}{\langle \nabla f \rangle^3} \Big[\langle \nabla f \rangle^2 \Delta f - D^2 f(\nabla f, \nabla f)\Big] = \mathcal{M}(f),$$

where $\langle \nabla f \rangle^2 = 1 + |\nabla f(x')|^2$. Written out more fully, the quantity in brackets above is

$$\text{(7.4)} \qquad \big(1 + |\nabla f|^2\big)\Delta f - \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} = \widetilde{\mathcal{M}}(f).$$

Thus the equation stating that a hypersurface $x_{n+1} = f(x')$ be a minimal submanifold of $\mathbb{R}^{n+1}$ is

$$\text{(7.5)} \qquad \widetilde{\mathcal{M}}(f) = 0.$$

In case $n = 2$, we have the minimal surface equation, which can also be written as

$$\text{(7.6)} \qquad \big(1 + |\partial_2 f|^2\big)\, \partial_1^2 f - 2\big(\partial_1 f \cdot \partial_2 f\big)\, \partial_1 \partial_2 f + \big(1 + |\partial_1 f|^2\big)\, \partial_2^2 f = 0.$$

It can be verified that this PDE also holds for a minimal surface in $\mathbb{R}^n$ described by $x'' = f(x')$, where $x'' = (x_3, \dots, x_n)$, if (7.6) is regarded as a system of $k$ equations in $k$ unknowns, $k = n - 2$, and $(\partial_1 f \cdot \partial_2 f)$ is the dot product of $\mathbb{R}^k$-valued functions. We continue to denote the left side of (7.6) by $\widetilde{\mathcal{M}}(f)$.

Proposition 6.12 can be translated immediately into the following existence theorem for the minimal surface equation:

**Proposition 7.1.** *Let $\mathcal{O}$ be a bounded, convex domain in $\mathbb{R}^2$ with smooth boundary. Let $g \in C^\infty(\partial\mathcal{O}, \mathbb{R}^k)$ be given. Then there is a solution*

$$\text{(7.7)} \qquad u \in C^\infty(\mathcal{O}, \mathbb{R}^k) \cap C(\overline{\mathcal{O}}, \mathbb{R}^k)$$

*to the boundary problem*

$$\text{(7.8)} \qquad \widetilde{\mathcal{M}}(u) = 0, \quad u\big|_{\partial\mathcal{O}} = g.$$

When $k = 1$, we also have uniqueness, as a consequence of the following:

**Proposition 7.2.** *Let $\mathcal{O}$ be any bounded domain in $\mathbb{R}^n$. Let $u_j \in C^\infty(\mathcal{O}) \cap C(\overline{\mathcal{O}})$ be real-valued solutions to*

$$\widetilde{\mathcal{M}}(u_j) = 0, \quad u_j = g_j \text{ on } \partial\mathcal{O}, \tag{7.9}$$

*for $j = 1, 2$. Then*

$$g_1 \le g_2 \text{ on } \partial\mathcal{O} \Longrightarrow u_1 \le u_2 \text{ on } \overline{\mathcal{O}}. \tag{7.10}$$

**Proof.** We prove this by deriving a linear PDE for the difference $v = u_2 - u_1$ and applying the maximum principle. In general,

$$\Phi(u_2) - \Phi(u_1) = Lv, \quad L = \int_0^1 D\Phi\bigl(\tau u_2 + (1-\tau)u_1\bigr)\, d\tau. \tag{7.11}$$

Suppose $\Phi$ is a second-order differential operator:

$$\Phi(u) = F(u, \partial u, \partial^2 u), \quad F = F(u, p, \zeta). \tag{7.12}$$

Then, as in (3.4),

$$D\Phi(u) = F_\zeta(u, \partial u, \partial^2 u)\, \partial^2 v + F_p(u, \partial u, \partial^2 u)\, \partial v + F_u(u, \partial u, \partial^2 u) v. \tag{7.13}$$

When $\Phi(u) = \widetilde{\mathcal{M}}(u)$ is given by (7.4), $F_u(u, \xi, \zeta) = 0$, and we have

$$D\widetilde{\mathcal{M}}(u)v = A(u)v + B(u)v, \tag{7.14}$$

where

$$A(u)v = \bigl(1 + |\nabla u|^2\bigr)\Delta v - \sum_{i,j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j} \tag{7.15}$$

is strongly elliptic, and $B(u)$ is a first-order differential operator. Consequently, we have

$$\widetilde{\mathcal{M}}(u_2) - \widetilde{\mathcal{M}}(u_1) = Av + Bv, \tag{7.16}$$

where $A = \int_0^1 A\bigl(\tau u_2 + (1-\tau)u_2\bigr)\, d\tau$ is strongly elliptic of order 2 at each point of $\mathcal{O}$, and $B$ is a first-order differential operator, which annihilates constants. If (7.9) holds, then $Av + Bv = 0$. Now (7.10) follows from the maximum principle, Proposition 2.1 of Chapter 5.

We have as of yet no estimates on $|\nabla u_j(x)|$ as $x \to \partial\mathcal{O}$, so $A$, which is elliptic in $\mathcal{O}$, could conceivably degenerate at $\partial\mathcal{O}$. To achieve a situation where the results of Chapter 5, §2, apply, we could note that the hypotheses of Proposition 7.2 imply that, for any $\varepsilon > 0$, $u_1 \le u_2 + \varepsilon$ on a neighborhood of $\partial\mathcal{O}$. Alternatively, one can check that the *proof* of Proposition 2.1 in Chapter 5 works even if the elliptic operator is allowed to degenerate at the boundary. Either way, the maximum principle then applies to yield (7.10).

While Proposition 7.2 is a sort of result that holds for a large class of second-order, scalar, elliptic PDE, the next result is much more special and has interesting consequences. It implies that the size of a solution to the minimal surface equation (7.8) can sometimes be controlled by the behavior of $g$ on *part* of the boundary.

**Proposition 7.3.** *Let $\mathcal{O} \subset \mathbb{R}^2$ be a domain contained in the annulus $r_1 < |x| < r_2$, and let $u \in C^2(\mathcal{O}) \cap C(\overline{\mathcal{O}})$ solve $\widetilde{\mathcal{M}}(u) = 0$. Set*

$$G(x; r) = r \ \cosh^{-1}\left(\frac{|x|}{r}\right), \text{ for } |x| > r, \quad G(x; r) \le 0. \tag{7.17}$$

*If*

$$u(x) \le G(x; r_1) + M \quad \text{on} \ \ \{x \in \partial\mathcal{O} : |x| > r_1\}, \tag{7.18}$$

*for some $M \in \mathbb{R}$, then*

$$u(x) \le G(x; r_1) + M \quad \text{on} \ \ \mathcal{O}. \tag{7.19}$$

Here, $z = G(x; r_1)$ defines the lower half of a catenoid, over $\{x \in \mathbb{R}^2 : |x| \ge r_1\}$. This function solves the minimal surface equation on $|x| > r_1$ and vanishes on $|x| = r_1$.

**Proof.** Given $s \in (r_1, r_2)$, let

$$\varepsilon(s) = \max_{s \le |x| \le r_2} \bigl|G(x; r_1) - G(x; s)\bigr|. \tag{7.20}$$

The hypothesis (7.18) implies that

$$u(x) \le G(x; s) + M + \varepsilon(s) \tag{7.21}$$

on $\{x \in \partial\mathcal{O} : |x| \ge s\}$. We claim that (7.21) holds for $x$ in

$$\mathcal{O}(s) = \mathcal{O} \cap \{x : s < |x| < r_2\}. \tag{7.22}$$

Once this is established, (7.19) follows by taking $s \searrow r_1$. To prove this, it suffices by Proposition 7.2 to show that (7.21) holds on $\partial\mathcal{O}(s)$. Since it holds on $\partial\mathcal{O}$, it remains to show that (7.21) holds for $x$ in

$$\mathcal{C}(s) = \mathcal{O} \cap \{x : |x| = s\}, \tag{7.23}$$

illustrated by a broken arc in Fig. 7.1. If not, then $u(x) - G(x; s)$ would have a maximum $M_1 > M + \varepsilon(s)$ at some point $p \in \mathcal{C}(s)$. By Proposition 7.1, we have $u(x) - G(x; s) \le M_1$ on $\mathcal{O}(s)$. However, $\nabla u(x)$ is bounded on a neighborhood of $p$, while

$$\frac{\partial}{\partial r} G(x; s) = -\infty \quad \text{on } |x| = s. \tag{7.24}$$

This implies that $u(x) - G(x; s) > M_1$, for all points in $\mathcal{O}(s)$ sufficiently near $p$. This contradiction shows that (7.21) must hold on $\mathcal{C}(s)$, and the proposition is proved.

FIGURE 7.1

One implication is that if $\mathcal{O} \subset \mathbb{R}^2$ is as illustrated in Fig. 7.1, it is not possible to solve the boundary problem (7.8) with $g$ prescribed arbitrarily on all of $\partial\mathcal{O}$. A more precise statement about domains $\mathcal{O} \subset \mathbb{R}^2$ for which (7.8) is always solvable is the following:

**Proposition 7.4.** *Let $\mathcal{O} \subset \mathbb{R}^2$ be a bounded, connected domain with smooth boundary. Then (7.8) has a solution for all $g \in C^\infty(\partial\mathcal{O})$ if and only if $\mathcal{O}$ is convex.*

**Proof.** The positive result is given in Proposition 7.1. Now, if $\mathcal{O}$ is not convex, let $p \in \partial\mathcal{O}$ be a point where $\mathcal{O}$ is concave, as illustrated in Fig. 7.2. Pick a disk $\mathcal{D}$ whose boundary $C$ is tangent to $\partial\mathcal{O}$ at $p$ and such that, near $p$, $C$ intersects the complement $\mathcal{O}^c$ only at $p$. Then apply Proposition 7.3 to the domain $\widetilde{\mathcal{O}} = \mathcal{O} \setminus \overline{\mathcal{D}}$, taking the origin to be the center of $\mathcal{D}$ and $r_1$ to be the radius of $\mathcal{D}$. We deduce that if $u$ solves $\widetilde{\mathcal{M}}(u) = 0$ on $\mathcal{O}$, then

$$u(x) \le M + G(x; r_1) \quad \text{on } \partial\mathcal{O} \setminus \mathcal{D} \Longrightarrow u(p) \le M, \tag{7.25}$$

which certainly restricts the class of functions $g$ for which (7.8) can be solved.

Note that the function $v(x) = G(x; r)$ defined by (7.17) also provides an example of a solution to the minimal surface equation (7.8) on an annular region

$$\mathcal{O} = \{x \in \mathbb{R}^2 : r < |x| < s\},$$

with smooth (in fact, locally constant) boundary values

$$v = 0 \text{ on } |x| = r, \quad v = -r\ \cosh^{-1}\ \frac{s}{r} \text{ on } |x| = s,$$

Figure 7.2

which is not a smooth function, or even a Lipschitz function, on $\overline{\mathcal{O}}$. This is another phenomenon that is different when $\mathcal{O}$ is convex. We will establish the following:

**Proposition 7.5.** *If $\mathcal{O} \subset \mathbb{R}^2$ is a bounded region with smooth boundary which is strictly convex (i.e., $\partial\mathcal{O}$ has positive curvature), and $g \in C^\infty(\partial\mathcal{O})$ is real-valued, then the solution to (7.8) is Lipschitz at each point $x_0 \in \partial\mathcal{O}$.*

**Proof.** Given $x_0 \in \partial\mathcal{O}$, we have $z_0 = \big(x_0, g(x_0)\big) \in \gamma \subset \mathbb{R}^3$, where $\gamma$ is the boundary of the minimal surface $M$ which is the graph of $z = u(x)$. The strict convexity hypothesis on $\mathcal{O}$ implies that there are two planes $\Pi_j$ in $\mathbb{R}^3$ through $z_0$, such that $\Pi_1$ lies below $\gamma$ and $\Pi_2$ above $\gamma$, and $\Pi_j$ are given by $z = \alpha_j \cdot (x - x_0) + g(x_0) = w_{jx_0}(x)$, $\alpha_j = \alpha_j(x_0) \in \mathbb{R}^3$. There is an estimate of the form

$$|\alpha_j(x_0)| \le K(x_0)\|g \circ \rho_{x_0}\|_{C^2}, \tag{7.26}$$

where $\rho_{x_0}$ is the radial projection (from the center of $\mathcal{O}$) of $\partial\mathcal{O}$ onto a circle $\mathcal{C}(x_0)$ containing $\mathcal{O}$ and tangent to $\partial\mathcal{O}$ at $x_0$, and $K(x_0)$ depends on the curvature of $\mathcal{C}(x_0)$. Now Proposition 7.2 applies to give

$$w_{1x_0}(x) \le u(x) \le w_{2x_0}(x), \quad x \in \overline{\mathcal{O}}, \tag{7.27}$$

since linear functions solve the minimal surface equation. This establishes the Lipschitz continuity, with the quantitative estimate

$$|u(x_0) - u(x)| \le A|x - x_0|, \quad x_0 \in \partial\mathcal{O}, \quad x \in \overline{\mathcal{O}}, \tag{7.28}$$

where

$$A = \sup_{x_0 \in \partial\mathcal{O}} |\alpha_1(x_0)| + |\alpha_2(x_0)|. \tag{7.29}$$

This result points toward an estimate on $|\nabla u(x)|,\ x \in \overline{\mathcal{O}}$, for a solution to (7.8). We begin the line of reasoning that leads to such an estimate, a line that applies to other situations. First, let's rederive the minimal surface equation, as the stationary condition for

$$I(u) = \int_{\mathcal{O}} F\big(\nabla u(x)\big)\, dx, \tag{7.30}$$

where

$$F(p) = \Big(1 + |p|^2\Big)^{1/2}, \tag{7.31}$$

so (7.30) gives the area of the graph of $z = u(x)$. The method used in Chapter 2, §1, yields the PDE

$$\sum A^{ij}(\nabla u)\, \partial_i \partial_j u = 0, \tag{7.32}$$

where

$$A^{ij}(p) = \frac{\partial^2 F}{\partial p_i \partial p_j}. \tag{7.33}$$

Compare this with (1.68) and (1.36) of Chapter 2. When $F(p)$ is given by (7.31), we have

$$A^{ij}(p) = \langle p\rangle^{-3}\Big(\delta_{ij}\langle p\rangle^2 - p_i p_j\Big), \tag{7.34}$$

so in this case (7.32) is equal to $-\mathcal{M}(u)$, defined by (7.3). Now, when $u$ is a sufficiently smooth solution to (7.32), we can apply $\partial_\ell = \partial/\partial x_\ell$ to this equation and obtain the PDE

$$\sum \partial_i A^{ij}(\nabla u)\, \partial_j w_\ell = 0, \tag{7.35}$$

for $w_\ell = \partial_\ell u$, not for all PDE of the form (7.32), but whenever $A^{ij}(p)$ is symmetric in $(i, j)$ and satisfies

$$\frac{\partial A^{ij}}{\partial p_m} = \frac{\partial A^{im}}{\partial p_j}, \tag{7.36}$$

which happens when $A^{ij}(p)$ has the form (7.33). If (7.35) satisfies the ellipticity condition

$$\sum A^{ij}\big(\nabla u(x)\big)\ \xi_i \xi_j \geq C(x)|\xi|^2, \quad C(x) > 0, \tag{7.37}$$

for $x \in \mathcal{O}$, then we can apply the maximum principle, to obtain the following:

**Proposition 7.6.** *Assume $u \in C^1(\overline{\mathcal{O}})$ is real-valued and satisfies the PDE (7.32), with coefficients given by (7.33). If the ellipticity condition (7.37) holds, then $\partial_\ell u(x)$ assumes its maximum and minimum values on $\partial\mathcal{O}$; hence*

$$\sup_{x \in \overline{\mathcal{O}}} |\nabla u(x)| = \sup_{x \in \partial\mathcal{O}} |\nabla u(x)|. \tag{7.38}$$

Combining this result with Proposition 7.5, we have the following:

**Proposition 7.7.** *Let $\mathcal{O} \subset \mathbb{R}^2$ be a bounded region with smooth boundary which is strictly convex, $g \in C^\infty(\partial\Omega)$ real-valued. If $u \in C^2(\mathcal{O}) \cap C^1(\overline{\mathcal{O}})$ is a solution to (7.8), then there is an estimate*

$$\|u\|_{C^1(\overline{\mathcal{O}})} \le C(\mathcal{O})\ \|g\|_{C^2(\partial\mathcal{O})}. \tag{7.39}$$

Note that the existence result of Proposition 7.1 does not provide us with the knowledge that $u$ belongs to $C^1(\overline{\mathcal{O}})$, and thus it will take further work to demonstrate that the estimate (7.39) actually holds for an arbitrary real-valued solution to (7.8) when $\mathcal{O} \subset \mathbb{R}^2$ is strictly convex and $g$ is smooth. We will be in a position to establish this result, and further regularity, after sufficient theory is developed in the next two sections. See in particular Theorem 10.4. For now, we can regard this as motivation to develop the tools in the following sections, on the regularity of solutions to elliptic boundary problems.

We next look at the Gauss curvature of a minimal surface $M$, given by $z = u(x),\ x \in \mathcal{O} \subset \mathbb{R}^2$. For a general $u$, the curvature is given by

$$K = \bigl(1 + |\nabla u|^2\bigr)^{-2} \det\Bigl(\frac{\partial^2 u}{\partial x_j \partial x_k}\Bigr). \tag{7.40}$$

See (4.29) in Appendix C. When $u$ satisfies the minimal surface equation, there are some other formulas for $K$, in terms of operations on

$$\Phi(x) = F(\nabla u)^{-1} = \bigl(1 + |\nabla u|^2\bigr)^{-1/2}, \tag{7.41}$$

which we will list, leaving their verification as an exercise:

$$K = -\frac{|\nabla\Phi|^2}{1 - \Phi^2}, \tag{7.42}$$

$$K = \frac{1}{2\Phi}\ \Delta\Phi, \tag{7.43}$$

$$K = \Delta\ \log(1 + \Phi). \tag{7.44}$$

Now if we alter the metric $g$ induced on $M$ via its imbedding in $\mathbb{R}^3$ by a conformal factor:

$$g' = (1 + \Phi)^2 g = e^{2v} g, \quad v = \log(1 + \Phi), \tag{7.45}$$

then, as in formula (1.30), we see that the Gauss curvature $k$ of $M$ in the new metric is

$$k = (-\Delta v + K)e^{-2v} = 0; \tag{7.46}$$

in other words, the metric $g' = (1 + \Phi)^2 g$ is flat! Using this observation, we can establish the following remarkable theorem of S. Bernstein:

**Theorem 7.8.** *If $u : \mathbb{R}^2 \to \mathbb{R}$ is an everywhere-defined $C^2$-solution to the minimal surface equation, then $u$ is a linear function.*

**Proof.** Consider the minimal surface $M$ given by $z = u(x),\ x \in \mathbb{R}^2$, in the metric $g' = (1+\Phi)^2 g$, which, as we have seen, is flat. Now $g' \ge g$, so this is a complete metric on $M$. Thus $(M, g')$ is isometrically equivalent to $\mathbb{R}^2$. Hence $(M, g)$ is conformally equivalent to $\mathbb{C}$.

On the other hand, the antipodal Gauss map

$$\widetilde{N} : M \longrightarrow S^2, \quad \widetilde{N} = \langle \nabla u \rangle^{-1} (\nabla u, -1), \tag{7.47}$$

is holomorphic; see Exercise 1 of §6. But the range of $\widetilde{N}$ is contained in the lower hemisphere of $S^2$, so if we take $S^2 = \mathbb{C} \cup \{\infty\}$ with the point at infinity identified with the "north pole" $(0,0,1)$, we see that $\widetilde{N}$ yields a bounded holomorphic function on $M \approx \mathbb{C}$. By Liouville's theorem, $\widetilde{N}$ must be constant. Thus $M$ is a flat plane in $\mathbb{R}^3$.

It turns out that Bernstein's theorem extends to $u : \mathbb{R}^n \to \mathbb{R}$, for $n \le 7$, by work of E. DeGiorgi, F. Almgren, and J. Simons, but not to $n \ge 8$.

## Exercises

1. If $D\widetilde{\mathcal{M}}(u)$ is the differential operator given by (7.14)–(7.15), show that its principal symbol satisfies

$$-\sigma_{D\widetilde{\mathcal{M}}(u)}(x, \xi) = \Big(1 + |p|^2\Big)|\xi|^2 - (p \cdot \xi)^2 \ge |\xi|^2, \tag{7.48}$$

   where $p = \nabla u(x)$.
2. Show that the formula (7.3) for $\mathcal{M}(f)$ is equivalent to

$$\mathcal{M}(f) = \sum_j \partial_j \Big( \langle \nabla f \rangle^{-1}\, \partial_j f \Big) = \operatorname{div}\Big( \langle \nabla f \rangle^{-1} \nabla f \Big). \tag{7.49}$$

3. Give a detailed demonstration of the estimate (7.26) on the slope of planes that can lie above and below the graph of $g$ over $\partial\mathcal{O}$ (assumed to have positive curvature), needed for the proof of Proposition 7.5. (*Hint*: In case $\partial\mathcal{O}$ is the unit circle $S^1$, consider the cases $g(\theta) = \cos^k \theta$.)
4. Establish the formulas (7.42)–(7.44) for the Gauss curvature of a minimal surface.

# 8. Elliptic regularity II (boundary estimates)

We establish estimates and regularity for solutions to nonlinear elliptic boundary problems. We treat completely nonlinear, second-order equations, obtaining $L^2$-Sobolev estimates for solutions assumed a priori to belong to $C^{2+r}(\overline{M}),\ r > 0$. We make note of improved estimates for solutions to quasi-linear, second-order equations. In §10 we will show how such results, when supplemented by the DeGiorgi-Nash-Moser theory, apply to the solvability of the Dirichlet problem for certain quasi-linear elliptic PDE.

Though we restrict attention to second-order equations, the analysis in this section extends readily to higher-order elliptic systems, such as we treated in §11 of Chapter 5. The exposition here is taken from [T].

Having looked at interior regularity in §4, we restrict attention to a collar neighborhood of the boundary $\partial M = X$, so we look at a PDE of the form

$$\partial_y^2 u = F(y, x, D_x^2 u, D_x^1 \partial_y u), \tag{8.1}$$

with $y \in [0,1], x \in X$. We set

$$v_1 = \Lambda u, \quad v_2 = \partial_y u, \tag{8.2}$$

and produce a first-order system for $v = (v_1, v_2)$,

$$\begin{aligned} \frac{\partial v_1}{\partial y} &= \Lambda v_2, \\ \frac{\partial v_2}{\partial y} &= F(y, x, D_x^2 \Lambda^{-1} v_1, D_x^1 v_2). \end{aligned} \tag{8.3}$$

An operator like $T = \Lambda$ or $T = D_x^2 \Lambda^{-1}$ does not map $C^{k+1+r}(I \times X)$ to $C^{k+r}(I \times X)$, but if we set

$$C^{k+r+}(I \times X) = \bigcup_{\varepsilon > 0} C^{k+r+\varepsilon}(I \times X), \tag{8.4}$$

then

$$T : C^{k+1+r+}(I \times X) \longrightarrow C^{k+r+}(I \times X). \tag{8.5}$$

Thus we will assume $u \in C^{2+r+}$. This implies $v \in C^{1+r+}$, and the arguments $D_x^2 \Lambda^{-1} v_1$ and $D_x^1 v_2$ appearing in (8.3) belong to $C^{r+}$. We will be able to drop the "+" in the statement of the main result.

Now if we treat $y$ as a parameter and apply the paradifferential operator construction developed in §10 of Chapter 13 to the family of operators on functions of $x$, we obtain

$$\begin{aligned} F(y, x, D_x^2 \Lambda^{-1} v_1, D_x^1 v_2) = {} & A_1(v; y, x, D_x) v_1 \\ & + A_2(v; y, x, D_x) v_2 + R(v), \end{aligned} \tag{8.6}$$

with (for fixed $y$) $R(v) \in C^\infty(X)$,

$$A_j(v; y, x, \xi) \in \mathcal{A}_0^r S_{1,1}^1 \subset C^r S_{1,0}^1 \cap S_{1,1}^1 \tag{8.7}$$

and

$$D_x^\beta A_j \in S_{1,1}^1, \text{ for } |\beta| \le r, \quad S_{1,1}^{1+(|\beta|-r)}, \text{ for } |\beta| > r, \tag{8.8}$$

provided $u \in C^{2+r+}$.

Note that if we write $F = F(y, x, \zeta, \eta)$, $\zeta_\alpha = D_x^\alpha u$ $(|\alpha| \le 2)$, $\eta_\alpha = D_x^\alpha \partial_y u$ $(|\alpha| \le 1)$, then we can set

$$B_1(v; y, x, \xi) = \sum_{|\alpha| \le 2} \frac{\partial F}{\partial \zeta_\alpha}(D_x^2 \Lambda^{-1} v_1, D_x^1 v_2) \xi^\alpha \langle \xi \rangle^{-1} \tag{8.9}$$

(suppressing the $y$- and $x$-arguments of $F$) and

$$B_2(v; y, x, \xi) = \sum_{|\alpha| \le 1} \frac{\partial F}{\partial \eta_\alpha}(D_x^2 \Lambda^{-1} v_1, D_x^1 v_2) \xi^\alpha. \tag{8.10}$$

Thus

$$v \in C^{1+r+} \Longrightarrow A_j - B_j \in C^r S^{1-r}_{1,1}. \tag{8.11}$$

Using (8.4), we can rewrite the system (8.3) as

$$\begin{aligned} \frac{\partial v_1}{\partial y} &= \Lambda v_2, \\ \frac{\partial v_2}{\partial y} &= A_1(x, D) v_1 + A_2(x, D) v_2 + R(v). \end{aligned} \tag{8.12}$$

We also write this as

$$\frac{\partial v}{\partial y} = K(v; y, x, D_x) v + R \quad (R \in C^\infty), \tag{8.13}$$

where $K(v; y, x, D_x)$ is a $2 \times 2$ matrix of first-order pseudodifferential operators. Let us denote the symbol obtained by replacing $A_j$ by $B_j$ as $\tilde{K}$, so

$$K - \tilde{K} \in C^r S^{1-r}_{1,1}. \tag{8.14}$$

The ellipticity condition can be expressed as

$$\text{spec } \tilde{K}(v; y, x, \xi) \subset \{z \in \mathbb{C} : |\text{Re } z| \ge C|\xi|\}, \tag{8.15}$$

for $|\xi|$ large. Hence we can make the same statement about the spectrum of the symbol $K$, for $|\xi|$ large, provided $v \in C^{1+r+}$ with $r > 0$.

In order to derive $L^2$-Sobolev estimates, we will construct a symmetrizer, in a fashion similar to §11 in Chapter 5. In particular, we will make use of Lemma 11.4 of Chapter 5. Let $\tilde{E} = \tilde{E}(v; y, x, \xi)$ denote the projection onto the $\{\text{Re } z > 0\}$ spectral space of $\tilde{K}$, defined by

$$\tilde{E}(y, x, \xi) = \frac{1}{2\pi i} \int_\gamma \left(z - \tilde{K}(y, x, \xi)\right)^{-1} dz, \tag{8.16}$$

where $\gamma$ is a curve enclosing that part of the spectrum of $\tilde{K}(y, x, \xi)$ contained in $\{\text{Re } z > 0\}$. Then the symbol

$$\tilde{A} = (2\tilde{E} - 1)\tilde{K} \in C^r S^1_{cl} \tag{8.17}$$

has spectrum in $\{\mathrm{Re}\, z > 0\}$. (The symbol class $C^r S^m_{cl}$ is defined as in (9.46) of Chapter 13.) Let $\tilde{P} \in C^r S^0_{cl}$ be a symmetrizer for the symbol $\tilde{A}$, constructed via Lemma 11.4 of Chapter 5, namely,

$$\tilde{P}(y, x, \xi) = \Phi\big(\tilde{A}(y, x, \xi)\big),$$

where $\Phi$ is as in (11.54)–(11.55) in Chapter 5. Thus $\tilde{P}$ and $(\tilde{P}\tilde{A} + \tilde{A}^*\tilde{P})$ are positive-definite symbols, for $|\xi| \geq 1$.

We now want to apply symbol smoothing to $\tilde{P}, \tilde{A}$, and $\tilde{E}$. It will be convenient to modify the construction slightly, and smooth in both $x$ and $y$. Thus we obtain various symbols in $S^m_{1,\delta}$, with the understanding that the symbol classes reflect estimates on $D_{y,x}$-derivatives. For example, we obtain (with $0 < \delta < 1$)

$$P(y, x, \xi) \in S^0_{1,\delta}; \quad P - \tilde{P} \in C^r S^{-r\delta}_{1,\delta} \tag{8.18}$$

by smoothing $\tilde{P}$, in $(y, x)$. We set

$$Q = \frac{1}{2}\big(P(y, x, D_x) + P(y, x, D_x)^*\big) + K\Lambda^{-1}, \tag{8.19}$$

with $K > 0$ picked to make the operator $Q$ positive-definite on $L^2(X)$. Similarly, define $A$ and $E$ by smoothing $\tilde{A}$ and $\tilde{E}$ in $(y, x)$, so

$$\begin{aligned} &A(y, x, \xi) \in S^1_{1,\delta}, \quad A - \tilde{A} \in C^r S^{1-r\delta}_{1,\delta}, \\ &E(y, x, \xi) \in S^0_{1,\delta}, \quad E - \tilde{E} \in C^r S^{-r\delta}_{1,\delta}, \end{aligned} \tag{8.20}$$

and we smooth $K$, writing

$$K = K_0 + K^b; \quad K_0 \in S^1_{1,\delta}, \quad K^b \in C^r S^{1-r\delta}_{1,\delta} \cap S^{1-r\delta}_{1,1}. \tag{8.21}$$

Consequently, on the symbol level,

$$\begin{aligned} &A = (2E - 1)K_0 + A^b, \quad A^b \in S^{1-r\delta}_{1,\delta}, \\ &PA + A^*P \geq C|\xi|, \quad \text{for } |\xi| \text{ large.} \end{aligned} \tag{8.22}$$

Let us note that the homogeneous symbols $\tilde{K}, \ \tilde{E}$, and $\tilde{A}$ commute, for each $(y, x, \xi)$; hence the commutators of the various symbols $K, \ E, \ A$ have order $\leq r\delta$ units less than the sum of the orders of these symbols; for example,

$$[E(y, x, \xi), K_0(y, x, \xi)] \in S^{1-r\delta}_{1,\delta}. \tag{8.23}$$

Using this symmetrizer construction, we will look for estimates for solutions to a system of the form (8.3) in the spaces $H_{k,s}(M) = H_{k,s}(I \times X)$, with norms

$$\|v\|^2_{k,s} = \sum_{j=0}^{k} \|\partial_y^j \Lambda^{k-j+s} v(y)\|^2_{L^2(I\times X)}. \tag{8.24}$$

We shall differentiate $(Q\Lambda^s Ev, \Lambda^s Ev)$ and $(Q\Lambda^s(1 - E)v, \Lambda^s(1 - E)v)$ with respect to $y$ (these expressions being $L^2(X)$-inner products) and sum the two resulting expressions, to obtain the desired a priori estimates, parallel to the treatment in §11 of Chapter 5.

Using (8.13), we have

$$\begin{aligned}\frac{d}{dy}(Q\Lambda^s Ev, \Lambda^s Ev) = 2\ \mathrm{Re}(Q\Lambda^s E(Kv+R), \Lambda^s Ev) \\ + (Q'\Lambda^s Ev, \Lambda^s Ev) \\ + 2\ \mathrm{Re}(Q\Lambda^s E'v, \Lambda^s Ev).\end{aligned} \tag{8.25}$$

Note that given $v \in C^{1+r+}, r > 0$, $Q'$ and $E'$ belong to $OPS^{\delta}_{1,\delta}$. Hence, for fixed $y$, each of the last two terms is bounded by

$$C\|v(y)\|^2_{H^{s+\delta/2}}. \tag{8.26}$$

Here and below, we will adopt the convention that $C = C(\|v\|_{C^{1+r+}})$, with a slight abuse of notation. Namely, $v \in C^{1+r+}$ belongs to $C^{1+r+\varepsilon}$ for some $\varepsilon > 0$, and we loosely use $\|v\|_{C^{1+r+}}$ instead of $\|v\|_{C^{1+r+\varepsilon}}$.

To analyze the first term on the right side of (8.25), we write

$$\begin{aligned}(Q\Lambda^s E(Kv+R), \Lambda^s Ev) = (Q\Lambda^s EK_0 v, \Lambda^s Ev) \\ + (Q\Lambda^s K^b v, \Lambda^s Ev) \\ + (Q\Lambda^s ER, \Lambda^s Ev),\end{aligned} \tag{8.27}$$

where the last term is harmless and, for fixed $y$,

$$|(Q\Lambda^s EK^b v, \Lambda^s Ev)| \le C\|v(y)\|^2_{H^{s+(1-r\delta)/2}}, \tag{8.28}$$

provided $s + (1-r\delta)/2 - (1-r\delta) > -(1-\delta)r$, that is,

$$s > \frac{1}{2} - r + \frac{1}{2}r\delta, \tag{8.29}$$

in view of (8.21).

Since $\tilde{E}(y,x,\xi)$ is a projection, we have $E(y,x,\xi)^2 - E(y,x,\xi) \in S^{-r\delta}_{1,\delta}$ and

$$\begin{aligned}E(y,x,D) - E(y,x,D)^2 = F(y,x,D) \in OPS^{-\sigma}_{1,\delta}, \\ \sigma = \ \min\,(r\delta, 1-\delta).\end{aligned} \tag{8.30}$$

Thus

$$QEK_0 = QAE + G; \quad G(y) \in OPS^{1-\sigma}_{1,\delta}. \tag{8.31}$$

Consequently, we can write the first term on the right side of (8.27) as

$$(QAE\Lambda^s v, \Lambda^s Ev) - (G\Lambda^s v, \Lambda^s Ev) + (Q[\Lambda^s, EK_0]v, \Lambda^s Ev). \tag{8.32}$$

The last two terms in (8.32) are bounded (for each $y$) by

$$C\|v(y)\|^2_{H^{s+(1-\sigma)/2}}. \tag{8.33}$$

As for the contribution of the first term in (8.32) to the estimation of (8.25), we have, for each $y$,

$$(QAE\Lambda^s v, \Lambda^s Ev) = (QA\Lambda^s Ev, \Lambda^s Ev) + (QA[E, \Lambda^s]v, \Lambda^s v), \tag{8.34}$$

the last term estimable by (8.33), and

$$2\ \mathrm{Re}(QA\Lambda^s Ev, \Lambda^s Ev) \geq C_1\|Ev(y)\|^2_{H^{s+1/2}} - C_2\|Ev(y)\|^2_{H^s}, \tag{8.35}$$

by (8.22) and Gårding's inequality. Keeping track of the various ingredients in the analysis of (8.25), we see that

$$\begin{aligned}\frac{d}{dy}(Q\Lambda^s Ev, \Lambda^s Ev) \geq C_1\|Ev(y)\|^2_{H^{s+1/2}}& \\ - C_2\|v(y)\|^2_{H^{s+(1-\sigma)/2}} - C_3\|R(y)\|^2_{H^s},&\end{aligned} \tag{8.36}$$

where $C_j = C_j(\|v\|_{C^{1+r+}}) > 0$.

A similar analysis gives

$$\begin{aligned}&\frac{d}{dy}(Q\Lambda^s(1-E)v, \Lambda^s(1-E)v) \\ &\leq -C_1\|(1-E)v(y)\|^2_{H^{s+1/2}} + C_2\|v(y)\|^2_{H^{s+(1-\sigma/2)}} + C_3\|R(y)\|^2_{H^s}.\end{aligned} \tag{8.37}$$

Putting together these two estimates yields

$$\begin{aligned}\frac{1}{2}C_1\|v(y)\|^2_{H^{s+1/2}} &\leq C_1\|Ev(y)\|^2_{H^{s+1/2}} + C_1\|(1-E)v(y)\|^2_{H^{s+1/2}} \\ &\leq \frac{d}{dy}(Q\Lambda^s Ev, \Lambda^s Ev) - \frac{d}{dy}(Q\Lambda^s(1-E)v, \Lambda^s(1-E)v) \\ &\quad + C_2\|v(y)\|^2_{H^{s+(1-\sigma)/2}} + C_3\|R(y)\|^2_{H^s}.\end{aligned} \tag{8.38}$$

Now standard arguments allow us to replace $H^{s+(1-\sigma)/2}$ by $H^t$, with $t << s$. Then integration over $y \in [0,1]$ gives

$$\begin{aligned}C_1\|v\|^2_{0,s+1/2} \leq \|\Lambda^s Ev(1)\|^2_{L^2} + \|\Lambda^s(1-E)v(0)\|^2_{L^2}& \\ + C_2\|v\|^2_{0,t} + C_3\|R\|^2_{0,s}.&\end{aligned} \tag{8.39}$$

Recalling that

$$\|v\|^2_{1,s} = \|\Lambda^{1+s}v\|^2_{L^2(M)} + \|\Lambda^s\partial_y v\|^2_{L^2(M)} \tag{8.40}$$

and using (8.13) to estimate $\partial_y v$, we have

$$\|v\|^2_{1,s-1/2} \leq C\Big[\|Ev(1)\|^2_{H^s} + \|(1-E)v(0)\|^2_{H^s} + \|v\|^2_{0,t} + \|R\|^2_{0,s}\Big], \tag{8.41}$$

with $C = C(\|v\|_{C^{1+r+}})$, provided that $v \in C^{1+r+}$ with $r > 0$ and that $s$ satisfies the lower bound (8.29). Let us note that

$$C_1\Big[\|\Lambda^s(1-E)v(1)\|^2_{L^2} + \|\Lambda^s Ev(0)\|^2_{L^2}\Big]$$

could have been included on the left side of (8.39), so we also have the estimate

$$\|(1-E)v(1)\|_{H^s}^2 + \|Ev(0)\|_{H^s}^2 \le \text{ right side of (8.41)}. \tag{8.42}$$

Having completed a first round of a priori estimates, we bring in a consideration of boundary conditions that might be imposed. Of course, the boundary conditions $Ev(1) = f_1, (1-E)v(0) = f_0$ are a possibility, but these are really a tool with which to analyze other, more naturally occurring boundary conditions. The "real" boundary conditions of interest include the Dirichlet condition on (8.1):

$$u(0) = f_0, \quad u(1) = f_1, \tag{8.43}$$

various sorts of (possibly nonlinear) conditions involving first-order derivatives:

$$G_j(x, D^1u) = f_j, \quad \text{at } y = j \quad (j = 0, 1), \tag{8.44}$$

and when (8.1) is itself a $K \times K$ system, other possibilities, which can be analyzed in the same spirit. Now if we write $D^1u = (u, \partial_x u, \partial_y u) = (\Lambda^{-1}v_1, \partial_x\Lambda^{-1}v_1, v_2)$, and use the paradifferential operator construction of Chapter 13, §10, we can write (8.44) as

$$H_j(v; x, D)v = g_j, \quad \text{at } y = j, \tag{8.45}$$

where, given $v \in C^{1+r+}$,

$$H_j(v; x, \xi) \in \mathcal{A}_0^{1+r} S_{1,1}^0 \subset C^{1+r} S_{1,0}^0 \cap S_{1,1}^0. \tag{8.46}$$

Of course, (8.43) can be written in the same form, with $H_j v = v_1$.

Now the following is the natural regularity hypothesis to make on (8.45); namely, that we have an estimate of the form

$$\begin{aligned}\sum_j \|v(j)\|_{H^s}^2 \le C\Big[\|Ev(0)\|_{H^s}^2 + \|(1-E)v(1)\|_{H^s}^2\Big] \\ + C\sum_j \Big[\|H_j(v; x, D)v(j)\|_{H^s}^2 + \|v(j)\|_{H^{s-1}}^2\Big].\end{aligned} \tag{8.47}$$

We then say the boundary condition is *regular*. If we combine this with (8.41) and (8.42), we obtain the following fundamental estimate:

**Proposition 8.1.** *If $v$ satisfies the elliptic system* (8.3), *together with the boundary condition* (8.45), *assumed to be regular, then*

$$\|v\|_{1,s-1/2}^2 \le C\Big[\sum_j \|g_j\|_{H^s}^2 + \|v\|_{0,t}^2 + \|R\|_{0,s}^2\Big], \tag{8.48}$$

*provided $v \in H_{1,s-1/2} \cap C^{1+r}, r > 0$, and $s$ satisfies* (8.29). *We can take $t << s$. In case* (8.44) *holds, we can replace $\|g_j\|_{H^s}$ by $\|f_j\|_{H^s}$, and in case* *the Dirichlet condition* (8.43) *holds and is regular, we can replace* $\|g_j\|_{H^s}$ *by* $\|f_j\|_{H^{s+1}}$ *in* (8.48).

Here, we have taken the opportunity to drop the "+" from $C^{1+r+}$; to justify this, we need only shift $r$ slightly. For the same reason, we can assume that, in (8.1), $u \in C^{2+r}$, for some $r > 0$. In the rest of this section, we assume for simplicity that $s - 1/2 \in \mathbb{Z}^+ \cup \{0\}$.

We can now easily obtain higher-order estimates, of the form

$$\|v\|^2_{k,s-1/2} \le C\Big[\sum_j \|g_j\|^2_{H^{s+k-1}} + \|v\|^2_{0,t} + \|R\|^2_{k-1,s}\Big], \tag{8.49}$$

for $t << s - 1/2$, by induction from

$$\|v\|^2_{k,s-1/2} = \|v\|^2_{k-1,s+1/2} + \|\partial_y v\|^2_{k-1,s-1/2},$$

plus substituting the right side of (8.3) for $\partial_y v$. This follows from the existence of Moser-type estimates:

$$\begin{aligned} &\|F(\cdot,\cdot,w_1,w_2)\|_{k,s-1/2} \\ &\le C\big(\|w_1\|_{L^\infty}, \|w_2\|_{L^\infty}\big)\big[\|w_1\|_{k,s-1/2} + \|w_2\|_{k,s-1/2}\big], \end{aligned} \tag{8.50}$$

for $k, k+s-1/2 > 0$. If $s - 1/2 \in \mathbb{Z}^+ \cup \{0\}$, such an estimate can be established by methods used in §3 of Chapter 13.

We also obtain a corresponding regularity theorem, via inclusion of Friedrich mollifiers in the standard fashion. Thus replace $\Lambda^s$ by $\Lambda^s_\varepsilon = \Lambda^s J_\varepsilon$ in (8.25) and repeat the analysis. One must keep in mind that $K^b$ must be applicable to $v(y)$ for the analogue of (8.28) to work. Given (8.21), we need $v(y) \in H^\sigma$ with $\sigma > 1 - r$. However, $v \in C^{1+r}$ already implies this. We thus have the following result.

**Theorem 8.2.** *Let* $v$ *be a solution to the elliptic system* (8.3), *satisfying the boundary conditions* (8.45), *assumed to be regular. Assume*

$$v \in C^{1+r}, \quad r > 0, \tag{8.51}$$

*and*

$$g_j \in H^{s+k-1}(X), \tag{8.52}$$

*with* $s - 1/2 \in \mathbb{Z}^+ \cup \{0\}$. *Then*

$$v \in H_{k,s-1/2}(I \times X). \tag{8.53}$$

In particular, taking $s = 1/2$, and noting that

$$H_{k,0}(M) = H^k(M), \tag{8.54}$$

we can specialize this implication to

$$g_j \in H^{k-1/2}(X) \Longrightarrow v \in H^k(I \times X), \tag{8.55}$$

for $k = 1, 2, 3, \ldots$, granted (8.51) (which makes the $k = 1$ case trivial).

Note that, in (8.36)–(8.38), one could replace the term $\|R(y)\|^2_{H^s}$ by the product $\|R(y)\|_{H^{s-1/2}} \cdot \|v(y)\|_{H^{s+1/2}}$; then an absorption can be performed in (8.38), and hence in (8.39)–(8.41) we can substitute $\|R\|^2_{0,s-1/2}$, and use $\|R\|^2_{k-1,s-1/2}$ in (8.49).

We note that Theorem 8.2 is also valid for solutions to a nonhomogeneous elliptic system, where $R$ in (8.13) can contain an extra term, belonging to $H_{k-1,s-1/2}$, and then the estimate (8.49), strengthened as indicated above, and consequent regularity theorem are still valid. If (8.1) is generalized to

$$\partial_y^2 u = F(D_x^2 u, D_x^1 \partial_y u) + f, \tag{8.56}$$

then a term of the form $(0, f)^t$ is added to (8.13).

In view of the estimate (8.11) comparing the symbol of $K$ with that obtained from the linearization of the original PDE (8.1), and the analogous result that holds for $H_j$, derived from $G_j$, we deduce the following:

**Proposition 8.3.** *Suppose that, at each point on $\partial M$, the linearization of the boundary condition of* (8.44) *is regular for the linearization of the PDE* (8.1)*. Assume $u \in C^{2+r}$, $r > 0$. Then the regularity estimate* (8.49) *holds. In particular, this holds for the Dirichlet problem, for any scalar (real) elliptic PDE of the form* (8.1).

