Harmonic Functions on Domains in $\mathbb{R}^n$
Topics

Michael Taylor

Contents

1. Introduction

These notes present material on harmonic functions on domains in Euclidean space. They have some overlap with results presented in Chapters 3 and 5 of [T], but are mostly complementary to those results. Topics treated here also have a bit of overlap with results on harmonic functions given in [T2].

We start with a discussion of the Dirichlet problem. Let $\Omega \subset \mathbb{R}^n$ be a nonempty open set. We will assume Ω is connected. If Ω is also bounded, the Dirichlet problem on Ω is the problem of solving

$$\Delta u = 0 \text{ on } \Omega, \quad u\big|_{\partial\Omega} = f, \quad u \in C^2(\Omega), \quad u \in C(\overline{\Omega}), \tag{1.1}$$

given $f \in C(\partial\Omega)$. If Ω is not bounded, we instead consider

$$\Delta u = 0 \text{ on } \Omega, \quad u\big|_{\partial\Omega} = f, \quad u \in C^2(\Omega), \quad u \in C_*(\overline{\Omega}), \tag{1.2}$$

given

$$f \in C_*(\partial\Omega). \tag{1.3}$$

Here, if $K \subset \mathbb{R}^n$ is a closed set,

$$C_*(K) = \{f \in C(K) : f(x) \to 0 \text{ as } x \to \infty\}. \tag{1.4}$$

(If K is also bounded, $C_*(K) = C(K)$.) Conditions for existence of solutions, and study of their properties, is a big topic for Math 751. In this introduction, we take care of the uniqueness issue.

Proposition 1.1. *If (1.1), or more generally (1.2), has a solution, it is unique.*

Proof. Suppose u and v both solve (1.2), with f as in (1.3). Consider $w = u - v$. Then

$$w\big|_{\partial\Omega} = 0, \quad w \in C_*(\overline{\Omega}), \tag{1.5}$$

and $\Delta w = 0$ on Ω. Now is w satisfies (1.5) and is not identically zero, $|w|$ must assume a maximum at some point $x_0 \in \Omega$. But since $\Delta w = 0$ on Ω, the strong maximum principle implies w is constant. Then (1.5) forces the constant to be 0.

In §§2–3, we will treat two cases, Ω is a half-space in §2, and Ω is a ball in §3. We look for explicit formulas yielding the solution to the Dirichlet problem in these two cases. In §2 we take two distinct approaches to these formulas, and use the uniqueness result of Proposition 1.1 to show that these approaches yield equivalent formulas. There results a nontrivial identity, namely, for $y > 0$,

$$(2\pi)^{-n} \int_{\mathbb{R}^n} e^{-y|\xi|} e^{ix\cdot\xi}\, d\xi = \frac{2}{A_n} \frac{y}{(y^2 + |x|^2)^{(n+1)/2}}, \tag{1.6}$$

established by different means in [T], Chapter 3, §5. Here A_n is the area of the unit sphere $S^n \subset \mathbb{R}^{n+1}$. (For $n = 1$, this is elementary, but not for $n \geq 2$.)

In §3 we obtain a formula for the solution to (1.1) on the unit ball $B \subset \mathbb{R}^n$, of the form

$$u(x) = C_n(1 - |x|^2) \int_{S^{n-1}} \frac{f(y)}{|x - y|^n}\, dS(y). \tag{1.7}$$

We have this for $n = 2$ by calculations involving Fourier series. Moving from this to (1.7) for $n \geq 3$ can be seen as motivated by the pattern in (1.6). We verify that this works, with $C_n = 1/A_{n-1}$. One essential tool in this verification is the mean value property for harmonic functions.

Sections 4–8 derive a number of results on harmonic functions on a domain $\Omega \subset \mathbb{R}^n$, using tools from §§1 and 3 as major tools. In §4 we show that if $u \in C^2(\Omega)$ is harmonic, then actually $u \in C^\infty(\Omega)$. We also show that if u_k are harmonic on Ω and $u_k \to u$ locally uniformly, then u is actually harmonic, and $\partial^\alpha u_k \to \partial^\alpha u$ locally uniformly, for all α. In §5, we recall the result that each harmonic function on Ω has the mean value property (MVP), and complement this with the converse: each continuous function on Ω with the MVP is actually smooth and harmonic. Sections 6–8 establish for harmonic functions on domains in $\mathbb{R}^n$ several results established for holomorphic functions on planar domains in Math 656:

Schwarz reflection principle,
Liouville theorem,
Removable singularity theorem.

Section 9 treats Harnack inequalities, and Section 10 applies them to some further Liouville theorems, which also have significant applications to complex function theory.