We next establish a strengthened version of Theorem 8.2 when $u$ solves a quasi-linear, second-order elliptic PDE, with a regular boundary condition. Thus we are looking at the special case of (8.1) in which

$$\begin{aligned} F(y, x, D_x^2 u, D_x^1 \partial_y u) = & -\sum_j B^j(x, y, D^1 u)\, \partial_j \partial_y u \\ & -\sum_{j,k} A^{jk}(x, y, D^1 u)\, \partial_j \partial_k u \\ & + F_1(x, y, D^1 u). \end{aligned} \tag{8.57}$$

All the calculations done above apply, but some of the estimates are better. This is because when we derive the equation (8.13), namely,

$$\frac{\partial v}{\partial y} = K(v; y, x, D_x)v + R \qquad (R \in C^\infty) \tag{8.58}$$

for $v = (v_1, v_2) = (\Lambda u, \partial_y u)$, (8.7) is improved to

$$u \in C^{1+r+} \Longrightarrow K \in \mathcal{A}_0^r S^1_{1,1} + S^{1-r}_{1,1} \qquad (r > 0). \tag{8.59}$$

Compare with (4.62). Under the hypothesis $u \in C^{1+r+}$, one has the result (8.17), $\tilde{A} \in C^r S^1_{cl}$, which before required $u \in C^{2+r+}$. Also (8.20)–(8.22) now hold for $u \in C^{1+r+}$. Thus all the a priori estimates, down through (8.49), hold, with $C = C(\|u\|_{C^{1+r+}})$. As before, we can delete the "+." One point that must be taken into consideration is that, for the estimates to work, one needs $v(y) \in H^\sigma$ with $\sigma > 1 - r$, and now this does not necessarily follow from the hypothesis $u \in C^{1+r}$. Hence we have the following regularity result. Compare the interior regularity established in Theorem 4.5.

**Theorem 8.4.** *Let $u$ satisfy a second-order, quasi-linear elliptic PDE with a regular boundary condition, of the form* (8.45), *for $v = (\Lambda u, \partial_y u)$. Assume that*

$$u \in C^{1+r} \cap H_{1,\sigma}, \quad r > 0, \quad r + \sigma > 1. \tag{8.60}$$

*Then, for $k = 0, 1, 2, \dots$,*

$$g_j \in H^{k-1/2}(X) \Longrightarrow v \in H^k(I \times X). \tag{8.61}$$

The Dirichlet boundary condition is regular (if the PDE is real and scalar), and

$$u(j) = f_j \in H^{k+s}(X) \Longrightarrow v \in H_{k,s-\frac{1}{2}}(I \times X) \tag{8.62}$$

if $s > (1-r)/2$. In particular,

$$\begin{aligned} u(j) = f_j \in H^{k+1/2}(X) &\Longrightarrow v \in H^k(I \times X) \\ &\Longrightarrow u \in H^{k+1}(I \times X). \end{aligned} \tag{8.63}$$

We consider now the further special case

$$\begin{aligned} F(y, x, D_x^2 u, D_x^1 \partial_y u) = &-\sum_j B^j(x, y, u)\, \partial_j \partial_y u \\ &- \sum_{j,k} A^{jk}(x, y, u)\, \partial_j \partial_k u + F_1(x, y, D^1 u). \end{aligned} \tag{8.64}$$

In this case, when we derive the system (8.58), we have the implication

$$u \in C^{r+}(\overline{M}) \Longrightarrow K \in \mathcal{A}_0^r S^1_{1,1} + S^{1-r}_{1,1} \quad (r > 0). \tag{8.65}$$

Similarly, under this hypothesis, we have $\tilde{A} \in C^r S^1_{cl}$, and so forth. Therefore we have the following:

**Proposition 8.5.** *If $u$ satisfies the PDE* (8.1) *with $F$ given by* (8.64), *then the conclusions of Theorem 8.4 hold when the hypothesis* (8.60) *is weakened to*

$$u \in C^r \cap H_{1,\sigma}, \quad r + \sigma > 1. \tag{8.66}$$

Note that associated to this regularity is an estimate. For example, if $u$ satisfies the Dirichlet boundary condition, we have, for $k \geq 2$,

$$\|u\|_{H^k(M)} \leq C_k(\|u\|_{C^r(\overline{M})})\big[\|u|_{\partial M}\|_{H^{k-1/2}(\partial M)} + \|u\|_{L^2(M)}\big], \tag{8.67}$$

where we have used Poincaré's inequality to replace the $H_{1,\sigma}$-norm of $u$ by the $L^2$-norm on the right.

Let us see to what extent the results obtained here apply to solutions to the minimal surface equation produced in §7. Recall the boundary problem (7.8):

$$\langle \nabla u \rangle^2 \Delta u - \sum_{i,j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j} = 0, \quad u = g \text{ on } \partial \mathcal{O}, \tag{8.68}$$

where $\mathcal{O}$ is a strictly convex region in $\mathbb{R}^2$, with smooth boundary. For this boundary problem, Theorem 8.4 applies, to yield the implication

$$g \in H^{k+1/2}(\partial \mathcal{O}) \Longrightarrow u \in H^{k+1}(\mathcal{O}), \quad k = 0, 1, 2, \dots, \tag{8.69}$$

*provided* we know that

$$u \in C^{1+r}(\overline{\mathcal{O}}) \cap H_{1,\sigma}(\mathcal{A}), \quad r > 0, \quad r + \sigma > 1, \tag{8.70}$$

where $\mathcal{A}$ is a collar neighborhood of $\partial \mathcal{O}$ in $\overline{\mathcal{O}}$. Now, while we know that solutions to the minimal surface equation are smooth inside $\mathcal{O}$ (having proved that minimal surfaces are real analytic), we so far have only *continuity* of a solution $u$ on $\overline{O}$, plus a Lipschitz bound on $u\big|_{\partial \mathcal{O}}$ and a hope of obtaining a bound in $C^1(\overline{\mathcal{O}})$. We therefore have a gap to close to be able to apply the results of this section to solutions of (8.68).

The material of the next two sections will close this gap. As we'll see, we will be able to treat (8.68), not only for dim $\mathcal{O} = 2$, but also for dim $\mathcal{O} = n > 2$. Also, the gap will be closed on a number of other quasi-linear elliptic PDE.

## Exercises

1. Suppose $u$ is a solution to a quasi-linear elliptic PDE of the form

$$\sum a_{jk}(x, u) \partial_j \partial_k u + b(x, u, \nabla u) = 0 \quad \text{on } \overline{M},$$

satisfying boundary conditions

$$B_0(x, u)u = g_0, \quad B_1(x, u, D)u = g_1, \quad \text{on } \partial M,$$

assumed to be regular. The operators $B_j$ have order $j$. Generalizing (8.67), show that, for any $r > 0$, $k \geq 2$, there is an estimate

$$\|u\|_{H^k(M)} \leq C_k \Big( \|u\|_{C^r(\overline{M})} \Big) \Big( \|g_0\|_{H^{k-1/2}(\partial M)} + \|g_1\|_{H^{k-3/2}(\partial M)} + \|u\|_{L^2(M)} \Big). \tag{8.71}$$

2. Extend Theorem 8.4 to nonhomogeneous, quasi-linear equations,

$$\sum a_{jk}(x, D^1 u) \, \partial_j \partial_k u + b(x, D^1 u) = h(x), \tag{8.72}$$

satisfying regular boundary conditions. If one uses the Dirichlet boundary condition, $u\big|_{\partial M} = g$, show that

(8.73)
$$\|u\|_{H^k(M)} \le C_k\Big(\|u\|_{C^{1+r}(\overline{M})}\Big)\Big(\|g\|_{H^{k-1/2}(\partial M)} + \|h\|_{H^{k-2}(M)} + \|u\|_{L^2(M)}\Big).$$

3. Give a proof of the mapping property (8.5).
4. Prove the Moser-type estimate (8.50), when $s - 1/2 = \ell \in \mathbb{Z}^+ \cup \{0\}$. (*Hint.* Rework Propositions 3.2–3.9 of Chapter 13, with $H^k$ replaced by $H_{k,\ell}$.)

## 9. Elliptic regularity III (DeGiorgi-Nash-Moser theory)

As noted at the end of §8, there is a gap between conditions needed on the solution of boundary problems for many nonlinear elliptic PDEs, in order to obtain higher-order regularity, and conditions that solutions constructed by methods used so far in this chapter have been shown to satisfy. One method of closing this gap, that has proved useful in many cases, involves the study of second-order, scalar, linear elliptic PDE, in divergence form, whose coefficients have no regularity beyond being bounded and measurable.

In this section we establish regularity for a class of PDE $Lu = f$, for second-order operators of the form (using the summation convention)

$$Lu = b^{-1}\partial_j\big(a^{jk}b\;\partial_k u\big), \tag{9.1}$$

where $(a^{jk}(x))$ is a positive-definite, bounded matrix and $0 < b_0 \le b(x) \le b_1$, $b$ scalar, and $a^{jk}, b$ are merely measurable. The breakthroughs on this were first achieved by DeGiorgi [DeG] and Nash [Na2]. We will present Moser's derivation of interior bounds and Hölder continuity of solutions to $Lu = 0$, from [Mo2], and then Morrey's analysis of the nonhomogeneous equation $Lu = f$ and proof of boundary regularity, from [Mor2]. Other proofs can be found in [GT] and [KS].

We make a few preliminary remarks on (9.1). We will use $a^{jk}$ to define an inner product of vectors:

$$\langle V, W\rangle = V_j a^{jk} W_k, \tag{9.2}$$

and use $b\,dx = dV$ as the volume element. In case $g_{jk}(x)$ is a metric tensor, if one takes $a^{jk} = g^{jk}$ and $b = g^{1/2}$, then (9.1) defines the Laplace operator. For a compactly supported function $w$,

$$(Lu, w) = -\int \langle \nabla u, \nabla w\rangle\; dV. \tag{9.3}$$

The behavior of $L$ on a nonlinear function of $u$, $v = f(u)$, plays an important role in estimates; we have

$$v = f(u) \Longrightarrow Lv = f'(u)Lu + f''(u)|\nabla u|^2, \tag{9.4}$$

where we set $|V|^2 = \langle V, V\rangle$. Also, taking $w = \psi^2 u$ in (9.3) gives the following important identity. If $Lu = g$ on an open set $\Omega$ and $\psi \in C_0^1(\Omega)$, then

$$\int \psi^2 |\nabla u|^2 \, dV = -2 \int \langle \psi \nabla u, u \nabla \psi\rangle \, dV - \int \psi^2 g u \, dV. \tag{9.5}$$

Applying Cauchy's inequality to the first term on the right yields the useful estimate

$$\frac{1}{2} \int \psi^2 |\nabla u|^2 \, dV \le 2 \int |u|^2 |\nabla \psi|^2 \, dV - \int \psi^2 g u \, dV. \tag{9.6}$$

Given these preliminaries, we are ready to present an approach to sup norm estimates known as "Moser iteration." Once this is done (in Theorem 9.3 below), we will then tackle Hölder estimates.

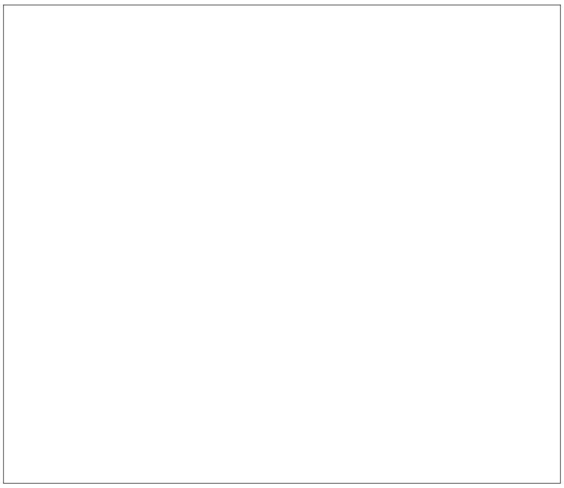

FIGURE 9.1

To implement Moser iteration, consider a nested sequence of open sets with smooth boundary

$$\Omega_0 \supset \cdots \supset \Omega_j \supset \Omega_{j+1} \supset \cdots \tag{9.7}$$

with intersection $\mathcal{O}$, as illustrated in Fig. 9.1. We will make the geometrical hypothesis that the distance of any point on $\partial\Omega_{j+1}$ to $\partial\Omega_j$ is $\sim Cj^{-2}$. We want to estimate the sup norm of a function $v$ on $\mathcal{O}$ in terms of its $L^2$-norm on $\Omega_0$, assuming

$$v > 0 \text{ is a subsolution of } L \quad (\text{i.e., } Lv \ge 0). \tag{9.8}$$

In view of (9.4), an example is

$$v = (1 + u^2)^{1/2}, \quad Lu = 0. \tag{9.9}$$

We will obtain such an estimate in terms of the Sobolev constants $\gamma(\Omega_j)$ and $C_j$, defined below. Ingredients for the analysis include the following two lemmas, the first being a standard Sobolev inequality.

**Lemma 9.1.** *For* $v \in H^1(\Omega_j),\ \kappa \le n/(n-2)$,

$$\|v^\kappa\|^2_{L^2(\Omega_j)} \le \gamma(\Omega_j)\big[\|\nabla v\|^{2\kappa}_{L^2(\Omega_j)} + \|v\|^{2\kappa}_{L^2(\Omega_j)}\big]. \tag{9.10}$$

The next lemma follows from (9.6) if we take $\psi = 1$ on $\Omega_{j+1}$, tending roughly linearly to 0 on $\partial\Omega_j$.

**Lemma 9.2.** *If $v > 0$ is a subsolution of $L$, then, with $C_j = C(\Omega_j, \Omega_{j+1})$,*

$$\|\nabla v\|_{L^2(\Omega_{j+1})} \le C_j \|v\|_{L^2(\Omega_j)}. \tag{9.11}$$

Under the geometrical conditions indicated above on $\Omega_j$, we can assume

$$\gamma(\Omega_j) \le \gamma_0, \quad C_j \le C(j^2+1). \tag{9.12}$$

Putting together the two lemmas, we see that when $v$ satisfies (9.8),

$$\begin{aligned} \|v^\kappa\|^2_{L^2(\Omega_{j+1})} &\le \gamma(\Omega_{j+1})\Big[C_j^{2\kappa}\|v\|^{2\kappa}_{L^2(\Omega_j)} + \|v\|^{2\kappa}_{L^2(\Omega_{j+1})}\Big] \\ &\le \gamma_0(C_j^{2\kappa}+1)\|v\|^{2\kappa}_{L^2(\Omega_j)}. \end{aligned} \tag{9.13}$$

Fix $\kappa \in (1, n/(n-2)]$. Now, if $v$ satisfies (9.8), so does

$$v_j = v^{\kappa^j}, \tag{9.14}$$

by (9.4). Note that $v_{j+1} = v_j^\kappa$. Now let

$$N_j = \|v\|_{L^{2\kappa^j}(\Omega_j)} = \|v_j\|^{1/\kappa^j}_{L^2(\Omega_j)}, \tag{9.15}$$

so

$$\|v\|_{L^\infty(\mathcal{O})} \le \limsup_{j\to\infty} N_j. \tag{9.16}$$

If we apply (9.13) to $v_j$, we have

$$\|v_{j+1}\|^2_{L^2(\Omega_{j+1})} \le \gamma_0(C_j^{2\kappa}+1)\|v_j\|^{2\kappa}_{L^2(\Omega_j)}. \tag{9.17}$$

Note that the left side is equal to $N_{j+1}^{2\kappa^{j+1}}$, and the norm on the right is equal to $N_j^{2\kappa^{j+1}}$. Thus (9.17) is equivalent to

$$N_{j+1}^2 \le \Big[\gamma_0(C_j^{2\kappa}+1)\Big]^{1/\kappa^{j+1}} N_j^2. \tag{9.18}$$

By (9.12), $C_j^{2\kappa}+1 \le C_0(j^{4\kappa}+1)$, so

$$
\begin{aligned}
\limsup_{j\to\infty} N_j^2 &\le \prod_{j=0}^{\infty} \Big[\gamma_0 C_0 (j^{4\kappa}+1)\Big]^{1/\kappa^{j+1}} N_0^2 \\
&\le (\gamma_0 C_0)^{1/(\kappa-1)} \Big[\exp \sum_{j=0}^{\infty} \kappa^{-j-1} \, \log(j^{4\kappa}+1)\Big] N_0^2 \qquad (9.19) \\
&\le K^2 N_0^2,
\end{aligned}
$$

for finite $K$. This gives Moser's sup-norm estimate:

**Theorem 9.3.** *If $v > 0$ is a subsolution of $L$, then*

$$\|v\|_{L^\infty(\mathcal{O})} \le K \|v\|_{L^2(\Omega_0)}, \tag{9.20}$$

*where $K = K(\gamma_0, C_0, n)$.*

Hölder continuity of a solution to $Lu = 0$ will be obtained as a consequence of the following "Harnack inequality." Let $B_\rho = \{x : |x| < \rho\}$.

**Proposition 9.4.** *Let $u \ge 0$ be a solution of $Lu = 0$ in $B_{2r}$. Pick $c_0 \in (0,\infty)$. Suppose*

$$\text{meas}\{x \in B_r : u(x) \ge 1\} > c_0^{-1} r^n. \tag{9.21}$$

*Then there is a constant $c > 0$ such that*

$$u(x) > c^{-1} \quad \text{in } B_{r/2}. \tag{9.22}$$

This will be established by examining $v = f(u)$ with

$$f(u) = \max\{-\log(u+\varepsilon), 0\}, \tag{9.23}$$

where $\varepsilon$ is chosen in $(0,1)$. Note that $f$ is convex, so $v$ is a subsolution. Our first goal will be to estimate the $L^2(B_r)$-norm of $\nabla v$. Once this is done, Theorem 9.3 will be applied to estimate $v$ from above (hence $u$ from below) on $B_{r/2}$.

We begin with a variant of (9.5), obtained by taking $w = \psi^2 f'(u)$ in (9.3). The identity (for smooth $f$) is

$$\int \psi^2 f'' |\nabla u|^2 \, dV + 2 \int \langle \psi f' \nabla u, \nabla \psi \rangle \, dV = -(Lu, \psi^2 f'). \tag{9.24}$$

This vanishes if $Lu = 0$. Applying Cauchy's inequality to the second integral, we obtain

$$\int \psi^2 \Big[f''(u) - \delta^2 f'(u)^2\Big] |\nabla u|^2 \, dV \le \frac{1}{\delta^2} \int |\nabla \psi|^2 \, dV. \tag{9.25}$$

Now the function $f(u)$ in (9.23) has the property that

$$h = -e^{-f} \text{ is a convex function;} \tag{9.26}$$

indeed, in this case $h(u) = \max\{-(u + \varepsilon), -1\}$. Thus

$$f'' - (f')^2 = e^f h'' \geq 0. \tag{9.27}$$

Thus $f''(u)|\nabla u|^2 \geq f'(u)^2|\nabla u|^2 = |\nabla v|^2$ if $v = f(u)$. Taking $\delta^2 = 1/2$ in (9.25), we obtain

$$\int \psi^2 |\nabla v|^2 \, dV \leq 4 \int |\nabla \psi|^2 \, dV, \tag{9.28}$$

after one overcomes the minor problem that $f'$ has a jump discontinuity. If we pick $\psi$ to $= 1$ on $B_r$ and go linearly to 0 on $\partial B_{2r}$, we obtain the estimate

$$\int_{B_r} |\nabla v|^2 \, dV \leq C r^{n-2}, \tag{9.29}$$

for $v = f(u)$, given that $Lu = 0$ and that (9.26) holds.

Now the hypothesis (9.21) implies that $v$ vanishes on a subset of $B_r$ of measure $> c_0^{-1} r^n$. Hence there is an elementary estimate of the form

$$r^{-n} \int_{B_r} v^2 \, dV \leq C r^{2-n} \int_{B_r} |\nabla v|^2 \, dV, \tag{9.30}$$

which is bounded from above by (9.29). Now Theorem 9.3, together with a simple scaling argument, gives

$$v(x)^2 \leq C r^{-n} \int_{B_r} v^2 \, dV \leq C_1^2, \quad x \in B_{r/2}, \tag{9.31}$$

so

$$u + \varepsilon \geq e^{-C_1}, \quad \text{for } x \in B_{r/2}, \tag{9.32}$$

for all $\varepsilon \in (0, 1)$. Taking $\varepsilon \to 0$, we have the proof of Proposition 9.4.

We remark that Moser obtained a stronger Harnack inequality in [Mo3], by a more elaborate argument. In that work, the hypothesis (9.21) is weakened to

$$\sup_{B_r} u(x) \geq 1. \tag{9.21a}$$

To deduce the Hölder continuity of a solution to $Lu = 0$ given Proposition 9.4 is fairly simple. Following [Mo2], who followed DeGiorgi, we have from (9.20) a bound

$$|u(x)| \leq K \tag{9.33}$$

on any compact subset $\mathcal{O}$ of $\Omega_0$, given $u \in H^1(\Omega_0)$, $Lu = 0$. Fix $x_0 \in \mathcal{O}$, such that $B_\rho(x_0) \subset \mathcal{O}$, and, for $r \le \rho$, let

$$\omega(r) = \sup_{B_r} u(x) - \inf_{B_r} u(x), \tag{9.34}$$

where $B_r = B_r(x_0)$. Clearly, $\omega(\rho) \le 2K$. Adding a constant to $u$, we can assume

$$\sup_{B_\rho} u(x) = -\inf_{B_\rho} u(x) = \frac{1}{2}\omega(\rho) = M. \tag{9.35}$$

Then $u_+ = 1 + u/M$ and $u_- = 1 - u/M$ are also annihilated by $L$. They are both $\ge 0$ and at least one of them satisfies the hypothesis (9.21), with $r = \rho/2$. If, for example, $u_+$ does, then Proposition 9.4 implies

$$u_+(x) > c^{-1} \quad \text{in } B_{\rho/4}, \tag{9.36}$$

so

$$-M\Big(1 - \frac{1}{c}\Big) \le u(x) \le M \quad \text{in } B_{\rho/4}. \tag{9.37}$$

Hence

$$\omega(\rho/4) \le \Big(1 - \frac{1}{2c}\Big)\omega(\rho), \tag{9.38}$$

which gives Hölder continuity:

$$\omega(r) \le \omega(\rho)\Big(\frac{r}{\rho}\Big)^\alpha, \quad \alpha = -\log_4\Big(1 - \frac{1}{2c}\Big). \tag{9.39}$$

We state the result formally.

**Theorem 9.5.** *If $u \in H^1(\Omega_0)$ solves $Lu = 0$, then for every compact $\mathcal{O}$ in $\Omega_0$, there is an estimate*

$$\|u\|_{C^\alpha(\mathcal{O})} \le C\|u\|_{L^2(\Omega_0)}. \tag{9.40}$$

It will be convenient to replace (9.40) by an estimate involving Morrey spaces, which are discussed in Appendix A at the end of this chapter. We claim that under the hypotheses of Theorem 9.5,

$$\nabla u\big|_{\mathcal{O}} \in M_2^p, \quad p = \frac{n}{1-\alpha}, \tag{9.41}$$

where the Morrey space $M_2^p$ consists of functions $f$ satisfying the $q = 2$ case of (A.2). The property (9.41) is stronger than (9.40), by Morrey's lemma (Lemma A.1). To see (9.41), if $B_R$ is a ball of radius $R$ centered at $y$, $B_{2R} \subset \Omega$, then let $c = u(y)$ and replace $u$ by $u(x) - c$ in (9.6), to get

$$\frac{1}{2}\int \psi^2 |\nabla u|^2 \, dV \le 2\int |u(x) - c|^2 |\nabla\psi|^2 \, dV.$$

Taking $\psi = 1$ on $B_R$, going linearly to 0 on $\partial B_{2R}$, gives

$$\int_{B_R} |\nabla u|^2 \, dV \le C\, R^{n-2+2\alpha}, \tag{9.42}$$

as needed to have (9.41).

So far we have dealt with the homogeneous equation, $Lu = 0$. We now turn to regularity for solutions to a nonhomogeneous equation. We will follow a method of Morrey, and Morrey spaces will play a very important role in this analysis. We take $L$ as in (9.1), with $a^{jk}$ measurable, satisfying

$$0 < \lambda_0 |\xi|^2 \le \sum a^{jk}(x)\xi_j\xi_k \le \lambda_1 |\xi|^2, \tag{9.43}$$

while for simplicity we assume $b, b^{-1} \in \operatorname{Lip}(\overline{\Omega})$. We consider a PDE

$$Lu = f. \tag{9.44}$$

It is clear that, for $u \in H_0^1(\Omega)$,

$$(Lu, u) \ge C \sum \|\partial_j u\|_{L^2}^2, \tag{9.45}$$

so we have an isomorphism

$$L : H_0^1(\Omega) \xrightarrow{\approx} H^{-1}(\Omega). \tag{9.46}$$

Thus, for any $f \in H^{-1}(\Omega)$, (9.44) has a unique solution $u \in H_0^1(\Omega)$. One can write such $f$ as

$$f = \sum \partial_j g_j, \quad g_j \in L^2(\Omega). \tag{9.47}$$

The solution $u \in H_0^1(\Omega)$ then satisfies

$$\|u\|_{H^1(\Omega)}^2 \le C \sum \|g_j\|_{L^2}^2. \tag{9.48}$$

Here $C$ depends on $\Omega, \lambda_0, \lambda_1$, and $b \in \operatorname{Lip}(\overline{\Omega})$.

One can also consider the boundary problem

$$Lv = 0 \text{ on } \Omega, \qquad v = w \text{ on } \partial\Omega, \tag{9.49}$$

given $w \in H^1(\Omega)$, where the latter condition means $v - w \in H_0^1(\Omega)$. Indeed, setting $v = u + w$, the equation for $u$ is $Lu = -Lw,\ u \in H_0^1(\Omega)$. Thus (9.49) is uniquely solvable, with an estimate

$$\|\nabla v\|_{L^2(\Omega)} \le C \|\nabla w\|_{L^2(\Omega)}, \tag{9.50}$$

where $C$ has a dependence as in (9.48).

Our present goal is to give Morrey's proof of the following local regularity result.

**Theorem 9.6.** *Suppose* $u \in H^1(\Omega)$ *solves* (9.44), *with* $f = \sum \partial_j g_j,\ g_j \in M_2^q(\Omega),\ q > n$, *that is,*

$$\int\limits_{B_r} |g_j|^2 \, dV \le K_1^2 \Big(\frac{r}{R}\Big)^{n-2+2\mu}, \quad \mu = 1 - \frac{n}{q} \in (0,1). \tag{9.51}$$

*Assume* $L$ *is of the form (9.1), where the coefficients* $a^{jk}$ *satisfy (9.43) and* $b, b^{-1} \in \mathrm{Lip}(\overline{\Omega})$. *Let* $\mathcal{O} \subset\subset \Omega$, *and assume* $\mu < \mu_0 = \alpha$, *for which Theorem 9.5 holds. Then* $u \in C^\mu(\mathcal{O})$; *more precisely,* $\nabla u \in M_2^q(\mathcal{O})$, *that is,*

$$\int\limits_{B_r} |\nabla u|^2 \, dV \le K_2^2 \Big(\frac{r}{R}\Big)^{n-2+2\mu}. \tag{9.52}$$

Morrey established this by using (9.48), (9.50), and an elegant dilation argument, in concert with Theorem 9.5. For this, suppose $B_R = B_R(y) \subset \Omega$ for each $y \in \mathcal{O}$. We can write $u = U + H$ on $B_R$, where

$$\begin{aligned} LU &= \sum \partial_j g_j \text{ on } B_R, \quad U \in H_0^1(B_R), \\ LH &= 0 \text{ on } B_R, \quad H - u \in H_0^1(B_R), \end{aligned} \tag{9.53}$$

and we have

$$\|\nabla U\|_{L^2(B_R)} \le C_1 \|g\|_{L^2(B_R)}, \quad \|\nabla H\|_{L^2(B_R)} \le C_2 \|\nabla u\|_{L^2(B_R)}, \tag{9.54}$$

where $\|g\|_{L^2}^2 = \sum \|g_j\|_{L^2}^2$. Let us set

$$\|F\|_r = \|F\|_{L^2(B_r)}. \tag{9.55}$$

Also let $\kappa(g_j, R)$ be the best constant $K_1$ for which (9.51) is valid for $0 < r \le R$. If $g_\tau(x) = g(\tau x)$, note that

$$\kappa(g_\tau, \tau^{-1} S) = \tau^{n/2} \kappa(g, S).$$

Now define

$$\begin{aligned} \varphi(r) = \sup\big\{ \|\nabla U\|_{rS} : U \in H_0^1(B_S),\, LU = \sum \partial_j g_j, \text{ on } B_S, \\ \kappa(g_j, S) \le 1, 0 < S \le R \big\}. \end{aligned} \tag{9.56}$$

Let us denote by $\varphi_S(r)$ the sup in (9.56) with $S$ fixed, in $(0, R]$. Then $\varphi_S(r)$ coincides with $\varphi_R(r)$, with $L$ replaced by the dilated operator, coming from the dilation taking $B_S$ to $B_R$. More precisely, the dilated operator is

$$L_S = b_S \, \partial_j \, a_S^{jk} \, b_S^{-1} \, \partial_k, \tag{9.57}$$

with

$$a_S^{jk}(x) = a^{jk}(SR^{-1}x), \quad b_S(x) = b(SR^{-1}x),$$

assuming 0 has been arranged to be the center of $B_R$. To see this, note that if $\tau = S/R$, $U_\tau(x) = \tau^{-1}U(\tau x)$, and $g_{j\tau}(x) = g_j(\tau x)$, then

$$LU = \sum \partial_j g_j \Longleftrightarrow L_S U_\tau = \sum \partial_j g_{j\tau}. \tag{9.58}$$

Also, $\nabla U_\tau(x) = (\nabla U)(\tau x)$, so $\|\nabla U_\tau\|_{S/\tau} = \tau^{n/2}\|\nabla U\|_S$.

Now for this family $L_S$, one has a *uniform* bound on $C$ in (9.48); hence $\varphi(r)$ is *finite* for $r \in (0, 1]$. We also note that the bounds in (9.40) and (9.42) are uniformly valid for this family of operators. Theorem 9.6 will be proved when we show that

$$\varphi(r) \le A\, r^{n/2-1+\mu}. \tag{9.59}$$

In fact, this will give the estimate (9.52) with $u$ replaced by $U$; meanwhile such an estimate with $u$ replaced by $H$ is a consequence of (9.42). Let $H$ satisfy (9.42) with $\alpha = \mu_0$. We take $\mu < \mu_0$.

Pick $S \in (0, R]$ and pick $g_j$ satisfying (9.51), with $R$ replaced by $S$ and $K_1$ by $K$. Write the $U$ of (9.53) as $U = U_S + H_S$ on $B_S$, where $U_S \in H_0^1(B_S)$, $LU_S = LU = \sum \partial_j g_j$ on $B_S$. Clearly, (9.51) implies

$$\int_{B_r} |g_j|^2\, dV \le K^2\Big(\frac{S}{R}\Big)^{n-2+2\mu}\Big(\frac{r}{S}\Big)^{n-2+2\mu}. \tag{9.60}$$

Thus, as in (9.54) (and recalling the definition of $\varphi$), we have

$$\begin{aligned} \|\nabla U_S\|_S &\le A_1 K\Big(\frac{S}{R}\Big)^{n/2-1+\mu}, \\ \|\nabla H_S\|_S &\le A_2\|\nabla U\|_S \le A_2 K\varphi\Big(\frac{S}{R}\Big). \end{aligned} \tag{9.61}$$

Now, suppose $0 < r < S < R$. Then, applying (9.42) to $H_S$, we have

$$\begin{aligned} \|\nabla U\|_r &\le \|\nabla U_S\|_r + \|\nabla H_S\|_r \\ &\le K\Big(\frac{S}{R}\Big)^{n/2-1+\mu}\varphi\Big(\frac{r}{S}\Big) + A_3 K\varphi\Big(\frac{S}{R}\Big)\Big(\frac{r}{S}\Big)^{n/2-1+\mu_0}. \end{aligned} \tag{9.62}$$

Therefore, setting $s = r/R,\ t = S/R$, we have the inequality

$$\varphi(s) \le t^{n/2-1+\mu}\varphi\Big(\frac{s}{t}\Big) + A_3\varphi(t)\Big(\frac{s}{t}\Big)^{n/2-1+\mu_0}, \tag{9.63}$$

valid for $0 < s < t \le 1$. Since it is clear that $\varphi(r)$ is monotone and finite on $(0, 1]$, it is an elementary exercise to deduce from (9.63) that $\varphi(r)$ satisfies an estimate of the form (9.59), as long as $\mu < \mu_0$. This proves Theorem 9.6.

Now that we have interior regularity estimates for the nonhomogeneous problem, we will be able to use a few simple tricks to establish regularity up to the boundary for solutions to the Dirichlet problem

$$Lu = \sum \partial_j g_j, \quad u = f \text{ on } \partial\Omega, \tag{9.64}$$

where $L$ has the form (9.1), $\overline{\Omega}$ is compact with smooth boundary, $f \in \text{Lip}(\partial\Omega)$, and $g_j \in L^q(\Omega)$, with $q > n$. First, extend $f$ to $f \in \text{Lip}(\overline{\Omega})$. Then $u = v + f$, where $v$ solves

$$Lv = \sum \partial_j h_j, \quad v = 0 \text{ on } \partial\Omega, \tag{9.65}$$

where

$$\partial_j h_j = \partial_j g_j - b^{-1}\partial_j \big(a^{jk} b \; \partial_k f\big). \tag{9.66}$$

We will assume $b \in \text{Lip}(\overline{\Omega})$; then $h_j$ can be chosen in $L^q$ also.

The class of equations (9.65) is invariant under smooth changes of variables (indeed, invariant under Lipschitz homeomorphisms with Lipschitz inverses, having the further property of preserving volume up to a factor in $\text{Lip}(\overline{\Omega})$). Thus make a change of variables to flatten out the boundary (locally), so we consider a solution $v \in H^1$ to (9.65) in $x_n > 0$, $|x| \le R$. We can even arrange that $b = 1$. Now extend $v$ to negative $x_n$, to be *odd* under the reflection $x_n \mapsto -x_n$. Also extend $a^{jk}(x)$ to be even when $j, k < n$ or $j = k = n$, and odd when $j$ or $k = n$ (but not both). Extend $h_j$ to be odd for $j < n$ and even for $j = n$. With these extensions, we continue to have (9.65) holding, this time in the ball $|x| \le R$. Thus interior regularity applies to this extension of $v$, yielding Hölder continuity. The following is hence proved.

**Theorem 9.7.** *Let $u \in H^1(\Omega)$ solve the PDE*

$$\sum b^{-1}\partial_j\big(a^{jk} b \; \partial_k u\big) = \sum \partial_j g_j \text{ on } \Omega, \quad u = f \text{ on } \partial\Omega. \tag{9.67}$$

*Assume $g_j \in L^q(\Omega)$ with $q > n = \dim \Omega$, and $f \in Lip(\partial\Omega)$. Assume that $b, b^{-1} \in Lip(\overline{\Omega})$ and that $(a^{jk})$ is measurable and satisfies the uniform ellipticity condition (9.43). Then $u$ has a Hölder estimate*

$$\|u\|_{C^\mu(\overline{\Omega})} \le C_1\Big(\sum \|g_j\|_{L^q(\Omega)} + \|f\|_{\text{Lip}(\partial\Omega)}\Big). \tag{9.68}$$

*More precisely, if $\mu = 1 - n/q \in (0,1)$ is sufficiently small, then $\nabla u$ belongs to the Morrey space $M_2^q(\Omega)$, and*

$$\|\nabla u\|_{M_2^q(\Omega)} \le C_2\Big(\sum \|g_j\|_{L^q(\Omega)} + \|f\|_{\text{Lip}(\partial\Omega)}\Big). \tag{9.69}$$

*In these estimates, $C_j = C_j(\Omega, \lambda_1, \lambda_2, b)$.*

So far in this section we have looked at differential operators of the form (9.1) in which $(a^{jk})$ is symmetric, but unlike the nondivergence case, where $a^{jk}(x)\,\partial_j\partial_k u = a^{kj}(x)\,\partial_j\partial_k u$, nonsymmetric cases do arise; we will see an example in §15. Thus we briefly describe the extension of the analysis of (9.1) to

$$Lu = b^{-1}\,\partial_j\big([a^{jk} + \omega^{jk}]b \; \partial_k u\big). \tag{9.70}$$

We make the same hypotheses on $a^{jk}(x)$ and $b(x)$ as before, and we assume $(\omega^{jk})$ is antisymmetric and bounded:

$$\omega^{jk}(x) = -\omega^{kj}(x), \quad \omega^{jk} \in L^\infty(\Omega). \tag{9.71}$$

We thus have both a positive symmetric form and an antisymmetric form defined at almost all $x \in \Omega$:

$$\langle V, W \rangle = V_j a^{jk}(x) W_k, \quad [V, W] = V_j \omega^{jk}(x) W_k. \tag{9.72}$$

We use the subscript $L^2$ to indicate the integrated quantities:

$$\langle v, w \rangle_{L^2} = \int \langle v, w \rangle \, dV, \quad [v, w]_{L^2} = \int [v, w] \, dV. \tag{9.73}$$

Then, in place of (9.3), we have

$$(Lu, w) = -\langle \nabla u, \nabla w \rangle_{L^2} - [\nabla u, \nabla w]_{L^2}. \tag{9.74}$$

The formula (9.4) remains valid, with $|\nabla u|^2 = \langle \nabla u, \nabla u \rangle$, as before. Instead of (9.5), we have

$$\int \psi^2 |\nabla u|^2 \, dV = -2\langle \psi \nabla u, u \nabla \psi \rangle_{L^2} - 2[\psi \nabla u, u \nabla \psi]_{L^2} - \int \psi^2 g u \, dV, \tag{9.75}$$

when $Lu = g$ on $\Omega$ and $\psi \in C_0^1(\Omega)$. This leads to a minor change in (9.6):

$$\frac{1}{2} \int \psi^2 |\nabla u|^2 \, dV \le (2 + C_0) \int |u|^2 |\nabla \psi|^2 \, dV - \int \psi^2 g u \, dV, \tag{9.76}$$

where $C_0$ is determined by the operator norm of $(\omega^{jk})$, relative to the inner product $\langle \ , \ \rangle$.

From here, the proofs of Lemmas 9.1 and 9.2, and that of Theorem 9.3, go through without essential change, so we have the sup-norm estimate (9.20). In the proof of the Harnack inequality, (9.24) is replaced by

$$\begin{aligned} \int \psi^2 f'' |\nabla u|^2 \, dV + 2\langle \psi f' \nabla u, \nabla \psi \rangle_{L^2} + 2[\psi f' \nabla u, \nabla \psi]_{L^2} \\ = -(Lu, \psi^2 f'). \end{aligned} \tag{9.77}$$

Hence (9.25) still works if you replace the factor $1/\delta^2$ by $(1+C_1)/\delta^2$, where again $C_1$ is estimated by the size of $(\omega^{jk})$. Thus Proposition 9.4 extends to our present case, and hence so does the key regularity result, Theorem 9.5. Let us record what has been noted so far:

**Proposition 9.8.** *Assume $Lu$ has the form (9.70), where $(a^{jk})$ and $b$ satisfy the hypotheses of Theorem 9.5, and $(\omega^{jk})$ satisfies (9.71). If $u \in H^1(\Omega_0)$ solves $Lu = 0$, then, for every compact $\mathcal{O} \subset \Omega_0$, there is an estimate*

$$\|u\|_{C^\alpha(\mathcal{O})} \le C \|u\|_{L^2(\Omega_0)}. \tag{9.78}$$

The Morrey space estimates go through as before, and the analysis of (9.64) is also easily modified to incorporate the change in $L$. Thus we have the following:

**Proposition 9.9.** *The boundary regularity of Theorem 9.7 extends to the operators $L$ of the form (9.70), under the hypothesis (9.71) on $(\omega^{jk})$.*

## Exercises

1. Given the strengthened form of the Harnack inequality, in which the hypothesis (9.21) is replaced by (9.21a), produce a shorter form of the argument in (9.33)–(9.40) for Hölder continuity of solutions to $Lu = 0$.
2. Show that in the statement of Theorem 9.7, $\sum \partial_j g_j$ in (9.67) can be replaced by
$$h + \sum \partial_j g_j, \quad g_j \in L^q(\Omega), \quad h \in L^p(\Omega), \quad q > n,\ p > \frac{n}{2}.$$
(*Hint*: Write $h = \sum \partial_j h_j$ for some $h_j \in L^q(\Omega)$.)
3. With $L$ given by (9.1), consider
$$L_1 = L + X, \quad X = \sum A_j(x)\,\partial_j.$$
Show that in place of (9.4) and (9.6), we have
$$v = f(u) \Longrightarrow L_1 v = f'(u) L_1 u + f''(u)|\nabla u|^2$$
and
$$\frac{1}{2}\int \psi^2 |\nabla u|^2\, dV \le \int \Big(4|\nabla\psi|^2 + 2A\psi^2\Big)|u|^2\, dV - \int \psi^2 u(L_1 u)\, dV,$$
where $A(x)^2 = \sum A_j(x)^2$.
Extend the sup-norm estimate of Theorem 9.3 to this case, given $A_j \in L^\infty(\Omega)$.
4. With $L$ given by (9.1), suppose $u$ solves
$$Lu + \sum \partial_j \Big(A_j(x)u\Big) + C(x)u = g \quad \text{on } \Omega \in \mathbb{R}^n.$$
Supppose we have
$$A_j \in L^q(\Omega),\ C \in L^p(\Omega),\ g \in L^p(\Omega), \quad p > \frac{n}{2},\ q > n,$$
and suppose we also have
$$\|u\|_{H^1(\Omega)} + \|u\|_{L^\infty(\Omega)} \le K, \quad u\Big|_{\partial\Omega} = f \in \text{Lip}(\partial\Omega).$$
Show that, for some $\mu > 0$, $u \in C^\mu(\overline{\Omega})$. (*Hint*: Apply Theorem 9.7, together with Exercise 2.)

# 10. The Dirichlet problem for quasi-linear elliptic equations

The primary goal in this section is to establish the existence of smooth solutions to the Dirichlet problem for a quasi-linear elliptic PDE of the form

$$\sum F_{p_j p_k}(\nabla u)\partial_j\partial_k u = 0 \text{ on } \Omega, \quad u = \varphi \text{ on } \partial\Omega. \tag{10.1}$$

More general equations will also be considered. As noted in (7.32), this is the PDE satisfied by a critical point of the function

$$I(u) = \int_\Omega F(\nabla u)\, dx \tag{10.2}$$

defined on the space

$$V^1_\varphi = \{u \in H^1(\Omega) : u = \varphi \text{ on } \partial\Omega\}.$$

Assume $\varphi \in C^\infty(\overline{\Omega})$. We assume $F$ is smooth and satisfies

$$A_1(p)|\xi|^2 \le \sum F_{p_j p_k}(p)\xi_j\xi_k \le A_2(p)|\xi|^2, \tag{10.3}$$

with $A_j : \mathbb{R}^n \to (0,\infty)$, continuous.

We use the method of continuity, showing that, for each $\tau \in [0,1]$, there is a smooth solution to

$$\Phi_\tau(D^2u) = 0 \text{ on } \Omega, \quad u = \varphi_\tau \text{ on } \partial\Omega, \tag{10.4}$$

where $\Phi_1(D^2u) = \Phi(D^2u)$ is the left side of (10.1) and $\varphi_1 = \varphi$. We arrange a situation where (10.4) is clearly solvable for $\tau = 0$. For example, we might take $\varphi_\tau \equiv \varphi$ and

$$\Phi_\tau(D^2u) = \tau\Phi(D^2u) + (1-\tau)\Delta u = \sum A^{jk}_\tau(\nabla u)\,\partial_j\partial_k u, \tag{10.5}$$

with

$$A^{jk}_\tau(p) = \partial_{p_j}\partial_{p_k}\Big[\tau F(p) + \frac{1}{2}(1-\tau)|p|^2\Big]. \tag{10.6}$$

Another possibility is to take

$$\Phi_\tau(D^2u) = \Phi(D^2u), \quad \varphi_\tau(x) = \tau\varphi(x), \tag{10.7}$$

since at $\tau = 0$ we have the solution $u = 0$ in this case.

Let $J$ be the largest interval containing $\{0\}$ such that (10.7) has a solution $u = u_\tau \in C^\infty(\overline{\Omega})$ for each $\tau \in J$. We will show that $J$ is all of $[0,1]$ by showing it is both open and closed in $[0,1]$. We will deal specifically with the method (10.5)–(10.6), but a similar argument can be applied to the method (10.7).

Demonstrating the openness of $J$ is the relatively easy part.

**Lemma 10.1.** *If $\tau_0 \in J$, then, for some $\varepsilon > 0$, $[\tau_0, \tau_0 + \varepsilon) \subset J$.*

**Proof.** Fix $k$ large and define

$$\Psi : [0,1] \times V^k_\varphi \longrightarrow H^{k-2}(\Omega) \tag{10.8}$$

by $\Psi(\tau,u)=\Phi_\tau(D^2u)$, where

$$V^k_\varphi=\{u\in H^k(\Omega):u=\varphi \text{ on } \partial\Omega\}. \tag{10.9}$$

This map is $C^1$, and its derivative with respect to the second argument is

$$D_2\Psi(\tau_0,u)v=Lv, \tag{10.10}$$

where

$$L:V^k_0=H^k\cap H^1_0\longrightarrow H^{k-2}(\Omega) \tag{10.11}$$

is given by

$$Lv=\sum\partial_jA^{jk}_{\tau_0}(\nabla u(x))\,\partial_kv. \tag{10.12}$$

$L$ is an elliptic operator with coefficients in $C^\infty(\overline{\Omega})$ when $u=u_{\tau_0}$, clearly an isomorphism in (10.11). Thus, by the inverse function theorem, for $\tau$ close enough to $\tau_0$, there will be $u_\tau$, close to $u_{\tau_0}$, such that $\Psi(\tau,u_\tau)=0$. Since $u_\tau\in H^k(\Omega)$ solves the regular elliptic boundary problem (10.4), if we pick $k$ large enough, we can apply the regularity result of Theorem 8.4 to deduce $u_\tau\in C^\infty(\overline{\Omega})$.

The next task is to show that $J$ is closed. This will follow from a sufficient a priori bound on solutions $u=u_\tau,\ \tau\in J$. We start with fairly weak bounds. First, the maximum principle implies

$$\|u\|_{L^\infty(M)}=\|\varphi\|_{L^\infty(\partial M)}, \tag{10.13}$$

for each $u=u_\tau,\ \tau\in J$.

Next we estimate derivatives. Each $w_\ell=\partial_\ell u$ satisfies

$$\sum\partial_jA^{jk}(\nabla u)\partial_kw_\ell=0, \tag{10.14}$$

where $A^{jk}(\nabla u)$ is given by (10.6); we drop the subscript $\tau$.

The next ingredient is a "boundary gradient estimate," of the form

$$|\nabla u(x)|\le K,\quad \text{for } x\in\partial\Omega, \tag{10.15}$$

As we have seen in the discussion of the minimal surface equation in §7, whether this holds depends on the nature of the PDE and the region $M$. For now, we will make (10.15) a hypothesis. Then the maximum principle applied to (10.14) yields a uniform bound

$$\|\nabla u\|_{L^\infty(\Omega)}\le K. \tag{10.16}$$

For the next step of the argument, we will suppose for simplicity that $\overline{\Omega}=\mathbb{T}^{n-1}\times[0,1]$, for the present, and discuss the modification of the argument for the general case later. Under this assumption, in addition to (10.14), we also have

$$w_\ell=\partial_\ell\varphi \text{ on } \partial\Omega,\quad \text{for } 1\le\ell\le n-1, \tag{10.17}$$

since $\partial_\ell$ is tangent to $\partial\Omega$ for $1 \le \ell \le n-1$.

Now we can say that Theorem 9.7 applies to $u_\ell = \partial_\ell u$, for $1 \le \ell \le n-1$. Thus there is an $r > 0$ for which we have bounds

$$\|w_\ell\|_{C^r(\overline{\Omega})} \le K, \quad 1 \le \ell \le n-1. \tag{10.18}$$

Let us note that Theorem 9.7 yields the bounds

$$\|\nabla w_\ell\|_{M_2^p(\Omega)} \le K', \quad 1 \le \ell \le n-1, \tag{10.19}$$

which are more precise than (10.18); here $1 - r = n/p$. Away from the boundary, such a property on *all* first derivatives of a solution to (10.1) leads to the applicability of Schauder estimates to establish interior regularity.

In the case of examining regularity at the boundary, more work is required since (10.18) does not include a derivative $\partial_n$ transverse to the boundary. Now, using (10.4), we can solve for $\partial_n^2 u$ in terms of $\partial_j \partial_k u$, for $1 \le j \le n,\ 1 \le k \le n-1$. This will lead to the estimate

$$\|u\|_{C^{r+1}(\overline{\Omega})} \le K, \tag{10.20}$$

as we will now show.

In order to prove (10.20), note that, by (10.19),

$$\partial_k \partial_\ell u \in M_2^p(\Omega), \quad \text{for } 1 \le \ell \le n-1,\ 1 \le k \le n, \tag{10.21}$$

where $p \in (n, \infty)$ and $r \in (0,1)$ are related by $1 - r = n/p$. Now the PDE (10.4) enables us to write $\partial_n^2 u$ as a linear combination of the terms in (10.21), with $L^\infty(\Omega)$-coefficients. Hence

$$\partial_n^2 u \in M_2^p(\Omega), \tag{10.22}$$

so

$$\nabla(\partial_n u) \in M_2^p(\Omega) \subset M^p(\Omega). \tag{10.23}$$

Morrey's lemma (Lemma A.1) states that

$$\nabla v \in M^p(\Omega) \Longrightarrow v \in C^r(\overline{\Omega}) \quad \text{if } r = 1 - \frac{n}{p} \in (0,1). \tag{10.24}$$

Thus

$$\partial_n u \in C^r(\overline{\Omega}), \tag{10.25}$$

and this together with (10.18) yields (10.20). From this, plus the Morrey space inclusions (10.21)–(10.22), we have the hypothesis (8.60) of Theorem 8.4, with $r > 0$ and $\sigma = 1$. Thus, by Theorem 8.4, and the associated estimate (8.73), we deduce estimates

$$\|u\|_{H^k(\Omega)} \le K_k, \tag{10.26}$$

for $k = 2, 3, \dots$. Therefore, if $[0, \tau_1) \subset J$, as $\tau_\nu \nearrow \tau_1$, we can pick a subsequence of $u_{\tau_\nu}$ converging weakly in $H^{k+1}(\Omega)$, hence strongly in $H^k(\Omega)$. If $k$ is picked large enough, the limit $u_1$ is an element of $H^{k+1}(\Omega)$, solving

(10.4) for $\tau = \tau_1$, and furthermore the regularity result Theorem 8.4 is applicable; hence $u_1 \in C^\infty(\overline{\Omega})$. This implies that $J$ is closed.

Hence we have a proof of the solvability of the boundary problem (10.1), for the special case $\overline{\Omega} = \mathbb{T}^{n-1} \times [0,1]$, granted the validity of the boundary gradient estimate (10.15).

As noted, to have $\partial_\ell, 1 \le \ell \le n-1$, tangent to $\partial M$, we required $\overline{\Omega} = \mathbb{T}^{n-1} \times [0,1]$. For $\overline{\Omega} \subset \mathbb{R}^n$, if $X = \sum b_\ell \partial_\ell$ is a smooth vector field tangent to $\partial\Omega$, then $u_X = Xu$ solves, in place of (10.14),

$$\sum \partial_j A^{jk}(\nabla u)\, \partial_k u_X = \sum \partial_j F_j, \tag{10.27}$$

with $F_j \in L^\infty$ calculable in terms of $\nabla u$. Thus Theorem 9.7 still applies, and the rest of the argument above extends easily. We have the following result.

**Theorem 10.2.** *Let $F : \mathbb{R}^n \to \mathbb{R}$ be a smooth function satisfying (10.3). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Let $\varphi \in C^\infty(\partial\Omega)$. Then the Dirichlet problem (10.1) has a unique solution $u \in C^\infty(\overline{\Omega})$, provided the boundary gradient estimate (10.15) is valid for all solutions $u = u_\tau$ to (10.4), for $\tau \in [0,1]$.*

**Proof.** Existence follows from the fact that $J$ is open and closed in $[0,1]$, and nonempty, as $0 \in J$. Uniqueness follows from the maximum principle argument used to establish Proposition 7.2.

Let us record a result that implies uniqueness.

**Proposition 10.3.** *Let $\Omega$ be any bounded domain in $\mathbb{R}^n$. Assume that $u_\nu \in C^\infty(\Omega) \cap C(\overline{\Omega})$ are real-valued solutions to*

$$G(\nabla u_\nu, \partial^2 u_\nu) = 0 \text{ on } \Omega, \quad u_\nu = g_\nu \text{ on } \partial\Omega, \tag{10.28}$$

*for $\nu = 1, 2$, where $G = G(p, \zeta)$, $\zeta = (\zeta_{jk})$. Then, under the ellipticity hypothesis*

$$\sum \frac{\partial G}{\partial \zeta_{jk}}(p, \zeta)\; \xi_j \xi_k \ge A(p)|\xi|^2 > 0, \tag{10.29}$$

*we have*

$$g_1 \le g_2 \text{ on } \partial\Omega \Longrightarrow u_1 \le u_2 \text{ on } \overline{\Omega}. \tag{10.30}$$

**Proof.** Same as Proposition 7.2. As shown there, $v = u_2 - u_1$ satisfies the identity $Lv = G(\nabla u_2, \partial^2 u_2) - G(\nabla u_1, \partial^2 u_1)$, and $L$ satisfies the conditions for the maximum principle, in the form of Proposition 2.1 of Chapter 5, given (10.29).

It is also useful to note that we can replace the first part of (10.28) by

$$G(\nabla u_2, \partial^2 u_2) \le G(\nabla u_1, \partial^2 u_1), \tag{10.31}$$

and the maximum principle still yields the conclusion (10.30).

Since the boundary gradient estimate was verified in Proposition 7.5 for the minimal surface equation whenever $\Omega \subset \mathbb{R}^2$ has strictly convex boundary, we have existence of smooth solutions in that case. In fact, the proof of Proposition 7.5 works when $\Omega \subset \mathbb{R}^n$ is strictly convex, so that $\partial\Omega$ has positive Gauss curvature everywhere. We hence have the following result.

**Theorem 10.4.** *If $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary that is strictly convex, then the Dirichlet problem*

$$\langle \nabla u \rangle^2 \Delta u - \sum_{j,k} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_k} \frac{\partial^2 u}{\partial x_j \partial x_k} = 0, \quad u = g \text{ on } \partial\Omega, \tag{10.32}$$

*for a minimal hypersurface, has a unique solution $u \in C^\infty(\overline{\Omega})$, given $g \in C^\infty(\partial\Omega)$.*

In Proposition 7.1, it was shown that when $n = 2$, the equation (10.32) has a solution $u \in C^\infty(\Omega) \cap C(\overline{\Omega})$, and Proposition 7.2 showed that such a solution must be unique. Hence in the case $n = 2$, Theorem 10.4 implies the regularity at $\partial\Omega$ for this solution, given $\varphi \in C^\infty(\partial\Omega)$.

We now look at other cases where the boundary gradient estimate can be verified, by extending the argument used in Proposition 7.5. Some terminology is useful. Let us be given a nonlinear operator $F(D^2u)$, and $g \in C^\infty(\partial\Omega)$. We say a function $B_+ \in C^2(\Omega)$ is an *upper barrier* at $y \in \partial\Omega$ (for $g$), provided

$$\begin{aligned} F(D^2 B_+) \le 0 \text{ on } \Omega, &\qquad B_+ \in C^1(\overline{\Omega}), \\ B_+ \ge g \text{ on } \partial\Omega, &\qquad B_+(y) = g(y). \end{aligned} \tag{10.33}$$

Similarly, we say $B_- \in C^2(\Omega)$ is a *lower barrier* at $y$ (for $g$), provided

$$\begin{aligned} F(D^2 B_-) \ge 0 \text{ on } \Omega, &\qquad B_- \in C^1(\overline{\Omega}), \\ B_- \le g \text{ on } \partial\Omega, &\qquad B_-(y) = g(y). \end{aligned} \tag{10.34}$$

An alternative expression is that $g$ has an upper (or lower) barrier at $y$. Note well the requirement that $B_\pm$ belong to $C^1(\overline{\Omega})$. We say $g$ has upper (resp., lower) barriers along $\partial\Omega$ if there are upper (resp., lower) barriers for $g$ at each $y \in \partial\Omega$, with uniformly bounded $C^1(\overline{\Omega})$-norms. The following result parallels Proposition 7.5.

**Proposition 10.5.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded region with smooth boundary. Consider a nonlinear differential operator of the form $F(D^2u) =$*

*$G(\nabla u, \partial^2 u)$, satisfying the ellipticity hypothesis (10.29). Assume that $g$ has upper and lower barriers along $\partial\Omega$, whose gradients are everywhere bounded by $K$. Then a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ to $F(D^2u) = 0$, $u = g$ on $\partial\Omega$, satisfies*

$$|u(y) - u(x)| \le 2K|y - x|, \quad y \in \partial\Omega, \quad x \in \overline{\Omega}. \tag{10.35}$$

*If $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, then*

$$|\nabla u(x)| \le 2K, \quad x \in \overline{\Omega}. \tag{10.36}$$

**Proof.** Same as Proposition 7.5. If $B_{\pm y}$ are the barriers for $g$ at $y \in \partial\Omega$, then

$$B_{-y}(x) \le u(x) \le B_{+y}(x), \quad x \in \overline{\Omega},$$

which readily yields (10.35). Note that $w_\ell = \partial_\ell u$ satisfies the PDE

$$\sum \frac{\partial G}{\partial \zeta_{jk}}\, \partial_j \partial_k w_\ell + \sum \frac{\partial G}{\partial p_j}\, \partial_j w_\ell = 0 \quad \text{on } \Omega, \tag{10.37}$$

so the maximum principle yields (10.36).

Now, behind the specific implementation of Proposition 7.5 is the fact that when $\partial\Omega$ is strictly convex and $g \in C^\infty(\partial\Omega)$, there are *linear* functions $B_{\pm y}$, satisfying $B_{-y} \le g \le B_{+y}$ on $\partial\Omega$, $B_{-y}(y) = g(y) = B_{+y}(y)$, with bounded gradients. Such functions $B_{\pm y}$ are annihilated by operators of the form (10.1). Therefore, we have the following extension of Theorem 10.4.

**Theorem 10.6.** *If $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary that is strictly convex, then the Dirichlet problem (10.1) has a unique solution $u \in C^\infty(\overline{\Omega})$, given $\varphi \in C^\infty(\partial\Omega)$, provided the ellipticity hypothesis (10.3) holds.*

We next consider the construction of upper and lower barriers when $F(D^2u) = \sum A^{jk}(\nabla u)\, \partial_j \partial_k u$ satisfies the *uniform* ellipticity condition

$$\lambda_0 |\xi|^2 \le \sum A^{jk}(p) \xi_j \xi_k \le \lambda_1 |\xi|^2, \tag{10.38}$$

for some $\lambda_j \in (0, \infty)$, independent of $p$. Given $z \in \mathbb{R}^n, R = |y - z|, \alpha \in (0, \infty)$, set

$$E_{y,z}(x) = e^{-\alpha r^2} - e^{-\alpha R^2}, \quad r^2 = |x - z|^2. \tag{10.39}$$

A calculation, used already in the derivation of maximum principles in §2 of Chapter 5, gives

$$\begin{aligned} &\sum A^{jk}(p)\, \partial_j \partial_k E_{y,z}(x) \\ &= e^{-\alpha r^2} \left[ 4\alpha^2 A^{jk}(p)(x_j - z_j)(x_k - z_k) - 2\alpha A^j{}_j(p) \right]. \end{aligned} \tag{10.40}$$

Under the hypothesis (10.38), we have

$$\sum A^{jk}(p)\,\partial_j\partial_k E_{y,z}(x) \geq 2\alpha e^{-\alpha r^2}\left[2\alpha\lambda_0|x-z|^2 - n\lambda_1\right]. \tag{10.41}$$

To make use of these functions, we proceed as follows. Given $y \in \partial\Omega$, pick $z = z(y) \in \mathbb{R}^n \setminus \overline{\Omega}$ such that $y$ is the closest point to $z$ on $\overline{\Omega}$. Given that $\overline{\Omega}$ is compact and $\partial\Omega$ is smooth, we can arrange that $|y-z| = R$, a positive constant, with the property that $R^{-1}$ is greater than twice the absolute value of any principal curvature of $\partial\Omega$ at any point. Note that, for any choice of $\alpha > 0$, $E_{y,z}(y) = 0$ and $E_{y,z}(x) < 0$ for $x \in \overline{\Omega} \setminus \{y\}$. From (10.41) we see that if $\alpha$ is picked sufficiently large (namely, $\alpha > n\lambda_1/2R^2\lambda_0$), then

$$\sum A^{jk}(p)\,\partial_j\partial_k E_{y,z}(x) > 0, \quad x \in \overline{\Omega}, \tag{10.42}$$

for all $p$, since $|x-z| \geq R$. Now, given $g \in C^\infty(\partial\Omega)$, we can find $K \in (0,\infty)$ such that, for all $x \in \partial\Omega$,

$$B_{\pm y}(x) = g(y) \mp K E_{y,z}(x) \Longrightarrow B_{-y}(x) \leq g(x) \leq B_{+y}(x). \tag{10.43}$$

Consequently, we have upper and lower barriers for $g$ along $\partial\Omega$. Therefore, we have the following existence theorem.

**Theorem 10.7.** *Let $\Omega \subset \mathbb{R}^n$ be any bounded region with smooth boundary. If the PDE (10.1) is uniformly elliptic, then (10.1) has a unique solution $u \in C^\infty(\overline{\Omega})$ for any $\varphi \in C^\infty(\partial\Omega)$.*

Certainly the equation (10.32) for minimal hypersurfaces is not uniformly elliptic. Here is an example of a uniformly elliptic equation. Take

$$F(p) = \left(\sqrt{1+|p|^2} - a\right)^2 = |p|^2 - 2a\sqrt{1+|p|^2} + 1 + a^2, \tag{10.44}$$

with $a \in (0,1)$. This models the potential energy of a stretched membrane, say a surface $S \subset \mathbb{R}^3$, given by $z = u(x)$, with the property that each point in $S$ is constrained to move parallel to the $z$-axis. Compare with (1.5) in Chapter 2.

It is also natural to look at the variational equation for a stretched membrane for which gravity also contributes to the potential energy. Thus we replace $F(p)$ in (10.44) by

$$F^\#(u,p) = F(p) + au, \tag{10.45}$$

where $a$ is a positive constant. This is of a form not encompassed by the class considered so far in this section. The PDE for $u$ in this case has the form

$$\operatorname{div} F^\#_p(u,\nabla u) - F^\#_u(u,\nabla u) = 0, \tag{10.46}$$

which, when $F^\#(u,p)$ has the form (10.45), becomes

$$\sum F_{p_j p_k}(\nabla u)\,\partial_j\partial_k u - a = 0. \tag{10.47}$$

We want to extend the existence argument to this case, to produce a solution $u \in C^\infty(\overline{\Omega})$, with given boundary data $\varphi \in C^\infty(\partial\Omega)$. Using the continuity method, we need estimates parallel to (10.13)–(10.20). Now, since $a > 0$, the maximum principle implies

$$\sup_{x\in\overline{\Omega}} u(x) = \sup_{y\in\partial\Omega} \varphi(y). \tag{10.48}$$

To estimate $\|u\|_{L^\infty}$, we also need control of $\inf_\Omega u(x)$. Such an estimate will follow if we obtain an estimate on $\|\nabla u\|_{L^\infty(\Omega)}$. To get this, note that the equation (10.14) for $w_\ell = \partial_\ell u$ continues to hold. Again the maximum principle applies, so the boundary gradient estimate (10.15) continues to imply (10.16). Furthermore, the construction of upper and lower barriers in (10.39)–(10.43) is easily extended, so one has such a boundary gradient estimate.

Now one needs to apply the DeGiorgi-Nash-Moser theory. Since (10.14) continues to hold, this application goes through without change, to yield (10.20), and the argument producing (10.26) also goes through as before. Thus Theorem 10.7 extends to PDE of the form (10.47).

One might consider more general force fields, replacing the potential energy function (10.45) by

$$F^\#(u,p) = F(p) + V(u). \tag{10.49}$$

Then the PDE for $u$ becomes

$$\sum F_{p_j p_k}(\nabla u)\partial_j\partial_k u - V'(u) = 0. \tag{10.50}$$

In this case, $w_\ell = \partial_\ell u$ satisfies

$$\sum \partial_j A^{jk}(\nabla u)\partial_k w_\ell - V''(u)w_\ell = 0. \tag{10.51}$$

This time, we won't start with an estimate on $\|u\|_{L^\infty}$, but we will aim directly for an estimate on $\|\nabla u\|_{L^\infty}$, which will serve to bound $\|u\|_{L^\infty}$, given that $u = \varphi$ on $\partial\Omega$.

The maximum principle applies to (10.51), to yield

$$\|\nabla u\|_{L^\infty(\Omega)} = \sup_{y\in\partial\Omega} |\nabla u(y)|, \quad \text{provided } V''(u) \ge 0. \tag{10.52}$$

Next, we check whether the barrier construction (10.39)–(10.43) yields a boundary gradient estimate in this case. Having (10.43) (with $g = \varphi$), we want

$$H(D^2 B_{+y}) \le H(D^2 u) \le H(D^2 B_{-y}) \quad \text{on } \Omega, \tag{10.53}$$

in place of (10.42), where $H(D^2 u)$ is given by the left side of (10.50), and we want this sequence of inequalities together with (10.43) to yield

$$B_{-y}(x) \le u(x) \le B_{+y}(x), \quad x \in \overline{\Omega}. \tag{10.54}$$

To obtain (10.53), note that we can arrange the left side of (10.42) to exceed a large constant, and also a large multiple of $E_{y,z}(x)$. Note that the middle quentity in (10.53) is zero, so we want $H(D^2B_{+y}) \le 0$ and $H(D^2B_{-y}) \ge 0$, on $\Omega$. We can certainly achieve this under the hypothesis that there is an estimate

$$|V'(u)| \le A_1 + A_2|u|. \tag{10.55}$$

In such a case, we have (10.53). To get (10.54) from this, we use the following extension of Proposition 10.3.

**Proposition 10.8.** *Let $\Omega \subset \mathbb{R}^n$ be bounded. Consider a nonlinear differential operator of the form*

$$H(x, D^2u) = G(x, u, \nabla u, \partial^2 u), \tag{10.56}$$

*where $G(x, u, p, \zeta)$ satisfies the ellipticity hypothesis (10.29), and*

$$\partial_u G(x, u, p, \zeta) \le 0. \tag{10.57}$$

*Then, given $u_\nu \in C^2(\Omega) \cap C(\overline{\Omega})$,*

$$H(D^2u_2) \le H(D^2u_1) \text{ on } \Omega,\ u_1 \le u_2 \text{ on } \partial\Omega \Longrightarrow u_1 \le u_2 \text{ on } \overline{\Omega}. \tag{10.58}$$

**Proof.** Same as Proposition 10.3. For the relevant maximum principle, replace Proposition 2.1 of Chapter 5 by Proposition 2.6 of that chapter.

To continue our analysis of the PDE (10.50), Proposition 10.8 applies to give (10.53) $\Rightarrow$ (10.54), provided $V''(u) \ge 0$. Consequently, we achieve a bound on $\|\nabla u\|_{L^\infty(\Omega)}$, and hence also on $\|u\|_{L^\infty(\Omega)}$, provided $V(u)$ satisfies the hypotheses stated in (10.52) and (10.55).

It remains to apply the DeGiorgi-Nash-Moser theory. In the simplified case where $\overline{\Omega} = \mathbb{T}^{n-1} \times [0,1]$, we obtain (10.18), this time by regarding (10.51) as a nonhomogeneous PDE for $w_\ell$, of the form (9.67), with one term $\partial_j g_j$, namely $\partial_\ell V'(u)$. The $L^\infty$-estimate we have on $u$ is more than enough to apply Theorem 9.7, so we again have (10.18)–(10.19). Next, the argument (10.21)–(10.23) goes through, so we again have (10.20) and the Morrey space inclusions (10.21)–(10.22). Hence the hypothesis (8.60) of Theorem 8.4 holds, with $r > 0$ and $\sigma = 1$. Theorem 8.4 yields

$$\|u\|_{H^k(\Omega)} \le K_k, \tag{10.59}$$

and a modification of the argument parallel to the use of (10.27) works for $\Omega \subset \mathbb{R}^n$.

The estimates above work for

$$\tau \sum F_{p_jp_k}(\nabla u)\partial_j\partial_k u - \tau V'(u) + (1-\tau)\Delta u = 0, \quad u\big|_{\partial\Omega} = \varphi, \tag{10.60}$$

for all $\tau \in [0,1]$. Also, each linearized operator is seen to be invertible, provided $V''(u) \ge 0$. Thus all the ingredients needed to use the method of continuity are in place. We have the following existence result.

**Proposition 10.9.** *Let $\Omega \subset \mathbb{R}^n$ be any bounded domain with smooth boundary. If the PDE*

$$\sum F_{p_j p_k}(\nabla u)\partial_j\partial_k u - V'(u) = 0, \quad u = \varphi \text{ on } \partial\Omega, \tag{10.61}$$

*is uniformly elliptic, and if $V'(u)$ satisfies*

$$|V'(u)| \le A_1 + A_2|u|, \quad V''(u) \ge 0, \tag{10.62}$$

*then (10.61) has a unique solution $u \in C^\infty(\overline{\Omega})$, given $\varphi \in C^\infty(\partial\Omega)$.*

Consider the case $V(u) = Au^2$. This satisfies (10.62) if $A \ge 0$ but not if $A < 0$. The case $A < 0$ corresponds to a repulsive force (away from $u = 0$) that increases linearly with distance. The physical basis for the failure of (10.61) to have a solution is that if $u(x)$ takes a large enough value, the repulsive force due to the potential $V$ cannot be matched by the elastic force of the membrane. If $F_{p_j p_k}(p)$ is independent of $p$ and $2A < 0$ is an eigenvalue of the linear operator $\sum F_{p_j p_k}\partial_j\partial_k$, then certainly (10.61) is not solvable.

On the other hand, if $V(u) = Au^2$ with $0 > A > -\ell_0$, where $\ell_0$ is less than the smallest eigenvalue of all operators $\sum A^{jk}\,\partial_j\partial_k$ with coefficients satisfying (10.38), then one can still hope to establish solvability for (10.61), in the uniformly elliptic case. We will not pursue the details on such existence results.

We now consider more general equations, of the form

$$H(D^2u) = \sum F_{p_j p_k}(\nabla u)\,\partial_j\partial_k u + g(x, u, \nabla u) = 0, \quad u\big|_{\partial\Omega} = \varphi. \tag{10.63}$$

Consider the family

$$H_\tau(D^2u) = \sum F_{p_j p_k}(\nabla u)\,\partial_j\partial_k u + \tau g(x, u, \nabla u) = 0, \quad u\big|_{\partial\Omega} = \tau\varphi. \tag{10.64}$$

We will prove the following:

**Proposition 10.10.** *Assume that the equation (10.63) satisfies the ellipticity condition (10.3) and that $\partial_u g(x, u, p) \le 0$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and let $\varphi \in C^\infty(\partial\Omega)$ be given. Assume that, for $\tau \in [0, 1]$, any solution $u = u_\tau$ to (10.64) has an a priori bound in $C^1(\overline{\Omega})$. Then (10.63) has a solution $u \in C^\infty(\overline{\Omega})$.*

**Proof.** For $w_\ell = \partial_\ell u$, we have, in place of (10.14),

$$\sum \partial_j A^{jk}(\nabla u)\,\partial_k w_\ell = -\partial_\ell g(x, u, \nabla u). \tag{10.65}$$

The $C^1$-bound on $u$ yields an $L^\infty$-bound on $g(x, u, \nabla u)$, so, as in the proof of Proposition 10.9, we can use Theorem 9.7 and proceed from there to obtain high-order Sobolev estimates on solutions to (10.64).

Thus the largest interval $J$ in $[0, 1]$ that contains $\tau = 0$ and such that (10.64) is solvable for all $\tau \in J$ is closed. The hypothesis $\partial_u g \le 0$ implies that the linearized equation at $\tau = \tau_0$ is uniquely solvable, so, as in Lemma 10.1, $J$ is open in $[0, 1]$, and the proposition is proved.

A simple example of (10.63) is the equation for a surface $z = u(x)$ of given constant mean curvature $H$:

$$\langle \nabla u \rangle^{-3} \Big[ \langle \nabla u \rangle^2 \Delta u - D^2 u(\nabla u, \nabla u) \Big] + nH = 0, \quad u = \varphi \text{ on } \partial\Omega, \tag{10.66}$$

which is of the form (10.63), with $F(p) = \big(1 + |p|^2\big)^{1/2}$ and $g(x, u, p) = nH$. Note that members of the family (10.64) are all of the same type in this case, namely equations for surfaces with mean curvature $\tau H$. We see that Proposition 10.3 applies to this equation. This implies uniqueness of solutions to (10.66), provided they exist, and also gives a tool to estimate $L^\infty$-norms, at least in some cases, by using equations of graphs of spheres of radius $1/H$ as candidates to bound $u$ from above and below. We can also use such functions to construct barriers, replacing the linear functions used in the proof of Proposition 7.5. This change means that the class of domains and boundary data for which upper and lower barriers can be constructed is different when $H \neq 0$ than it is in the minimal surface case $H = 0$.

Note that if $u$ solves (10.66), then $w_\ell = \partial_\ell u$ solves a PDE of the form (10.14). Thus the maximum principle yields $\|\nabla u\|_{L^\infty(\Omega)} = \sup_{\partial\Omega} |\nabla u(y)|$. Consequently, we have the solvability of (10.66) whenever we can construct barriers to prove the boundary gradient estimate.

The methods for constructing barriers described above do not exhaust the results one can obtain on boundary gradient estimates, which have been pushed quite far. We mention a result of H. Jenkins and J. Serrin. They have shown that the Dirichlet problem (10.66) for surfaces of constant mean curvature $H$ is solvable for arbitrary $\varphi \in C^\infty(\partial\Omega)$ if and only if the mean curvature $\varkappa(y)$ of $\partial\Omega \subset \mathbb{R}^n$ satisfies

$$\varkappa(y) \geq \frac{n}{n-1} |H|, \quad \forall y \in \partial\Omega. \tag{10.67}$$

In the special case $n = 2, H = 0$, this implies Proposition 7.3 in this chapter. See [GT] and [Se2] for proofs of this and extensions, including variable mean curvature $H(x)$, as well as extensive general discussions of boundary gradient estimates. We will have a little more practice constructing barriers and deducing boundary gradient estimates in §§13 and 15 of this chapter. See the proofs of Lemma 13.12 and of the estimate (15.54).

Results discussed above extend to more general second-order, scalar, quasi-linear PDE. In particular, Proposition 10.10 can be extended to all equations of the form

$$\sum a_{jk}(x, u, \nabla u)\, \partial_j \partial_k u + b(x, u, \nabla u) = 0, \quad u\big|_{\partial\Omega} = \varphi. \tag{10.68}$$

Let $\varphi \in C^\infty(\partial\Omega)$ be given. As long as it can be shown that, for each $\tau \in [0,1]$, a solution to

$$\sum a_{jk}(x,u,\nabla u)\,\partial_j\partial_k u + \tau b(x,u,\nabla u) = 0, \quad u\big|_{\partial\Omega} = \tau\varphi, \tag{10.69}$$

has an a priori bound in $C^1(\overline{\Omega})$, then (10.68) has a solution $u \in C^\infty(\overline{\Omega})$. This result, due to O. Ladyzhenskaya and N. Ural'tseva, is proved in [GT] and [LU]. These references, as well as [Se2], also discuss conditions under which one can establish a boundary gradient estimate for solutions to such PDE, and when one can pass from that to a $C^1(\overline{\Omega})$-estimate on solutions. The DeGiorgi-Nash-Moser estimates are still a major analytical tool in the proof of this general result, but further work is required beyond what was used to prove Proposition 10.10.

## Exercises

1. Carry out the construction of barriers for the equation of a surface of constant mean curvature mentioned below (10.66) and thus obtain some existence results for this equation. Compare these results with the result of Jenkins and Serrin, stated in (10.67).

Exercises 2–4 deal with quasi-linear elliptic equations of the form

$$\sum \partial_j A^{jk}(x,u)\partial_k u = 0 \quad \text{on } \Omega, \quad u\Big|_{\partial\Omega} = \varphi. \tag{10.70}$$

Assume there are positive functions $A_j$ such that

$$A_1(u)|\xi|^2 \le \sum A^{jk}(x,u)\xi_j\xi_k \le A_2(u)|\xi|^2.$$

2. Fix $\varphi \in C^\infty(\partial\Omega)$. Consider the operator $\Phi(u) = v$, the solution to

$$\sum \partial_j A^{jk}(x,u)\partial_k v = 0, \quad v\Big|_{\partial\Omega} = \varphi.$$

   Show that, for some $r > 0$,

$$\Phi : C(\overline{\Omega}) \longrightarrow C^r(\overline{\Omega}),$$

   continuously. Use the Schauder fixed-point theorem to deduce that $\Phi$ has a fixed point in $\{u \in C(\overline{\Omega}) : \sup\ |u| \le \sup\ |\varphi|\} \cap C^r(\overline{\Omega})$.
3. Show that this fixed point lies in $C^\infty(\overline{\Omega})$.
4. Examine whether solutions to (10.70) are unique.
5. Extend results on (10.1) to the case

$$\sum \partial_j F_{p_j}(x,\nabla u) = 0, \quad u\Big|_{\partial\Omega} = \varphi, \tag{10.71}$$

   arising from the search for critical points of $I(u) = \int_\Omega F(x,\nabla u)\,dx$, generalizing the case considered in (10.2).

In Exercises 6–9, we consider a PDE of the form

$$\sum \partial_j a^j(x,u,\nabla u) + b(x,u) = 0 \quad \text{on} \quad \Omega. \tag{10.72}$$

We assume $a^j$ and $b$ are smooth in their arguments and

$$|a^j(x,u,p)| \le C(u)\langle p\rangle, \quad |\nabla_p a^j(x,u,p)| \le C(u).$$

We make the ellipticity hypothesis

$$\sum \frac{\partial a^j}{\partial p_k}(x,u,p)\xi_j\xi_k \ge A(u)|\xi|^2, \quad A(u) > 0.$$

6. Show that if $u \in H^1(\Omega) \cap L^\infty(\Omega)$ solves (10.72), then $u$ solves a PDE of the form

$$\sum \partial_j A^{jk}(x)\partial_k u + \partial_j c^j(x,u) + b(x,u) = 0,$$

with

$$A^{jk} \in L^\infty, \quad \sum A^{jk}(x)\xi_j\xi_k \ge A|\xi|^2.$$

(*Hint*: Start with

$$a^j(x,u,p) = a^j(x,u,0) + \sum_k \widetilde{A}^{jk}(x,u,p)p_k, \quad \widetilde{A}^{jk}(x,u,p) = \int_0^1 \frac{\partial a^j}{\partial p_k}(x,u,sp)\,ds.)$$

7. Deduce that if $u \in H^1(\Omega) \cap L^\infty(\Omega)$ solves (10.72), then $u$ is Hölder continuous on the interior of $\Omega$.
8. If $\Omega$ is a smooth, bounded region in $\mathbb{R}^n$ and $u \in H^1(\Omega) \cap L^\infty(\Omega)$ satisfies (10.72) and $u\big|_{\partial\Omega} = \varphi \in C^1(\partial\Omega)$, show that $u$ is Hölder continuous on $\overline{\Omega}$ and that $\nabla u \in M_2^q(\Omega)$, for some $q > n$.
9. If $u \in C^2(\overline{\Omega})$ satisfies (10.72), show that $u_\ell = \partial_\ell u$ satisfies

$$\begin{aligned}\partial_j a^j_{p_k}(x,u,\nabla u)\,\partial_k u_\ell + \partial_j \Big[a^j_u(x,u,\nabla u)u_\ell\Big]&\\ +\partial_j a^j_{x_\ell}(x,u,\nabla u) + b_u(x,u)u_\ell + b_{x_\ell}(x,u) = 0.&\end{aligned}$$

Discuss obtaining estimates on $u$ in $C^{1+r}(\overline{\Omega})$, given estimates on $u$ in $C^1(\overline{\Omega})$.

## 11. Direct methods in the calculus of variations

We study the existence of minima (or other stationary points) of functionals of the form

$$I(u) = \int_\Omega F(x,u,\nabla u)\,dV(x), \tag{11.1}$$

on some set of functions, such as $\{u \in B : u = g \text{ on } \partial\Omega\}$, where $B$ is a suitable Banach space of functions on $\Omega$, possibly taking values in $\mathbb{R}^N$, and $g$ is a given smooth function on $\partial\Omega$. We assume $\overline{\Omega}$ is a compact Riemannian manifold with boundary and

$$F : \mathbb{R}^N \times (\mathbb{R}^N \otimes T^*\overline{\Omega}) \longrightarrow \mathbb{R} \text{ is continuous.} \tag{11.2}$$

Let us begin with a fairly direct generalization of the hypotheses (1.3)–(1.8) made in §1. Thus, let

$$V = \{u \in H^1(\Omega, \mathbb{R}^N) : u = g \text{ on } \partial\Omega\}. \tag{11.3}$$

For now, we assume that, for each $x \in \overline{\Omega}$,

$$F(x, \cdot, \cdot) : \mathbb{R}^N \times (\mathbb{R}^N \otimes T_x^* \overline{\Omega}) \longrightarrow \mathbb{R} \text{ is convex,} \tag{11.4}$$

where the domain has its natural linear structure. We also assume

$$A_0|\xi|^2 - B_0|u| - C_0 \le F(x, u, \xi), \tag{11.5}$$

for some positive constants $A_0,\ B_0,\ C_0$, and

$$|F(x, u, \xi) - F(x, v, \zeta)| \le C\big(|u - v| + |\xi - \zeta|\big)\big(|\xi| + |\zeta| + 1\big). \tag{11.6}$$

These hypotheses will be relaxed below.

**Proposition 11.1.** *Assume $\Omega$ is connected, with nonempty boundary. Assume $I(u) < \infty$ for some $u \in V$. Under the hypotheses (11.2)–(11.6), $I$ has a minimum on $V$.*

**Proof.** As in the situation dealt with in Proposition 1.2, we see that $I : V \to \mathbb{R}$ is Lipschitz continuous, bounded below, and convex. Thus, if $\alpha_0 = \inf_V \ I(u)$, then

$$K_\varepsilon = \{u \in V : \alpha_0 \le I(u) \le \alpha_0 + \varepsilon\} \tag{11.7}$$

is, for each $\varepsilon \in (0, 1]$, a nonempty, closed, convex subset of $V$. Hence $K_\varepsilon$ is weakly compact in $H^1(\Omega, \mathbb{R}^N)$. Hence $\bigcap_{\varepsilon>0} K_\varepsilon = K_0 \ne \emptyset$, and inf $I(u)$ is assumed on $K_0$.

We will state a rather general result whose proof is given by the argument above.

**Proposition 11.2.** *Let $V$ be a closed, convex subset of a reflexive Banach space $W$, and let $\Phi : V \to \mathbb{R}$ be a continuous map, satisfying:*

$$\inf_V \ \Phi = \alpha_0 \in (-\infty, \infty), \tag{11.8}$$

$$\exists\, b > \alpha_0 \text{ such that } \Phi^{-1}\big([\alpha_0, b]\big) \text{ is bounded in } W, \tag{11.9}$$

$$\forall\, y \in (\alpha_0, b], \quad \Phi^{-1}\big([\alpha_0, y]\big) \text{ is convex.} \tag{11.10}$$

*Then there exists $v \in V$ such that $\Phi(v) = \alpha_0$.*

As above, the proof comes down to the observation that, for $0 < \varepsilon \le b - \alpha_0$, $K_\varepsilon$ is a nested family of subsets of $W$ that are compact when $W$

has the weak topology. This result encompasses such generalizations of Proposition 11.1 as the following. Given $p \in (1,\infty)$, $g \in C^\infty(\partial\Omega,\mathbb{R}^N)$, let

$$V = \{u \in H^{1,p}(\Omega,\mathbb{R}^N) : u = g \text{ on } \partial\Omega\}. \tag{11.11}$$

We continue to assume (11.4), but replace (11.5) and (11.6) by

$$A_0|\xi|^p - B_0|u| - C_0 \le F(x,u,\xi), \tag{11.12}$$

for some positive $A_0, B_0, C_0$, and

$$|F(x,u,\xi) - F(x,v,\zeta)| \le C\big(|u-v| + |\xi-\zeta|\big)\big(|\xi| + |\zeta| + 1\big)^{p-1}. \tag{11.13}$$

Then we have the following:

**Proposition 11.3.** *Assume $\Omega$ is connected, with nonempty boundary. Take $p \in (1,\infty)$, and assume $I(u) < \infty$ for some $u \in V$. Under the hypotheses (11.2), (11.4), and (11.11)–(11.13), $I$ has a minumum on $V$.*

It is useful to extend Propositions 11.1 and 11.3, replacing (11.4) by a hypothesis of convexity only in the last set of variables.

**Proposition 11.4.** *Make the hypotheses of Proposition 11.1, or more generally of Proposition 11.3, but weaken (11.4) to the hypothesis that*

$$F(x,u,\cdot) : \mathbb{R}^N \otimes T_x^*\overline{\Omega} \longrightarrow \mathbb{R} \text{ is convex,} \tag{11.14}$$

*for each $(x,u) \in \overline{\Omega} \times \mathbb{R}^N$. Then $I$ has a minimum on $V$.*

**Proof.** Let $\alpha_0 = \inf_V \ I(u)$. The hypothesis (11.12) plus Poincaré's inequality imply that $\alpha_0 > -\infty$ and that

$$B = \{u \in V : I(u) \le \alpha_0 + 1\} \text{ is bounded in } H^{1,p}(\Omega,\mathbb{R}^N). \tag{11.15}$$

Pick $u_j \in B$ so that $I(u_j) \to \alpha_0$. Passing to a subsequence, we can assume

$$u_j \to u \text{ weakly in } H^{1,p}(\Omega,\mathbb{R}^N). \tag{11.16}$$

Hence $u_j \to u$ strongly in $L^p(\Omega,\mathbb{R}^N)$. We want to show that

$$I(u) = \alpha_0. \tag{11.17}$$

To this end, set

$$\Phi(u,v) = \int_\Omega F(x,u,v)\, dV(x). \tag{11.18}$$

With $v_j = \nabla u_j$, we have

$$\Phi(u_j,v_j) \to \alpha_0. \tag{11.19}$$

Also $v_j \to v = \nabla u$ weakly in $L^p(\Omega,\mathbb{R}^N \otimes T^*)$.

We can conclude that $I(u) \le \alpha_0$, and hence (11.17) holds if we show that

$$\Phi(u, v) \le \alpha_0. \tag{11.20}$$

Now, by hypothesis (11.13) we have

$$\begin{aligned} |\Phi(u_j, v_j) - \Phi(u, v_j)| &\le C \int_\Omega |u_j - u| \big(|v_j| + 1\big)^{p-1} \, dV(x) \\ &\le C' \|u_j - u\|_{L^p(\Omega)}, \end{aligned} \tag{11.21}$$

so

$$\Phi(u, v_j) \longrightarrow \alpha_0. \tag{11.22}$$

This time, by (11.5), (11.6), and (11.14) we have that, for each $\varepsilon \in (0, 1]$,

$$\mathcal{K}_\varepsilon = \{w \in L^p(\Omega, \mathbb{R}^N \otimes T^*) : \Phi(u, w) \le \alpha_0 + \varepsilon\} \tag{11.23}$$

is a closed, convex subset of $L^p(\Omega, \mathbb{R}^N \otimes T^*)$. Hence $\mathcal{K}_\varepsilon$ is weakly compact, provided it is nonempty. Furthermore, by (11.22), $v_j \in \mathcal{K}_{\varepsilon_j}$ with $\varepsilon_j \to 0$, so we have $v \in \mathcal{K}_0$. This implies (11.20), so Proposition 11.4 is proved.

The following extension of Proposition 11.4 applies to certain constrained minimization problems.

**Proposition 11.5.** *Let $p \in (1, \infty)$, and let $F(x, u, \xi)$ satisfy the hypotheses of Proposition 11.4. Then, if $S$ is any subset of $V$ (given by (11.11)) that is closed in the weak topology of $H^{1,p}(\Omega, \mathbb{R}^N)$, it follows that $I\big|_S$ has a minimum in $S$.*

**Proof.** Let $\alpha_0 = \inf_S \ I(u)$, and take $u_j \in S, \ I(u_j) \to \alpha_0$. Since (11.15) holds, we can take a subsequence $u_j \to u$ weakly in $H^{1,p}(\Omega, \mathbb{R}^N)$, so $u \in S$. We want to show that $I(u) = \alpha_0$. Indeed, if we form $\Phi(u, v)$ as in (11.18), then the argument involving (11.19)–(11.23) continues to hold, and our assertion is proved.