2. The Dirichlet problem on a half-space

Here we take

$$\Omega = \mathbb{R}^{n+1}_+ = \{(y,x) : y > 0, x \in \mathbb{R}^n\}. \tag{2.1}$$

Our problem is to solve

$$\begin{gathered}(\partial_j^2 + \Delta_x)u(x,y) = 0, \quad y > 0,\ x \in \mathbb{R}^n, \\ u(0,x) = f(x),\end{gathered} \tag{2.2}$$

such that

$$u \in C^2(\mathbb{R}^{n+1}_+) \cap C_*(\overline{\mathbb{R}}^{n+1}_+), \tag{2.3}$$

given

$$f \in C_*(\mathbb{R}^n). \tag{2.4}$$

We will establish the existence of a solution to (2.2) by finding a formula for u.

Our first approach uses Fourier analysis, taking

$$u(y,x) = (2\pi)^{-n/2} \int e^{-y|\xi|} \hat{f}(\xi) e^{ix\cdot\xi}\, d\xi. \tag{2.5}$$

For this, we modify the hypothesis (2.4) to

$$f \in \mathrm{FL}^1(\mathbb{R}^n), \tag{2.6}$$

i.e.,

$$f \in \mathcal{S}'(\mathbb{R}^n), \quad \hat{f} \in L^1(\mathbb{R}^n). \tag{2.7}$$

Note that if f satisfies (2.7), then $f = \mathcal{F}^*\hat{f}$. Now the Riemann-Lebesgue lemma says

$$\mathcal{F}, \mathcal{F}^* : L^1(\mathbb{R}^n) \longrightarrow C_*(\mathbb{R}^n), \tag{2.8}$$

so

$$\mathrm{FL}^1(\mathbb{R}^n) \subset C_*(\mathbb{R}^n). \tag{2.9}$$

Given (2.6), we see from the dominated convergence theorem that

$$e^{-y|\xi|}\hat{f} \in C([0,\infty)_y, L^1(\mathbb{R}^n)), \tag{2.10}$$

hence, by (2.8),

$$u(y,\cdot) = \mathcal{F}^*(e^{-y|\xi|}\hat{f}) \in C([0,\infty)_y, C_*(\mathbb{R}^n)). \tag{2.11}$$

Furthermore,

$$\begin{aligned} \sup_x |u(y,x)| &\le (2\pi)^{-n/2} \int e^{-y|\xi|}|\hat{f}(\xi)|\,d\xi \\ &\to 0, \quad \text{as } \ y \searrow \infty, \end{aligned} \tag{2.12}$$

also by the dominated convergence theorem. The two properties (2.11)–(2.12) imply

$$u \in C_*(\overline{\mathbb{R}}^{n+1}_+). \tag{2.13}$$

In addition, we have, for $y > 0$,

$$\begin{aligned} \partial_y u(y,x) &= -(2\pi)^{-n/2} \int |\xi| e^{-y|\xi|}\hat{f}(\xi) e^{ix\cdot\xi}\,d\xi, \\ \partial_y^2 u(y,x) &= (2\pi)^{-n/2} \int |\xi|^2 e^{-y|\xi|}\hat{f}(\xi) e^{ix\cdot\xi}\,d\xi \\ &= -\Delta_x u(y,x), \end{aligned} \tag{2.14}$$

so u, given by (2.5), satisfies the conditions (2.2)–(2.3), as long as f satisfies (2.6). Let us denote the solution operator by PI,

$$\mathrm{PI}\, f(y,x) = (2\pi)^{-n/2} \int e^{-y|\xi|}\hat{f}(\xi) e^{ix\cdot\xi}\,d\xi. \tag{2.15}$$

We have

$$\mathrm{PI} : \mathrm{FL}^1(\mathbb{R}^n) \longrightarrow C_*(\overline{\mathbb{R}}^{n+1}_+) \cap \{u \in C^2(\mathbb{R}^{n+1}_+) : \Delta u = 0\}. \tag{2.16}$$

Now, since $u = \mathrm{PI}\, f$ is harmonic on $\mathbb{R}^{n+1}_+$, the strong maximum principle applies. In light of (2.11)–(2.12), this implies

$$\sup_{y,x} |\,\mathrm{PI}\, f(y,x)| = \sup_x |f(x)|, \tag{2.17}$$

when $f \in \mathrm{FL}^1(\mathbb{R}^n)$. Using this, we can establish the following.

Proposition 2.1. *The map* PI *in (2.15)–(2.16) has a unique continuous linear extension to*

$$\mathrm{PI} : C_*(\mathbb{R}^n) \longrightarrow C_*(\overline{\mathbb{R}}^{n+1}_+). \tag{2.18}$$

Proof. To see this, note that

$$\mathrm{FL}^1(\mathbb{R}^n) \text{ is dense in } C_*(\mathbb{R}^n). \tag{2.19}$$

Indeed, since $\mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ and $\mathcal{F}, \mathcal{F}^* : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, we get (2.19) from

$$\mathcal{S}(\mathbb{R}^n) \subset \mathrm{FL}^1(\mathbb{R}^n) \ \text{ and } \ \mathcal{S}(\mathbb{R}^n) \ \text{ is dense in } \ C_*(\mathbb{R}^n). \tag{2.20}$$

To proceed, given $f \in C_*(\mathbb{R}^n)$, produce $f_\nu \in \mathrm{FL}^1(\mathbb{R}^n)$ such that $\sup |f - f_\nu| \le 2^{-\nu}$. Then $\mathrm{PI}\, f_\nu$ is well defined in $C_*(\overline{\mathbb{R}}^{n+1}_+)$, and

$$\sup_{\mathbb{R}^{n+1}_+} |\,\mathrm{PI}\, f_\nu - \mathrm{PI}\, f_\mu| \le \sup_{\mathbb{R}^n} |f_\nu - f_\mu|, \tag{2.21}$$

so $(\mathrm{PI}\, f_\nu)$ is a Cauchy sequence in $C_*(\overline{\mathbb{R}}^{n+1}_+)$. As such, it has a unique limit u in $C_*(\overline{\mathbb{R}}^{n+1}_+)$, and the extension takes $\mathrm{PI}\, f = u$.

REMARK. A little later we will see that $\mathrm{PI}\, f$ is harmonic on $\mathbb{R}^{n+1}_+$, for all $f \in C_*(\mathbb{R}^n)$.

As one attack on a further formula for $\mathrm{PI}\, f$, we get from (2.5) that, if $f, \hat{f} \in L^1(\mathbb{R}^n)$, i.e., $f \in \mathcal{A}(\mathbb{R}^n)$, then

$$\begin{aligned} u(y,x) &= (2\pi)^{-n} \iint f(z) e^{-y|\xi|} e^{i(x-z)\cdot\xi} \, dz \, d\xi \\ &= \int f(z) P_n^{\#}(y, x-z) \, dz, \end{aligned} \tag{2.22}$$

where

$$P_n^{\#}(y,x) = (2\pi)^{-n} \int e^{-y|\xi|} e^{ix\cdot\xi} \, d\xi. \tag{2.23}$$

Here we have used the absolute summability of the double integral in (2.22) to interchange order of integration (using Fubini's theorem). We claim that $P_n^{\#}(y,x)$ is equal to

$$P_n(y,x) = c_n \frac{y}{(y^2+|x|^2)^{(n+1)/2}}, \tag{2.24}$$

with

$$c_n = \pi^{-(n+1)/2}\Gamma\Big(\frac{n+1}{2}\Big). \tag{2.25}$$

This is established in Chapter 3, §5 of [T]. The derivation is not at all straightforward, except in the case $n = 1$. It is straightforward to get from (2.23) that

$$P_1^{\#}(y,z) = P_1(y,z) = \frac{1}{\pi}\,\frac{y}{y^2+x^2}. \tag{2.26}$$

For $n \geq 2$, the derivation of (2.24) given there goes through a "subordination identity," whose proof in turn is somewhat sophisticated. Here, we provide another route to the identity

$$\mathrm{PI}\, f(y,x) = \int f(z)P_n(y,x-z)\,dz, \tag{2.27}$$

with $P_n(y,x)$ given by (2.24).

Even without looking at (2.24), we might expect the formula for $P_n(y,x)$ to have a structure somewhat parallel to (2.26). To get it, we note that the fundamental solution to the Laplacian $\partial_y^2 + \Delta_x$ on $\mathbb{R}^{n+1}$ is equal to a constant times

$$E_n(y,x) = (y^2+|x|^2)^{(2-(n+1))/2} = (y^2+|x|^2)^{(1-n)/2}. \tag{2.28}$$

Now

$$\partial_y E_n(y,x) = (1-n)y(y^2+|x|^2)^{-(n+1)/2}, \tag{2.29}$$

which is indeed a constant times the right side of (2.24). Note that since E_n is harmonic on $\mathbb{R}^{n+1}\setminus(0,0)$, so is $\partial_y E_n$, so we deduce that the function $P_n(y,x)$, given by (2.24), is harmonic on $\mathbb{R}^{n+1}\setminus(0,0)$. Furthermore, for $y > 0$, this function is an integrable function of $x \in \mathbb{R}^n$. In fact, for $y > 0$, we have

$$\begin{aligned} P_n(y,x) &= y^{-n}Q_n\Big(\frac{x}{y}\Big), \\ Q_n(x) &= \frac{c_n}{(1+|x|^2)^{(n+1)/2}}. \end{aligned} \tag{2.30}$$