For example, if $X \subset \mathbb{R}^N$ is a closed subset, we could take

$$S = \{u \in V : u(x) \in X \text{ for a.e. } x \in \Omega\}, \tag{11.24}$$

and Proposition 11.5 applies. As a specific example, $X$ could be a compact Riemannian manifold, isometrically imbedded in $\mathbb{R}^N$, and we could take $p = 2, \ F(x, u, \nabla u) = |\nabla u|^2$. The resulting minimum of $I(u)$ is a harmonic map of $\overline{\Omega}$ into $X$. If $u : \Omega \to X$ is a harmonic map, it satisfies the PDE

$$\Delta u - \Gamma(u)(\nabla u, \nabla u) = 0, \tag{11.25}$$

where $\Gamma(u)(\nabla u, \nabla u)$ is a certain quadratic form in $\nabla u$. See §2 of Chapter 15 for a derivation.

A generalization of the notion of harmonic map arises in the study of "liquid crystals." One takes

$$F(x,u,\nabla u) = a_1|\nabla u|^2 + a_2(\text{div } u)^2 + a_3(u\cdot\text{curl } u)^2 + a_4|u\times\text{curl } u|^2, \tag{11.26}$$

where the coefficients $a_j$ are positive constants, and then one minimizes the functional $\int_\Omega F(x,u,\nabla u)\,dV(x)$ over a set $S$ of the form (11.24), with $X = S^2 \subset \mathbb{R}^3$, namely, over

$$S = \{u \in H^1(\Omega,\mathbb{R}^3) : |u(x)| = 1 \text{ a.e. on } \Omega,\ u = g \text{ on } \partial\Omega\}. \tag{11.27}$$

In this case, $F(x,u,\xi)$ has the form

$$F(x,u,\xi) = \sum_{j,\alpha} b_{j\alpha}(u)\xi_{j\alpha}^2, \quad b_{j,\alpha}(u) \ge a_1 > 0,$$

where each coefficient $b_{j\alpha}(u)$ is a polynomial of degree 2 in $u$. Clearly, this function is convex in $\xi$. The function $F(x,u,\xi)$ does not satisfy (11.6); hence, in going through the argument establishing Proposition 11.4, we would need to replace the $p = 2$ case of (11.22) by

$$|\Phi(u_j,v_j) - \Phi(u,v_j)| \le C \int_\Omega |u_j - u|\cdot|v_j|^2\,dV(x). \tag{11.28}$$

The following result covers integrands of the form (11.26), as well as many others. It assumes a slightly bigger lower bound on $F$ than the previous results, but it greatly relaxes the hypotheses on how rapidly $F$ can vary.

**Theorem 11.6.** *Assume $\Omega$ is connected, with nonempty boundary. Take $p \in (1,\infty)$, and set*

$$V = \{u \in H^{1,p}(\Omega,\mathbb{R}^N) : u = g \text{ on } \partial\Omega\}.$$

*Assume $I(u) < \infty$ for some $u \in V$. Assume that $F(x,u,\xi)$ is smooth in its arguments and satisfies the convexity condition (11.14) in $\xi$ and the lower bound*

$$A_0|\xi|^p \le F(x,u,\xi), \tag{11.29}$$

*for some $A_0 > 0$. Then $I$ has a minimum on $V$.*

*Also, if $S$ is a subset of $V$ that is closed in the weak topology of $H^{1,p}(\Omega,\mathbb{R}^N)$, then $I\big|_S$ has a minimum in $S$.*

**Proof.** Clearly, $\alpha_0 = \inf_S I(u) \ge 0$. With $B$ as in (11.15), pick $u_j \in B\cap S$ so that

$$I(u_j) \to \alpha_0, \quad u_j \to u \text{ weakly in } H^{1,p}(\Omega,\mathbb{R}^N). \tag{11.30}$$

Passing to a subsequence, we can assume $u_j \to u$ a.e. on $\Omega$. We need to show that

$$\int_\Omega F(x,u,\nabla u)\, dV \le \alpha_0. \tag{11.31}$$

By Egorov's theorem, we can pick measurable sets $E_\nu \supset E_{\nu+1} \supset \cdots$ in $\Omega$, of measure $< 2^{-\nu}$, such that $u_j \to u$ uniformly on $\Omega \setminus E_\nu$. We can also arrange that

$$|u(x)| + |\nabla u(x)| \le C \cdot 2^\nu, \quad \text{for} \quad x \in \Omega \setminus E_\nu. \tag{11.32}$$

Now, we have

$$\begin{aligned}\int_{\Omega\setminus E_\nu} F(x,u,\nabla u)\, dV = &\int_{\Omega\setminus E_\nu} F(x,u_j,\nabla u_j)\, dV \\ &+ \int_{\Omega\setminus E_\nu} \big[F(x,u_j,\nabla u) - F(x,u_j,\nabla u_j)\big]\, dV \\ &+ \int_{\Omega\setminus E_\nu} \big[F(x,u,\nabla u) - F(x,u_j,\nabla u)\big]\, dV.\end{aligned} \tag{11.33}$$

To estimate the second integral on the right side of (11.33), we use the convexity hypothesis to write

$$F(x,u_j,\nabla u) - F(x,u_j,\nabla u_j) \le D_\xi F(x,u_j,\nabla u) \cdot (\nabla u - \nabla u_j). \tag{11.34}$$

Now, for each $\nu$,

$$D_\xi F(x,u_j,\nabla u) \longrightarrow D_\xi F(x,u,\nabla u), \quad \text{uniformly on} \quad \Omega \setminus E_\nu, \tag{11.35}$$

while $\nabla u - \nabla u_j \to 0$ weakly in $L^p(\Omega,\mathbb{R}^n)$, so

$$\lim_{j\to\infty} \int_{\Omega\setminus E_\nu} \big[F(x,u_j,\nabla u) - F(x,u_j,\nabla u_j)\big]\, dV = 0. \tag{11.36}$$

Estimating the last integral in (11.33) is easy, since

$$F(x,u,\nabla u) - F(x,u_j,\nabla u) \longrightarrow 0, \quad \text{uniformly on} \quad \Omega \setminus E_\nu. \tag{11.37}$$

Thus, from our analysis of (11.33), we have

$$\int_{\Omega\setminus E_\nu} F(x,u,\nabla u)\, dV \le \limsup_{j\to\infty} \int_{\Omega\setminus E_\nu} F(x,u_j,\nabla u_j)\, dV \le \alpha_0, \tag{11.38}$$

for all $\nu$, and taking $\nu \to \infty$ gives (11.31). The theorem is proved.

There are a number of variants of the results above. We mention one:

**Proposition 11.7.** *Assume that $F$ is smooth in $(x, u, \xi)$, that*

$$F(x, u, \xi) \geq 0, \tag{11.39}$$

*and that*

$$F(x, u, \cdot) : \mathbb{R}^N \otimes T_x^*\overline{\Omega} \longrightarrow \mathbb{R} \text{ is convex,} \tag{11.40}$$

*for each $x, u$. Suppose*

$$u_\nu \to u \text{ weakly in } H^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^N). \tag{11.41}$$

*Then*

$$I(u) \leq \liminf_{\nu \to \infty} I(u_\nu). \tag{11.42}$$

For a proof, and other extensions, see [Gia] or [Dac]. It is a result of J. Serrin [Se1] that, in the case where $u$ is real-valued, the hypothesis (11.41) can be weakened to

$$u_\nu, u \in H^{1,1}_{\text{loc}}(\Omega), \quad u_\nu \to u \text{ in } L^1_{\text{loc}}(\Omega). \tag{11.43}$$

In [Mor2] there is an attempt to extend Serrin's result to systems, but it was shown by [Eis] that such an extension is false.

In [Dac] there is also a discussion of a replacement for convexity, due to Morrey, called "quasi-convexity." For other contexts in which the convexity hypothesis is absent, and one often looks not for a minimizer but some sort of saddle point, see [Str2] and [Gia2].

In this section we have obtained solutions to extremal problems, but these solutions lie in Sobolev spaces with rather low regularity. The problem of higher regularity for such solutions is considered in §12.

## Exercises

1. In Theorem 11.6, take $p > n = \dim \Omega = N$, and consider

$$S = \{u \in V : \det Du = 1, \text{ a.e. on } \Omega\}.$$

Show that $S$ is closed in the weak topology of $H^{1,p}(\Omega, \mathbb{R}^n)$ and hence that Theorem 11.6 applies. (*Hint*: See (6.35)–(6.36) of Chapter 13.)

2. In Theorem 11.6, take $p \in (1, \infty)$, $\Omega \subset \mathbb{R}^n$, $N = 1$. Let $h \in C^\infty(\overline{\Omega})$, and consider

$$S = \{u \in V : u \geq h \text{ on } \Omega\}.$$

Show that $S$ is closed in the weak topology of $H^{1,p}(\Omega)$ and hence that Theorem 11.6 applies.

Say $I\big|_S$ achieves its minimum at $u$, and suppose you are given that $u \in C(\Omega)$, so

$$\mathcal{O} = \{x \in \Omega : u(x) > h(x)\}$$

is open. Assume also that $\partial F/\partial \xi_j$ and $\partial F/\partial u$ satisfy convenient bounds. Show that, on $\mathcal{O}$, $u$ satisfies the PDE

$$\sum_j \partial_j F_{\xi_j}(x, u, \nabla u) + F_u(x, u, \nabla u) = 0.$$

For more on this sort of variational problem, see [KS].

## 12. Quasi-linear elliptic systems

Here we (partially) extend the study of the scalar equation (10.1) to a study of an $N \times N$ system

$$A^{jk}_{\alpha\beta}(\nabla u)\partial_j \partial_k u^\beta = 0 \ \text{ on } \ \Omega, \quad u = \varphi \ \text{ on } \ \partial\Omega, \tag{12.1}$$

where $\varphi \in C^\infty(\partial\Omega, \mathbb{R}^N)$ is given. The hypothesis of strong ellipticity used previously is

$$\sum A^{jk}_{\alpha\beta}(p) v_\alpha v_\beta \xi_j \xi_k \geq C|v|^2 |\xi|^2, \quad C > 0, \tag{12.2}$$

but many nonlinear results require that $A^{jk}_{\alpha\beta}(p)$ satisfy the *very strong ellipticity hypothesis*:

$$\sum A^{jk}_{\alpha\beta}(p) \zeta_{j\alpha} \zeta_{k\beta} \geq \kappa |\zeta|^2, \quad \kappa > 0. \tag{12.3}$$

We mention that, in much of the literature, (12.3) is called strong ellipticity and (12.2) is called the "Legendre-Hadamard condition."

In the case when (12.1) arises from minimizing the function

$$I(u) = \int_\Omega F(\nabla u)\, dx, \tag{12.4}$$

we have

$$A^{jk}_{\alpha\beta}(p) = \partial_{p_{j\alpha}} \partial_{p_{k\beta}} F(p). \tag{12.5}$$

In such a case, (12.3) is the statement that $F(p)$ is a uniformly strongly convex function of $p$. If (12.5) holds, (12.1) can be written as

$$\sum_j \partial_j G^j_\alpha(\nabla u) = 0 \ \text{ on } \ \Omega, \quad u = \varphi \ \text{ on } \ \partial\Omega; \quad G^j_\alpha(p) = \partial_{p_{j\alpha}} F(p). \tag{12.6}$$

We will assume

$$\begin{gathered} a_0|p|^2 - b_0 \leq F(p) \leq a_1|p|^2 + b_1, \\ |G^j_\alpha(p)| \leq C_0 \langle p \rangle, \qquad |A^{jk}_{\alpha\beta}(p)| \leq C_1. \end{gathered} \tag{12.7}$$

These are called "controllable growth conditions."

If (12.5) holds, then

$$\begin{aligned}\partial_j G^j_\alpha(\nabla u) - \partial_j G^j_\alpha(\nabla v) &= \partial_j \mathcal{A}^{jk}_{\alpha\beta}(x)\partial_k(u^\beta - v^\beta),\\ \mathcal{A}^{jk}_{\alpha\beta}(x) &= \int_0^1 A^{jk}_{\alpha\beta}\big(s\nabla u + (1-s)\nabla v\big)\, ds.\end{aligned} \tag{12.8}$$

This leads to a uniqueness result:

**Proposition 12.1.** *Assume $\Omega \subset \mathbb{R}^n$ is a smoothly bounded domain, and assume that (12.3) and (12.7) hold. If $u, v \in H^1(\Omega, \mathbb{R}^N)$ both solve (12.6), then $u = v$ on $\Omega$.*

**Proof.** By (12.8), we have

$$\int\limits_\Omega \mathcal{A}^{jk}_{\alpha\beta}(x)\; \partial_j(u^\alpha - v^\alpha)\; \partial_k(u^\beta - v^\beta)\; dx = 0, \tag{12.9}$$

so (12.3) implies $\partial_j(u - v) = 0$, which immediately gives $u = v$.

Let $X = \sum b^\ell \partial_\ell$ be a smooth vector field on $\overline{\Omega}$, tangent to $\partial\Omega$. If we knew that $u \in H^2(\Omega)$, we could deduce that $u_X = Xu$ is the unique solution in $H^1(\Omega, \mathbb{R}^N)$ to

$$\sum \partial_j A^{jk}(\nabla u)\, \partial_k u_X = \sum \partial_j f^j + g, \quad u_X = X\varphi \;\text{ on }\; \partial\Omega, \tag{12.10}$$

where

$$\begin{aligned} f^j &= A^{jk}(\nabla u)(\partial_k b^\ell)(\partial_\ell u) + (\partial_\ell b^j) G^\ell_\alpha(\nabla u),\\ g &= -(\partial_\ell \partial_j b^\ell) G^j_\alpha(\nabla u).\end{aligned} \tag{12.11}$$

Under the growth hypothesis (12.7), $|f^j(x)| \le C|\nabla u(x)|$, so $\|f^j\|_{L^2(\Omega)} \le C\|\nabla u\|_{L^2(\Omega)}$. Similarly, $\|g\|_{L^2(\Omega)} \le C\|\nabla u\|_{L^2(\Omega)} + C$. Hence, we can say that (12.10) has a unique solution, satisfying

$$\|u_X\|_{H^1(\Omega)} \le C\big(\|u\|_{H^1(\Omega)} + \|\varphi\|_{H^2(\Omega)} + 1\big). \tag{12.12}$$

It is unsatisfactory to hypothesize that $u$ belong to $H^2(\Omega)$, so we replace the differentiation of (12.6) by taking difference quotients. Let $\mathcal{F}^t_X$ denote the flow on $\overline{\Omega}$ generated by $X$, and set $u_h = u \circ \mathcal{F}^h_X$. Then $u_h$ extremizes a functional

$$I_h(u_h) = \int\limits_\Omega F_h(x, \nabla u_h)\; dx, \tag{12.13}$$

where $F_h(x,p)$ depends smoothly on $(h, x, p)$ and $F_0(x,p) = F(p)$. (In fact, (12.13) is simply (12.4), after a coordinate change.) Thus $u_h$ satisfies the PDE

$$\partial_j(\partial_{p_{j\alpha}} F_h)(x, \nabla u_h) = 0, \quad u_h = \varphi_k \text{ on } \partial\Omega. \tag{12.14}$$

Applying the fundamental theorem of calculus to the difference of (12.14) and (12.6), we have

$$\partial_j \mathcal{A}^{jk}_{\alpha\beta h}(x)\,\partial_k\Big(\frac{u^\beta_h - u^\beta}{h}\Big) = \partial_j H^j_{\alpha h}(x, \nabla u_h), \tag{12.15}$$

where $\mathcal{A}^{jk}_{\alpha\beta h}(x)$ is as in (12.8), with $v = u_h$, and

$$H^j_{\alpha h}(x,p) = \int_0^h \frac{d}{ds}(\partial_{p_{j\alpha}} F_s)(x,p)\; ds. \tag{12.16}$$

As in the analysis of (12.10), we have

$$\|h^{-1}(u_h - u)\|_{H^1(\Omega)} \le C\big(\|u\|_{H^1(\Omega)} + \|\varphi\|_{H^2(\Omega)} + 1\big). \tag{12.17}$$

Taking $h \to 0$, we have $u_X \in H^1(\Omega, \mathbb{R}^N)$, with the estimate (12.12).

From here, a standard use of ellipticity, parallel to the argument in (10.21)–(10.25), gives an $H^1$-bound on a transversal derivative of $u$; hence $u \in H^2(\Omega, \mathbb{R}^n)$, and

$$\|u\|_{H^2(\Omega)} \le C\big(\|u\|_{H^1(\Omega)} + \|\varphi\|_{H^2(\Omega)} + 1\big). \tag{12.18}$$

As in the scalar case, one of the keys to the further analysis of a solution to (12.6) is an examination of regularity for solutions to linear elliptic systems with $L^\infty$-coefficients. Thus we consider linear operators of the form

$$Lu = b(x)^{-1} \sum_{j,k=1}^{n} \partial_j\big(A^{jk}(x) b(x)\,\partial_k u\big), \tag{12.19}$$

Compare with (9.1). Here $u$ takes values in $\mathbb{R}^N$ and each $A^{jk}$ is an $N \times N$ matrix, with real-valued entries $A^{jk}_{\alpha\beta} \in L^\infty(\Omega)$. We assume $A^{jk}_{\alpha\beta} = A^{kj}_{\beta\alpha}$. As in (12.3), we make the hypothesis

$$\lambda_1 |\zeta|^2 \ge \sum A^{jk}_{\alpha\beta}(x)\zeta_{j\alpha}\zeta_{k\beta} \ge \lambda_0|\zeta|^2, \quad \lambda_0 > 0, \tag{12.20}$$

of very strong ellipticity. Thus $A^{jk}_{\alpha\beta}$ defines a positive-definite inner product $\langle\ ,\ \rangle$ on $T^* \otimes \mathbb{R}^N$. We also assume

$$0 < C_0 \le b(x) \le C_1. \tag{12.21}$$

Then $b(x)\,dx = dV$ defines a volume element, and, for $\varphi \in C^1_0(\Omega, \mathbb{R}^N)$,

$$(Lu, \varphi) = -\int_\Omega \langle \nabla u, \nabla \varphi\rangle\; dV. \tag{12.22}$$

We will establish the following result of [Mey].

**Proposition 12.2.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, let $f_j \in L^q(\Omega, \mathbb{R}^N)$ for some $q > 2$, and let $u$ be the unique solution in*

*$H_0^{1,2}(\Omega)$ to*

$$Lu = \sum \partial_j f_j. \tag{12.23}$$

*Assume $L$ has the form (12.19), with coefficients $A^{jk} \in L^\infty(\Omega)$, satisfying (12.20), and $b \in C^\infty(\overline{\Omega})$, satisfying (12.21). Then $u \in H^{1,p}(\Omega)$, for some $p > 2$.*

**Proof.** We define the affine map

$$T : H_0^{1,p}(\Omega) \longrightarrow H_0^{1,p}(\Omega) \tag{12.24}$$

as follows. Let $\Delta$ be the Laplace operator on $\overline{\Omega}$, endowed with a smooth Riemannian metric whose volume element is $dV = b(x)\,dx$, and adjust $\lambda_0, \lambda_1$ so (12.20) holds when $|\zeta|^2$ is computed via the inner product $(\ ,\ )$ on $T^* \otimes \mathbb{R}^N$ associated with this metric, so that

$$(\Delta u, \varphi) = -\int_\Omega (\nabla u, \nabla \varphi)\ dV. \tag{12.25}$$

Then we define $Tw = v$ to be the unique solution in $H_0^{1,2}(\Omega)$ to

$$\Delta v = \Delta w - \lambda_1^{-1} Lw + \lambda_1^{-1} \sum \partial_j f_j. \tag{12.26}$$

The mapping property (12.24) holds for $2 \le p \le q$, by the $L^p$-estimates of Chapter 13. In fact, if $\Delta v = \sum \partial_j g_j,\ v \in H_0^{1,2}(\Omega)$, then

$$\|\nabla v\|_{L^p(\Omega)} \le C(p) \|g\|_{L^p(\Omega)}. \tag{12.27}$$

If we fix $r > 2$, then, for $2 \le p \le r$, interpolation yields such an estimate, with

$$C(p) = C(r)^\theta, \quad \frac{1-\theta}{2} + \frac{\theta}{r} = \frac{1}{p}, \quad \text{i.e.,}\ \ \theta = \frac{r}{p}\frac{p-2}{r-2}. \tag{12.28}$$

Hence $C(p) \searrow 1$, as $p \searrow 2$. Now we see that $Tw_1 - Tw_2 = v_1 - v_2$ satisfies

$$\Delta(v_1 - v_2) = \big(\Delta - \lambda_1^{-1} L\big)(w_1 - w_2) = \nabla g, \tag{12.29}$$

where

$$g_j^\alpha = \partial_j (w_1^\alpha - w_2^\alpha) - \lambda_1^{-1} A_{\alpha\beta}^{jk}\, \partial_k (w_1^\beta - w_2^\beta), \tag{12.30}$$

and hence, under our hypotheses,

$$\|g\|_{L^p(\Omega)} \le \Big(1 - \frac{\lambda_0}{\lambda_1}\Big) \|\nabla (w_1 - w_2)\|_{L^p(\Omega)}, \tag{12.31}$$

so

$$\|\nabla (v_1 - v_2)\|_{L^p(\Omega)} \le C(p) \Big(1 - \frac{\lambda_0}{\lambda_1}\Big) \|\nabla (w_1 - w_2)\|_{L^p(\Omega)}, \tag{12.32}$$

for $2 \le p \le q$. We see that, for some $p > 2$, $C(p)\big(1 - \lambda_0/\lambda_1\big) < 1$; hence $T$ is a contraction on $H^{1,p}(\Omega)$ in such a case. Thus $T$ has a unique fixed point. This fixed point is $u$, so we have $u \in H_0^{1,p}(\Omega)$, as claimed.

**Corollary 12.3.** *With hypotheses as in Proposition 12.2, given a function* $\psi \in H^{1,q}(\Omega)$, *the unique solution* $u \in H^{1,2}(\Omega)$ *satisfying (12.23) and*

$$u = \psi \quad \text{on} \quad \partial\Omega \tag{12.33}$$

*also belongs to* $H^{1,p}(\Omega)$, *for some* $p > 2$.

**Proof.** Apply Proposition 12.2 to $u - \psi$.

Let us return to the analysis of a solution $u \in H^1(\Omega, \mathbb{R}^N)$ to the nonlinear system (12.6), under the hypotheses of Proposition 12.1. Since we have established that $u \in H^2(\Omega, \mathbb{R}^N)$, we have a bound

$$\|\nabla u\|_{L^q(\Omega)} \leq A, \quad q > 2. \tag{12.34}$$

In fact, this holds with $q = 2n/(n-2)$ if $n \geq 3$, and for all $q < \infty$ if $n = 2$. As above, if $X = \sum b^\ell \, \partial_\ell$ is a smooth vector field on $\overline{\Omega}$, tangent to $\partial\Omega$, then $u_X = Xu$ is the unique solution in $H^1(\Omega, \mathbb{R}^N)$ to (12.10), and we can now say that $f^j \in L^q(\Omega)$. Thus Corollary 12.3 gives

$$Xu \in H^{1,p}(\Omega), \quad \text{for some } p > 2, \tag{12.35}$$

with a bound, and again a standard use of ellipticity gives an $H^{1,p}$-bound on a transversal derivative of $u$. We have established the following result.

**Theorem 12.4.** *If* $u \in H^1(\Omega, \mathbb{R}^N)$ *solves (12.6) on a smoothly bounded domain* $\Omega \in \mathbb{R}^n$, *and if the very strong ellipticity hypothesis (12.3) and the controllable growth hypothesis (12.7) hold, then* $u \in H^{2,p}(\Omega, \mathbb{R}^N)$, *for some* $p > 2$, *and*

$$\|u\|_{H^{2,p}(\Omega)} \leq C\big(\|\nabla u\|_{L^2(\Omega)} + \|\varphi\|_{H^{2,q}(\Omega)} + 1\big). \tag{12.36}$$

The case $n = \dim \Omega = 2$ of this result is particularly significant, since, for $p > n$, $H^{1,p}(\Omega) \subset C^r(\overline{\Omega})$, $r > 0$. Thus, under the hypotheses of Theorem 12.4, we have $u \in C^{1+r}(\overline{\Omega})$, for some $r > 0$, if $n = 2$. Then the material of §8 applies to (12.1), so we have the following:

**Proposition 12.5.** *If* $u \in H^1(\Omega, \mathbb{R}^N)$ *solves (12.6) on a smoothly bounded domain* $\Omega \subset \mathbb{R}^2$, *and the hypotheses (12.3) and (12.7) hold, then* $u \in C^\infty(\overline{\Omega})$, *provided* $\varphi \in C^\infty(\partial\Omega)$.

When $n = 2$, we then have existence of a unique smooth solution to (12.1), given $\varphi \in C^\infty(\partial\Omega)$. In fact, we have two routes to such existence. We could obtain a minimizer $u \in H^1(\Omega, \mathbb{R}^N)$ for (12.4), subject to the condition that $u\big|_{\partial\Omega} = \varphi$, by the results of §11, and then apply Proposition 12.5 to deduce smoothness.

Alternatively, we could apply the continuity method, to solve

$$A^{jk}_{\alpha\beta}(\nabla u)\partial_j\partial_k u^\beta = 0 \quad \text{on} \quad \Omega, \qquad u = \tau\varphi \quad \text{on} \quad \partial\Omega. \tag{12.37}$$

This is clearly solvable for $\tau = 0$, and the proof that the biggest $\tau$-interval $J \subset [0,1]$, containing 0, on which (12.37) has a unique solution $u \in C^\infty(\overline{\Omega})$, is both open and closed is accomplished along lines similar to arguments in §10. However, unlike in §10, we do not need to establish a sup-norm bound on $\nabla u$, or even on $u$; we make do with an $H^1$-norm bound, which can be deduced from (12.3) as follows.

If $\mathcal{A}^{jk}_{\alpha\beta}(x)$ is given by (12.8), with $v = \varphi$, we have

$$\begin{aligned}\int_\Omega & \mathcal{A}^{jk}_{\alpha\beta}(x)\ \partial_k(u^\beta - \varphi^\beta)\ \partial_j(u^\alpha - \varphi^\alpha)\ dx \\ &= \int_\Omega \partial_j G^j_\alpha(\nabla\varphi)(u^\alpha - \varphi^\alpha)\ dx,\end{aligned} \tag{12.38}$$

for a solution to (12.37) (in case $\tau = 1$). Hence

$$\kappa\|\nabla(u-\varphi)\|^2_{L^2(\Omega)} \le C\|u-\varphi\|_{L^2(\Omega)}. \tag{12.39}$$

Note the different exponents. We have $\|u-\varphi\|^2_{L^2(\Omega)} \le C_2\|\nabla(u-\varphi)\|^2_{L^2(\Omega)}$, by Poincaré's inequality, so

$$\|u-\varphi\|_{L^2(\Omega)} \le \frac{C}{\kappa C_2}. \tag{12.40}$$

Plugging this back into (12.39) gives

$$\|\nabla(u-\varphi)\|^2_{L^2(\Omega)} \le \frac{C^2}{\kappa^2 C_2}, \tag{12.41}$$

which implies the desired $H^1$-bound on $u$.

Once we have the $H^1$-bound on $u = u_\tau$, (12.36) gives an $H^{2,p}$-bound for some $p > 2$, hence a bound in $C^{1+r}(\overline{\Omega})$, for some $r > 0$. Then the results of §8 give bounds in higher norms, sufficient to show that $J$ is closed.

Proposition 12.5 does not in itself imply all the results of §10 when $\dim \Omega = 2$, since the hypotheses (12.3) and (12.7) imply that (12.1) is uniformly elliptic. For example, the minimal surface equation is not covered by Proposition 12.5. However, it is a simple matter to prove the following result, which does (essentially) contain the $n = 2$ case of Theorem 10.2.

**Proposition 12.6.** *Assume $A^{jk}_{\alpha\beta}(p)$ is smooth in $p$ and satisfies*

$$A^{jk}_{\alpha\beta}(p)\zeta_{j\alpha}\zeta_{k\beta} \ge C(p)|\zeta|^2, \quad C(p) > 0. \tag{12.42}$$

*Let $\Omega \subset \mathbb{R}^2$ be a smoothly bounded domain. Then the Dirichlet problem (12.1) has a unique solution $u \in C^\infty(\overline{\Omega})$, provided one has an a priori bound*

$$\|\nabla u_\tau\|_{L^\infty(\Omega)} \le K, \tag{12.43}$$

*for all smooth solutions $u = u_\tau$ to (12.37), for $\tau \in [0,1]$.*

**Proof.** Use the method of continuity, as above. To prove that $J$ is closed, simply modify $F(p)$ on $\{p : |p| \geq K+1\}$ to obtain $\widetilde{F}(p)$, satisfying (12.3) and (12.7). The solution $u_\tau$ to (12.1) for $\tau \in J$ also solves the modified equation, for which (12.36) works, so as above we have strong norm bounds on $u_\tau$ as $\tau$ approaches an endpoint of $J$.

Recall that, for scalar equations, (12.43) follows from a boundary gradient estimate, via the maximum principle. The maximum principle is not available for general elliptic $N \times N$ systems, even under the very strong ellipticity hypothesis, so (12.43) is then a more severe hypothesis.

Moving beyond the case $n = 2$, we need to confront the fact that solutions to elliptic PDE of the form (12.1) need not be smooth everywhere. A number of examples have been found; we give one of J. Necas [Nec], where $A^{jk}_{\alpha\beta}(p)$ in (12.1) has the form (12.5), satisfying (12.3), such that $F(p)$ satisfies $|D^\alpha F(p)| \leq C_\alpha \langle p\rangle^{-|\alpha|}|p|^2,\ \forall\, \alpha \geq 0$. Namely, take

$$\begin{aligned} F(\nabla u) = {} & \frac{1}{2}\frac{\partial u^{ij}}{\partial x_k}\frac{\partial u^{ij}}{\partial x_k} + \frac{\mu}{2}\frac{\partial u^{ij}}{\partial x_i}\frac{\partial u^{kk}}{\partial x_j} \\ & + \lambda \frac{\partial u^{ij}}{\partial x_i}\frac{\partial u^{ak}}{\partial x_a}\frac{\partial u^{\ell b}}{\partial x_\ell}\frac{\partial u^{jk}}{\partial x_b}\langle \nabla u\rangle^{-2}, \end{aligned} \tag{12.44}$$

where $u$ takes values in $M_{n\times n} \approx \mathbb{R}^{n^2}$, and we set

$$\lambda = 2\frac{n^3-1}{n(n-1)(n^3-n+1)}, \qquad \mu = \frac{4+n\lambda}{n^2-n+1}. \tag{12.45}$$

Since $\lambda, \mu \to 0$ as $n \to \infty$, we have ellipticity for sufficiently large $n$. But for any $n$,

$$u^{ij}(x) = \frac{x_i x_j}{|x|} \tag{12.46}$$

is a solution to (12.1). Thus $u$ is Lipschitz but not $C^1$ on every neighborhood of $0 \in \mathbb{R}^n$. See [Gia] for other examples. Also, when one looks at more general classes of nonlinear elliptic systems, there are examples of singular solutions even in the case $n = 2$; this is discussed further in §12B.

We now discuss some results known as *partial regularity*, to the effect that solutions $u \in H^1(\Omega, \mathbb{R}^N)$ to (12.1) can be singular only on relatively *small* subsets of $\Omega$.

We will measure how small the singular set is via the Hausdorff $s$-dimensional measure $\mathcal{H}^s$, which is defined for $s \in [0,\infty)$ as follows. First, given $\rho > 0,\ S \subset \mathbb{R}^n$, set

$$h^*_{s,\rho}(S) = \inf\Big\{\sum_{j\geq 1}\big(\operatorname{diam} Y_j\big)^s : S \subset \bigcup_{j\geq 1} Y_j,\ \operatorname{diam} Y_j \leq \rho\Big\}. \tag{12.47}$$

Here diam $Y_j = \sup\{|x-y| : x,y \in Y_j\}$. Each set function $h^*_{s,\rho}$ is an outer measure on $\mathbb{R}^n$. As $\rho$ decreases, $h^*_{s,\rho}(S)$ increases. Set

$$h^*_s(S) = \lim_{\rho\to 0} h^*_{s,\rho}(S). \tag{12.48}$$

Then $h^*_s(S)$ is an outer measure. It is seen to be a *metric* outer measure, that is, if $A, B \subset \mathbb{R}^n$ and $\inf\{|x-y| : x \in A, y \in B\} > 0$, then $h^*_s(A\cup B) = h^*_s(A) + h^*_s(B)$. It follows by a fundamental theorem of Caratheodory that every Borel set in $\mathbb{R}^n$ is $h^*_s$-measurable. For any $h^*_s$-measurable set $A$, we set

$$\mathcal{H}^s(A) = \gamma_s h^*_s(A), \quad \gamma_s = \frac{\pi^{s/2}2^{-s}}{\Gamma(\frac{s}{2}+1)}, \tag{12.49}$$

the factor $\gamma_s$ being picked so that if $k \le n$ is an integer and $S \subset \mathbb{R}^n$ is a smooth, $k$-dimensional surface, then $\mathcal{H}^k(S)$ is exactly the $k$-dimensional surface area of $S$. Treatments of Hausdorff measure can be found in [EG], [Fed], and [Fol].

Our next goal will be to establish the following result. Assume $n \ge 3$.

**Theorem 12.7.** *If $\Omega \subset \mathbb{R}^n$ is a smoothly bounded domain and $u \in H^1(\Omega, \mathbb{R}^N)$ solves (12.1), then there exists an open $\Omega_0 \subset \Omega$ such that $u \in C^\infty(\Omega_0)$ and*

$$\mathcal{H}^r(\Omega \setminus \Omega_0) = 0, \quad \text{for some } r < n-2. \tag{12.50}$$

We know from Theorem 12.4 that $u \in H^{2,p}(\Omega, \mathbb{R}^N)$, for some $p > 2$. Hence (12.10) holds for derivatives of $u$; in particular,

$$u_\ell = \partial_\ell u \Longrightarrow u_\ell \in H^{1,p}(\Omega, \mathbb{R}^N) \tag{12.51}$$

and

$$\partial_j A^{jk}(\nabla u)\partial_k u_\ell = 0, \quad 1 \le \ell \le n. \tag{12.52}$$

Regarding this as an elliptic system for $v = (\partial_1 u, \dots, \partial_n u)$, we see that to establish Theorem 12.7, it suffices to prove the following:

**Proposition 12.8.** *Assume that $v \in H^{1,p}(\Omega, \mathbb{R}^M)$, for some $p > 2$, and that $v$ solves the system*

$$\partial_j A^{jk}(x,v)\, \partial_k v = 0, \tag{12.53}$$

*where $A^{jk}_{\alpha\beta}(x,v)$ is uniformly continuous in $(x,v)$ and satisfies*

$$\lambda_1|\zeta|^2 \ge A^{jk}_{\alpha\beta}(x,v)\zeta_{j\alpha}\zeta_{k\beta} \ge \lambda_0|\zeta|^2, \quad \lambda_0 > 0. \tag{12.54}$$

*Then there is an open $\Omega_0 \subset \Omega$ such that $v$ is Hölder continuous on $\Omega_0$, and (12.50) holds.*

In turn, we will derive Proposition 12.8 from the following more precise result:

**Proposition 12.9.** *Under the hypotheses of Proposition 12.8, consider the subset* $\Sigma \subset \Omega$ *defined by*

$$x \in \Sigma \Longleftrightarrow \liminf_{R\to 0} R^{-n} \int\limits_{B_R(x)} |v(y) - v_{x,R}|^2 \, dy > 0, \tag{12.55}$$

*where*

$$v_{x,R} = Avg_{B_R(x)} \, v = \frac{1}{Vol \, B_R(x)} \int\limits_{B_R(x)} v(y) \, dy. \tag{12.56}$$

*Then*

$$\mathcal{H}^r(\Sigma) = 0, \quad \text{for some } \; r < n-2, \tag{12.57}$$

*and* $\Sigma$ *contains a closed subset* $\widetilde{\Sigma}$ *of* $\Omega$ *such that* $v$ *is Hölder continuous on* $\Omega_0 = \Omega \setminus \widetilde{\Sigma}$.

Note that every point of continuity of $v$ belongs to $\Omega \setminus \Sigma$; it follows from Proposition 12.9 that $v$ is Hölder continuous on a neighborhood of every point of continuity, under the hypotheses of Proposition 12.8. As Lemma 12.11 will show, for this fact we need assume only that $u \in H^{1,2}$, instead of $u \in H^{1,p}$ for some $p > 2$.

Let us first prove that $\Sigma$, defined by (12.55), has the property (12.57). First, by Poincaré's inequality,

$$\Sigma \subset \Big\{ x \in \Omega : \liminf_{R\to 0} R^{2-n} \int\limits_{B_R(x)} |\nabla v(y)|^2 \, dy > 0 \Big\}. \tag{12.58}$$

Since $\nabla v \in L^p(\Omega)$ for some $p > 2$, Hölder's inequality implies

$$\Sigma \subset \Big\{ x \in \Omega : \liminf_{R\to 0} R^{p-n} \int\limits_{B_R(x)} |\nabla v(y)|^p \, dy > 0 \Big\}. \tag{12.59}$$

Therefore, (12.57) is a consequence of the following.

**Lemma 12.10.** *Given* $w \in L^1(\Omega)$, $0 \le s < n$, *let*

$$E_s = \Big\{ x \in \Omega : \limsup_{r\to 0} r^{-s} \int\limits_{B_r(x)} |w(y)| \, dy > 0 \Big\}. \tag{12.60}$$

*Then*

$$\mathcal{H}^{s+\varepsilon}(E_s) = 0, \quad \forall \, \varepsilon > 0. \tag{12.61}$$

It is actually true that $\mathcal{H}^s(E_s) = 0$ (see [EG] and [Gia]), but to shorten the argument we will merely prove the weaker result (12.61), which will suffice for our purposes. In fact, we will show that

$$\mathcal{H}^s(E_{s\delta}) < \infty, \quad \forall\, \delta > 0, \tag{12.62}$$

where

$$E_{s\delta} = \Big\{ x \in \Omega : \limsup_{r\to 0}\ r^{-s} \int\limits_{B_r(x)} |w(y)|\ dy \geq \delta \Big\}.$$

This implies that $\mathcal{H}^{s+\varepsilon}(E_{s\delta}) = 0$, $\forall\, \varepsilon > 0$, and since $E_s = \bigcup_n E_{s,1/n}$, this yields (12.61).

As a tool in the argument, we use the following:

**Vitali covering lemma.** *Let $\mathcal{C}$ be a collection of closed balls in $\mathbb{R}^n$ (with positive radius) such that diam $B < C_0 < \infty$, for all $B \in \mathcal{C}$. Then there exists a countable family $\mathcal{F}$ of disjoint balls in $\mathcal{C}$ such that*

$$\bigcup_{B\in\mathcal{F}} \widehat{B} \supset \bigcup_{B\in\mathcal{C}} B, \tag{12.63}$$

*where $\widehat{B}$ is a ball concentric with $B$, with 5 times its radius.*

**Sketch of proof.** Take $\mathcal{C}_j = \{B \in \mathcal{C} : 2^{-j}C_0 \leq \text{ diam } B < 2^{1-j}C_0\}$. Let $\mathcal{F}_1$ be a maximal disjoint collection of balls in $\mathcal{C}_1$. Inductively, let $\mathcal{F}_k$ be a maximal disjoint set of balls in

$$\{B \in \mathcal{C}_k : B \text{ disjoint from all balls in } \mathcal{F}_1, \dots, \mathcal{F}_{k-1}\}.$$

Then set $\mathcal{F} = \bigcup \mathcal{F}_k$. One can then verify (12.63).

To begin the proof of (12.62), note that, for each $\rho > 0$, $E_{s\delta}$ is covered by a collection $\mathcal{C}$ of balls $B_x$ of radius $r_x < \rho$, such that

$$\int\limits_{B_x} |w(y)|\ dy \geq \delta r_x^s. \tag{12.64}$$

Thus there is a collection $\mathcal{F}$ of disjoint balls $B_\nu$ in $\mathcal{C}$ (of radius $r_\nu$) such that (12.63) holds. In particular, $\{\widehat{B}_\nu\}$ covers $E_{s\delta}$, so

$$h^*_{s,5\rho}(E_{s\delta}) \leq C_n \sum_\nu r_\nu^s \leq \frac{C_n}{\delta} \int\limits_{\bigcup B_\nu} |w(y)|\ dy \leq \frac{C_n}{\delta} \|w\|_{L^1(\Omega)}, \tag{12.65}$$

where $C_n$ is independent of $\rho$. This proves (12.62) and hence Lemma 12.10.

Thus we have (12.57) in Proposition 12.9. To prove the other results stated in that proposition, we will establish the following:

**Lemma 12.11.** *Given $\tau \in (0,1)$, there exist constants*

$$\varepsilon_0 = \varepsilon_0(\tau, n, M, \lambda_0^{-1}\lambda_1), \quad R_0 = R_0(\tau, n, M, \lambda_0^{-1}\lambda_1),$$

*and furthermore there exists a constant*

$$A_0 = A_0(n, M, \lambda_0^{-1}\lambda_1),$$

*independent of* $\tau$*, such that the following holds. If* $u \in H^1(\Omega, \mathbb{R}^M)$ *solves (12.53) and if, for some* $x_0 \in \Omega$ *and some*

$$R < R_0(x_0) = \min(R_0, dist(x_0, \partial\Omega)),$$

*we have*

$$U(x_0, R) < \varepsilon_0^2, \tag{12.66}$$

*where*

$$U(x_0, R) = R^{-n} \int_{B_R(x_0)} |u(y) - u_{x_0,R}|^2 \, dy, \tag{12.67}$$

*then*

$$U(x_0, \tau R) \leq 2A_0\tau^2 U(x_0, R). \tag{12.68}$$

Let us show how this result yields Proposition 12.9. Pick $\alpha \in (0, 1)$, and choose $\tau \in (0, 1)$ such that $2A_0\tau^{2-2\alpha} = 1$. Suppose $x_0 \in \Omega$ and $R < \min(R_0, \operatorname{dist}(x_0, \partial\Omega))$, and suppose (12.66) holds. Then (12.68) implies

$$U(x_0, \tau R) \leq \tau^{2\alpha} U(x_0, R).$$

In particular, $U(x_0, \tau R) < U(x_0, R) < \varepsilon_0^2$, so inductively the implication (12.66) $\Rightarrow$ (12.68) yields

$$U(x_0, \tau^k R) \leq \tau^{2\alpha k} U(x_0, R).$$

Hence, for $\rho < R$,

$$U(x_0, \rho) \leq C\Big(\frac{\rho}{R}\Big)^{2\alpha} U(x_0, R). \tag{12.69}$$

Note that, for fixed $R > 0$, $U(x_0, R)$ is continuous in $x_0$, so if (12.66) holds at $x_0$, then we have $U(x, R) < \varepsilon_0^2$ for every $x$ in some neighborhood $B_r(x_0)$ of $x_0$, and hence

$$U(x, \rho) \leq C\Big(\frac{\rho}{R}\Big)^{2\alpha} U(x, R), \quad x \in B_r(x_0);$$

that is, we have

$$\int_{B_\rho(x)} |u(y) - u_{x,\rho}|^2 \, dy \leq C\rho^{n+2\alpha} \tag{12.70}$$

uniformly for $x \in B_r(x_0)$. This implies, by Proposition A.2,

$$u \in C^\alpha\big(B_r(x_0)\big). \tag{12.71}$$

In fact, we can say more. Extending some of the preliminary results of §9, we have, for a solution $u \in H^1(\Omega)$ of (12.53), estimates of the form

$$\|\nabla u\|^2_{L^2(B_{\rho/2}(x))} \le C\rho^{-2} \int\limits_{B_\rho(x)} |u(y) - u_{x,\rho}|^2\, dy; \tag{12.72}$$

see Exercise 2 below. Consequently, (12.70) implies

$$\nabla u\big|_{B_r(x_0)} \in M_2^q\big(B_r(x_0)\big), \quad q = \frac{n}{1-\alpha}. \tag{12.73}$$

which by Morrey's lemma implies (12.71). Thus, granted Lemma 12.11, Proposition 12.9 is proved, with

$$\Omega_0 = \{x_0 \in \Omega : \inf_{R<R_0(x_0)} U(x_0,R) < \varepsilon_0^2\}, \tag{12.74}$$

since clearly $\Sigma \supset \Omega \setminus \Omega_0 = \widetilde{\Sigma}$.

The proof of Lemma 12.11 (following the exposition in [Gia]) evolved from work of E. DeGiorgi [DeG2] and F. Almgren [Alm2] on regularity for minimal surfaces. It consists of blowing up small neighborhoods of $x_0$ and obtaining a limiting PDE for a limit of the resulting dilations of $u$. As a preliminary to the proof of Lemma 12.11, we first identify the constant $A_0$.

**Lemma 12.12.** *There is a constant $A_0 = A_0(n, M, \lambda_1/\lambda_0)$ such that whenever $b^{jk}_{\alpha\beta}$ are constants satisfying*

$$\lambda_1|\zeta|^2 \ge \sum b^{jk}_{\alpha\beta}\zeta_{j\alpha}\zeta_{k\beta} \ge \lambda_0|\zeta|^2, \quad \lambda_0 > 0, \tag{12.75}$$

*the following holds. If $u \in H^1\big(B_1(0), \mathbb{R}^M\big)$ solves*

$$\partial_j b^{jk}_{\alpha\beta}\partial_k u^\beta = 0 \quad \text{on} \ \ B_1(0), \tag{12.76}$$

*then, for all $\rho \in (0,1)$,*

$$U(0,\rho) \le A_0\rho^2 U(0,1). \tag{12.77}$$

**Proof.** For $\rho \in (0, 1/2]$, we have

$$U(0,\rho) \le \rho^{2-n} \int\limits_{B_\rho(0)} |\nabla u(y)|^2\, dy \le C_n\rho^2\|\nabla u\|^2_{L^\infty(B_{1/2}(0))}. \tag{12.78}$$

On the other hand, regularity for the constant-coefficient, elliptic PDE (12.76) readily yields an estimate

$$\|\nabla u\|^2_{L^\infty(B_{1/2}(0))} \le B_0\|\nabla u\|^2_{L^2(B_{3/4}(0))} \le B_1\|u - u_{0,1}\|^2_{L^2(B_1(0))}, \tag{12.79}$$

with $B_j = B_j(n, M, \lambda_1/\lambda_0)$, from which (12.77) easily follows.

We now tackle the proof of Lemma 12.11. If the conclusion (12.68) is false, then there exist $\tau \in (0,1)$ and $x_\nu \in \Omega$, $\varepsilon_\nu \to 0$, $R_\nu \to 0$, and

$u_\nu \in H^1(\Omega, \mathbb{R}^M)$, solving (12.53), such that

$$U_\nu(x_\nu, R_\nu) = \varepsilon_\nu^2, \quad U_\nu(x_\nu, \tau R_\nu) > 2A_0\tau^2\varepsilon_\nu^2. \tag{12.80}$$

To implement the dilation argument mentioned above, we set

$$v_\nu(x) = \varepsilon_\nu^{-1}\big[u_\nu(x_\nu + R_\nu x) - u_{\nu x_\nu, R_\nu}\big]. \tag{12.81}$$

Then $v_\nu$ solves

$$\partial_j A^{jk}_{\alpha\beta}\big(x_\nu + R_\nu x, \varepsilon_\nu v_\nu(x) + u_{\nu x_\nu, R_\nu}\big)\, \partial_k v_\nu^\beta = 0 \quad \text{on} \quad B_1(0). \tag{12.82}$$

If we set

$$\begin{aligned} V_\nu(0,\rho) &= \rho^{-n} \int\limits_{B_\rho(0)} |v_\nu(y) - v_{\nu 0,\rho}|^2 \, dy \\ &= \varepsilon_\nu^{-2}\rho^{-n}R_\nu^{-n} \int\limits_{B_{\rho R_\nu}(x_\nu)} |u_\nu(y) - u_{\nu x_\nu, R_\nu}|^2 \, dy, \end{aligned} \tag{12.83}$$

we have (since $v_{\nu 0,1} = 0$)

$$V_\nu(0,1) = \|v_\nu\|^2_{L^2(B_1(0))} = 1, \quad V_\nu(0,\tau) > 2A_0\tau^2. \tag{12.84}$$

Passing to a subsequence, we can assume that

$$v_\nu \to v \text{ weakly in } L^2\big(B_1(0), \mathbb{R}^M\big), \quad \varepsilon_\nu v_\nu \to 0 \text{ a.e. in } B_1(0). \tag{12.85}$$

Also

$$A^{jk}_{\alpha\beta}(x_\nu, u_{\nu x_\nu, R_\nu}) \longrightarrow b^{jk}_{\alpha\beta}, \tag{12.86}$$

an array of constants satisfying (12.75). The uniform continuity of $A^{jk}_{\alpha\beta}$ then implies

$$A^{jk}_{\alpha\beta}\big(x_\nu + R_\nu x, \varepsilon_\nu v_\nu(x) + u_{\nu x_\nu, R_\nu}\big) \longrightarrow b^{jk}_{\alpha\beta} \ \text{ a.e. in } \ B_1(0). \tag{12.87}$$

Now, as in (12.72), the fact that $v_\nu$ solves (12.82) implies

$$\|v_\nu\|_{H^1(B_\rho(0))} \le C_\rho, \quad \forall\, \rho < 1. \tag{12.88}$$

Hence, passing to a further subsequence if necessary, we have

$$\begin{aligned} v_\nu &\longrightarrow v \ \text{ strongly in } \ L^2_{\text{loc}}\big(B_1(0)\big), \\ \nabla v_\nu &\longrightarrow \nabla v \ \text{ weakly in } \ L^2_{\text{loc}}\big(B_1(0)\big). \end{aligned} \tag{12.89}$$

Since the functions in (12.87) are uniformly bounded on $B_1(0)$, these results imply that we can pass to the limit in (12.82), to conclude that

$$\partial_j b^{jk}_{\alpha\beta}\, \partial_k v^\beta = 0 \quad \text{on} \quad B_1(0). \tag{12.90}$$

Then Lemma 12.12 implies

$$V(0,\tau) \le A_0\tau^2 V(0,1), \tag{12.91}$$

which is $\leq A_0\tau^2$ by (12.85). On the other hand, (12.89) implies

$$V(0,\tau) \geq 2A_0\tau^2 \tag{12.92}$$

if (12.80) holds. This contradiction proves Lemma 12.11.

Hence the proof of Proposition 12.9 is complete, so we have Theorem 12.7.

Theorem 12.7 can be extended to a result on partial regularity up to the boundary (see [Gia]).

There is a condition more general than strong convexity on the integrand in (12.4), known as "quasi-convexity," under which extrema for (12.4) have been shown to possess partial regularity of the sort established in Theorem 12.7 (see [Ev3]).

There are also some results on regularity everywhere for stationary points of (12.4) when $\Omega$ has dimension $\geq 3$. A notable result of [U] is that such solutions are smooth on $\Omega$ provided $F(\nabla u)$ in (12.4), in addition to being strongly convex in $\nabla u$ and satisfying the controllable growth conditions, depends only on $|\nabla u|^2$. A proof can also be found in [Gia].

## Exercises

In Exercises 1–3, we consider an $N \times N$ system

$$\sum \partial_j A^{jk}_{\alpha\beta}(x)\partial_k u^\beta = \sum \partial_j f^\alpha_j \quad \text{on } B_1 = \{x \in \mathbb{R}^n : |x| < 1\}, \tag{12.93}$$

under the very strong ellipticity hypothesis (12.20). Assume $f_j \in L^2(B_1)$.

1. Show that, with $C = C(\lambda_0, \lambda_1)$,

$$\|\nabla u\|_{L^2(B_{1/2})} \leq C\|u\|_{L^2(B_1)} + C\sum \|f_j\|_{L^2(B_1)}. \tag{12.94}$$

   (*Hint*: Extend (9.6).)

2. Let $\delta_r v(x) = v(rx)$. Show that, for $r \in (0,1]$,

$$\|\delta_r(\nabla u)\|_{L^2(B_{1/2})} \leq Cr^{-1}\|\delta_r(u-\overline{u})\|_{L^2(B_1)} + C\sum \|\delta_r f_j\|_{L^2(B_1)}, \tag{12.95}$$

   where $\overline{u} = \operatorname{Avg}_{B_1} u$. (*Hint*: First apply a dilation argument to (12.94). Then apply the result to $u - \overline{u}$.) This sort of estimate is called a "Caccioppoli inequality."

3. Deduce from Exercise 2 that if $u \in H^1(\Omega)$ solves (12.93), then

(12.96)
$$\|\delta_r(\nabla u)\|_{L^2(B_{1/2})} \leq C\|\delta_r(\nabla u)\|_{L^q(B_1)} + C\sum \|\delta_r f_j\|_{L^2(B_1)}, \quad q = \frac{2n}{n+2} < 2.$$

   This sort of estimate is sometimes called a "reverse Hölder inequality."

4. Deduce from (12.95) that if $u \in H^1(\Omega)$ solves (12.93), then, for $0 < r < 1$,

$$u \in C^r(B_1),\ f_j \in M^p_2(B_1),\ p = \frac{n}{1-r} \Longrightarrow \nabla u \in M^p_2(B_{1/2}). \tag{12.97}$$

   Compare (9.41)–(9.42).

5. Let $C(p)$ be the constant in (12.27), in case $\Omega = B_1$. Show that if $C(n)\Big(1 - \lambda_0/\lambda_1\Big) < 1$, then a solution $u \in H^1_0(\Omega)$ to (12.93) is Hölder continuous on

$\overline{B}_1$, provided $f_j \in L^q(B_1)$ for some $q > n$. Consider the problem of obtaining precise estimates on $C(p)$ in this case.

## 12B. Further results on quasi-linear systems

Regularity questions can become more complex when lower-order terms are added to systems of the form (12.1). In fact, there are extra complications even for solutions to a semilinear system of the form

$$Lu + B(x, u, \nabla u) = f, \tag{12b.1}$$

where $L$ is a second-order, linear elliptic differential operator and $B(x, u, p)$ is smooth in its arguments. One limitation on what one could possibly prove is given by the following example of J. Frehse [Freh], namely that

$$u_1(x) = \sin \log \log |x|^{-1}, \quad u_2(x) = \cos \log \log |x|^{-1} \tag{12b.2}$$

provides a bounded, weak solution to the $2 \times 2$ system

$$\begin{aligned} \Delta u_1 + \frac{2(u_1 + u_2)}{1 + |u|^2} |\nabla u|^2 = 0, \\ \Delta u_2 + \frac{2(u_2 - u_1)}{1 + |u|^2} |\nabla u|^2 = 0, \end{aligned} \tag{12b.3}$$

belonging to $H^1(B)$, for any ball $B \subset \mathbb{R}^2$, centered at the origin, of radius $r < 1$. Evidently, $u$ is not continuous at the origin; one can also see that $\nabla u$ does not belong to $L^p(B)$ for any $p > 2$. (After all, that would force $u$ to be Hölder continuous.) Thus Theorem 12.4 and Proposition 12.5 do not extend to this case.

The following result shows that if a weak solution to such a semilinear system as (12b.1) has any Hölder continuity, then higher-order regularity results hold.

**Proposition 12B.1.** *Assume $u \in H^1$ solves (12b.1) and $B(x, u, p)$ is a smooth function of its arguments, satisfying*

$$|B(x, u, p)| \le C \langle p \rangle^2. \tag{12b.4}$$

*Then, given $r > 0, \ s > -1$,*

$$u \in C^r, \ f \in C^s_* \Longrightarrow u \in C^{s+2}_*. \tag{12b.5}$$

**Proof.** Write

$$u = Ef - EB(x, u, \nabla u), \quad \text{mod } C^\infty, \tag{12b.6}$$

where $E \in OPS^{-2}_{1,0}$ is a parametrix for the elliptic operator $L$. We have $Ef \in C^{s+2}_*$, and, since $u \in H^1 \Rightarrow B(x, u, \nabla u) \in L^1$, we have

$$EB(x, u, \nabla u) \in H^{2-\sigma, 1+\varepsilon}, \quad \forall\, \varepsilon > 0, \ \ \sigma > \frac{n\varepsilon}{1+\varepsilon}.$$

If $s \geq 0$, this implies

$$u \in H^{2-\sigma,1+\varepsilon} \cap H^{r-\sigma,p}, \tag{12b.7}$$

for all $p < \infty$, hence

$$u \in [H^{2-\sigma,1+\varepsilon}, H^{r-\sigma,p}]_\theta, \quad \forall\, \theta \in (0,1). \tag{12b.8}$$

Results on such interpolation spaces follow from (6.30) of Chapter 13. If we set $\theta = 1/2$ and take $p$ large enough, we have

$$u \in H^{1+r/2-\sigma,2+2\varepsilon}, \quad \forall\, \varepsilon \in (0,1), \quad \sigma > \frac{n\varepsilon}{1+\varepsilon}. \tag{12b.9}$$

On the other hand, if we set $\theta = (1-\sigma)/(2-r)$, (assuming $r < 1$), we have

$$u \in H^{1,2q}, \quad \forall\, q < \frac{1-\frac{1}{2}r}{1-r}, \tag{12b.10}$$

hence

$$B(x,u,\nabla u) \in L^q, \quad \forall\, q < \frac{1-\frac{1}{2}r}{1-r}, \quad \text{e.g., } q = 1 + \frac{r}{2}. \tag{12b.11}$$

Another look at (12b.6) now yields

$$u \in H^{2,1+r/2} \cap H^{r-\sigma,p}, \quad \forall p < \infty, \tag{12b.12}$$

provided $s \geq 0$, which is an improvement of (12b.7). We can iterate this argument until we get (12b.5), provided $s \geq 0$.

If instead we merely assume $s > -1$, then, instead of (12b.7), we deduce from (12b.6) and $EB(x,u,\nabla u) \in H^{2-\sigma,1+\varepsilon}$ that

$$EB(x,u,\nabla u) \in H^{2-\sigma,1+\varepsilon} \cap H^{r-\sigma,p} \tag{12b.13}$$

and hence (parallel to (12b.8)–(12b.11)) that

$$\begin{aligned} EB(x,u,\nabla u) &\in \bigcap_{\theta\in(0,1)} [H^{2-\sigma,1+\varepsilon}, H^{r-\sigma,p}]_\theta \\ &\subset H^{1+r/2-\sigma,2} \cap H^{1,2+r}, \end{aligned} \tag{12b.14}$$

so another look at (12b.6) gives

$$u \in H^{1,2+r},$$

hence

$$B(x,u,\nabla u) \in L^{1+r/2}, \tag{12b.15}$$

so

$$EB(x,u,\nabla u) \in H^{2,1+r/2} \cap H^{r-\sigma,p}, \tag{12b.16}$$

and we can iterate this argument until (12b.5) is proved.

Note that Proposition 12B.1 applies to the semilinear system (11.25) for a harmonic map $u : \Omega \to X$, where $X$ is a submanifold of $\mathbb{R}^N$:

$$\Delta u - \Gamma(u)(\nabla u, \nabla u) = 0. \tag{12b.17}$$

On the other hand, there are quasi-linear equations with a somewhat similar structure that also arise naturally in geometry, such as the system (4.94) satisfied by the metric tensor, in harmonic coordinates, when the Ricci tensor is given. This system has the following form, more general than (12b.1):

$$\sum \partial_j a^{jk}(x, u) \partial_k u + B(x, u, \nabla u) = f. \tag{12b.18}$$

We assume that $a^{jk}(x, u)$ and $B(x, u, p)$ are smooth in their arguments and that (12b.4) holds. Recall that we have established one regularity result for such a system in §4, namely, if $n = \dim \Omega$ and $n < q < p < \infty$, then

$$u \in H^{1,q}, \ f \in H^{s,p} \Longrightarrow u \in H^{s+2,p} \tag{12b.19}$$

if $s \geq -1$. Here, we want to weaken the hypothesis that $u \in H^{1,q}$ for some $q > n$, which of course implies $u \in C^r, \ r = 1 - n/q$. We will establish the following:

**Proposition 12B.2.** *Assume that $u \in H^1$ solves (12b.18) and that $B(x, u, p)$ satisfies (12b.4). Also assume $u \in C^r$ for some $r > 0$. Then*

$$f \in L^1 \Longrightarrow u \in H^{2-\sigma, 1+\varepsilon}, \quad \forall\, \varepsilon \in (0, 1), \quad \sigma > \frac{n\varepsilon}{1+\varepsilon}, \tag{12b.20}$$

*and, if $1 < p < \infty$,*

$$f \in L^p \Longrightarrow u \in H^{2,p}. \tag{12b.21}$$

*More generally, for $s \geq 0$,*

$$f \in H^{s,p} \Longrightarrow u \in H^{s+2,p}. \tag{12b.22}$$

To begin the proof, as in the demonstration of Proposition 4.9, we write

$$\sum a^{jk}(x, u)\, \partial_k u = A_j(u; x, D) u, \tag{12b.23}$$

mod $C^\infty$, with

$$u \in C^r \Longrightarrow A_j(u; x, \xi) \in C^r S^1_{1,0} \cap S^1_{1,1} + S^{1-r}_{1,1}. \tag{12b.24}$$

Hence, given $\delta \in (0, 1)$,

$$\begin{gathered} A_j(u; x, \xi) = A^\#_j(x, \xi) + A^b_j(x, \xi), \\ A^\#_j(x, \xi) \in S^1_{1,\delta}, \qquad A^b_j(x, \xi) \in S^{1-r\delta}_{1,1}. \end{gathered} \tag{12b.25}$$

Thus we can write

$$\sum \partial_j a^{jk}(x, \xi)\, \partial_k u = P^\# u + P^b u, \tag{12b.26}$$

with

$$P^{\#} = \sum \partial_j A_j^{\#}(x,D) \in OPS^2_{1,\delta}, \quad \text{elliptic} \tag{12b.27}$$

and

$$P^b = \sum \partial_j A_j^b(x,D). \tag{12b.28}$$

Then we let

$$E^{\#} \in OPS^{-2}_{1,\delta} \tag{12b.29}$$

be a parametrix for $P^{\#}$, and we have

$$u = -E^{\#}P^b u + E^{\#}B(x,u,\nabla u) + E^{\#}f, \tag{12b.30}$$

mod $C^\infty$, and if $u \in C^r$,

$$P^b : H^{\sigma,p} \longrightarrow H^{\sigma-2+r\delta,p}, \quad P^b : C^\sigma_* \longrightarrow C^{\sigma-2+r\delta}_*, \tag{12b.31}$$

provided $1 < p < \infty$ and $\sigma - 2 + r\delta > -1$, so

$$\sigma > 1 - r\delta. \tag{12b.32}$$

Therefore, our hypotheses on $u$ imply

$$E^{\#}P^b u \in H^{1+r\delta,2}. \tag{12b.33}$$

Now, if $u \in H^1(\Omega)$, then (12b.4) implies

$$B(x,u,\nabla u) \in L^1, \tag{12b.34}$$

so, for small $\varepsilon > 0,\ \sigma > n\varepsilon/(1+\varepsilon)$,

$$E^{\#}B(x,u,\nabla u) \in H^{2-\sigma,1+\varepsilon}. \tag{12b.35}$$

Hence we have (12b.30), mod $C^\infty$, with

$$\begin{gathered} E^{\#}P^b u \in H^{1+r\delta,2}, \qquad E^{\#}B(x,u,\nabla u) \in H^{2-\sigma,1+\varepsilon}, \\ E^{\#}f \in H^{2-\sigma,1+\varepsilon}. \end{gathered} \tag{12b.36}$$

This implies

$$u \in H^{1+r\delta,1+\varepsilon},$$

hence, by (12b.31),

$$E^{\#}P^b u \in H^{1+2r\delta,1+\varepsilon}. \tag{12b.37}$$

Another look at (12b.30) gives

$$\begin{aligned} u \in H^{1+2r\delta,1+\varepsilon} \quad & \text{if } \ 1+2r\delta \le 2-\sigma, \\ H^{2-\sigma,1+\varepsilon} \quad & \text{if } \ 1+2r\delta \ge 2-\sigma. \end{aligned} \tag{12b.38}$$

If the first of these alternatives holds, then

$$E^{\#}P^b u \in H^{1+3r\delta,1+\varepsilon}.$$

We continue until the conclusion of (12b.20) is achieved.

Given that $u \in C^r$ and that (12b.20) holds, by interpolation we have

$$u \in \left[H^{2-\sigma,1+\varepsilon}, H^{r-\sigma,p}\right]_\theta, \quad \forall\, \theta \in (0,1), \tag{12b.39}$$

using $C^r_* \subset H^{r-\sigma,p}$, $\forall\, \sigma > 0, p < \infty$. If we take $\theta = 1/2$ we get

$$u \in H^{1+r/2-\sigma,q}, \quad \frac{1}{q} = \frac{1}{2+2\varepsilon} + \frac{1}{2p},$$

hence, taking $p$ arbitrarily large, we have

$$u \in H^{1+r/2-\sigma,2+2\varepsilon}, \quad \forall\, \varepsilon \in (0,1), \quad \sigma > \frac{n\varepsilon}{1+\varepsilon}. \tag{12b.40}$$

Note that this is an improvement of the original hypothesis that $u \in H^{1,2}$. On the other hand, if we take $\theta = (1-\sigma)/(2-r)$, we get

$$u \in H^{1,2q}, \quad \forall\, q < \frac{1-\frac{1}{2}r}{1-r}, \tag{12b.41}$$

so

$$B(x,u,\nabla u) \in L^q, \quad \forall\, q < \frac{1-\frac{1}{2}r}{1-r}. \tag{12b.42}$$

Hence

$$E^\# B(x,u,\nabla u) \in H^{2,q}. \tag{12b.43}$$

Meanwhile, by (12b.40),

$$E^\# P^b u \in H^{1+r/2+r\delta-\sigma,2}. \tag{12b.44}$$

On the other hand, if we set

$$q = 1 + \frac{r}{2}, \tag{12b.45}$$

which satisfies the condition in (12b.41), we can take $\theta \approx r/(2+r)$ in (12b.39) and get

$$u \in H^{\mu,q}, \quad \forall\, \mu < \frac{4+r^2}{2+r}, \tag{12b.46}$$

hence

$$E^\# P^b u \in H^{\rho,q}, \quad \forall\, \rho < \frac{4+r^2}{2+r} + r\delta. \tag{12b.47}$$

Note that

$$\frac{4+r^2}{2+r} + r\delta = 2 - r + r\delta + r^2 - \frac{1}{4}r^3 + \cdots, \tag{12b.48}$$

which is $> 2$, for any given $r \in (0,1)$, if $\delta$ is taken close enough to 1. Now, another look at (12b.30) establishes the following special case of (12b.21):

$$1 < p \le 1 + \frac{r}{2}, \quad f \in L^p(\Omega) \Longrightarrow u \in H^{2,p}. \tag{12b.49}$$

Under the hypotheses that $u \in C^r$ and that (12b.49) holds, we have, parallel to (12b.39),

$$u \in \big[H^{2,p}, H^{r-\sigma,Q}\big]_\theta, \quad \forall\, \theta \in (0,1), \tag{12b.50}$$

for all $\sigma > 0,\ Q < \infty$. As before, we can take $\theta \approx 1/(2-r)$ and get

$$u \in H^{1,2q}, \quad \forall\, q < \frac{1-\frac{1}{2}r}{1-r}p. \tag{12b.51}$$

Hence, parallel to (12b.43), and as before using $1+r/2 < (1-r/2)/(1-r)$, we have

$$E^{\#}B(x,u,\nabla u) \in H^{2,(1+r/2)p}. \tag{12b.52}$$

Similarly, if we take $\theta \approx r/(2+r)$ in (12b.50), we get

$$u \in H^{\mu,(1+r/2)p}, \quad \forall\, \mu < \frac{4+r^2}{2+r}, \tag{12b.53}$$

and hence

$$E^{\#}P^b u \in H^{\rho,(1+r/2)p}, \quad \forall\, \rho < \frac{4+r^2}{2+r} + r\delta.$$

As before, given $r \in (0,1)$, we can choose $\delta$ close enough to 1 that $\rho > 2$. Another look at (12b.30) establishes that

$$1 < p \le \Big(1 + \frac{r}{2}\Big)^2, \quad f \in L^p(\Omega) \Longrightarrow u \in H^{2,p}. \tag{12b.54}$$

Now we can iterate this argument repeatedly, and since, for all $r > 0$, we have $(1+r/2)^k \to \infty$ as $k \to \infty$, we obtain (12b.21).

We next want to weaken the requirement of Hölder continuity on $u$.

**Proposition 12B.3.** *Let $u \in H^1(\Omega)$ solve (12b.18). Assume the very strong ellipticity condition*

$$a^{jk}_{\alpha\beta}(x,u)\zeta_{j\alpha}\zeta_{k\beta} \ge \lambda_0|\zeta|^2, \quad \lambda_0 > 0. \tag{12b.55}$$

*Also assume $B(x,u,\nabla u)$ is a quadratic form in $\nabla u$. Assume furthermore that $u$ is continuous on $\Omega$. Then, locally, if $p > n/2$,*

$$f \in M^p_2 \Longrightarrow \nabla u \in M^q_2, \quad \text{for some } q > n. \tag{12b.56}$$

*Hence $u \in C^r$, for some $r > 0$.*

To begin, given $x_0 \in \Omega$, shrink $\Omega$ down to a smaller neighborhood, on which

$$|u(x) - u_0| \le E, \tag{12b.57}$$

for some $u_0 \in \mathbb{R}^M$ (if (12b.18) is an $M \times M$ system). We will specify $E$ below. With the same notation as in (12.22), write

$$\big(\partial_j a^{jk}(x,u)\,\partial_k u, w\big)_{L^2} = -\int \langle \nabla u, \nabla w\rangle \, dx, \tag{12b.58}$$

so $a^{jk}_{\alpha\beta}(x,u)$ determines an inner product on $T^*_x \otimes \mathbb{R}^M$ for each $x \in \Omega$, in a fashion that depends on $u$, perhaps, but one has bounds on the set of inner products so arising. Now, if we let $\psi \in C_0^\infty(\Omega)$ and $w = \psi(x)^2(u - u_0)$, and take the inner product of (12b.18) with $w$, we have

$$\begin{aligned}\int \psi^2|\nabla u|^2\,dx + 2\int \psi(\nabla u)(\nabla\psi)(u-u_0)\,dx \\ -\int \psi^2(u-u_0)B(x,u,\nabla u)\,dx \\ = -\int \psi^2 f(u-u_0)\,dx.\end{aligned} \tag{12b.59}$$

Hence we obtain the inequality

$$\begin{aligned}&\int \psi^2\big[|\nabla u|^2 - |u-u_0|\cdot|B(x,u,\nabla u)| - \delta^2|\nabla u|^2\big]\,dx \\ &\le \frac{1}{\delta^2}\int |\nabla\psi|^2|u-u_0|^2\,dx + \int \psi^2|f|\cdot|u-u_0|\,dx,\end{aligned} \tag{12b.60}$$

for any $\delta \in (0,1)$. Now, for some $A < \infty$, we have

$$|B(x,u,\nabla u)| \le A|\nabla u|^2. \tag{12b.61}$$

Then we choose $E$ in (12b.57) so that

$$EA \le 1 - a < 1. \tag{12b.62}$$

Then take $\delta^2 = a/2$, and we have

$$\frac{a}{2}\int \psi^2|\nabla u|^2\,dx \le \frac{2}{a}\int |\nabla\psi|^2\cdot|u-u_0|^2\,dx + \int \psi^2|f|\cdot|u-u_0|\,dx. \tag{12b.63}$$

Now, given $x \in \Omega$, for $R <$ dist$(x, \partial\Omega)$, define $U(x,R)$ as in (12.67) by

$$U(x,R) = R^{-n}\int_{B_R(x)} |u(y) - u_{x,R}|^2\,dy, \tag{12b.64}$$

where, as before, $u_{x,R}$ is the mean value of $u\big|_{B_R(x)}$. The following result is analogous to Lemma 12.11. Let $A_0$ be the constant produced by Lemma 12.12, applied to the present case, and pick $\rho$ such that $A_0\rho^2 \le 1/2$.

**Lemma 12B.4.** *Let $\overline{\mathcal{O}} \subset\subset \Omega$. There exist $R_0 > 0$, $\vartheta < 1$, and $C_0 < \infty$ such that if $x \in \overline{\mathcal{O}}$ and $r \le R_0$, then either*

$$U(x,r) \le C_0 r^{2(2-n/p)}, \tag{12b.65}$$

*or*

$$U(x,\rho r) \le \vartheta U(x,r). \tag{12b.66}$$

**Proof.** If not, there exist $x_\nu \in \overline{\mathcal{O}}$, $R_\nu \to 0$, $\vartheta_\nu \to 1$, and $u_\nu \in H^1(\Omega, \mathbb{R}^M)$ solving (12b.18) such that

$$U_\nu(x_\nu, R_\nu) = \varepsilon_\nu^2 > C_0 R_\nu^{2(2-n/p)} \tag{12b.67}$$

and

$$U_\nu(x_\nu, \rho R_\nu) > \vartheta_\nu U_\nu(x_\nu, R_\nu). \tag{12b.68}$$

The hypothesis that $u$ is continuous implies $\varepsilon_\nu \to 0$. We want to obtain a contradiction.

As in (12.81), set

$$v_\nu(x) = \varepsilon_\nu^{-1}\big[u_\nu(x_\nu + R_\nu x) - u_{\nu x_\nu, R_\nu}\big]. \tag{12b.69}$$

Then $v_\nu$ solves

$$\begin{aligned} &\partial_j a^{jk}_{\alpha\beta}\big(x_\nu + R_\nu x, \varepsilon_\nu v_\nu(x) + u_{\nu x_\nu, R_\nu}\big)\, \partial_k v^\beta_\nu \\ &+ \varepsilon_\nu B\big(x_\nu + R_\nu x, \varepsilon_\nu v_\nu(x) + u_{\nu x_\nu, R_\nu}, \nabla v_\nu(x)\big) = \frac{R_\nu^2}{\varepsilon_\nu} f. \end{aligned} \tag{12b.70}$$

Note that, by the hypothesis (12b.67),

$$\frac{R_\nu^2}{\varepsilon_\nu} < \frac{1}{C_0} R_\nu^{n/p}. \tag{12b.71}$$

Now set

$$V_\nu(0, r) = r^{-n} \int\limits_{B_r(0)} |v_\nu(y) - v_{\nu 0, r}|^2 \, dy. \tag{12b.72}$$

Then, as in (12.84), we have

$$V_\nu(0,1) = \|v_\nu\|^2_{L^2(B_1(0))} = 1, \quad V_\nu(0,\rho) > \vartheta_\nu. \tag{12b.73}$$

Passing to a subsequence, we can assume that

$$v_\nu \to v \ \text{ weakly in } \ L^2\big(B_1(0), \mathbb{R}^M\big), \quad \varepsilon_\nu v_\nu \to 0 \ \text{ a.e. in } B_1(0). \tag{12b.74}$$

Also, as in (12.87), there is an array of constants $b^{jk}_{\alpha\beta}$ such that

$$a^{jk}_{\alpha\beta}\big(x_\nu + R_\nu x, \varepsilon_\nu v_\nu(x) + u_{\nu x_\nu, R_\nu}\big) \longrightarrow b^{jk}_{\alpha\beta} \quad \text{a.e. in } B_1(0), \tag{12b.75}$$

and this is bounded convergence.

We next need to estimate the $L^2$-norm of $\nabla v_\nu$, which will take just slightly more work than it did in (12.88).

Substituting $\varepsilon_\nu v_\nu\big((x-x_\nu)/R_\nu\big)+u_{\nu x_\nu,R_\nu}$ for $u_\nu(x)$ in (12b.63), and replacing $u_0$ by $u_{\nu x_\nu,R_\nu}$, we have

$$\begin{aligned}&\frac{a}{2}\int\psi^2\Big|\nabla v_\nu\Big(\frac{x-x_\nu}{R_\nu}\Big)\Big|^2\,dx\\ &\le\frac{2}{a}\int R_\nu^2|\nabla\psi|^2\,\Big|v_\nu\Big(\frac{x-x_\nu}{R_\nu}\Big)\Big|^2\,dx\\ &\quad+\frac{R_\nu^2}{\varepsilon_\nu}\int\psi^2|f|\cdot\Big|v_\nu\Big(\frac{x-x_\nu}{R_\nu}\Big)\Big|\,dx,\end{aligned}\tag{12b.76}$$

for $\psi\in C_0^\infty\big(B_{R_\nu}(x_\nu)\big)$. Actually, for this new value of $u_0$, the estimate (12b.57) might change to $|u(x)-u_0|\le 2E$, so at this point we strengthen the hypothesis (12b.62) to

$$2EA\le 1-a<1,\tag{12b.77}$$

in order to get (12b.76). Since $R_\nu^2/\varepsilon_\nu\le R_\nu^{n/p}/C_0$, we have, for $\Psi(x)=\psi(x_\nu+R_\nu x)\in C_0^\infty\big(B_1(0)\big)$,

$$\frac{a}{2}\int\Psi^2|\nabla v_\nu|^2\,dx\le\frac{2}{a}\int|\nabla\Psi|^2|v_\nu|^2\,dx+\frac{R_\nu^{n/p}}{C_0}\int\Psi^2|F|\cdot|v_\nu|\,dx,\tag{12b.78}$$

where $F(x)=f(x_\nu+R_\nu x)$.

Since $\|v_\nu\|_{L^2(B_1(0))}=1$, if $\Psi\le 1$, we have

$$\int\Psi^2|F|\cdot|v_\nu|\,dx\le\Big(\int_{B_1(0)}|F|^2\,dx\Big)^{1/2}\le C_1R_\nu^{-n/p}\tag{12b.79}$$

if $f\in M_2^p$, so we have

$$\frac{a}{2}\int\Psi^2|\nabla v_\nu|^2\,dx\le\frac{2}{a}\int|\nabla\Psi|^2|v_\nu|^2\,dx+\frac{C_1}{C_0}\|f\|_{M_2^p}.\tag{12b.80}$$

This implies that $v_\nu$ is bounded in $H^1\big(B_\rho(0)\big)$ for each $\rho<1$. Now, as in (12.89), we can pass to a further subsequence and obtain

$$\begin{aligned}v_\nu&\longrightarrow v\ \text{ strongly in }\ L^2_{\text{loc}}\big(B_1(0)\big),\\ \nabla v_\nu&\longrightarrow\nabla v\ \text{ weakly in }\ L^2_{\text{loc}}\big(B_1(0)\big).\end{aligned}\tag{12b.81}$$

Thus, as in (12.90), we can pass to the limit in (12b.70), to obtain

$$\partial_j b^{jk}_{\alpha\beta}\partial_k v^\beta=0\quad\text{on }B_1(0).\tag{12b.82}$$

Also, by (12b.73),

$$V(0,1)=\|v\|_{L^2(B_1(0))}\le 1,\quad V(0,\rho)\ge 1.\tag{12b.83}$$

This contradicts Lemma 12.12, which requires $V(0,\rho)\le(1/2)V(0,1)$.

Now that we have Lemma 12B.4, the proof of Proposition 12B.3 is easily completed, by estimates similar to those in (12.69)–(12.73).

We can combine Propositions 12B.2 and 12B.3 to obtain the following:

**Corollary 12B.5.** *Let $u \in H^1(\Omega) \cap C(\Omega)$ solve (12b.18). If the very strong ellipticity condition (12b.53) holds and $B(x,u,\nabla u)$ is a quadratic form in $\nabla u$, then, given $p \geq n/2,\ q \in (1,\infty),\ s \geq 0$,*

$$f \in M_2^p \cap H^{s,q} \Longrightarrow u \in H^{s+2,q}. \tag{12b.84}$$

We mention that there are improvements of Proposition 12B.3, in which the hypothesis that $u$ is continuous is relaxed to the hypothesis that the local oscillation of $u$ is sufficiently small (see [HW]). For a number of results in the case when the hypothesis (12b.4) is strengthened to

$$|B(x,u,p)| \leq C\langle p\rangle^a,$$

for some $a < 2$, see [Gia]. Extensions of Corollary 12B.5, involving Morrey space estimates, can be found in [T2].

Corollary 12B.5 implies that any harmonic map (satisfying (12b.17)) is smooth wherever it is continuous. An example of a discontinuous harmonic map from $\mathbb{R}^3$ to the unit sphere $S^2 \subset \mathbb{R}^3$ is

$$u(x) = \frac{x}{|x|}. \tag{12b.85}$$

It has been shown by F. Helein [Hel2] that any harmonic map $u : \Omega \to M$ from a two-dimensional manifold $\Omega$ into a compact Riemannian manifold $M$ is smooth. Here we will give the proof of Helein's first result of this nature:

**Proposition 12B.6.** *Let $\Omega$ be a two-dimensional Riemannian manifold and let*

$$u : \Omega \longrightarrow S^m \tag{12b.86}$$

*be a harmonic map into the standard unit sphere $S^m \subset \mathbb{R}^{m+1}$. Then $u \in C^\infty(\Omega)$.*

**Proof.** We are assuming that $u \in H^1_{\text{loc}}(\Omega)$, that $u$ satisfies (12b.86), and that the components $u_j$ of $u = (u_1, \dots, u_{m+1})$ satisfy

$$\Delta u_j + u_j|\nabla u|^2 = 0. \tag{12b.87}$$

Here, $\Delta u_j$ and $|\nabla u|^2 = \sum |\nabla u_\ell|^2$ are determined by the Riemannian metric on $\Omega$, but the property of being a harmonic map is invariant under conformal changes in this metric (see Chapter 15, §2, for more on this), so we may as well take $\Omega$ to be an open set in $\mathbb{R}^2$, and $\Delta = \partial_1^2 + \partial_2^2$ the standard Laplace operator. Now $|u(x)|^2 = 1$ a.e. on $\Omega$ implies

$$\sum_{j=1}^{m+1} u_j(\partial_i u_j) = 0, \quad i = 1, 2, \tag{12b.88}$$

and putting this together with (12b.87) gives

$$\Delta u_j = -\sum_{k=1}^{m+1} (u_j \nabla u_k - u_k \nabla u_j) \cdot \nabla u_k, \quad \forall\ j. \tag{12b.89}$$

On the other hand, a calculation gives

$$\operatorname{div}(u_j \nabla u_k - u_k \nabla u_j) = \sum_\ell \partial_\ell (u_j \partial_\ell u_k - u_k \partial_\ell u_j) = 0, \tag{12b.90}$$

for all $j$ and $k$. Furthermore, since $u \in H^1_{\text{loc}}(\Omega) \cap L^\infty(\Omega)$,

$$u_j \nabla u_k - u_k \nabla u_j \in L^2_{\text{loc}}(\Omega), \quad \nabla u_k \in L^2_{\text{loc}}(\Omega). \tag{12b.91}$$

Now Proposition 12.14 of Chapter 13 implies

$$\sum_k (u_j \nabla u_k - u_k \nabla u_j) \cdot \nabla u_k = f_j \in \mathfrak{H}^1_{\text{loc}}(\Omega), \tag{12b.92}$$

where $\mathfrak{H}^1_{\text{loc}}(\Omega)$ is the local Hardy space, discussed in §12 of Chapter 13. Also, by Corollary 12.12 of Chapter 13, when $\dim \Omega = 2$,

$$\Delta u_j = -f_j \in \mathfrak{H}^1_{\text{loc}}(\Omega) \Longrightarrow u_j \in C(\Omega). \tag{12b.93}$$

Now that we have $u \in C(\Omega)$, Proposition 12B.6 follows from Corollary 12B.5.

If $\dim \Omega > 2$, there are results on partial regularity for harmonic maps $u : \Omega \to M$, for energy-minimizing harmonic maps [SU] and for "stationary" harmonic maps; see [Ev4] and [Bet]. See also [Si2], for an exposition. On the other hand, there is an example due to T. Riviere [Riv] of a harmonic map for which there is no partial regularity.

We mention another system of the type (12b.1), the $3 \times 3$ system

$$\Delta u = 2H u_x \times u_y \ \text{ on } \ \Omega, \quad u = g \ \text{ on } \ \partial\Omega. \tag{12b.94}$$

Here $H$ is a real constant, $\Omega$ is a bounded open set in $\mathbb{R}^2$, and $g \in C^\infty(\overline{\Omega}, \mathbb{R}^3)$. We seek $u : \overline{\Omega} \to \mathbb{R}^3$. This equation arises in the study of surfaces in $\mathbb{R}^3$ of constant mean curvature $H$. In fact, if $\Sigma \subset \mathbb{R}^3$ is a surface and $u : \Omega \to \Sigma$ a conformal map (using, e.g., isothermal coordinates) then, by (6.10) and (6.15), $\Sigma$ has constant mean curvature $H$ if and only if (12b.94) holds. In one approach to the analogue of the Plateau problem for surfaces of mean curvature $H$, the problem (12b.94) plays a role parallel to that played by $\Delta u = 0$ in the study of the Plateau problem for minimal surfaces (the $H = 0$ case) in §6. For this reason, in some articles (12b.94) is called the "equation of prescribed mean curvature," though that term is a bit of a misnomer.

The equation (12b.94) is satisfied by a critical point of the functional

$$J(u) = \int_\Omega \Big\{ \frac{1}{2} |\nabla u|^2 + \frac{2}{3} H (u \cdot u_x \times u_y) \Big\}\, dx\, dy, \tag{12b.95}$$

acting on the space

$$V = \{u \in H^1(\Omega, \mathbb{R}^3) : u = g \text{ on } \partial\Omega\}. \tag{12b.96}$$

That $J$ is well defined and smooth on $V$ follows from the following estimate of Rado:

$$|V(u) - V(g)|^2 \le \frac{1}{32\pi}\big(\|\nabla u\|_{L^2}^2 + \|\nabla g\|_{L^2}^2\big)^3, \tag{12b.97}$$

provided $u = g$ on $\partial\Omega$, where

$$V(u) = \int_\Omega (u \cdot u_x \times u_y)\, dx\, dy. \tag{12b.98}$$

The boundary problem (12b.94) is not solvable for all $g$, though it is known to be solvable provided

$$|H| \cdot \|g\|_{L^\infty} \le 1. \tag{12b.99}$$

We refer to [Str1] for a discussion of this and also a treatment of the Plateau problem for surfaces of mean curvature $H$, using (12b.94). Here we merely mention that given $u \in H^1(\Omega, \mathbb{R}^3)$, solving (12b.94), the fact that

$$u \in C(\overline{\Omega}, \mathbb{R}^3) \tag{12b.100}$$

then follows from Corollary 12.12 and Proposition 12.14 of Chapter 13, just as in (12b.93). Hence Corollary 12B.5 is applicable. This result, established by [Wen], was an important precursor to Proposition 12.13 of Chapter 13.

## 13. Elliptic regularity IV (Krylov-Safonov estimates)

In this section we obtain estimates for solutions to second-order elliptic equations of the form

$$Lu = f, \quad Lu = a^{jk}(x)\,\partial_j\partial_k u + b^j(x)\,\partial_j u + c(x)u, \tag{13.1}$$

on a domain $\Omega \subset \mathbb{R}^n$. We assume that $a^{jk}$, $b^j$, and $c$ are real-valued and that $a^{jk} \in L^\infty(\Omega)$, with

$$\lambda|\xi|^2 \le a^{jk}(x)\xi_j\xi_k \le \Lambda|\xi|^2, \tag{13.2}$$

for certain $\lambda, \Lambda \in (0, \infty)$. We define

$$\mathcal{D} = \det\,(a^{jk}), \quad \mathcal{D}_* = \mathcal{D}^{1/n}. \tag{13.3}$$

A. Alexandrov [Al] proved that if $|b|/\mathcal{D}_* \in L^n(\Omega)$ and $c \le 0$ on $\Omega$, then

$$u \in C(\overline{\Omega}) \cap H^{2,n}_{\text{loc}}(\Omega), \quad Lu \ge f \text{ on } \Omega, \tag{13.4}$$

implies

$$\sup_{x\in\Omega} u(x) \le \sup_{y\in\partial\Omega} u^+(y) + C\|\mathcal{D}_*^{-1} f\|_{L^n(\Omega)}, \tag{13.5}$$

where $C = C\big(n, \text{diam}\ \Omega, \|b/\mathcal{D}_*\|_{L^n}\big)$. We will not make use of this and will not include a proof, but we will establish the following result of I. Bakelman [B], essentially a more precise version of (13.5) for the special case $b^j = c = 0$ (under stronger regularity hypotheses on $u$). It is used in some proofs of (13.5) (see [GT]).

To formulate this result, set

$$\Gamma^+ = \{y \in \Omega : u(x) \le u(y) + p \cdot (x - y), \forall\ x \in \Omega, \\ \text{for some } p = p(y) \in \mathbb{R}^n\}. \tag{13.6}$$

If $u \in C^1(\Omega)$, then $y$ belongs to $\Gamma^+$ if and only if the graph of $u$ lies everywhere *below* its tangent plane at $\big(y, u(y)\big)$. If $u \in C^2(\Omega)$, then $u$ is concave on $\Gamma^+$, that is, $(\partial_j \partial_k u) \le 0$ on $\Gamma^+$.

**Proposition 13.1.** *If $u \in C^2(\Omega) \cap C(\overline{\Omega})$, we have*

$$\sup_{x\in\Omega}\ u(x) \le \sup_{y\in\partial\Omega}\ u(y) + \frac{d}{nV_n^{1/n}} \big\|\mathcal{D}_*^{-1}(a^{jk}\partial_j\partial_k u)\big\|_{L^n(\Gamma^+)}, \tag{13.7}$$

*where $d =$ diam $\Omega$, and $V_n$ is the volume of the unit ball in $\mathbb{R}^n$.*

To establish this, we use the matrix inequality

$$(\det\ A)(\det\ B) \le \Big(\frac{1}{n}\ \text{Tr}\ AB\Big)^n, \tag{13.8}$$

for positive, symmetric, $n \times n$ matrices $A$ and $B$. (See the exercise at the end of this section for a proof.) Setting

$$A = -H(u) = -\big(\partial_j\partial_k u(x)\big), \quad B = \big(a^{jk}(x)\big), \quad x \in \Gamma^+, \tag{13.9}$$

where $H(u)$ is the Hessian matrix, as in (3.7a), we have

$$|\det\ H(u)| \le \mathcal{D}^{-1}\Big(-\frac{1}{n} a^{jk}\, \partial_j\partial_k u\Big)^n \quad \text{on } \Gamma^+. \tag{13.10}$$

Thus Proposition 13.1 follows from

**Lemma 13.2.** *For $u \in C^2(\Omega) \cap C(\overline{\Omega})$, we have*

$$\sup_{x\in\Omega}\ u(x) \le \sup_{y\in\partial\Omega}\ u(y) + \frac{d}{V_n^{1/n}}\Big(\int_{\Gamma^+} |\det\ H(u)|\ dx\Big)^{1/n}. \tag{13.11}$$

**Proof.** Replacing $u$ by $u - \sup_{\partial\Omega}\ u$, it suffices to assume $u \le 0$ on $\partial\Omega$. Define $\chi(\Omega)$ to be $\bigcup_{y\in\Omega} \chi(y)$, where

$$\chi(y) = \{p \in \mathbb{R}^n : u(x) \le u(y) + p \cdot (x - y), \forall\ x \in \Omega\}, \tag{13.12}$$

so $\chi(y) \ne \emptyset \Leftrightarrow y \in \Gamma^+$. Also, if $u \in C^1(\Omega)$ (as we assume here),

$$\chi(y) = \{Du(y)\}, \quad \text{for } y \in \Gamma^+. \tag{13.13}$$

Thus the Lebesgue measure of $\chi(\Omega)$ is given by

$$\mathcal{L}^n\big(\chi(\Omega)\big) = \mathcal{L}^n\big(\chi(\Gamma^+)\big) = \mathcal{L}^n\big(Du(\Gamma^+)\big) \le \int_{\Gamma^+} |\det H(u)|\, dx. \tag{13.14}$$

Thus it suffices to show that if $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ and $u \le 0$ on $\partial\Omega$, then

$$\sup_{x\in\Omega} u(x) \le \frac{d}{V_n^{1/n}} \mathcal{L}^n\big(\chi(\Omega)\big). \tag{13.15}$$

This is basically a comparison result. Assume $\sup u > 0$ is attained at $x_0$. Let $W_1$ be the function on $\overline{\Omega}$ whose graph is the cone with apex at $\big(x_0, u(x_0)\big)$ and base $\partial\Omega \times \{0\}$. Then, if $\chi_{W_1}(y)$ denotes the function (13.12) with $u$ replaced by $W_1$, we have

$$\chi_u(\Omega) \supset \chi_{W_1}(\Omega). \tag{13.16}$$

Similarly, if $W_2$ is the function on $B_d(x_0)$ whose graph is the cone with apex at $\big(x_0, u(x_0)\big)$ and base $\{x : |x - x_0| = d\} \times \{0\}$, then

$$\chi_{W_1}(\Omega) \supset \chi_{W_2}\big(B_d(x_0)\big). \tag{13.17}$$

Finally, the inequality

$$\sup\ W_2 \le \frac{d}{V_n^{1/n}} \mathcal{L}^n\big(\chi_{W_2}(B_d(x_0))\big) \tag{13.18}$$

is elementary, so we have (13.15), and hence Lemma 13.2 is proved.

We now make the assumption that

$$\frac{\Lambda}{\lambda} \le \gamma, \quad \Big(\frac{|b|}{\lambda}\Big)^2 \le \nu, \quad \frac{|c|}{\lambda} \le \nu, \tag{13.19}$$

and establish the following local maximum principle, following [GT].

**Proposition 13.3.** *Let $u \in H^{2,n}(\Omega)$, $Lu \ge f$, $f \in L^n(\Omega)$. Then, for any ball $B = B_{2R}(y) \subset \Omega$ and any $p \in (0, n]$, we have*

$$\sup_{x\in B_R(y)} u(x) \le C \left\{ \Big( \frac{1}{Vol(B)} \int_B (u^+)^p\, dx \Big)^{1/p} + \frac{R}{\lambda} \|f\|_{L^n(B)} \right\}, \tag{13.20}$$

*where $C = C(n, \gamma, \nu R^2, p)$.*

**Proof.** Translating and dilating, we can assume without loss of generality that $0 \in \Omega$ and $B = B_1(0)$. We will also assume that $u \in C^2(\Omega) \cap H^{2,n}(\Omega)$, since if (13.20) is established in this case, the more general case follows by a simple approximation argument.

Given $\beta \ge 1$, define

$$\eta(x) = \big(1 - |x|^2\big)^\beta, \quad \text{for } |x| \le 1. \tag{13.21}$$

Setting $v = \eta u$ on $B$, we have

$$\begin{aligned} a^{jk}\,\partial_j\partial_k v &= \eta a^{jk}\,\partial_j\partial_k u + 2a^{jk}(\partial_j\eta)(\partial_k u) + u a^{jk}\,\partial_j\partial_k\eta \\ &\ge \eta(f - b^j\,\partial_j u - cu) + 2a^{jk}(\partial_j\eta)(\partial_k u) + u a^{jk}\,\partial_j\partial_k\eta. \end{aligned} \tag{13.22}$$

Let $\Gamma_v^+$ be as in (13.6), but with $u$ replaced by $v$, and $\Omega$ replaced by $B$. Clearly, $u \ge 0$ on $\Gamma_v^+$. We have

$$|Dv| \le \frac{v}{1-|x|} \quad \text{on} \quad \Gamma_v^+, \tag{13.23}$$

so

$$\begin{aligned} |Du| = \eta^{-1}|Dv - uD\eta| &\le \frac{1}{\eta}\Bigl(\frac{v}{1-|x|} + u|D\eta|\Bigr) \\ &\le 2(1+\beta)\eta^{-1/\beta}u \quad \text{on } \Gamma_v^+. \end{aligned} \tag{13.24}$$

Hence

$$\begin{aligned} -a^{jk}\,\partial_j\partial_k v &\le \Bigl\{\bigl(16\beta^2 + 2\beta\eta\bigr)\Lambda\eta^{-2/\beta} + 2\beta|b|\eta^{-1/\beta} + c\Bigr\}v + \eta f \\ &\le C\lambda\eta^{-2/\beta}v + f, \end{aligned} \tag{13.25}$$

on $\Gamma_v^+$, where $C = C(n,\beta,\gamma,\nu)$. Of course, $a^{jk}\partial_j\partial_k v \le 0$ on $\Gamma_v^+$. If $\beta \ge 2$, we have, upon applying Proposition 13.1 to $v$,

$$\begin{aligned} \sup_B\ v &\le C\Bigl(\bigl\|\eta^{-2/\beta}v^+\bigr\|_{L^n(B)} + \frac{1}{\lambda}\|f\|_{L^n(B)}\Bigr) \\ &\le C_1\left\{\bigl(\sup_B\ v^+\bigr)^{1-2/\beta}\bigl\|(u^+)^{2/\beta}\bigr\|_{L^n(B)} + \frac{1}{\lambda}\|f\|_{L^n(B)}\right\}. \end{aligned} \tag{13.26}$$

Choose $\beta = 2n/p \ge 2$, so we have

$$\sup_B\ v \le C_1\left\{\bigl(\sup_B\ v^+\bigr)^{1-p/n}\|u^+\|_{L^p(B)}^{p/n} + \frac{1}{\lambda}\|f\|_{L^n(B)}\right\}. \tag{13.27}$$

(Here we allow $p < 1$, in which case $\|\cdot\|_{L^p}$ is not a norm, but (13.27) is still valid.) Using the elementary inequality

$$a^{1-p/n}b^{p/n} \le \varepsilon a + \varepsilon^{-\left(n/p-1\right)}b, \quad \forall\ \varepsilon \in (0,\infty), \tag{13.28}$$

and taking $a = \sup_B\ v^+$, $b = \|u^+\|_{L^p(B)}$, and $\varepsilon = 1/2C_1$, we have (the $R = 1$ case of) (13.20), so Proposition 13.3 is proved.

Replacing $u$ by $-u$, we have an estimate on $\sup_{B_R(y)}\ (-u)$ when $Lu \le f$. Thus, when $Lu = f$ and the hypotheses of Proposition 13.3 hold, we have

$$\sup_{B_R(y)}\ |u| \le C\left\{\Bigl(\frac{1}{\mathrm{Vol}(B)}\int\limits_B |u|^p\ dx\Bigr)^{1/p} + \frac{R}{\lambda}\|f\|_{L^n(B)}\right\}. \tag{13.29}$$

Next we establish a “weak Harnack inequality” of [KrS], which will lead to results on Hölder continuity of solutions of $Lu = f$. This result will also be applied directly in the next section, to results on solutions to certain completely nonlinear equations.

**Proposition 13.4.** *Assume* $u \in H^{2,n}(\Omega)$, $Lu \le f$ *in* $\Omega$, $f \in L^n(\Omega)$, *and* $u \ge 0$ *on a ball* $B = B_{2R}(y) \subset \Omega$. *Then*

$$\left(\frac{1}{\mathrm{Vol}(B_R)} \int_{B_R} u^p \, dx\right)^{1/p} \le C\Big(\inf_{B_R} u + \frac{R}{\lambda}\|f\|_{L^n(B)}\Big), \tag{13.30}$$

*for some positive* $p = p(n, \gamma, \nu R^2)$ *and* $C = C(n, \gamma, \nu R^2)$.

As before, there is no loss of generality in assuming $B = B_1(0)$. Also, replacing $L$ and $f$ by $\lambda^{-1}L$ and $\lambda^{-1}f$, we can assume $\lambda = 1$.

To begin the proof, take $\varepsilon > 0$ and set

$$\begin{gathered} \overline{u} = u + \varepsilon + \|f\|_{L^n(B)}, \quad w = \log \frac{1}{\overline{u}}, \\ v = \eta w, \qquad g = \frac{f}{\overline{u}}, \end{gathered} \tag{13.31}$$

where $\eta$ is given by (13.21). Note that $w$ is large (positive) where $\overline{u}$ is small. We have

$$\begin{aligned} -a^{jk}\,\partial_j\partial_k v &= -\eta a^{jk}\,\partial_j\partial_k w - 2a^{jk}(\partial_j\eta)(\partial_k w) - w a^{jk}\,\partial_j\partial_k\eta \\ &\le \eta\big[-a^{jk}(\partial_j w)(\partial_k w) + b^j\partial_j w + |c| + g\big] \\ &\qquad - 2a^{jk}(\partial_j\eta)(\partial_k w) - w a^{jk}\,\partial_j\partial_k\eta \\ &\le \frac{2}{\eta}a^{jk}(\partial_j\eta)(\partial_k\eta) - w a^{jk}\,\partial_j\partial_k\eta + (|b|^2 + |c| + g)\eta, \end{aligned} \tag{13.32}$$

where the last inequality is obtained via Cauchy's inequality, applied to the inner product $\langle V, W\rangle = V_j a^{jk} W_k$.

Now the form of $\eta$ implies that $a^{jk}\,\partial_j\partial_k\eta \ge 0$ provided $2(\beta-1)a^{jk}x_j x_k + a^{jj}|x|^2 \ge a^{jj}$, and hence

$$2\beta|x|^2 \ge n\Lambda \Longrightarrow a^{jk}\,\partial_j\partial_k\eta \ge 0. \tag{13.33}$$

Thus, if $\alpha \in (0,1)$, then

$$\beta \ge \frac{n\gamma}{2\alpha}, \quad |x| \ge \alpha \Longrightarrow a^{jk}\,\partial_j\partial_k\eta \ge 0. \tag{13.34}$$

Hence, on the set $B^+ = \{x \in B : w(x) > 0\}$, we have

$$\begin{aligned} -a^{jk}\,\partial_j\partial_k v &\le 4\beta^2\big(1 - |x|^2\big)^{\beta-2}|x|^2 + v\chi_{B_\alpha} \sup_{B_\alpha}\left(-\frac{a^{jk}\,\partial_j\partial_k\eta}{\eta}\right) \\ &\qquad + \big(|b|^2 + |c| + g\big)\eta \\ &\le 4\beta^2\Lambda + |b|^2 + |c| + g + \frac{2n\beta\Lambda}{1-\alpha^2}v\chi_{B_\alpha}. \end{aligned} \tag{13.35}$$

Note that $\|g\|_{L^n(B)} \le 1$. Thus Proposition 13.1 yields

$$\sup_B v \le C\big(1 + \|v^+\|_{L^n(B_\alpha)}\big), \tag{13.36}$$

with $C = C(n, \alpha, \gamma, \nu)$.

Note that if $u$ satisfies the hypotheses of Proposition 13.4 and $t \in (0, \infty)$, then $u/t$ satisfies $L(u/t) \le f/t$, and the analogue of $w$ in (13.31) is $w - k$, where $k = \log(1/t)$. The function $g$ in (13.31) is unchanged, and, working through (13.32)–(13.36), we obtain the following extension of (13.36):

$$\sup_B \eta(w-k) \le C\big(1 + \|\eta(w-k)^+\|_{L^n(B_\alpha)}\big), \quad \forall\, k \in \mathbb{R}, \tag{13.37}$$

with constants independent of $k$.

The next stage in the proof of Proposition 13.4 will involve a decomposition into cubes of the sort used for Calderon-Zygmund estimates in §5 of Chapter 13. To set up some notation, given $y \in \mathbb{R}^n$, $R > 0$, let $Q_R(y)$ denote the open cube centered at $y$, of edge $2R$:

$$Q_R(y) = \{x \in \mathbb{R}^n : |x_j - y_j| < R,\ 1 \le j \le n\}. \tag{13.38}$$

If $\alpha < 1/\sqrt{n}$, then $Q_\alpha = Q_\alpha(0) \subset\subset B$.

The cube decomposition we will use in the proof of Lemma 13.5 below can be described in general as follows. Let $Q_0$ be a cube in $\mathbb{R}^n$, let $\varphi \ge 0$ be an element of $L^1(Q_0)$, and suppose $\int_{Q_0} \varphi\, dx \le t\mathcal{L}^n(Q_0)$, $t \in (0, \infty)$. Bisecting the edges of $Q_0$, we subdivide it into $2^n$ subcubes. Those subcubes that satisfy $\int_Q \varphi\, dx \le t\mathcal{L}^n(Q)$ are similarly subdivided, and this process is repeated indefinitely. Let $\mathcal{F}$ denote the set of subcubes so obtained that satisfy

$$\int_Q \varphi\, dx > t\mathcal{L}^n(Q);$$

we do not further subdivide these cubes. For each $Q \in \mathcal{F}$, denote by $\widetilde{Q}$ the subcube whose subdivision gives $Q$. Since $\mathcal{L}^n(\widetilde{Q})/\mathcal{L}^n(Q) = 2^n$, we see that

$$t < \frac{1}{\mathcal{L}^n(Q)} \int_Q \varphi\, dx \le 2^n t, \quad \forall\, Q \in \mathcal{F}. \tag{13.39}$$

Also, setting $F = \bigcup_{Q \in \mathcal{F}} Q$ and $G = Q_0 \setminus F$, we have

$$\varphi \le t, \quad \text{a.e. in } G. \tag{13.40}$$

This subdivision was also done in the proof of Lemma 5.5 in Chapter 13. Let us also set $\widetilde{F} = \bigcup_{Q \in \mathcal{F}} \widetilde{Q}$; since $Q \in \mathcal{F} \Rightarrow \widetilde{Q} \notin \mathcal{F}$, we have

$$\int_{\widetilde{F}} \varphi\, dx \le t\mathcal{L}^n(\widetilde{F}). \tag{13.41}$$

In particular, when $\varphi$ is the characteristic function $\chi_\Gamma$ of a measurable subset $\Gamma$ of $Q_0$, of measure $\le t \cdot \mathcal{L}^n(Q_0)$, we deduce from (13.40)–(13.41) that

$$\mathcal{L}^n(\Gamma) = \mathcal{L}^n(\Gamma \cap \widetilde{F}) \le t\mathcal{L}^n(\widetilde{F}). \tag{13.42}$$

We have the following measure-theoretic result:

**Lemma 13.5.** *Let $Q_0$ be a cube in $\mathbb{R}^n$, $w \in L^1(Q_0)$, and, for $k \in \mathbb{R}$, set*

$$\Gamma_k = \{x \in Q_0 : w(x) \le k\}. \tag{13.43}$$

*Suppose there are positive constants $\delta < 1$ and $C$ such that*

$$\sup_{Q_0 \cap Q_{3r}(z)} (w - k) \le C \tag{13.44}$$

*whenever $k$ and $Q = Q_r(z) \subset Q_0$ satisfy*

$$\mathcal{L}^n(\Gamma_k \cap Q) \ge \delta \mathcal{L}^n(Q). \tag{13.45}$$

*Then, for all $k \in \mathbb{R}$,*

$$\sup_{Q_0} (w - k) \le C \left( 1 + \frac{\log\big(\mathcal{L}^n(\Gamma_k)/\mathcal{L}^n(Q_0)\big)}{\log\ \delta} \right). \tag{13.46}$$

**Proof.** We show by induction that

$$\sup_{Q_0} (w - k) \le mC, \tag{13.47}$$

for any $m \in \mathbb{Z}^+$ and $k \in \mathbb{R}$ such that $\mathcal{L}^n(\Gamma_k) \ge \delta^m \mathcal{L}^n(Q_0)$. This is true by hypothesis if $m = 1$. Suppose that it holds for $m = M \in \mathbb{Z}^+$ and that $\mathcal{L}^n(\Gamma_k) \ge \delta^{M+1}\mathcal{L}^n(Q_0)$. Define $\widetilde{\Gamma}_k$ by

$$\widetilde{\Gamma}_k = \bigcup \big\{ Q_{3r}(z) \cap Q_0 : \mathcal{L}^n\big(Q_r(z) \cap \Gamma_k\big) \ge \delta\ \mathcal{L}^n\big(Q_r(z)\big) \big\}. \tag{13.48}$$

Applying the estimate (13.42), with $t = \delta$, we see that either $\widetilde{\Gamma}_k = Q_0$ or

$$\mathcal{L}^n(\widetilde{\Gamma}_k) \ge \delta^{-1}\mathcal{L}^n(\Gamma_k) \ge \delta^M\ \mathrm{vol}(Q_0), \tag{13.49}$$

and hence, replacing $k$ by $k + C$, we obtain

$$\sup_{Q_0}(w - k) \le (M + 1)C, \tag{13.50}$$

which verifies (13.47) for $m = M + 1$.

Now, the estimate (13.46) follows by choosing $m$ appropriately, and the lemma is proved.

Returning to the estimation of the functions defined in (13.31), we see that (13.36) implies

$$\sup_B\ v \le C\big(1 + \|v^+\|_{L^n(Q_\alpha)}\big) \le C\Big(1 + \big[\mathrm{vol}(Q_\alpha^+)\big]^{1/n}\ \sup_B\ v^+\Big), \tag{13.51}$$

where $Q_\alpha = Q_\alpha(0)$, as stated below (13.38), and

$$Q_\alpha^+ = \{x \in Q_\alpha : v(x) > 0\} = \{x \in Q_\alpha : \overline{u}(x) < 1\}.$$

Hence, if $C$ is the constant in (13.36),

$$\frac{\text{vol}(Q_\alpha^+)}{\text{vol}(Q_\alpha)} \le \Bigl(\frac{1}{4\alpha C}\Bigr)^n = \theta \Longrightarrow \sup_B \ v \le 2C. \tag{13.52}$$

Now choose $\alpha = 1/3n$, and take $\theta = (4\alpha C)^{-n}$, as in (13.52). Using the coordinate change $x \mapsto \alpha(x-z)/r$, we obtain for any cube $Q = Q_r(z)$ such that $B_{3nr}(z) \subset B$, the implication

$$\frac{\text{vol}(Q^+)}{\text{vol}(Q)} \le \theta \Longrightarrow \sup_{Q_{3r}(z)} \ w \le C(n,\gamma,\nu). \tag{13.53}$$

With $\alpha$ and $\theta$ as specified above, take $\delta = 1-\theta,\ Q_0 = Q_\alpha(0)$, and note that the estimate (13.53) holds also when $w$ is replaced by $w-k$, and $Q^+$ is replaced by the set $\{x \in Q : w(x) - k > 0\}$, as a consequence of (13.37). Let

$$\mu(t) = \mathcal{L}^n\bigl(\{x \in Q_0 : \overline{u}(x) > t\}\bigr). \tag{13.54}$$

Setting $k = \log\ 1/t$, we have from Lemma 13.5 the estimate

$$\mu(t) \le C\bigl(\inf_{Q_0}\ t^{-1}\overline{u}\bigr)^\kappa, \quad \forall\, t > 0, \tag{13.55}$$

where $C = C(n,\gamma,\nu),\ \kappa = \kappa(n,\gamma,\nu)$. Replacing the cube $Q_0$ by the inscribed ball $B_\alpha(0),\ \alpha = 1/3n$, and using the identity

$$\int_{Q_0} (\overline{u})^p \ dx = p\int_0^\infty t^{p-1}\mu(t)\ dt, \tag{13.56}$$

we have

$$\int_{B_\alpha} (\overline{u})^p \ dx \le C\bigl(\inf_{B_\alpha}\ \overline{u}\bigr)^p, \quad \text{for } p = \frac{\kappa}{2}. \tag{13.57}$$

The inequality (13.30) then follows by letting $\varepsilon \to 0$ if we use a covering argument to extend (13.57) to arbitrary $\alpha < 1$ (especially, $\alpha = 1/2$) and use the coordinate transformation $x \mapsto (x-y)/2R$. Thus Proposition 13.4 is established.

Putting together (13.29) and (13.30), we have the following.

**Corollary 13.6.** *Assume* $u \in H^{2,n}(\Omega),\ Lu = f$ *on* $\Omega,\ f \in L^n(\Omega)$, *and* $u \ge 0$ *on a ball* $B = B_{4R}(y) \subset \Omega$. *Then*

$$\sup_{B_R(y)}\ u(x) \le C_1\Bigl(\inf_{B_{2R}(y)}\ u + \frac{R}{\lambda}\|f\|_{L^n(B_{4R})}\Bigr), \tag{13.58}$$

*for some $C_1 = C_1(n, \gamma, \nu R^2)$. In particular, if $u \geq 0$ on $\Omega$,*

$$Lu = 0 \Longrightarrow \sup_{B_R(y)} u(x) \leq C_1 \inf_{B_{2R}(y)} u(x). \tag{13.59}$$

We can use this to establish Hölder estimates on solutions to $Lu = f$. We will actually apply Corollary 13.6 to $L_1 = a^{jk}\,\partial_j\partial_k + b^j\partial_j$, so $L_1 u = f_1 = f - cu$. Suppose that

$$a = \inf_{B_{4R}(y)} u \leq \sup_{B_{4R}(y)} u = b. \tag{13.60}$$

Then $v = (u-a)/(b-a)$ is $\geq 0$ on $B_{4R}(y)$, and $L_1 v = f_1/(b-a)$, so Corollary 13.6 yields

$$\sup_{B_R(y)} \frac{u-a}{b-a} \leq C_1\Big(\inf_{B_{2R}(y)} \frac{u-a}{b-a} + \frac{R}{\lambda}\frac{1}{b-a}\|f - cu\|_{L^n(B_{4R})}\Big). \tag{13.61}$$

Without loss of generality, we can assume $C_1 \geq 1$. Now given this, one of the following two cases must hold:

(i) $$C_1 \inf_{B_{2R}(y)} \frac{u-a}{b-a} \geq \frac{1}{2} \sup_{B_R(y)} \frac{u-a}{b-a},$$

(ii) $$C_1 \inf_{B_{2R}(y)} \frac{u-a}{b-a} < \frac{1}{2} \sup_{B_R(y)} \frac{u-a}{b-a}.$$

If case (i) holds, then either

$$\sup_{B_R(y)} \frac{u-a}{b-a} \leq \frac{1}{2} \quad \text{or} \quad \inf_{B_{2R}(y)} \frac{u-a}{b-a} \geq \frac{1}{4C_1},$$

and hence (since we are assuming $C_1 \geq 1$)

$$\text{(i)} \Longrightarrow \operatorname*{osc}_{B_R(y)} u \leq \Big(1 - \frac{1}{4C_1}\Big) \operatorname*{osc}_{B_{4R}(y)} u. \tag{13.62}$$

If case (ii) holds, then

$$\sup_{B_R(y)} \frac{u-a}{b-a} \leq \frac{2R}{\lambda}\frac{1}{b-a}\|f - cu\|_{L^n(B_{4R})},$$

so

$$\text{(ii)} \Longrightarrow \operatorname*{osc}_{B_R(y)} u \leq \frac{2R}{\lambda}\|f - cu\|_{L^n(B_{4R})}, \tag{13.63}$$

which is bounded by $C_2 R$ in view of the sup-norm estimate (13.29). Consequently, under the hypotheses of Corollary 13.6, we have

$$\operatorname*{osc}_{B_R(y)} u \leq \max\Big(C_2 R, \Big(1 - \frac{1}{C_1}\Big) \operatorname*{osc}_{B_{4R}(y)} u\Big), \tag{13.64}$$

with $C_1 = C_1(n, \gamma, \nu R_0^2)$, $C_2 = C_2(n, \gamma, \nu R_0^2)\big[\|f\|_{L^n(\Omega)} + \|u\|_{L^n(\Omega)}\big]$, given $B_{4R_0}(y) \subset \Omega$, $R \le R_0$. Therefore, we have the following:

**Theorem 13.7.** *Assume $u \in H^{2,n}(\Omega)$, $Lu = f$, and $f \in L^n(\Omega)$. Given $\mathcal{O} \subset\subset \Omega$, there is a positive $\mu = \mu(\mathcal{O}, \Omega, n, \gamma, \nu)$ such that*

$$\|u\|_{C^\mu(\mathcal{O})} \le A\big(\|u\|_{L^n(\Omega)} + \|f\|_{L^n(\Omega)}\big), \tag{13.65}$$

*with $A = A(\mathcal{O}, \Omega, n, \gamma, \nu)$.*

Some boundary regularity results follow fairly easily from the methods developed above. For the present, assume $\Omega$ is a smoothly bounded region in $\mathbb{R}^n$, that

$$u \in H^{2,n}(\Omega) \cap C(\overline{\Omega}), \quad u\big|_{\partial\Omega} \le 0, \tag{13.66}$$

and that $Lu = f$ on $\Omega$. Let $B = B_{2R}(y)$ be a ball centered at $y \in \partial\Omega$. Then, extending (13.20), we have, for any $p \in (0, n]$,

$$\sup_{\Omega \cap B_R(y)} u \le C\left\{\Big(\frac{1}{\operatorname{vol}(B)} \int\limits_{B\cap\Omega} (u^+)^p\, dx\Big)^{1/p} + \frac{R}{\lambda}\|f\|_{L^n(B\cap\Omega)}\right\}, \tag{13.67}$$

with $C = C(n, \gamma, \nu R^2, p)$. To establish this, extend $u$ to be 0 on $B \setminus \Omega$. This extended function might not belong to $H^{2,n}(B)$, but the proof of Proposition 13.3 can still be seen to apply, given the following observation:

**Lemma 13.8.** *Assume that $u$ satisfies the hypotheses of Proposition 13.1 and that $\Omega \subset \widetilde{\Omega}$, and set $u = 0$ on $\widetilde{\Omega} \setminus \Omega$. Then*

$$\sup_{\widetilde{\Omega}} u \le \sup_{\partial\widetilde{\Omega}} u + \frac{\widetilde{d}}{nV_n^{1/n}}\big\|\mathcal{D}_*^{-1}(a^{jk}\,\partial_j\partial_k u)\big\|_{L^n(\widetilde{\Gamma}^+)}, \tag{13.68}$$

*where $\widetilde{d} =$ diam $\widetilde{\Omega}$, and $\widetilde{\Gamma}^+$ is the upper contact set of $u$, defined as in (13.6), with $\Omega$, replaced by $\widetilde{\Omega}$.*

Note that if $u(x) > 0$ anywhere on $\Omega$, then $\widetilde{\Gamma}^+ \subset \Gamma^+$.

The following result extends Proposition 13.4.

**Proposition 13.9.** *Assume $u \in H^{2,n}(\Omega)$, $Lu = f$ on $\Omega$, $u \ge 0$ on $B \cap \Omega$. Set*

$$m = \inf_{B\cap\partial\Omega} u, \tag{13.69}$$

*and*

$$\begin{aligned} \widetilde{u}(x) = \min\big(m, u(x)\big), &\quad x \in B \cap \Omega, \\ m, &\quad x \in B \setminus \Omega. \end{aligned} \tag{13.70}$$

*Then*

$$\left(\frac{1}{\text{vol}(B_R)}\int\limits_{B_R}(\widetilde{u})^p\,dx\right)^{1/p}\le C\Big(\inf_{\Omega\cap B_R}\ u+\frac{R}{\lambda}\|f\|_{L^n(B\cap\Omega)}\Big),\tag{13.71}$$

*for some positive* $p=p(n,\gamma,\nu R^2)$ *and* $C=C(n,\gamma,\nu R^2)$.

**Proof.** One adapts the proof of Proposition 13.4, with $u$ replaced by $\widetilde{u}$. One gets an estimate of the form (13.53), with $w$ replaced by $w-k$, $k\ge -\log m$. From there, one gets an estimate of the form (13.55), for $0<t\le m$. But $\mu(t)=0$ for $t>m$, so (13.71) follows as before.

This leads as before to a Hölder estimate:

**Proposition 13.10.** *Assume* $u\in H^{2,n}(\Omega)$, $Lu=f\in L^n(\Omega)$, $u\big|_{\partial\Omega}=\varphi\in C^\beta(\partial\Omega)$, *and* $\beta>0$. *Then there is a positive* $\mu=\mu(\Omega,n,\gamma,\nu,\beta)$ *such that*

$$\|u\|_{C^\mu(\overline{\Omega})}\le A\Big(\|u\|_{L^n(\Omega)}+\|f\|_{L^n(\Omega)}+\|\varphi\|_{C^\beta(\partial\Omega)}\Big),\tag{13.72}$$

*with* $A=A(\Omega,n,\gamma,\nu,\beta)$.

We next establish another type of boundary estimate, which will also be very useful in applications in the following sections. The following result is due to [Kry2]; we follow the exposition in [Kaz] of a proof of L. Caffarelli.

**Proposition 13.11.** *Assume* $u\in C^2(\overline{\Omega})$ *satisfies*

$$Lu=f,\quad u\big|_{\partial\Omega}=0.\tag{13.73}$$

*Assume*

$$\|u\|_{L^\infty(\Omega)}+\|\nabla u\|_{L^\infty(\Omega)}+\|f\|_{L^\infty(\Omega)}\le K.\tag{13.74}$$

*Then there is a Hölder estimate for the normal derivative of* $u$ *on* $\partial\Omega$:

$$\|\partial_\nu u\|_{C^\alpha(\partial\Omega)}\le CK,\tag{13.75}$$

*for some positive* $\alpha=\alpha(\Omega,n,\nu,\lambda,\Lambda,K)$ *and* $C=C(\Omega,n,\nu,\lambda,\Lambda)$.

To prove this, we can flatten out a portion of the boundary. After having done so, absorb the terms $b^j(x)\partial_j u+c(x)u$ into $f$. It suffices to assume that

$$Lu=f\ \text{ on }\ B^+,\quad Lu=a^{jk}(x)\,\partial_j\partial_k u,\tag{13.76}$$

where

$$B^+=\{x\in\mathbb{R}^n:|x|<4,\ x_n\ge 0\},$$

and that

(13.77) $$u = 0 \quad \text{on} \quad \Sigma = \{x \in \mathbb{R}^n : |x| < 4,\ x_n = 0\},$$

and to show that there is an estimate

(13.78) $$\|\partial_n u\|_{C^\alpha(\Gamma)} \le CK, \quad C = C(n, \lambda, \Lambda),$$

where $K$ is as in (13.74), with $\Omega$ replaced by $B^+$, $\alpha = \alpha(n, \lambda, \Lambda, K) > 0$, and

(13.79) $$\Gamma = \{x \in \Sigma : |x| \le 1\}.$$

Note that, for $(x', 0) \in \Sigma$,

(13.80) $$\partial_n u(x', 0) = v(x', 0),$$

where

(13.81) $$v(x) = x_n^{-1} u(x).$$

FIGURE 13.1

Let us fix some notation. Given $R \le 1$ and $\delta = \lambda/9n\Lambda < 1/2$, let

(13.82) $$\begin{aligned} Q(R) &= \big\{x \in B^+ : |x'| \le R,\ 0 \le x_n \le \delta R\big\}, \\ Q^+(R) &= \big\{x \in Q(R) : \frac{1}{2}\delta R \le x_n \le \delta R\big\} \end{aligned}$$

(see Fig. 13.1). Then set

(13.83) $$m_R = \inf_{Q(R)} v, \quad M_R = \sup_{Q(R)} v,$$

so $\operatorname{osc}_{Q(R)} v = M_R - m_R$. Before proving Proposition 13.11, we establish two lemmas.

**Lemma 13.12.** *Under the hypotheses (13.76) and (13.77), if also $u \geq 0$ on $Q(R)$, then*

$$\inf_{Q^+(R)} v \leq \frac{2}{\delta} \inf_{Q(R/2)} v + \frac{R}{\lambda} \sup |f|. \tag{13.84}$$

**Proof.** Let $\gamma = \inf\{v(x) : |x'| \leq R,\ x_n = \delta R\}$, and set

$$z(x) = \gamma x_n \Big(\delta - \frac{2\delta}{R^2}|x'|^2 + \frac{1}{R} x_n\Big) - \frac{1}{2\lambda} x_n \big(\delta R - x_n\big) \sup |f|. \tag{13.85}$$

Given $\delta \in (0, 1/2]$, we have the following behavior on $\partial Q(R)$:

$$\begin{aligned} &z(x) = 0, \quad \text{for } \ x = (x', 0), && \text{(bottom)},\\ &z(x) < 0 \quad \text{on } \ \{x \in Q(R) : |x'| = R\}, && \text{(side)},\\ &z(x) < 2\gamma\delta^2 R < \gamma\delta R \quad \text{on } \ \{x \in Q(R) : x_n = \delta R\} && \text{(top)}. \end{aligned} \tag{13.86}$$

Also,

$$Lz \leq -\sup \ |f| \leq f \quad \text{on } \ Q(R) \quad \text{if } \ \delta = \frac{\lambda}{9n\Lambda}. \tag{13.87}$$

Since $u \geq 0$ on $Q(R)$ and $u = x_n v \geq \gamma\delta R$ on the top of $Q(R)$, we have

$$L(u - z) \geq 0 \ \text{ on } \ Q(R), \quad u \geq z \ \text{ on } \ \partial Q(R). \tag{13.88}$$

Thus, by the maximum principle, $u \geq z$ on $Q(R)$, so $v(z) \geq z(x)/x_n$ on $Q(R)$. Hence

$$\inf_{Q(R/2)} v \geq \frac{\delta}{2}\Big(\gamma - \frac{R}{\lambda} \sup \ |f|\Big). \tag{13.89}$$

Since $\gamma \geq \inf_{Q^+(R)} v$, this yields (13.84).

**Lemma 13.13.** *If $u$ satisfies (13.76) and (13.77) and $u \geq 0$ on $Q(2R)$, then*

$$\sup_{Q^+(R)} v \leq C\Big(\inf_{Q^+(R)} v + R \sup \ |f|\Big), \tag{13.90}$$

*with $C = C(n, \lambda, \Lambda, K)$.*

**Proof.** By (13.58), if $x \in Q^+(R),\ r = \delta R/8$, we have

$$\sup_{B_r(x)} u \leq C\Big(\inf_{B_r(x)} u + r^2 \sup \ |f|\Big). \tag{13.91}$$

Since $\delta R/2 \leq x_n \leq \delta R$ on $Q^+(R)$, (13.90) follows from this plus a simple covering argument.

We now prove Proposition 13.11. The various factors $C_j$ will all have the form $C_j = C_j(n, \lambda, \Lambda, K)$. If we apply (13.90), with $u$ replaced by

$u - m_{2R}x_n \geq 0$, on $Q(2R)$, we obtain

$$\sup_{Q^+(R)} (v - m_{2R}) \leq C_1 \Big( \inf_{Q^+(R)} (v - m_{2R}) + R \ \sup \ |f| \Big). \tag{13.92}$$

By Lemma 13.12, this is

$$\begin{aligned} &\leq C_2 \Big( \inf_{Q(R/2)} (v - m_{2R}) + R \ \sup \ |f| \Big) \\ &= C_2 \big( m_{R/2} - m_{2R} + R \ \sup \ |f| \big). \end{aligned} \tag{13.93}$$

Reasoning similarly, with $u$ replaced by $M_{2R}x_n - u \geq 0$ on $Q(2R)$, we have

$$\sup_{Q^+(R)} (M_{2R} - v) \leq C_2 \big( M_{2R} - M_{R/2} + R \ \sup \ |f| \big). \tag{13.94}$$

Summing these two inequalities yields

$$M_{2R} - m_{2R} \leq C_3 \big[ (M_{2R} - m_{2R}) - (M_{R/2} - m_{R/2}) + R \ \sup \ |f| \big], \tag{13.95}$$

which implies

$$\operatorname*{osc}_{Q(R/2)} v \leq \vartheta \operatorname*{osc}_{Q(2R)} v + R \ \sup \ |f|, \tag{13.96}$$

with $\vartheta = 1 - 1/C_3 < 1$. This readily implies the Hölder estimate (13.78), proving Proposition 13.11.

## Exercises

1. Prove the matrix inequality (13.8). (*Hint*: Set $C = A^{1/2} \geq 0$ and reduce (13.8) to

$$\frac{1}{n} \operatorname{Tr} X \geq (\det X)^{1/n}, \tag{13.97}$$

for $X = CBC \geq 0$. This is equivalent to the inequality

$$\frac{1}{n}(\lambda_1 + \cdots + \lambda_n) \geq (\lambda_1 \cdots \lambda_n)^{1/n}, \quad \lambda_j > 0, \tag{13.98}$$

which is called the *arithmetic-geometric mean inequality*. It can be deduced from the facts that $\log x$ is concave and that *any* concave function $\varphi$ satisfies

$$\varphi\Big(\frac{1}{n}(\lambda_1 + \cdots + \lambda_n)\Big) \geq \frac{1}{n}\Big[\varphi(\lambda_1) + \cdots + \varphi(\lambda_n)\Big].) \tag{13.99}$$

## 14. Regularity for a class of completely nonlinear equations

In this section we derive Hölder estimates on the second derivatives of real-valued solutions to nonlinear PDE of the form

$$F(x, D^2u) = 0, \tag{14.1}$$

satisfying the following conditions. First we require uniform strong ellipticity:

$$\lambda|\xi|^2 \le \partial_{\zeta_{jk}} F(x, u, \nabla u, \partial^2 u)\xi_j\xi_k \le \Lambda|\xi|^2, \tag{14.2}$$

with $\lambda, \Lambda \in (0, \infty)$, constants. Next, we require that $F$ be a *concave* function of $\zeta$:

$$\partial_{\zeta_{jk}}\partial_{\zeta_{\ell m}} F(x, u, p, \zeta)\Xi_{jk}\Xi_{\ell m} \le 0, \quad \Xi_{jk} = \Xi_{kj}, \tag{14.3}$$

provided $\zeta = \partial^2 u(x),\ p = \nabla u(x)$.

As an example, consider

$$F(x, u, p, \zeta) = \log\ \det \zeta - f(x, u, p). \tag{14.4}$$

Then $(D_\zeta F)\Xi = \mathrm{Tr}(\zeta^{-1}\Xi)$, so the quantity (14.3) is equal to

$$-\mathrm{Tr}\big(\zeta^{-1}\Xi\zeta^{-1}\Xi\big) = -\ \mathrm{Tr}\big(\zeta^{-1/2}\Xi\zeta^{-1}\Xi\zeta^{-1/2}\big), \quad \Xi^t = \Xi, \tag{14.5}$$

provided the real, symmetric, $n \times n$ matrix $\zeta$ is positive-definite, and $\zeta^{-1/2}$ is the positive-definite square root of $\zeta^{-1}$. Then the function (14.4) satisfies (14.3), on the region where $\zeta$ is positive-definite. It also satisfies (14.2) for $\partial^2 u(x) = \zeta \in \mathcal{K}$, any compact set of positive-definite, real, $n \times n$ matrices. In particular, if $\mathcal{F}$ is a bounded set in $C^2(\overline{\Omega})$ such that $(\partial_j\partial_k u)$ is positive-definite for each $u \in \mathcal{F}$, and (14.1) holds, with $|f(x, u, \nabla u)| \le C_0$, then (14.2) holds, uniformly for $u \in \mathcal{F}$.

We first establish interior estimates on solutions to (14.1). We will make use of results of §13 to establish these estimates, following [Ev], with simplifications of [GT]. To begin, let $\mu \in \mathbb{R}^n$ be a unit vector and apply $\partial_\mu$ to (14.1), to get

$$F_{\zeta_{ij}}\, \partial_i\partial_j\partial_\mu u + F_{p_i}\, \partial_i\partial_\mu u + F_u\, \partial_\mu u + \mu^i\, \partial_{x_i} F = 0. \tag{14.6}$$

Then apply $\partial_\mu$ again, to obtain

$$\begin{aligned} F_{\zeta_{ij}}\, \partial_i\partial_j\partial_\mu^2 u &+ (\partial_{\zeta_{ij}}\partial_{\zeta_{k\ell}} F)(\partial_i\partial_j\partial_\mu u)(\partial_k\partial_\ell\partial_\mu u) \\ &+ A_\mu^{ij}(x, D^2u)\, \partial_i\partial_j\partial_\mu u + B_\mu(x, D^2u) = 0, \end{aligned} \tag{14.7}$$

where

$$A_\mu^{ij}(x, D^2u) = 2(\partial_{\zeta_{ij}}\partial_{p_k} F)(\partial_k\partial_\mu u) + 2(\partial_{\zeta_{ij}}\partial_u F)(\partial_\mu u) + 2\mu^k(\partial_{\zeta_{ij}}\partial_{x_k} F),$$

and $B_\mu(x, D^2u)$ also involves first- and second-order derivatives of $F$.

Given the concavity of $F$, we have the differential inequality

$$F_{\zeta_{ij}}\, \partial_i\partial_j\partial_\mu^2 u \ge -A_\mu^{ij}\, \partial_i\partial_j\partial_\mu u - B_\mu, \tag{14.8}$$

where $A^{ij}_\mu = A^{ij}_\mu(x, D^2u),\ B_\mu = B_\mu(x, D^2u)$. If we set

$$h_\mu = \frac{1}{2}\Bigl(1 + \frac{\partial^2_\mu u}{1+M}\Bigr), \quad M = \sup_\Omega\ |\partial^2 u|, \tag{14.9}$$

then (14.8) implies

$$-F_{\zeta_{ij}}\,\partial_i\partial_j h_\mu \le \frac{C}{1+M}\bigl(A_0|\partial^3 u| + B_0\bigr), \tag{14.10}$$

where

$$A_0 = A_0\bigl(\|u\|_{C^2(\overline{\Omega})}\bigr), \quad B_0 = B_0\bigl(\|u\|_{C^2(\overline{\Omega})}\bigr). \tag{14.11}$$

Now let $\{\mu_k : 1 \le k \le N\}$ be a collection of unit vectors, and set

$$h_k = h_{\mu_k}, \quad v = \sum_{k=1}^N h_k^2. \tag{14.12}$$

Use $h_k$ in (14.10), multiply this by $h_k$, and sum over $k$, to obtain

$$\sum_{k=1}^N F_{\zeta_{ij}}(\partial_i h_k)(\partial_j h_k) - \frac{1}{2}F_{\zeta_{ij}}\,\partial_i\partial_j v \le \frac{C}{1+M}\bigl(A_0|\partial^3 u| + B_0\bigr). \tag{14.13}$$

Make sure that $\{\mu_k : 1 \le k \le N\}$ contains the set

$$\mathfrak{U} = \{e_j : 1 \le j \le n\} \cup \{2^{-1/2}(e_i \pm e_j) : 1 \le i < j \le n\}, \tag{14.14}$$

where $\{e_j\}$ is the standard basis of $\mathbb{R}^n$. Consequently,

$$|\partial^3 u|^2 = \sum_{i,j,\ell} |\partial_i\partial_j\partial_\ell u|^2 \le 4(1+M)^2 \sum_{k=1}^N |\partial h_k|^2. \tag{14.15}$$

The ellipticity condition (14.2) implies

$$\sum_{k=1}^N F_{\zeta_{ij}}(\partial_i h_k)(\partial_j h_k) \ge \lambda \sum_{k=1}^N |\partial h_k|^2. \tag{14.16}$$

Now, take $\varepsilon \in (0,1)$, and set

$$w_k = h_k + \varepsilon v. \tag{14.17}$$

We have

$$\varepsilon\lambda \sum_{k=1}^N |\partial h_k|^2 - \frac{1}{2}F_{\zeta_{ij}}\,\partial_i\partial_j w_k \le C\left\{A_0\Bigl(\sum_{k=1}^N |\partial h_k|^2\Bigr)^{1/2} + \frac{B_0}{1+M}\right\}. \tag{14.18}$$

Thus, by Cauchy's inequality,

$$F_{\zeta_{ij}}\partial_i\partial_j w_k \ge -\lambda\overline{\mu}, \quad \overline{\mu} = \frac{C_n}{\lambda}\Bigl(\frac{A_0^2}{\lambda\varepsilon} + \frac{B_0}{1+M}\Bigr). \tag{14.19}$$

We now prepare to apply Proposition 13.4. Let $B_R \subset B_{2R}$ be concentric balls in $\Omega$, and set

$$W_{ks} = \sup_{B_{sR}} w_k, \quad M_{ks} = \sup_{B_{sR}} h_k, \quad m_{ks} = \inf_{B_{sR}} h_k,$$
$$\omega(sR) = \sum_{k=1}^{N} \operatorname*{osc}_{B_{sR}} h_k = \sum_{k=1}^{N} (M_{ks} - m_{ks}). \tag{14.20}$$

Applying Proposition 13.4 to (14.19), we have

$$\left( \frac{1}{\text{vol } B_R} \int_{B_R} (W_{k2} - w_k)^p \, dx \right)^{1/p} \leq C(W_{k2} - W_{k1} + \overline{\mu} R^2), \tag{14.21}$$

where $p = p(n, \Lambda/\lambda) > 0$, $C = C(n, \Lambda/\lambda)$. Denote the left side of (14.21) by

$$\Phi_{p,R}(W_{k2} - w_k).$$

Note that

$$W_{k2} - w_k \geq M_{k2} - h_k - 2\varepsilon\omega(2R),$$
$$W_{k2} - W_{k1} \geq M_{k2} - M_{k1} + 2\varepsilon\omega(2R). \tag{14.22}$$

Hence

$$\Phi_{p,R}(M_{k2} - h_k) \leq C\{M_{k2} - M_{k1} + \varepsilon\omega(2R) + \overline{\mu} R^2\}. \tag{14.23}$$

Consequently,

$$\Phi_{p,R}\Big(\sum_k (M_{k2} - h_k)\Big) \leq N^{1/p} \sum_k \Phi_{p,R}(M_{k2} - h_k)$$
$$\leq \{(1 + \varepsilon)\omega(2R) - \omega(R) + \overline{\mu} R^2\}. \tag{14.24}$$

We want a complementary estimate on $\Phi_{p,R}(h_\ell - m_{\ell 2})$. We exploit the concavity of $F$ in $\zeta$ again to obtain

$$\begin{aligned} F_{\zeta_{ij}}\big(y, D^2u(y)\big)\big(\partial_i\partial_j u(y) - \partial_i\partial_j u(x)\big) & \\ \leq F\big(y, Du(y), \partial^2 u(x)\big) - F\big(y, Du(y), \partial^2 u(y)\big) & \\ = F\big(y, Du(y), \partial^2 u(x)\big) - F\big(x, Du(x), \partial^2 u(x)\big) & \\ \leq D_0 |x - y|, & \end{aligned} \tag{14.25}$$

where

$$D_0 = D_0\big(\|u\|_{C^2(\overline{\Omega})}\big). \tag{14.26}$$

The equality in (14.25) follows from $F(x, D^2u) = 0$. At this point, we impose a special condition on the unit vectors $\mu_k$ used to define $h_k$ above. The following is a result of [MW]:

**Lemma 14.1.** *Given* $0 < \lambda < \Lambda < \infty$, *let* $\mathcal{S}(\lambda, \Lambda)$ *denote the set of positive-definite, real,* $n \times n$ *matrices with spectrum in* $[\lambda, \Lambda]$. *Then there exist* $N \in \mathbb{Z}^+$ *and* $\lambda^* < \Lambda^*$ *in* $(0, \infty)$, *depending only on* $n, \lambda$, *and* $\Lambda$, *and unit vectors* $\mu_k \in \mathbb{R}^n$, $1 \le k \le N$, *such that*

$$\{\mu_k : 1 \le k \le N\} \supset \mathfrak{U}, \tag{14.27}$$

*where* $\mathfrak{U}$ *is defined by (14.14), and such that every* $A \in \mathcal{S}(\lambda, \Lambda)$ *can be written in the form*

$$A = \sum_{k=1}^{N} \beta_k P_{\mu_k}, \quad \beta_k \in [\lambda^*, \Lambda^*], \tag{14.28}$$

*where* $P_{\mu_k}$ *is the orthogonal projection of* $\mathbb{R}^n$ *onto the linear span of* $\mu_k$.

**Proof.** Let the set of real, symmetric, $n \times n$ matrices be denoted as $\mathrm{Symm}(n) \approx \mathbb{R}^{n(n+1)/2}$. Note that $A \in \mathrm{Symm}(n)$ belongs to $\mathcal{S}(\lambda, \Lambda)$ if and only if

$$\lambda |v|^2 \le v \cdot Av \le \Lambda |v|^2, \quad \forall\, v \in \mathbb{R}^n.$$

Thus $\mathcal{S}(\lambda, \Lambda)$ is seen to be a compact, convex subset of $\mathrm{Symm}(n)$. Also, $\mathcal{S}(\lambda, \Lambda)$ is contained in the *interior* of $\mathcal{S}(\lambda_1, \Lambda_1)$ if $0 < \lambda_1 < \lambda < \Lambda < \Lambda_1$.

It suffices to prove the lemma in the case $\Lambda = 1/2n$. Suppose $0 < \lambda < 1/2n$. By the spectral theorem for elements of $\mathrm{Symm}(n)$, $\mathcal{S}(\lambda/2, 1/2n)$ is contained in the interior of the convex hull $CH(\mathcal{P})$ of the set

$$\mathcal{P} = \{0\} \cup \{P_\mu : \mu \in S^{n-1} \subset \mathbb{R}^n\}.$$

Thus, there exists a *finite* subset $\mathfrak{A} \supset \mathfrak{U}$ of unit vectors such that $\mathcal{S}(\lambda/2, 1/2n)$ is contained in the interior of $CH(\mathcal{P}_0)$, with $\mathcal{P}_0 = \{0\} \cup \{P_\mu : \mu \in \mathfrak{A}\}$. Write $\mathfrak{A}$ as $\{\mu_k : 1 \le k \le N\}$. Then any element of $\mathcal{S}(\lambda/2, 1/2n)$ has a representation of the form $\sum_{k=1}^{N} \widetilde{\beta}_k P_{\mu_k}$, with $\widetilde{\beta}_k \in [0, 1]$.

Now, if we take $A \in \mathcal{S}(\lambda, 1/2n)$, it follows that

$$A - \sum_{k=1}^{N} \frac{\lambda}{2N} P_{\mu_k} \in \mathcal{S}\Big(\frac{\lambda}{2}, \frac{1}{2n}\Big),$$

so $A = \sum_{k=1}^{N} \big(\widetilde{\beta}_k + \lambda/2N\big) P_{\mu_k}$ has the form (14.28), with $\beta_k = \widetilde{\beta}_k + \lambda/2N \in \big[\lambda/2N, 2\big]$. This proves the lemma.

If we choose the set $\{\mu_k : 1 \le k \le N\}$ of unit vectors to satisfy the condition of Lemma 14.1, then

$$\begin{aligned} &F_{\zeta_{ij}}\big(y, D^2u(y)\big)\big(\partial_i\partial_j u(y) - \partial_i\partial_j u(x)\big) \\ &\quad = \sum_{k=1}^{N} \beta_k(y)\big(\partial^2_{\mu_k} u(y) - \partial^2_{\mu_k} u(x)\big) \\ &\quad = 2(1+M)\sum_{k=1}^{N} \beta_k(y)\big(h_k(y) - h_k(x)\big), \end{aligned} \tag{14.29}$$

with $\beta_k(y) \in [\lambda^*, \Lambda^*]$. Consequently, for $x \in B_{2R},\ y \in B_R$, we have from (14.25) that

$$\sum_{k=1}^{N} \beta_k(y)\big(h_k(y) - h_k(x)\big) \le C\lambda\widetilde{\mu}R, \quad \widetilde{\mu} = \frac{D_0}{\lambda(1+M)}. \tag{14.30}$$

Hence, for any $\ell \in \{1, \dots, N\}$,

$$\begin{aligned} h_\ell(y) - m_{\ell 2} &\le \frac{1}{\lambda^*}\Big\{C\lambda\widetilde{\mu}R + \Lambda^* \sum_{k\ne\ell}\big(M_{k2} - h_k(y)\big)\Big\} \\ &\le C\Big\{\widetilde{\mu}R + \sum_{k\ne\ell}\big(M_{k2} - h_k(y)\big)\Big\}, \end{aligned} \tag{14.31}$$

where $C = C(n, \Lambda/\lambda)$. We can use (14.24) to estimate the right side of (14.31), obtaining

$$\Phi_{p,R}(h_\ell - m_{\ell 2}) \le C\big\{(1+\varepsilon)\omega(2R) - \omega(R) + \widetilde{\mu}R + \overline{\mu}R^2\big\}. \tag{14.32}$$

Setting $\ell = k$, adding (14.32) to (14.23), and then summing over $k$, we obtain

$$\omega(2R) \le C\big\{(1+\varepsilon)\omega(2R) - \omega(R) + \widetilde{\mu}R + \overline{\mu}R^2\big\}, \tag{14.33}$$

and hence

$$\omega(R) \le \Big(1 - \frac{1}{C} + \varepsilon\Big)\omega(2R) + \big(\widetilde{\mu}R + \overline{\mu}R^2\big). \tag{14.34}$$

Now $C$ is independent of $\varepsilon$, though $\overline{\mu}$ is not. Thus fix $\varepsilon = 1/2C$, to obtain

$$\omega(R) \le \Big(1 - \frac{1}{2C}\Big)\omega(2R) + \big(\widetilde{\mu}R + \overline{\mu}R^2\big). \tag{14.35}$$

From this it follows that if $B_{2R_0} \subset \Omega$ and $R \le R_0$, we have

$$\operatorname*{osc}_{B_R} \partial^2 u \le C\Big(\frac{R}{R_0}\Big)^{\alpha}(1+M)\big(1 + \widetilde{\mu}R_0 + \overline{\mu}R_0^2\big), \tag{14.36}$$

where $C$ and $\alpha$ are positive constants depending only on $n$ and $\Lambda/\lambda$. We have proved the following interior estimate:

**Proposition 14.2.** *Let $u \in C^4(\overline{\Omega})$ satisfy (14.1), and assume that (14.2) and (14.3) hold. Then, for any $\mathcal{O} \subset\subset \Omega$, there is an estimate*

$$\|\partial^2 u\|_{C^\alpha(\mathcal{O})} \le C\big(\mathcal{O}, \Omega, n, \lambda, \Lambda, \|F\|_{C^2}, \|u\|_{C^2(\overline{\Omega})}\big). \tag{14.37}$$

In fact, examining the derivation of (14.36), we can specify the dependence on $\mathcal{O}, \Omega$. If $\mathcal{O}$ is a ball, and $|x - y| \ge \rho$ for all $x \in \mathcal{O},\ y \in \partial\Omega$, then

$$\|\partial^2 u\|_{C^\alpha(\mathcal{O})} \le C\big(n, \lambda, \Lambda, \|F\|_{C^2}, \|u\|_{C^2(\overline{\Omega})}\big)\rho^{-\alpha}. \tag{14.38}$$

We now tackle global estimates on $\overline{\Omega}$ for solutions to the Dirichlet problem for (14.1). We first obtain estimates for $\partial^2 u\big|_{\partial\Omega}$.

**Lemma 14.3.** *Under the hypotheses of Proposition 14.2, if $u\big|_{\partial\Omega} = \varphi$, there is an estimate*

$$\|\partial^2 u\|_{C^\alpha(\partial\Omega)} \le C\big(\Omega, n, \lambda, \Lambda, \|F\|_{C^2}, \|u\|_{C^2(\overline{\Omega})}, \|\varphi\|_{C^3(\partial\Omega)}\big). \tag{14.39}$$

**Proof.** Let $Y = b^\ell(x)\partial_\ell$ be a smooth vector field tangent to $\partial\Omega$, and consider $v = Yu$, which solves the boundary problem

$$F_{\zeta_{ij}} \partial_i\partial_j v = G(x), \quad v\big|_{\partial\Omega} = Y\varphi, \tag{14.40}$$

where

$$\begin{aligned} G(x) = 2F_{\zeta_{ij}}(\partial_i B^\ell)(\partial_j\partial_\ell u) + F_{\zeta_{ij}}(\partial_i\partial_j b^\ell)\partial_\ell u \\ + F_{p_i}(\partial_i b^\ell)(\partial_\ell u) - F_{p_i}\partial_i v - F_u v - b^\ell \partial_{x_\ell} F. \end{aligned} \tag{14.41}$$

The hypotheses give a bound on $\|G\|_{L^\infty(\Omega)}$ in terms of the right side of (14.39). If $\psi \in C^2(\overline{\Omega})$ denotes an extension of $Y\varphi$ from $\partial\Omega$ to $\overline{\Omega}$, then Proposition 13.11, applied to $v - \psi$, yields an estimate

$$\|\partial_\nu Yu\|_{C^\alpha(\partial\Omega)} \le C, \tag{14.42}$$

where $C$ is of the form (14.39). On the other hand, the ellipticity of (14.1) allows one to solve for $\partial_\nu^2 u\big|_{\partial\Omega}$ in terms of quantities estimated in (14.42), plus $u\big|_{\partial\Omega}$ and $\nabla u\big|_{\partial\Omega}$, and second-order tangential derivatives of $u$, so (14.39) is proved.

We now want to estimate $|\partial_\gamma^2 u(x) - \partial_\gamma^2 u(x_0)|$, given $x_0 \in \partial\Omega,\ x \in \overline{\Omega},\ \gamma \in \mathbb{R}^n$ a unit vector. For simplicity, we will strengthen the concavity hypothesis (14.3) to *strong concavity*:

$$\partial_{\zeta_{jk}}\partial_{\zeta_{\ell m}} F(x, u, p, \zeta)\Xi_{jk}\Xi_{\ell m} \le -\lambda_0|\Xi|^2, \quad \Xi = \Xi^t, \tag{14.43}$$

for some $\lambda_0 > 0$, when $\zeta = \partial^2 u,\ p = \nabla u$. Then we can improve (14.8) to

$$F_{\zeta_{ij}}\, \partial_i\partial_j(\partial_\gamma^2 u) \le -A_\gamma^{ij}\, \partial_i\partial_j\partial_\gamma u - B_\gamma - \lambda_0|\partial^2\partial_\gamma u|^2 \le -C_1, \tag{14.44}$$

by Cauchy's inequality, where

$$C_1 = C_1\big(n, \lambda, \Lambda, \lambda_0, \|A_\gamma(x, D^2u)\|_{L^\infty}, \|B_\gamma(x, D^2u)\|_{L^\infty}\big).$$

Now the function

$$W(x) = C_2|x - x_0|^\alpha \quad (0 < \alpha < 1) \tag{14.45}$$

is concave on $\mathbb{R}^n \setminus \{x_0\}$, and if $C_2$ is sufficiently large, compared to $C_1 \cdot \operatorname{diam}(\Omega)^{2-\alpha}/\lambda$, we have

$$LW \le -C_1, \quad Lv = F_{\zeta_{ij}}\, \partial_i\partial_j v. \tag{14.46}$$

Hence, by the maximum principle,

$$\partial_\gamma^2 u \le \partial_\gamma^2(x_0) + W \text{ on } \partial\Omega \Longrightarrow \partial_\gamma^2 u \le \partial_\gamma^2 u(x_0) + W \text{ on } \Omega. \tag{14.47}$$

Now the estimate (14.39) implies that the hypothesis of (14.47) is satisfied, provided that also $C_2 \ge \|\partial^2 u\|_{C^\alpha(\partial\Omega)}$, so we have the one-sided estimate given by the conclusion of (14.47).

For the reverse estimate, use (14.25), with $y = x_0$, together with (14.29), to write

$$\sum_{k=1}^N \beta_k(x_0)\big(\partial_{\mu_k}^2 u(x_0) - \partial_{\mu_k}^2 u(x)\big) \le D_0|x - x_0|. \tag{14.48}$$

Recall that $\beta_k(x_0) \in [\lambda^*, \Lambda^*],\ \lambda^* > 0$. This together with (14.47) implies

$$|\partial_{\mu_k}^2 u(x) - \partial_{\mu_k}^2 u(x_0)| \le C_3|x - x_0|^\alpha, \tag{14.49}$$

with $C_3$ of the form (14.39), and we can express any $\partial_j\partial_\ell u$ as a linear combination of the $\partial_{\mu_k}^2 u$, to obtain the following:

**Lemma 14.4.** *If we have the hypotheses of Lemma 14.3, and we also assume (14.43), then there is an estimate*

$$|\partial^2 u(x) - \partial^2 u(x_0)| \le C|x - x_0|^\alpha, \quad x_0 \in \partial\Omega, \quad x \in \overline{\Omega}, \tag{14.50}$$

*with*

$$C = C\big(\Omega, n, \lambda, \Lambda, \lambda_0, \|F\|_{C^2}, \|u\|_{C^2(\overline{\Omega})}, \|\varphi\|_{C^3(\partial\Omega)}\big). \tag{14.51}$$

We now put (14.38) and (14.50) together to obtain a Hölder estimate for $\partial^2 u$ on $\overline{\Omega}$. To estimate $|\partial^2 u(x) - \partial^2 u(y)|$, given $x,\ y \in \overline{\Omega}$, suppose $\operatorname{dist}(x, \partial\Omega) + \operatorname{dist}(y, \partial\Omega) = 2\rho$, and consider two cases:

(i) $|x - y| < \rho^2$,
(ii) $|x - y| \ge \rho^2$.

In case (i), we can use (14.38) to deduce that

$$|\partial^2 u(x) - \partial^2 u(y)| \le C|x - y|^\alpha \rho^{-\alpha} \le C|x - y|^{\alpha/2}. \tag{14.52}$$

In case (ii), let $x' \in \partial\Omega$ minimize the distance from $x$ to $\partial\Omega$, and let $y' \in \partial\Omega$ minimize the distance from $y$ to $\partial\Omega$. Thus

$$\begin{aligned} |x - x'| \le 2\rho \le 2|x-y|^{1/2}, \qquad |y - y'| \le 2\rho \le 2|x-y|^{1/2}, \\ |x'-y'| \le |x-y| + |x'-x| + |y'-y| \le |x-y| + 4|x-y|^{1/2}. \end{aligned} \tag{14.53}$$

Thus

$$\begin{aligned} |\partial^2 u(x) - \partial^2 u(y)| &\le |\partial^2 u(x) - \partial^2 u(x')| + |\partial^2 u(x') - \partial^2 u(y')| \\ &\quad + |\partial^2 u(y') - \partial^2 u(y)| \\ &\le \widetilde{C}|x-x'|^\alpha + \widetilde{C}|x'-y'|^\alpha + \widetilde{C}|y'-y|^\alpha \\ &\le C|x-y|^{\alpha/2}. \end{aligned} \tag{14.54}$$

In (14.52) and (14.54), $C$ has the form (14.51). Taking $r = \alpha/2$, we have the following global estimate:

**Proposition 14.5.** *Let $u \in C^4(\overline{\Omega})$ satisfy (14.1), with $u\big|_{\partial\Omega} = \varphi$. Assume the ellipticity hypothesis (14.2) and the strong concavity hypothesis (14.43). Then there is an estimate*

$$\|u\|_{C^{2+r}(\overline{\Omega})} \le C\big(\Omega, n, \lambda, \Lambda, \lambda_0, \|F\|_{C^2}, \|u\|_{C^2(\overline{\Omega})}, \|\varphi\|_{C^3(\partial\Omega)}\big), \tag{14.55}$$

*for some $r > 0$, depending on the same quantities as $C$.*

Now that we have this estimate, the continuity method yields the following existence result. For $\tau \in [0,1]$, consider a family of boundary problems

$$F_\tau(x, D^2 u) = 0 \ \text{ on } \ \Omega, \quad u\big|_{\partial\Omega} = \varphi_\tau. \tag{14.56}$$

Assume $F_\tau$ and $\varphi_\tau$ are smooth in all variables, including $\tau$. Also, assume that the ellipticity condition (14.2) and the strong concavity condition (14.43) hold, uniformly in $\tau$, for any smooth solution $u_\tau$.

**Theorem 14.6.** *Assume there is a uniform bound in $C^2(\overline{\Omega})$ for any solution $u_\tau \in C^\infty(\overline{\Omega})$ of (14.56). Also assume that $\partial_u F_\tau \le 0$. Then, if (14.56) has a solution in $C^\infty(\overline{\Omega})$ for $\tau = 0$, it has a smooth solution for $\tau = 1$.*

With some more work, one can replace the strong concavity hypothesis (14.43) by (14.3); see [CKNS].

There is an interesting class of elliptic PDE, known as *Bellman equations*, for which $F(x, u, p, \zeta)$ is concave but not strongly concave in $\zeta$, and also it is Lipschitz but not $C^\infty$ in its arguments; see [Ev2] for an analysis.

Verifying the hypothesis in Theorem 14.6 that $u_\tau$ is bounded in $C^2(\overline{\Omega})$ can be a nontrivial task. We will tackle this, for Monge-Ampere equations, in the next section.

## Exercises

1. Discuss the Dirichlet problem for

$$\Delta u + \partial_1^2 u + \frac{1}{2}\Big(1 + (\Delta u)^2\Big)^{1/2} = \sigma e^u,$$

for $\sigma \geq 0$.

# 15. Monge-Ampere equations

Here we look at equations of Monge-Ampere type:

$$\det H(u) - F(x, u, \nabla u) = 0 \text{ on } \Omega, \quad u = \varphi \text{ on } \partial\Omega, \tag{15.1}$$

where $\Omega$ is a smoothly bounded domain in $\mathbb{R}^n$, which we will assume to be strongly convex. As in (3.7a), $H(u) = (\partial_j \partial_k u)$ is the Hessian matrix. We assume $F(x, u, \nabla u) > 0$, say $F(x, u, \nabla u) = \exp f(x, u, \nabla u)$, and look for a convex solution to (15.1). It is convenient to set

$$G(u) = \log \det H(u) - f(x, u, \nabla u), \tag{15.2}$$

so (15.1) is equivalent to $G(u) = 0$ on $\Omega$, $u = \varphi$ on $\partial\Omega$. Note that

$$DG(u)v = g^{jk}\, \partial_j \partial_k v - (\partial_{p_j} f)(x, u, \nabla u)\, \partial_j v - (\partial_u f)(x, u, \nabla u) v, \tag{15.3}$$

where $(g^{jk})$ is the inverse matrix of $(\partial_j \partial_k u)$, which we will also denote as $(g_{jk})$. We will assume

$$(\partial_u f)(x, u, p) \geq 0, \tag{15.4}$$

this hypothesis being equivalent to $(\partial_u F)(x, u, p) \geq 0$.

The hypotheses made above do not suffice to guarantee that (15.1) has a solution. Consider the following example:

$$\det H(u) - K\big(1 + |\nabla u|^2\big)^2 = 0 \text{ on } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{15.5}$$

where $\Omega$ is a domain in $\mathbb{R}^2$. Compare with (3.41). Let $K$ be a positive constant. If there is a convex solution $u$, the surface $\Sigma = \{(x, u(x)) : x \in \Omega\}$ is a surface in $\mathbb{R}^3$ with Gauss curvature $K$. If $\Omega$ is convex, then the Gauss map $N : \Sigma \to S^2$ is one-to-one and the image $N(\Sigma)$ has area equal to $K \cdot \text{Area}(\Omega)$. But $N(\Sigma)$ must be contained in a hemisphere of $S^2$, so we must have $K \cdot \text{Area}(\Omega) \leq 2\pi$. We deduce that if $K \cdot \text{Area}(\Omega) > 2\pi$, then (15.5) has no solution.

To avoid this obstruction to existence, we hypothesize that there exists $u^b \in C^\infty(\overline{\Omega})$, which is convex and satisfies

$$\log \det H(u^b) - f(x, u^b, \nabla u^b) \geq 0 \text{ on } \Omega, \quad u^b = \varphi \text{ on } \partial\Omega. \tag{15.6}$$

We call $u^b$ a *lower solution* to (15.1). Note that the first part of (15.6) is equivalent to $\det H(u^b) \geq F(x, u^b, \nabla u^b)$. In such a case, we will use the method of continuity and seek a convex $u_\sigma \in C^\infty(\overline{\Omega})$ solving

$$\begin{aligned} &\log\ \det H(u_\sigma) - f(x, u_\sigma, \nabla u_\sigma) \\ &= (1-\sigma)\big[\log\ \det H(u^b) - f(x, u^b, \nabla u^b)\big] \\ &= (1-\sigma)h(x), \end{aligned} \tag{15.7}$$

for $\sigma \in [0,1]$ and $u_\sigma = \varphi$ on $\partial\Omega$. Note that $u_0 = u^b$ solves (15.7) for $\sigma = 0$. If such $u_\sigma$ exists for all $\sigma \in [0,1]$, then $u = u_1$ is the desired solution to (15.1).

Let $J$ be the largest interval in $[0,1]$, containing 0, such that (15.7) has a convex solution $u_\sigma \in C^\infty(\overline{\Omega})$ for all $\sigma \in J$. Since the linear operator in (15.3) is elliptic and invertible (by the maximum principle) under the hypothesis (15.4), the same sort of argument used in the proof of Lemma 10.1 shows that $J$ is open, and the real work is to show that $J$ is closed. In this case, we need to obtain bounds on $u_\sigma$ in $C^{2+\mu}(\overline{\Omega})$, for some $\mu > 0$, in order to apply the regularity theory of §8 and conclude that $J$ is closed.

**Lemma 15.1.** *Given $\sigma \le \tau \in J$, we have*

$$u^b \le u_\sigma \le u_\tau \quad \text{on } \Omega. \tag{15.8}$$

**Proof.** The operator $G(u)$ satisfies the hypotheses of Proposition 10.8; since $u^b = u_\sigma = u_\tau$ on $\partial\Omega$, (15.8) follows.

In particular, taking $\sigma = \tau$, we have *uniqueness* of the solution $u_\sigma \in C^\infty(\overline{\Omega})$ to (15.7).

Next we record some estimates that are simple consequences of convexity alone:

**Lemma 15.2.** *Assume $\Omega$ is convex. For any $\sigma \in J$,*

$$u_\sigma \le \sup_{\partial\Omega} \varphi \quad \text{on } \Omega \tag{15.9}$$

*and*

$$\sup_{x\in\Omega} |\nabla u_\sigma(x)| \le \sup_{y\in\partial\Omega} |\nabla u_\sigma(y)|. \tag{15.10}$$

Thus we will have a bound on $u_\sigma$ in $C^1(\overline{\Omega})$ if we bound $\nabla u_\sigma$ on $\partial\Omega$. Since $u_\sigma\big|_{\partial\Omega} = \varphi \in C^\infty(\partial\Omega)$, it remains to bound the normal derivative $\partial_\nu u_\sigma$ on $\partial\Omega$. Assume $\partial_\nu$ points out of $\Omega$. Then (15.8) implies

$$\partial_\nu u_\sigma(y) \le \partial_\nu u^b(y), \quad \forall\ y \in \partial\Omega. \tag{15.11}$$

On the other hand, a lower bound on $\partial_\nu u_\sigma(y)$ follows from convexity alone. In fact, if $\nu(y)$ is the outward normal to $\partial\Omega$ at $y$, say $\widetilde{y} = y - \ell(y)\nu(y)$ is the other point in $\partial\Omega$ through which the normal line passes. Then convexity of $u_\sigma$ implies

$$u_\sigma\big(sy + (1-s)\widetilde{y}\big) \le s\varphi(y) + (1-s)\varphi(\widetilde{y}), \tag{15.12}$$

for $0 \le s \le 1$. Noting that $\ell(y) = |y - \widetilde{y}|$, we have

$$\partial_\nu u_\sigma(y) \ge \frac{\varphi(\widetilde{y}) - \varphi(y)}{|\widetilde{y} - y|}.$$

Thus we have the next result:

**Lemma 15.3.** *If $\Omega$ is convex, then, for any $\sigma \in J$,*

$$\sup_{\overline{\Omega}} |\nabla u_\sigma| \le \mathit{Lip}^1(\varphi) + \sup_{\overline{\Omega}} |\nabla u^b|. \tag{15.13}$$

Here, $\mathrm{Lip}^1(\varphi)$ denotes the Lipschitz constant of $\varphi$:

$$\mathrm{Lip}^1(\varphi) = \sup_{y,y' \in \partial\Omega} \frac{|\varphi(y) - \varphi(y')|}{|y - y'|}. \tag{15.14}$$

We now look for $C^2$-bounds on solutions to (15.7). For notational simplicity, we write (15.7) as

$$\log\, \det H(u) - f(x, u, \nabla u) = 0, \quad u\big|_{\partial\Omega} = \varphi, \tag{15.15}$$

where the second term on the left is

$$f_\sigma(x, u, \nabla u) = f(x, u, \nabla u) + (1-\sigma)h(x),$$

and we drop the $\sigma$. By (15.4) and (15.6), we have $f(x, u, p) > 0$ and $(\partial_u f)(x, u, p) \le 0$.

Since $u$ is convex, it suffices to estimate pure second derivatives $\partial_\gamma^2 u$ from above. Following [CNS], who followed [LiP2], we make use of the function

$$w = e^{\beta|\nabla u|^2/2}\, \partial_\gamma^2 u,$$

where $\beta$ is a constant that will be chosen later. Suppose this is maximized, among all unit $\gamma \in \mathbb{R}^n$, $x \in \overline{\Omega}$, at $\gamma = \gamma_0$, $x = x_0$. Rotating coordinates, we can assume $\big(g_{jk}(x_0)\big) = \big(\partial_j\partial_k u(x_0)\big)$ is in diagonal form and $\gamma_0 = (1, 0, \dots, 0)$. Set $u_{11} = \partial_1^2 u$, so we take

$$w = e^{\beta|\nabla u|^2/2} u_{11} = \psi(\nabla u) u_{11}. \tag{15.16}$$

We now derive some identities and inequalities valid on all of $\Omega$.

Differentiating (15.15), we obtain

$$\begin{aligned} &g^{ij}\, \partial_i\partial_j\partial_\ell u = \partial_\ell f(x, u, \nabla u), \\ &g^{ij}\, \partial_i\partial_j u_{11} = g^{i\ell} g^{jm} (\partial_i\partial_j\partial_1 u)(\partial_k\partial_m\partial_1 u) + \partial_1^2 f, \end{aligned} \tag{15.17}$$

where $(g^{ij})$ is the inverse matrix to $(g_{ij}) = (\partial_i\partial_j u)$, as above. Also, a calculation gives

(15.18)
$$\begin{aligned} w^{-1}\,\partial_i w &= (\log\psi)_{p_k}\,\partial_i\partial_k u + u_{11}^{-1}(\partial_i\partial_1^2 u),\\ w^{-1}\,\partial_i\partial_j w &= w^{-2}(\partial_i w)(\partial_j w) + (\log\psi)_{p_k p_\ell}(\partial_i\partial_k u)(\partial_j\partial_\ell u)\\ &\quad + (\log\psi)_{p_k}(\partial_i\partial_j\partial_k u) + u_{11}^{-1}\,\partial_i\partial_j u_{11} - u_{11}^{-2}(\partial_i\partial_1^2 u)(\partial_j\partial_1^2 u). \end{aligned}$$

Forming $w^{-1}\,g^{ij}\partial_i\partial_j w$ and using (15.17) to rewrite the term $u_{11}^{-1}g^{ij}\,\partial_i\partial_j u_{11}$, we obtain

$$\begin{aligned} &\psi^{-1}g^{ij}\,\partial_i\partial_j w\\ &\ge u_{11}\Big[(\log\psi)_{p_k p_\ell}g^{ij}(\partial_i\partial_k u)(\partial_j\partial_\ell u) + (\log\psi)_{p_k}g^{ij}\,\partial_i\partial_j\partial_k u\Big]\\ &\quad + g^{ik}\,g^{i\ell}(\partial_i\partial_j\partial_1 u)(\partial_k\partial_\ell\partial_1 u) - u_{11}^{-1}g^{ij}(\partial_i\partial_1^2 u)(\partial_j\partial_1^2 u) + \partial_1^2 f. \end{aligned} \tag{15.19}$$

Now we have $(\log\psi)_{p_k} = \beta p_k$ and $(\log\psi)_{p_k p_\ell} = \beta\delta^{k\ell}$, and hence

$$(\log\psi)_{p_k p_\ell}g^{ij}(\partial_i\partial_k u)(\partial_j\partial_\ell u) = \beta\delta^{k\ell}\,\delta^j{}_k(\partial_j\partial_\ell u) = \beta\Delta u. \tag{15.20}$$

Let us assume the following bounds hold on $f(x,u,p)$:

$$|(\nabla f)(x,u,p)| \le \mu, \quad |(\partial^2 f)(x,u,p)| \le \mu. \tag{15.21}$$

Using the first identity in (15.17), we have

$$\begin{aligned} &u_{11}(\log\psi)_{p_k}g^{ij}\,\partial_i\partial_j\partial_k u + \partial_1^2 f\\ &\ge f_{p_i}(w^{-1}\partial_i w)u_{11} - C\big[1 + |\partial^2 u|^2 + \beta(1+|\partial^2 u|)\big], \end{aligned} \tag{15.22}$$

with $C = C\big(\mu, \|\nabla u\|_{L^\infty(\Omega)}\big)$.

Now, let us look at $x_0$, where, recall, $e^{\beta|\nabla u|^2/2}\partial_1^2 u$ is maximal, among all values of $e^{\beta|\nabla u(x)|^2/2}\partial_\gamma^2 u(x)$. If $x_0 \in \Omega$ (i.e., $x_0 \notin \partial\Omega$), then $\partial_i w(x_0) = 0$ and the left side of (15.19) is $\le 0$ at $x_0$. Furthermore, due to the diagonal nature of $(g^{ij})$ at $x_0$, we easily verify that $g^{11}g^{ij}\zeta_{i1}\zeta_{j1} \le g^{ij}g^{k\ell}\zeta_{ik}\zeta_{j\ell}$, and hence

$$u_{11}^{-1}g^{ij}(\partial_i\partial_1^2 u)(\partial_j\partial_1^2 u) \le g^{ik}g^{j\ell}(\partial_i\partial_j\partial_1 u)(\partial_k\partial_\ell\partial_1 u), \tag{15.23}$$

at $x_0$. Thus the evaluation of (15.19) at $x_0$ implies the estimate

$$0 \ge \beta(\partial_1^2 u)(\Delta u) - \mu - C\big[1 + |\partial^2 u|^2 + \beta(1+|\partial^2 u|)\big] \tag{15.24}$$

if $x_0 \notin \partial\Omega$. Hence, with $X = \partial_1^2 u(x_0)$,

$$(\beta - C_1)X^2 \le \beta C_2(1+X) + \mu, \tag{15.25}$$

where $C_1$ and $C_2$ depend on $\mu$ and $\|\nabla u\|_{L^\infty}$, but not on $\beta$. Taking $\beta$ large, we obtain a bound on $X$:

$$\partial_1^2 u(x_0) \le C\big(\mu, \|\nabla u\|_{L^\infty(\Omega)}\big) \quad \text{if } x_0 \notin \partial\Omega. \tag{15.26}$$

On the other hand, if sup $w$ is achieved on $\partial\Omega$, we have

$$\sup_{x,\gamma} |\partial_\gamma^2 u(x)| \le \sup_{\partial\Omega} |\partial^2 u| \cdot \exp\bigl(\beta \|\nabla u\|_{L^\infty}\bigr).$$

This establishes the following.

**Lemma 15.4.** *If $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ solves (15.15) and the hypotheses above hold, then*

$$\sup_{\overline{\Omega}} |\partial^2 u| \le C\bigl(\mu, \|\nabla u\|_{L^\infty(\Omega)}\bigr) \Bigl[1 + \sup_{\partial\Omega} |\partial^2 u|\Bigr]. \tag{15.27}$$

To estimate $\partial^2 u$ at a boundary point $y \in \partial\Omega$, suppose coordinates are rotated so that $\nu(y)$ is parallel to the $x_n$-axis. Pick vector fields $Y_j$, tangent to $\partial\Omega$, so that $Y_j(y) = \partial_j,\ 1 \le j \le n-1$. Then we easily get

$$|\partial_j \partial_k u(y)| \le |Y_j Y_k \varphi(y)| + C|\nabla u(y)|, \quad 1 \le j,\ k \le n-1. \tag{15.28}$$

In fact, for later reference, we note the following. Suppose $Y_j$ is the vector field tangent to $\partial\Omega$, equal to $\partial_j$ at $y$, and obtained by parallel transport along geodesics emanating from $y$. If $Y_k = b_k^\ell \partial_\ell$, then

$$\begin{aligned} Y_j Y_k u(y) &= \partial_j \partial_k u(y) + \bigl(\partial_j b_k^\ell(y)\bigr)\, \partial_\ell u(y) \\ &= \partial_j \partial_k u(y) + \bigl(\nabla^0_{\partial_j} Y_k\bigr) u(y), \end{aligned} \tag{15.29}$$

where $\nabla^0$ is the standard flat connection on $\mathbb{R}^n$. If $\nabla$ is the Levi-Civita connection on $\partial\Omega$, we have $\nabla_{\partial_j} Y_k = 0$ at $y$, hence $\nabla^0_{\partial_j} Y_k = -\widetilde{II}(\partial_j, \partial_k)\, \partial_\nu$ at $y$, where $\partial_\nu = -N$ is the outward-pointing normal and $\widetilde{II}$ is the second fundamental form of $\partial\Omega$; see §4 of Appendix C. Hence

$$\partial_j \partial_k u(y) = Y_j Y_k u(y) + \widetilde{II}(\partial_j, \partial_k)\, \partial_\nu u(y), \quad 1 \le j,\ k \le n-1. \tag{15.30}$$

Later it will be important to note that strong convexity of $\partial\Omega$ implies positive definiteness of $\widetilde{II}$.

We next need to estimate $\partial_n Y_k u(y),\ 1 \le k \le n-1$. If $Y_k = b_k^\ell(x)\, \partial_\ell$, then $v_k = Y_k u$ satisfies the equation

$$g^{ij}\, \partial_i \partial_j v_k - f_{p_i}\, \partial_i v_k = A(x) + g^{ij} B_{ij}(x), \tag{15.31}$$

where

$$\begin{aligned} A(x) &= 2\partial_i b_k^i + f_{x_\ell} b_k^\ell + f_u v_k + f_{p_i} (\partial_i b_k^\ell)\, \partial_\ell u, \\ B_{ij}(x) &= (\partial_i \partial_j b_k^\ell)\, \partial_\ell u, \end{aligned} \tag{15.32}$$

and $v_k\big|_{\partial\Omega} = Y_k \varphi$. This follows by multiplying the first identity in (15.17) by $b_k^\ell$ and summing over $\ell$; one also makes use of the identity $g^{ij}\, \partial_j \partial_\ell u = \delta^i{}_\ell$.

We first derive a boundary gradient estimate for $v_k = Y_k u$ when (15.15) takes the simpler form

$$\log\ \det H(u) - f(x, u) = 0, \quad u\big|_{\partial\Omega} = \varphi; \tag{15.33}$$

that is, $\nabla u$ is not an argument of $f$. Here, we follow [Au]. We assume $\varphi \in C^\infty(\overline{\Omega})$, set

$$w_k = Y_k(u - \varphi) = v_k - Y_k\varphi, \tag{15.34}$$

then let $\alpha$ and $\beta$ be real numbers, to be fixed below, and set

$$\widetilde{w}_k = w_k + \alpha h + \beta(u - \varphi). \tag{15.35}$$

Here, $h \in C^\infty(\overline{\Omega})$ is picked to vanish on $\partial\Omega$ and satisfy a strong convexity condition:

$$(\partial_i\partial_j h) \geq I, \quad h\big|_{\partial\Omega} = 0. \tag{15.36}$$

The hypothesis that $\overline{\Omega}$ is strongly convex is equivalent to the existence of such a function.

Now, a calculation using (15.31) (and noting that in this case $f_{p_i} = 0$) gives

$$g^{ij}\,\partial_i\partial_j\widetilde{w}_k = A(x) + n\beta + g^{ij}\widetilde{B}_{ij}(x), \quad \widetilde{w}_k\big|_{\partial\Omega} = 0, \tag{15.37}$$

where $A(x)$ is as in (15.32) (with the last term equal to zero), and

$$\widetilde{B}_{ij}(x) = B_{ij}(x) - \partial_i\partial_j Y_k\varphi + \alpha\,\partial_i\partial_j h - \beta\,\partial_i\partial_j\varphi. \tag{15.38}$$

We now choose $\alpha$ and $\beta$. Pick $\beta = \beta_0$, so large that $A(x) + n\beta_0 \geq 0$. This done, pick $\alpha = \alpha_0$, so large that $(\widetilde{B}_{ij}) \geq 0$. Then $\widetilde{w}_{k0}$, defined by (15.34) with $\alpha = \alpha_0$, $\beta = \beta_0$, satisfies

$$g^{ij}\,\partial_i\partial_j\widetilde{w}_{k0} \geq 0, \quad \widetilde{w}_{k0}\big|_{\partial\Omega} = 0. \tag{15.39}$$

Similarly, pick $\beta = \beta_1$ sufficiently negative that $A(x) + n\beta_1 \leq 0$, and then pick $\alpha = \alpha_1$ sufficiently negative that $(\widetilde{B}_{ij}) \leq 0$. Then, $\widetilde{w}_{k1}$, defined by (15.35) with $\alpha = \alpha_1$ and $\beta = \beta_1$, satisfies

$$g^{ij}\partial_i\partial_j\widetilde{w}_{k1} \leq 0, \quad \widetilde{w}_{k1}\big|_{\partial\Omega} = 0. \tag{15.40}$$

The maximum principle implies $\widetilde{w}_{k0} \leq 0$ and $\widetilde{w}_{k1} \geq 0$; hence

$$Y_k\varphi - \alpha_1 h - \beta_1(u - \varphi) \leq Y_k u \leq Y_k\varphi - \alpha_0 h - \beta_0(u - \varphi). \tag{15.41}$$

Thus, if $\partial_\nu$ denotes the normal derivative at $\partial\Omega$,

$$|\partial_\nu Y_k u| \leq (\alpha_0 - \alpha_1)|\partial_\nu h| + (\beta_0 - \beta_1)|\partial_\nu u - \partial_\nu\varphi| + |\partial_\nu Y_k\varphi|, \tag{15.42}$$

when $u$ solves (15.33).

In view of the example (15.5), for a surface with Gauss curvature $K$, we have ample motivation to estimate the normal derivative of $Y_k u$ when $u$ solves the more general equation (15.15). We now tackle this, following [CNS].

Generally, if $w_k = Y_k(u - \varphi)$, (15.31) yields

$$\begin{aligned} &g^{ij}\,\partial_i\partial_j w_k - f_{p_i}\,\partial_i w_k \\ &\qquad = \big[A(x) + f_{p_i}\partial_i Y_k\varphi\big] + g^{ij}\big[B_{ij}(x) - \partial_i\partial_j Y_k\varphi\big] = \Phi(x). \end{aligned} \tag{15.43}$$

Note that, given a bound for $u$ in $C^1(\overline{\Omega})$, we have

$$|\Phi(x)| \le C + Cg^{jj}, \tag{15.44}$$

where $g^{jj}$ is the trace of $(g^{ij})$.

Translate coordinates so that $y = 0$. Recall that we assume $\nu(y)$ is parallel to the $x_n$-axis. Assume $x_n \ge 0$ on $\overline{\Omega}$. As above, assume $h \in C^\infty(\overline{\Omega})$ satisfies (15.36). Take $\mu \in (0, 1/4)$ and $M \in (0,\infty)$, and set $h_\mu(x) = h(x) - \mu|x|^2$. We have

$$\begin{aligned}
&(g^{ij}\,\partial_i\partial_j - f_{p_i}\,\partial_i)(h_\mu + Mx_n^2) \\
&\quad = g^{ij}\,\partial_i\partial_j h_\mu - f_{p_i}\,\partial_i h_\mu + 2Mg^{nn} - 2Mf_{p_n}x_n \\
&\quad \ge \Bigl(\frac{1}{2}g^{jj} + 2Mg^{nn}\Bigr) - \bigl(Mf_{p_n}x_n + f_{p_i}\,\partial_i h_\mu\bigr).
\end{aligned} \tag{15.45}$$

The arithmetic-geometric mean inequality implies

$$\bigl(M\sigma_1\cdots\sigma_n\bigr)^{1/n} \le \frac{1}{n}\Bigl(\sum_{j<n}\sigma_j + M\sigma_n\Bigr),$$

and if the eigenvalues of $(g^{ij})$ are $\sigma_n \le \cdots \le \sigma_1$, we have $g^{nn} \ge \sigma_n$, and hence

$$\bigl[M\,\det(g^{ij})\bigr]^{1/n} \le \frac{1}{n}\bigl(g^{jj} + Mg^{nn}\bigr). \tag{15.46}$$

Given a positive lower bound on $\det(g^{ij}) = 1/F(x,u,\nabla u)$, we have

$$\frac{1}{2}g^{jj} + 2Mg^{nn} \ge cg^{jj} + c_1M^{1/n}. \tag{15.47}$$

Hence (15.45) implies

$$(g^{ij}\,\partial_i\partial_j - f_{p_i}\,\partial_i)(h_\mu + Mx_n^2) \ge cg^{jj} + c_1M^{1/n} - c_2 - c_3Mx_n. \tag{15.48}$$

At this point, fix $M$ sufficiently large that $c_1M^{1/n} \ge 1 + c_2$, so that

$$(g^{ij}\,\partial_i\partial_j - f_{p_i}\,\partial_i)(h_\mu + Mx_n^2) \ge 1 + cg^{jj} - c_3Mx_n \quad \text{on } \Omega. \tag{15.49}$$

Now, let

$$\mathcal{O}_\varepsilon = \{x \in \Omega : 0 < x_n < \varepsilon\},$$

as illustrated in Fig. 15.1. We can then pick $\varepsilon$ sufficiently small that (e.g., with $\mu = 1/8$)

$$(g^{ij}\,\partial_i\partial_j - f_{p_i}\,\partial_i)(h_\mu + Mx_n^2) \ge cg^{jj} + \frac{1}{2} \quad \text{on } \mathcal{O}_\varepsilon. \tag{15.50}$$

Note that the function $h$ has the property $\nabla h \ne 0$ on $\partial\Omega$. Thus, after possibly further shrinking $\varepsilon$, we have

$$h_\mu + Mx_n^2 \le \begin{cases} 0 & \text{on } \partial\mathcal{O}_\varepsilon \cap \partial\Omega, \\ -c_4 < 0 & \text{on } \Omega \cap \{x_n = \varepsilon\}. \end{cases} \tag{15.51}$$

FIGURE 15.1

With $\varepsilon > 0$ so fixed, we can then pick $A$ sufficiently large (depending on $\|u\|_{C^1(\overline{\Omega})}$) that $c_4 A \geq \|Y_k u\|_{L^\infty(\Omega)}$; hence

$$\begin{aligned} w_k + A(h_\mu + Mx_n^2) &\leq 0, \\ w_k - A(h_\mu + Mx_n^2) &\geq 0 \end{aligned} \tag{15.52}$$

on $\partial\mathcal{O}_\varepsilon$. We can also pick $A$ so large that (by (15.50) and (15.43)–(15.44))

$$\begin{aligned} (g^{ij}\partial_i\partial_j - f_{p_i}\partial_i)\big(w_k + A(h_\mu + Mx_n^2)\big) &\geq 0, \\ (g^{ij}\partial_i\partial_j - f_{p_i}\partial_i)\big(w_k - A(h_\mu + Mx_n^2)\big) &\leq 0 \end{aligned} \tag{15.53}$$

on $\mathcal{O}_\varepsilon$. The maximum principle then implies that (15.52) holds on $\mathcal{O}_\varepsilon$. Thus

$$|\partial_n Y_k u(y)| \leq A|\partial_n h_\mu(y)|. \tag{15.54}$$

This completes our estimation of $\partial_n Y_k u(y)$, begun at (15.31).

We prepare to tackle the estimation of $\partial_n^2 u(y)$. A key ingredient will be a positive *lower* bound on $\partial_j^2 u(y)$, for $1 \leq j \leq n-1$. In order to get this, we make a further (temporary) hypothesis, namely that there is a strictly convex function $u^\# \in C^\infty(\overline{\Omega})$ satisfying

$$\log\det\, H(u^\#) - f(x, u^\#, \nabla u^\#) \leq 0 \text{ on } \Omega, \quad u^\#\big|_{\partial\Omega} = \varphi. \tag{15.55}$$

The function $u^\#$ is called an *upper solution* to (15.1). The proof of (15.8) yields

$$u^b \leq u_\sigma \leq u_\tau \leq u^\# \quad \text{on} \ \ \Omega, \tag{15.56}$$

for $\sigma \leq \tau \in J$. In the present context, where we have dropped the $\sigma$ and where $u \in C^\infty(\overline{\Omega})$ is a solution to (15.15), this means $u^b \leq u \leq u^\#$ on $\Omega$. Consequently, complementing (15.11), we have

$$\partial_\nu u \geq \partial_\nu u^\# \quad \text{on} \ \ \partial\Omega. \tag{15.57}$$

Now let $Y_j$ be the vector field tangent to $\partial\Omega$, equal to $\partial_j$ at $y$, used in (15.30). We have

$$\partial_j^2 u(y) = Y_j^2 u(y) + \kappa_j \partial_\nu u(y), \quad \kappa_j = \widetilde{II}(\partial_j, \partial_j) > 0, \tag{15.58}$$

for $1 \le j \le n-1$, by (15.30), assuming $\partial\Omega$ is strongly convex. There is a similar identity for $\partial_j^2 u^\#(y)$. Since $u = u^\# = \varphi$ on $\partial\Omega$, subtraction yields

$$\partial_j^2 u(y) = \partial_j^2 u^\#(y) + \kappa_j \big[\partial_\nu u(y) - \partial_\nu u^\#(y)\big] \ge \partial_j^2 u^\#(y), \tag{15.59}$$

for $1 \le j \le n-1$, the inequality following from (15.57). Since $u^\#$ is assumed to be a given strongly convex function, this yields a positive lower bound:

$$\partial_j^2 u(y) \ge K_0 > 0, \quad 1 \le j \le n-1. \tag{15.60}$$

Now we can get an upper bound on $\partial_n^2 u(y)$. Rotating the $x_1 \dots x_{n-1}$ coordinate axes, we can assume $\big(\partial_j \partial_k u(y)\big)_{1 \le j,k \le n-1}$ is diagonal. Then, at $y$,

$$\det H(u) = (\partial_n^2 u) \prod_{j=1}^{n-1} (\partial_j^2 u) + \varkappa(\partial^2 u), \tag{15.61}$$

where $\varkappa$ is an $n$-linear form in $\partial^2 u(y)$ that does *not* contain $\partial_n^2 u(y)$. Since $\det H(u) = f(x, u, \nabla u)$ and we have estimates on $\nabla u$, as well as $\partial_j \partial_k u(y)$ for $\partial_j \partial_k \ne \partial_n^2$, we deduce that

$$K_0^{n-1} \partial_n^2 u(y) \le K_1. \tag{15.62}$$

This completes the estimation of $\|u\|_{C^2(\overline{\Omega})}$.

Once we have a bound in $C^2(\overline{\Omega})$ for solutions to (15.15), we can apply Theorem 14.6 to deduce the existence of a solution $u \in C^\infty(\overline{\Omega})$ to (15.1). We thus have the following:

**Proposition 15.5.** *Let $\Omega \subset \mathbb{R}^n$ be a smoothly bounded, open set with strongly convex boundary. Consider the Dirichlet problem (15.1), with $\varphi \in C^\infty(\partial\Omega)$. Assume $F(x, u, p)$ is a smooth function of its arguments satisfying*

$$F(x, u, p) > 0, \quad \partial_u F(x, u, p) \ge 0.$$

*Furthermore, assume (15.1) has a lower solution $u^b$, and an upper solution $u^\# \in C^\infty(\overline{\Omega})$. Then (15.1) has a unique convex solution $u \in C^\infty(\overline{\Omega})$.*

After a little more work, we will show that we need not assume the existence of an upper solution $u^\#$. Note that $u^\#$ was not needed for the estimates of

$$s_0 = \sup\ |u|, \quad s_1 = \sup\ |\nabla u|$$

in Lemmas 15.1–15.3. Thus, if we take a constant $a$ satisfying

$$0 < a < \inf \{F(x,u,p) : x \in \overline{\Omega},\ |u| \le s_0,\ |p| \le s_1\},$$

then any smooth, strongly convex $u^\#$ satisfying

$$\det H(u^\#) \le a \text{ on } \Omega, \quad u^\#\big|_{\partial\Omega} = \varphi, \tag{15.63}$$

will serve as an upper solution to (15.1). Thus, for arbitrary $a > 0$, we want to produce $u^\# \in C^\infty(\overline{\Omega})$, which is strongly convex and satisfies (15.63). For this purpose, it is more than sufficient to have the following result, which is of interest in its own right.

**Proposition 15.6.** *Let $\Omega \subset \mathbb{R}^n$ be a smoothly bounded, open set with strongly convex boundary. Let $\varphi \in C^\infty(\partial\Omega)$ be given and assume $F \in C^\infty(\overline{\Omega})$ is positive. Then there is a unique convex solution $u \in C^\infty(\overline{\Omega})$ to*

$$\det H(u) = F(x), \quad u\big|_{\partial\Omega} = \varphi. \tag{15.64}$$

**Proof.** First, note that (15.64) always has a lower solution. In fact, if you extend $\varphi$ to an element of $C^\infty(\overline{\Omega})$ and let $h \in C^\infty(\overline{\Omega})$ be as in (15.36), then $u^b = \varphi + \tau h$ will work, for sufficiently large $\tau$.

Following the proof of Proposition 15.5, we see that to establish Proposition 15.6, it suffices to obtain an a priori estimate in $C^2(\overline{\Omega})$ for a solution to (15.64). All the arguments used above to establish Proposition 15.5 apply in this case, up to the use of $u^\#$, in (15.55)–(15.59), to establish the estimate (15.60), namely,

$$\partial_j^2 u(y) \ge K_0 > 0, \quad 1 \le j \le n-1. \tag{15.65}$$

Recall that $y$ is an arbitrarily selected point in $\partial\Omega$, and we have rotated coordinates so that the normal $\nu(y)$ to $\partial\Omega$ is parallel to the $x_n$-axis. If we establish (15.65) in this case, without using the hypothesis that an upper solution exists, then the rest of the previous argument giving an estimate in $C^2(\overline{\Omega})$ will work, and Proposition 15.6 will be proved.

We establish (15.65), following [CNS], via a certain barrier function. It suffices to treat the case $j = 1$. We can also assume that $y$ is the origin in $\mathbb{R}^n$ and that, near $y$, $\partial\Omega$ is given by

$$x_n = \rho(x') = \sum_{j=1}^{n-1} B_j x_j^2 + O(|x'|^3), \quad B_j > 0, \tag{15.66}$$

where $x' = (x_1, \dots, x_{n-1})$.

Note that adding a linear term to $u$ leaves the left side of (15.64) unchanged and also has no effect on $\partial_j^2 u$. Thus, without loss of generality, we can assume that

$$u(0) = 0, \quad \partial_j u(0) = 0, \quad 1 \le j \le n-1. \tag{15.67}$$

We have, on $\partial\Omega$,

$$u = \varphi = \frac{1}{2}\sum_{j,k<n} \gamma_{jk}x_jx_k + \varkappa_3(x') + O(|x|^4), \tag{15.68}$$

where $\varkappa_3(x')$ is a polynomial, homogeneous of degree 3 in $x'$.

Now consider

$$\tilde{u}(x) = u(x) - \lambda x_n, \quad \lambda = B_1^{-1}\gamma_{11}. \tag{15.69}$$

This function satisfies $\det H(\tilde{u}) = F(x)$. Looking at $\tilde{u}\big|_{\partial\Omega} = \varphi - \lambda\rho(x')$, we see that the coefficients of $x_1^2$ cancel out here. We claim there is an estimate of the form

$$\tilde{u}\big|_{\partial\Omega} \le \sum_{1<j\le n} a_{1j}x_1x_j + C\Big(\sum_{1<k<n} x_k^2 + |x|^4\Big). \tag{15.70}$$

Indeed, in light of our remark about the disappearance of $x_1^2$, we need only worry about a multiple of $x_1^3$, which can be dominated on $\partial\Omega$ by a term of the form $a_{1n}x_1x_n$ plus a multiple of the quantity in parentheses in (15.70).

The barrier function will take the form

$$W(x) = \frac{1}{2B}\sum_{1<j\le n} (a_{1j}x_1 + Bx_j)^2 + \delta|x|^2 - \varepsilon x_n. \tag{15.71}$$

Take $B >> C$, then fix $\delta > 0$ small, and take $\varepsilon << \delta$. We can do this in such a fashion as to arrange

$$W \ge \tilde{u} \quad \text{on} \ \ \partial\Omega. \tag{15.72}$$

Note that $2\delta$ is the smallest eigenvalue of $H(W)$, and all the other eigenvalues are bounded above independently of $\delta \in (0,1)$, so choosing $\delta$ small enough gives

$$\det H(W) < F(x) \quad \text{on} \ \ \Omega. \tag{15.73}$$

Then $W$ is an upper barrier for $\tilde{u}$; the maximum principle yields

$$\tilde{u} \le W \quad \text{on} \ \ \Omega. \tag{15.74}$$

Consequently,

$$\partial_n\tilde{u}(0) \le \partial_n W(0) = -\varepsilon. \tag{15.75}$$

As noted above, our construction (15.69) yields

$$\partial_1^2\tilde{u}\big(x', \rho(x')\big) = 0, \quad \text{at} \ \ x' = 0, \tag{15.76}$$

that is, $\partial_1^2\tilde{u} + (\partial_n\tilde{u})\partial_1^2\rho = 0$, at $x' = 0$. Hence

$$\partial_1^2 u(0) = \partial_1^2\tilde{u}(0) = -\partial_n\tilde{u}(0)\cdot\partial_1^2\rho(0) \ge \varepsilon\partial_1^2\rho(0). \tag{15.77}$$

This proves the $j = 1$ case of (15.65), as needed, so Proposition 15.6 is proved.

In light of the comments made after the statement of Proposition 15.5, we have

**Corollary 15.7.** *In Proposition 15.5, the hypothesis that there exists an upper solution* $u^\#$ *can be omitted.*

There are some results for Monge-Ampere equations on nonconvex domains; see [GS] and [HRS].

In addition to the Monge-Ampere equations studied here, there are *complex* Monge-Ampere equations, whose study has been very important in complex function theory and differential geometry; see [Au], [BT], [CKNS], [Fef], and [Yau1].

## Exercises

1. Let $\Omega \subset \mathbb{R}^2$ be a strongly convex, smoothly bounded region. Let us assume that $F \in C^\infty(\overline{\Omega})$, $\varphi \in C^\infty(\partial\Omega)$, and $F > 0$. Show that
$$\det H(u) = F(x) \quad \text{on} \quad \Omega, \quad u\Big|_{\partial\Omega} = \varphi,$$
has exactly two solutions in $C^\infty(\overline{\Omega})$, one convex and one concave.
2. Suppose the hypothesis $\partial_u F(x, u, p) \geq 0$ in Proposition 15.5 is dropped. Establish the existence of solutions, using the Leray-Schauder theory.
3. Given $\Omega$ as in Proposition 15.5, $\varphi \in C^\infty(\partial\Omega)$, show that there exists $K_0 > 0$ such that, for all $K \in (0, K_0)$, there is a unique convex solution $u_K \in C^\infty(\overline{\Omega})$ to
$$\det H(u_K) = K\Big(1 + |\nabla u_K|^2\Big)^{(n+2)/2} \quad \text{on} \quad \Omega, \quad u_K\Big|_{\partial\Omega} = \varphi. \tag{15.78}$$
(*Hint*: Show that the convex solution to (15.64), with $F = 1$, yields a lower solution for (15.78), provided $K > 0$ is sufficiently small.)
Note that the graph of $u_K$ is a surface with Gauss curvature $K$.
4. With $u_K$ as in Exercise 3, show that there is $u_0 \in \text{Lip}^1(\overline{\Omega})$ such that
$$u_K \nearrow u_0 \quad \text{as} \quad K \searrow 0. \tag{15.79}$$
In what sense can you say that $u_0$ solves
$$\det H(u_0) = 0 \quad \text{on} \quad \Omega, \quad u_0\Big|_{\partial\Omega} = \varphi? \tag{15.80}$$
See [RT] and [TU] for more on (15.80).

# 16. Elliptic equations in two variables

We have seen in §12 that results on quasi-linear, uniformly elliptic equations for real-valued functions on a domain $\Omega$ are obtained more easily when $\dim \Omega = 2$ than when $\dim \Omega \geq 3$ and have extensions to systems that do not work in higher dimensions. Here we will obtain results on completely nonlinear equations for functions of two variables which are more general than those established in §14 for functions of $n$ variables. The key is the following result of Morrey on linear equations with bounded measurable coefficients, whose conclusion is stronger than that of Theorem 13.7:

**Theorem 16.1.** *Assume $u \in C^2(\Omega)$ and $Lu = f$ on $\Omega \subset \mathbb{R}^2$, where*

$$Lu = \sum_{j,k=1}^{2} a^{jk}(x)\,\partial_j\partial_k u. \tag{16.1}$$

*Assume $a^{jk} = a^{kj}$ are measurable on $\Omega$ and*

$$\lambda|\xi|^2 \le a^{jk}(x)\xi_j\xi_k \le \Lambda|\xi|^2, \tag{16.2}$$

*for some $\lambda, \Lambda \in (0,\infty)$. Pick $p > 2$. Then, for $\mathcal{O} \subset\subset \Omega$, there is a $\mu > 0$ such that*

$$\|u\|_{C^{1+\mu}(\mathcal{O})} \le C\big[\|u\|_{H^1(\Omega)} + \|f\|_{L^p(\Omega)}\big], \tag{16.3}$$

*where $C = C(\mathcal{O}, \Omega, p, \lambda, \Lambda)$.*

**Proof.** Let $V_j = \partial_j u$. Then these functions satisfy the divergence-form equations

$$\begin{aligned} &\partial_1\Big(\frac{a^{11}}{a^{22}}\partial_1 V_1 + 2\frac{a^{12}}{a^{22}}\partial_2 V_1\Big) + \partial_2\big(\partial_2 V_1\big) = \partial_1\Big(\frac{f}{a^{22}}\Big), \\ &\partial_1\big(\partial_1 V_2\big) + \partial_2\Big(\frac{a^{22}}{a^{11}}\partial_2 V_2 + 2\frac{a^{12}}{a^{11}}\partial_1 V_2\Big) = \partial_2\Big(\frac{f}{a^{11}}\Big). \end{aligned} \tag{16.4}$$

Proposition 9.8 applies to each of these equations, yielding

$$\|V_j\|_{C^\mu(\mathcal{O})} \le C\big[\|V_j\|_{L^2(\Omega)} + \|f\|_{L^p(\Omega)}\big]. \tag{16.5}$$

This yields the desired estimate (16.3).

Morrey's original proof of Theorem 16.1 came earlier than the DeGiorgi-Nash-Moser estimate used in the proof above. Instead, he used estimates on quasi-conformal mappings (see [Mor2]).

We apply Theorem 16.1 to estimates for real-valued solutions to equations of the form

$$F(x, u, \nabla u, \partial^2 u) = f \quad \text{on } \Omega \subset \mathbb{R}^2, \tag{16.6}$$

where $F = F(x, u, p, \zeta)$ is a smooth function of its arguments satisfying the ellipticity condition

$$\begin{gathered} \lambda|\xi|^2 \le \sum \frac{\partial F}{\partial \zeta_{jk}}(x, u, p, \zeta)\xi_j\xi_k \le \Lambda|\xi|^2, \\ 0 < \lambda = \lambda(u, p, \zeta), \quad \Lambda = \Lambda(u, p, \zeta). \end{gathered} \tag{16.7}$$

For $h > 0,\ \ell = 1, 2$, set

$$V_{\ell h}(x) = h^{-1}\big(u(x + he_\ell) - u(x)\big). \tag{16.8}$$

Then $V_{\ell h}$ satisfies the equation

$$\sum_{j,k} a^{jk}_{\ell h}(x)\partial_j\partial_k V_{\ell h} = g_{\ell h}(x) \tag{16.9}$$

on $\Omega_h = \{x \in \Omega : \operatorname{dist}(x, \mathbb{R}^2 \setminus \Omega) > h\}$, where the coefficients $a^{jk}_{\ell h}(x)$ are given by

$$a^{jk}_{\ell h}(x) = \int_0^1 \frac{\partial F}{\partial \zeta_{jk}}\big(x + she_\ell, \dots, s\partial^2\tau_{\ell h}u + (1 - s)\partial^2 u\big)\, ds, \tag{16.10}$$

with $\tau_{\ell h}u(x) = u(x + he_\ell)$, and the functions $g_{\ell h}(x)$ are given by

$$\begin{aligned} g_{\ell h}(x) = &-\sum_j \left[\int_0^1 \frac{\partial F}{\partial p_j}\big(x + she_\ell, \dots, s\partial^2\tau_{\ell h}u + (1 - s)\partial^2 u\big)\, ds\right] \partial_j V_{\ell k} \\ &- \int_0^1 \frac{\partial F}{\partial u}\big(x + she_\ell, \dots, s\partial^2\tau_{\ell h} + (1 - s)\partial^2 u\big)\, ds\ V_{\ell h} \\ &- \int_0^1 \frac{\partial F}{\partial x_\ell}\big(x + she_\ell, \dots, s\partial^2\tau_{\ell h}u + (1 - s)\partial^2 u\big)\, ds \\ &+ h^{-1}\big(f(x + he_\ell) - f(x)\big). \end{aligned} \tag{16.11}$$

Theorem 16.1 then yields an estimate

$$\|V_{\ell h}\|_{C^{1+\mu}(\mathcal{O})} \le C\big[\|V_{\ell h}\|_{L^2(\Omega)} + \|g_{\ell h}\|_{L^p(\Omega)}\big], \tag{16.12}$$

with $C = C(\mathcal{O}, \Omega, p, \lambda, \Lambda, \|u\|_{C^2(\overline{\Omega})})$. Note that

$$\|g_{\ell h}\|_{L^p(\Omega)} \le C\big(\|u\|_{C^2(\overline{\Omega})}\big) + \big\|h^{-1}(\tau_{\ell h}f - f)\big\|_{L^p(\Omega)}. \tag{16.13}$$

Letting $h \to 0$, we have the following:

**Theorem 16.2.** *Assume that $\Omega \subset \mathbb{R}^2$, that $u \in C^2(\overline{\Omega})$ solves (16.6), that the ellipticity condition (16.7) holds, and that $f \in H^{1,p}(\Omega)$, for some $p > 2$. Then, given $\mathcal{O} \subset\subset \Omega$, there is a $\mu > 0$ such that $u \in C^{2+\mu}(\mathcal{O})$ and*

$$\|u\|_{C^{2+\mu}(\mathcal{O})} \le C\big[1 + \|f\|_{H^{1,p}(\Omega)}\big], \tag{16.14}$$

*where*

$$C = C\big(\mathcal{O}, \Omega, p, \lambda, \Lambda, \|u\|_{C^2(\overline{\Omega})}\big). \tag{16.15}$$

For estimates up to the boundary, we use the following complement to Theorem 16.1:

**Proposition 16.3.** *If $u \in C^2(\overline{\Omega})$ and the hypotheses of Theorem 16.1 hold, then there is an estimate*

$$\|u\|_{C^{1+\mu}(\overline{\Omega})} \le C\big[\|u\|_{H^{1,p}(\Omega)} + \|\varphi\|_{C^2(\partial\Omega)} + \|f\|_{L^p(\Omega)}\big], \tag{16.16}$$

*where $\varphi = u\big|_{\partial\Omega}$ and $C = C(\Omega, p, \lambda, \Lambda)$.*

**Proof.** Given $y \in \partial\Omega$, locally flatten $\partial\Omega$ near $y$, using a coordinate change, transforming it to the $x_1$-axis. In the new coordinates, $u$ satisfies an elliptic equation of the form

$$\widetilde{a}^{jk}\partial_j\partial_k u = f - \widetilde{b}^j\partial_j u = \widetilde{f}. \tag{16.17}$$

Then $\widetilde{V}_1 = \partial_1 u$ satisfies an analogue of the first equation in (16.4), while $\widetilde{V}_1 = \partial_1\varphi$ on the flattened part of $\partial\Omega$. Thus Proposition 9.9 (or rather the local version mentioned at the end of §9) yields an estimate on $\widetilde{V}_1$ in $C^\mu(U \cap \overline{\Omega})$, for some neighborhood $U$ of $y$ in $\mathbb{R}^2$.

Thus, for any smooth vector field $X$ on $\mathbb{R}^2$, tangent to $\partial\Omega$, we have an estimate on $\|Xu\|_{C^\mu(\overline{\Omega})}$ by the right side of (16.16). Furthermore, by Proposition 9.9, there is a Morrey space estimate

$$\|\nabla Xu\|_{M_2^q(\Omega)} \le \text{ RHS}, \tag{16.18}$$

for some $q > 2$, where "RHS" stands for the right side of (16.16). We may as well assume $q \le p$, so $\widetilde{f} \in L^p(\Omega) \subset M_2^q(\Omega)$. Then (16.17) and (16.18) together imply

$$\|\partial_j\partial_k u\|_{M_2^q(\Omega)} \le \text{ RHS}, \tag{16.19}$$

for all $j, k \le 2$, which in turn implies (16.16).

We now establish the following:

**Theorem 16.4.** *Assume that $\Omega \subset \mathbb{R}^2$ and that $u \in C^3(\overline{\Omega})$ solves (16.6), with the ellipticity condition (16.7), with $f \in H^{1,p}(\Omega)$ for some $p > 2$, and $u\big|_{\partial\Omega} = \varphi$. Then, for some $\mu > 0$, there is an estimate*

$$\|u\|_{C^{2+\mu}(\overline{\Omega})} \le C\big[1 + \|\varphi\|_{C^3(\partial\Omega)} + \|f\|_{H^{1,p}(\Omega)}\big], \tag{16.20}$$

*where*

$$C = C\big(\Omega, p, \lambda, \Lambda, \|u\|_{C^2(\overline{\Omega})}\big). \tag{16.21}$$

**Proof.** If $X = b^\ell\partial_\ell$ is a smooth vector field in $\mathbb{R}^2$, tangent to $\partial\Omega$, then $Xu$ satisfies

$$\begin{aligned} F_{\zeta_{jk}}\,\partial_j\partial_k(Xu) = &\; - F_{p_j}\,\partial_j(Xu) - F_u\; Xu + F_{\zeta_{jk}}(\partial_j\partial_k b^\ell)(\partial_\ell u) \\ &+ 2F_{\zeta_{jk}}(\partial_j b^\ell)(\partial_k\partial_\ell u) + F_{p_j}(\partial_j b^\ell)(\partial_\ell u) + Xf, \end{aligned} \tag{16.22}$$

and $Xu = X\varphi$ on $\partial\Omega$. Thus Proposition 16.3 applies. We have a $C^{1+\mu}(\overline{\Omega})$-estimate on $Xu$, and even better, a Morrey space estimate:

$$\|\partial_j \partial_k Xu\|_{M_2^q(\Omega)} \leq \text{ RHS}, \tag{16.23}$$

for some $q > 2$, and for all $j, k \leq 2$, where "RHS" now stands for the right side of (16.20).

The proof is almost done. Parallel to (16.22), we have, for any $\ell$,

$$F_{\zeta_{jk}}\, \partial_j \partial_k \partial_\ell u = -F_{p_j}\, \partial_j \partial_\ell u - F_u\, \partial_\ell u + \partial_\ell f. \tag{16.24}$$

Thus we can solve for $\partial_j \partial_k \partial_\ell u$ in terms of functions of the form $\partial_j \partial_k Xu$ and other terms estimable in the $M_2^q(\Omega)$-norm by the right side of (16.20). Hence we have (16.20), and even the stronger estimate

$$\|\partial^3 u\|_{M_2^q(\Omega)} \leq \text{ RHS}. \tag{16.25}$$

From this result the continuity method readily gives the following:

**Theorem 16.5.** *Let $\Omega$ be a smoothly bounded domain in $\mathbb{R}^2$. Let the function $F_\sigma(x, u, p, \zeta)$ depend smoothly on all its arguments, for $\sigma \in [0, 1]$, and let $\varphi_\sigma \in C^\infty(\overline{\Omega})$ have smooth dependence on $\sigma$. Assume that, for each $\sigma \in [0, 1]$,*

$$\partial_u F_\sigma(x, u, p, \zeta) \leq 0$$

*and that the ellipticity condition (16.7) holds. Also assume that, for any solution $u_\sigma \in C^\infty(\overline{\Omega})$ to the equation*

$$F_\sigma(x, u_\sigma, \nabla u_\sigma, \partial^2 u_\sigma) = 0 \text{ on } \Omega, \quad u_\sigma\big|_{\partial\Omega} = \varphi_\sigma, \tag{16.26}$$

*there is a $C^2(\overline{\Omega})$-bound:*

$$\|u_\sigma\|_{C^2(\overline{\Omega})} \leq K. \tag{16.27}$$

*If (16.26) has a solution in $C^\infty(\overline{\Omega})$ for $\sigma = 0$, then it has a solution in $C^\infty(\overline{\Omega})$ for $\sigma = 1$.*

## Exercises

1. In the proof of Theorem 16.1, can you replace the use of Proposition 9.8 by a result analogous to Proposition 12.5?
2. Suppose that, in (16.7), $\lambda$ and $\Lambda$ are independent of $\zeta$. Obtain a variant of Theorem 16.5 in which (16.27) is weakened to a bound in $C^1(\overline{\Omega})$.

## A. Morrey spaces

Given $f \in L^1_{\text{loc}}(\mathbb{R}^n),\ p \in [1, \infty)$, one says $f \in M^p(\mathbb{R}^n)$ provided that

$$R^{-n} \int_{B_R} |f(x)|\, dx \le C\, R^{-n/p}, \tag{A.1}$$

for all balls $B_R$ of radius $R \le 1$ in $\mathbb{R}^n$. More generally, if $1 \le q \le p$ and $f \in L^q_{\text{loc}}(\mathbb{R}^n)$, we will say $f \in M^p_q(\mathbb{R}^n)$ provided that, for all such $B_R$,

$$R^{-n} \int_{B_R} |f(x)|^q\, dx \le C\, R^{-nq/p}. \tag{A.2}$$

The spaces $M^p_q(\mathbb{R}^n)$ are called *Morrey spaces*. If we set $\delta_R f(x) = f(Rx)$, the left side of (A.2) is equal to $\int_{B_1} |\delta_R f(x)|^q\, dx$, so an equivalent condition is

$$\|\delta_R f\|_{L^q(B_1)} \le C' R^{-n/p}, \tag{A.3}$$

for all balls $B_1$ of radius 1, and for all $R \in (0, 1]$. It follows from Hölder's inequality that

$$L^p_{\text{unif}}(\mathbb{R}^n) = M^p_p(\mathbb{R}^n) \subset M^p_q(\mathbb{R}^n) \subset M^p(\mathbb{R}^n).$$

We can give an equivalent characterization of $M^p$ in terms of the heat kernel. Let $p_r(\xi) = e^{-|r\xi|^2}$. Then, given $f \in L^1_{\text{unif}}(\mathbb{R}^n)$,

$$f \in M^p(\mathbb{R}^n) \Longleftrightarrow p_r(D)|f| \le C\, r^{-n/p}, \tag{A.4}$$

for $0 < r \le 1$. To see the implication $\Rightarrow$, given $x \in \mathbb{R}^n$ write $f = f_1 + f_2$, where $f_1$ is the restriction of $f$ to the unit ball $B_1(x)$ centered at $x$, and $f_2$ is the restriction of $f$ to the complement. That $p_r(D)|f_1|(x) \le C r^{-n/p}$, for $r \in (0, 1]$, follows easily from the characterization (A.1) and the formula

$$p_r(D)\delta_x(y) = (4\pi r^2)^{-n/2}\ e^{-|x-y|^2/4r^2},$$

while this formula also implies that $p_r(D)|f_2|(x)$ is rapidly decreasing as $r \searrow 0$. The implication $\Leftarrow$ is similarly easy to verify. Note that

$$f \text{ satisfies (A.4)} \Longrightarrow |p_r(D)f| \le C r^{-n/p}. \tag{A.5}$$

Recall the Zygmund spaces $C^r_*(\mathbb{R}^n),\ r \in \mathbb{R}$, introduced in §8 of Chapter 13, with norms defined as follows. Let $\Psi_0(\xi) \in C_0^\infty(\mathbb{R}^n)$ be equal to 1 for $|\xi| \le 1$, set $\Psi_k(\xi) = \Psi_0(2^{-k}\xi)$, and let $\psi_k(\xi) = \Psi_k(\xi) - \Psi_{k-1}(\xi)$. The set $\{\psi_k(\xi)\}$ is a Littlewood-Paley partition of unity. One sets

$$\|f\|_{C^r_*} = \sup_k\ 2^{kr} \|\psi_k(D)f\|_{L^\infty}. \tag{A.6}$$

For $r \in (0, \infty) \setminus \mathbb{Z}^+,\ C^r_*$ coincides with the Hölder space $C^r$, and $C^1_*$ is the classical Zygmund space. As shown in Chapter 13, one has, for all

$m,\ r \in \mathbb{R}$,

$$P \in OPS^m_{1,0} \Longrightarrow P : C^r_* \longrightarrow C^{r-m}_*. \tag{A.7}$$

The following relation exists between Zygmund spaces and Morrey spaces. From (A.4)–(A.5) we readily obtain the inclusion

$$M^p(\mathbb{R}^n) \subset C_*^{-n/p}(\mathbb{R}^n). \tag{A.8}$$

From this we deduce a result known as *Morrey's lemma*:

**Lemma A.1.** *If $p > n$, then, for $f \in \mathcal{S}'(\mathbb{R}^n)$,*

$$\nabla f \in M^p(\mathbb{R}^n) \Longrightarrow f \in C^r_{\text{loc}}(\mathbb{R}^n), \quad r = 1 - \frac{n}{p} \in (0,1). \tag{A.9}$$

**Proof.** We can write

$$f = \sum_{j=1}^n B_j(\partial_j f) + Rf, \quad B_j \in OPS^{-1}(\mathbb{R}^n), \quad R \in OPS^{-\infty}(\mathbb{R}^n). \tag{A.10}$$

Then (A.7)–(A.8) imply that $B_j\, \partial_j f \in C^r_*(\mathbb{R}^n)$, if the hypothesis of (A.9) holds.

If $\Omega \subset \mathbb{R}^n$ is a bounded region, we say $f \in M^p_q(\Omega)$ if $\tilde{f} \in M^p_q(\mathbb{R}^n)$, where $\tilde{f}(x) = f(x)$ for $x \in \Omega,\ 0$ for $x \notin \Omega$. If $\partial\Omega$ is smooth, it is easy to extend (A.9) to the implication (for $p > n$):

$$\nabla f \in M^p(\Omega) \Longrightarrow f \in C^r(\overline{\Omega}), \quad r = 1 - \frac{n}{p} \in (0,1), \tag{A.11}$$

via a simple reflection argument (across $\partial\Omega$).

One also considers homogeneous versions of Morrey spaces. If $p \in (1,\infty)$ and $1 \le q \le p,\ f \in L^q_{\text{loc}}(\mathbb{R}^n)$, we say $f \in \mathcal{M}^p_q(\mathbb{R}^n)$ provided (A.2) holds for all $R \in (0,\infty)$, not just for $R \le 1$. Note that if we set

$$\|f\|_{\mathcal{M}^p_q} = \sup_R\ R^{n/p}\Bigl(R^{-n} \int\limits_{B_R} |f(x)|^q\ dx\Bigr)^{1/q}, \tag{A.12}$$

where $R$ runs over $(0,\infty)$ and $B_R$ over all balls of radius $R$, then

$$\|\delta_r\ f\|_{\mathcal{M}^p_q} = r^{-n/p}\ \|f\|_{\mathcal{M}^p_q}, \tag{A.13}$$

where $\delta_r f(x) = f(rx)$. This is the same type of scaling as the $L^p(\mathbb{R}^n)$-norm. It is clear that compactly supported elements of $M^p_q(\mathbb{R}^n)$ and of $\mathcal{M}^p_q(\mathbb{R}^n)$ coincide. In a number of references, including [P2], $\mathcal{M}^p_q$ is denoted $\mathcal{L}_{q,\lambda}$, with $\lambda = n\bigl(1 - q/p\bigr)$.

The following refinement of Morrey's lemma is due to S. Campanato.

**Proposition A.2.** *Given* $p \in [1, \infty)$, $s \in (0,1)$, *assume that* $u \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ *and that, for each ball* $B_R(x)$ *with* $R \le 1$, *there exists* $\alpha \in \mathbb{C}$ *such that*

$$\int_{B_R(x)} |u(y) - \alpha|^p \, dy \le C R^{n+ps}. \tag{A.14}$$

*Then*

$$u \in C^s_{\mathrm{loc}}(\mathbb{R}^n). \tag{A.15}$$

**Proof.** Pick $\varphi \in C_0^\infty(\mathbb{R}^n)$ to be a radial function, supported on $|x| \le 1$, such that $\widehat{\varphi}(\xi) \ge 0$, and let $\psi = \Delta\varphi$, so $\int \psi \, dx = 0$. It suffices to show that

$$\big|(\psi_R * u)(x)\big| \le C R^s, \quad R \le 1, \tag{A.16}$$

where $\psi_R(x) = R^{-n}\psi(R^{-1}x)$. Note that, for fixed $x$, $R$, $\alpha = \alpha\big(B_R(x)\big)$, we have

$$(\psi_R * u)(x) = \psi_R * (u - \alpha)(x), \tag{A.17}$$

so

$$\begin{aligned} &\big|(\psi_R * u)(x)\big| \\ &\le \|\psi_R\|_{L^{p'}(B_R(0))} \|u - \alpha\|_{L^p(B_R(x))} \\ &\le \Big( \int_{B_R(0)} R^{-np'} |\psi(R^{-1}y)|^{p'} \, dy \Big)^{1/p'} \Big( \int_{B_R(x)} |u(y) - \alpha|^p \, dy \Big)^{1/p} \\ &\le C \; R^{-n} \cdot R^{n/p'} \cdot R^{n/p} \cdot R^s = R^s, \end{aligned} \tag{A.18}$$

as desired.

## B. Leray-Schauder fixed-point theorems

We will demonstrate several fixed-point theorems that are useful for nonlinear PDE. The first, known as *Schauder's fixed-point theorem*, is an infinite dimensional extension of Brouwer's fixed-point theorem, which we recall.

**Proposition B.1.** *If* $K$ *is a compact, convex set in a finite-dimensional vector space* $V$, *and* $F : K \to K$ *is a continuous map, then* $F$ *has a fixed point.*

This was proved in §19 of Chapter 1, specifically when $K$ was the closed unit ball in $\mathbb{R}^n$. Now, given any compact convex $K \subset V$, if we translate it, we can assume $0 \in K$. Let $W$ denote the smallest vector space in $V$

that contains $K$; say $\dim_{\mathbb{R}} W = n$. Thus there is a basis of $W$, of the form $E \subset K$. Clearly, the convex hull of $E$ has nonempty interior in $W$. From here, it is easily established that $K$ is homeomorphic to the closed unit ball in $\mathbb{R}^n$.

A quicker reduction to the case of a ball goes like this. Put an inner product on $V$, and say a ball $B \subset V$ contains $K$. Let $\psi : B \to K$ map a point $x$ to the point in $K$ closest to $x$. Then consider a fixed point of $F \circ \psi : B \to K \subset B$.

The following is Schauder's generalization:

**Theorem B.2.** *If $K$ is a compact, convex set in a Banach space $V$, and $F : K \to K$ is a continuous map, then $F$ has a fixed point.*

**Proof.** Whether or not $V$ has a countable dense set, $K$ certainly does; say $\{v_j : j \in \mathbb{Z}^+\}$ is dense in $K$. For each $n \geq 1$, let $V_n$ be the linear span of $\{v_1, \dots, v_n\}$ and $K_n \subset K$ the closed, convex hull of $\{v_1, \dots, v_n\}$. Thus $K_n$ is a compact, convex subset of $V_n$, a linear space of dimension $\leq n$.

We define continuous maps $Q_n : K \to K_n$ as follows. Cover $K$ by balls of radius $\delta_n$ centered at the points $v_j$, $1 \leq j \leq n$. Let $\{\varphi_{nj} : 1 \leq j \leq n\}$ be a partition of unity subordinate to this cover, satisfying $0 \leq \varphi_j \leq 1$. Then set

$$Q_n(v) = \sum_{j=1}^{n} \varphi_{nj}(v) v_j, \quad Q_n : K \to K_n. \tag{B.1}$$

Since $\varphi_{nj}(v) = 0$ unless $\|v - v_j\| \leq \delta_n$, it follows that

$$\|Q_n(v) - v\| \leq \delta_n. \tag{B.2}$$

The denseness of $\{v_j : j \in \mathbb{Z}^+\}$ in $K$ implies we can take $\delta_n \to 0$ as $n \to \infty$.

Now consider the maps $F_n : K_n \to K_n$, given by $F_n = Q_n \circ F\big|_{K_n}$. By Proposition B.1, each $F_n$ has a fixed point $x_n \in K_n$. Now

$$Q_n F(x_n) = x_n \Longrightarrow \|F(x_n) - x_n\| \leq \delta_n. \tag{B.3}$$

Since $K$ is compact, $(x_n)$ has a limit point $x \in K$ and (B.3) implies $F(x) = x$, as desired.

It is easy to extend Theorem B.2 to the case where $V$ is a Fréchet space, using a translation-invariant distance function. In fact, a theorem of Tychonov extends it to general locally convex $V$.

The following slight extension of Theorem B.2 is technically useful:

**Corollary B.3.** *Let $E$ be a closed, convex set in a Banach space $V$, and let $F : E \to E$ be a continuous map such that $F(E)$ is relatively compact. Then $F$ has a fixed point.*

**Proof.** The closed, convex hull $K$ of $F(E)$ is compact; simply consider $F\big|_K$, which maps $K$ to itself.

**Corollary B.4.** *Let $B$ be the open unit ball in a Banach space $V$. Let $F : \overline{B} \to V$ be a continuous map such that $F(\overline{B})$ is relatively compact and $F(\partial B) \subset B$. Then $F$ has a fixed point.*

**Proof.** Define a map $G : \overline{B} \to \overline{B}$ by

$$G(x) = F(x) \quad \text{if } \ \|F(x)\| \le 1, \quad G(x) = \frac{F(x)}{\|F(x)\|} \quad \text{if } \ \|F(x)\| \ge 1.$$

Then $G : \overline{B} \to \overline{B}$ is continuous and $G(\overline{B})$ is relatively compact. Corollary B.3 implies that $G$ has a fixed point; $G(x) = x$. The hypothesis $F(\partial B) \subset B$ implies $\|x\| < 1$, so $F(x) = G(x) = x$.

The following Leray-Schauder theorem is the one we directly apply to such results as Theorem 1.10. The argument here follows [GT].

**Theorem B.5.** *Let $V$ be a Banach space, and let $F : [0,1] \times V \to V$ be a continuous, compact map, such that $F(0,v) = v_0$ is independent of $v \in V$. Suppose there exists $M < \infty$ such that, for all $(\sigma,x) \in [0,1] \times V$,*

$$F(\sigma,x) = x \Longrightarrow \|x\| < M. \tag{B.4}$$

*Then the map $F_1 : V \to V$ given by $F_1(v) = F(1,v)$ has a fixed point.*

**Proof.** Without loss of generality, we can assume $v_0 = 0$ and $M = 1$. Let $B$ be the open unit ball in $V$. Given $\varepsilon \in (0,1]$, define $G_\varepsilon : \overline{B} \to V$ by

$$\begin{aligned} G_\varepsilon(x) = F\Big(\frac{1-\|x\|}{\varepsilon}, \frac{x}{\|x\|}\Big) \quad & \text{if } \ 1-\varepsilon \le \|x\| \le 1, \\ F\Big(1, \frac{x}{1-\varepsilon}\Big) \quad & \text{if } \ \|x\| \le 1-\varepsilon. \end{aligned}$$

Note that $G_\varepsilon(\partial B) = 0$. For each $\varepsilon \in (0,1]$, Corollary B.4 applies to $G_\varepsilon$. Hence each $G_\varepsilon$ has a fixed point $x(\varepsilon)$. Let $x_k = x(1/k)$, and set

$$\begin{aligned} \sigma_k = k\big(1-\|x_k\|\big) \quad & \text{if } \ 1 - \frac{1}{k} \le \|x_k\| \le 1, \\ 1 \quad & \text{if } \ \|x_k\| \le 1 - \frac{1}{k}, \end{aligned}$$

so $\sigma_k \in (0,1]$ and $F(\sigma_k, x_k) = x_k$. Passing to a subsequence, we have $(\sigma_k, x_k) \to (\sigma, x)$ in $[0,1] \times \overline{B}$, since the map $F$ is compact.

We claim $\sigma = 1$. Indeed, if $\sigma < 1$, then $\|x_k\| \ge 1 - 1/k$ for large $k$, hence $\|x\| = 1$ and $F(\sigma,x) = x$, contradicting (B.4) (with $M = 1$). Thus $\sigma_k \to 1$ and we have $F(1,x) = x$, as desired.

There are more general results, involving Leray-Schauder "degree theory," which can be found in [Schw], [Ni6], and [Deim].

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