Clearly

$$\int P_n(y,x)\,dx = \int Q_n(x)\,dx, \quad \forall\, y > 0. \tag{2.31}$$

We pick c_n (below) so that this integral is equal to 1. We deduce from these calculations that, if $f \in C_*(\mathbb{R}^n)$, then

$$u(y,x) = \int f(z)P_n(y,x-z)\,dz \tag{2.32}$$

is harmonic on $\mathbb{R}^{n+1}_+$, and

$$u(y,x) \longrightarrow f(x), \quad \text{uniformly, as } \ y \searrow 0. \tag{2.33}$$

Consequently we can readily verify that, if $f \in C_*(\mathbb{R}^n)$, then u, defined by (2.32), with P_n given by (2.24), solves the Dirichlet problem (2.2)–(2.3). In light of the uniqueness result, Proposition 1.1, we hence have

$$\mathrm{PI}\, f(y,x) = \int f(z) P_n(y, x-z)\, dz, \tag{2.34}$$

with $P_n(y,x)$ given by (2.24).

It remains to compute c_n so that $\int Q_n(x)\, dx = 1$, and verify (2.25). We use spherical polar coordinates on $\mathbb{R}^n$ to write

$$\begin{aligned} \frac{1}{c_n}\int Q_n(x)\, dx &= \int (1+|x|^2)^{-\alpha}\, dx \qquad \Big(\alpha = \frac{n+1}{2}\Big) \\ &= \int_{S^{n-1}} \int_0^\infty (1+r^2)^{-\alpha} r^{n-1}\, dr\, dS(\omega) \\ &= A_{n-1} \int_0^\infty (1+r^2)^{-\alpha} r^{n-1}\, dr \\ &= \frac{A_{n-1}}{2} \int_0^\infty (1+t)^{-\alpha} t^{n/2-1}\, dt. \end{aligned} \tag{2.35}$$

This last integral is given by Euler's beta function, defined for $x, y > 0$ by

$$\begin{aligned} B(x,y) &= \int_0^\infty (1+u)^{-x-y} u^{x-1}\, du \\ &= \int_0^1 s^{x-1} (1-s)^{y-1}\, ds. \end{aligned} \tag{2.36}$$

We have

$$\frac{1}{c_n}\int Q_n(x)\, dx = \frac{A_{n-1}}{2} B\Big(\frac{n}{2}, \frac{1}{2}\Big). \tag{2.37}$$

Now the classical evaluation of $B(x,y)$ (cf. [T], Chapter 3, Appendix A) is

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{2.38}$$

Hence, recalling that $A_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$, we have

$$\frac{1}{c_n}\int Q_n(x)\, dx = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2})} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n+1}{2})} = \frac{\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})}, \tag{2.39}$$

and we have (2.25).

In view of the harmonicity on $\mathbb{R}^{n+1}_+$ of u, given by (2.36), for $f \in C_*(\mathbb{R}^n)$, we have the following result, advertised in the remark below (2.21).

Corollary 2.2. *The extension* PI *given in Proposition 2.1 has the property that*

$$\mathrm{PI} : C_*(\mathbb{R}^n) \longrightarrow C_*(\overline{\mathbb{R}}^{n+1}_+) \cap \{u \in C^2(\mathbb{R}^{n+1}_+) : \Delta u = 0\}. \tag{2.40}$$

3. The Dirichlet problem on a ball

Let $B \subset \mathbb{R}^n$ denote the unit ball $\{x \in \mathbb{R}^n : |x| < 1\}$. (For $n = 2$, we use the notation D, for the unit disk.) The Dirichlet problem on B is

$$\Delta u = 0 \text{ on } B, \quad u\big|_{\partial B} = f, \tag{3.1}$$

for

$$u \in C^2(B) \cap C(\overline{B}), \quad \text{given } f \in C(\partial B). \tag{3.2}$$

In the course of studying Fourier series, we produced the formula

$$u(re^{i\theta}) = \frac{1 - r^2}{2\pi} \int_{-\pi}^{\pi} \frac{f(e^{i\varphi})}{1 - 2r\cos(\theta - \varphi) + r^2}\, d\varphi, \tag{3.3}$$

when $n = 2$. If we switch notation to

$$x = re^{i\theta} \in D, \quad y = e^{i\varphi} \in S^1 = \partial D, \quad ds(y) = d\varphi \text{ (arclength)}, \tag{3.4}$$

and note that

$$\begin{aligned} |x - y|^2 &= (re^{i\theta} - e^{i\varphi})(re^{-i\theta} - e^{-i\varphi}) \\ &= r^2 + 1 - r(e^{i(\theta-\varphi)} + e^{-i(\theta-\varphi)}) \\ &= 1 - 2r\cos(\theta - \varphi) + r^2, \end{aligned} \tag{3.5}$$

we can rewrite (3.3) as

$$u(x) = \frac{1 - |x|^2}{2\pi} \int\limits_{\partial D} \frac{f(y)}{|x - y|^2}\, ds(y), \tag{3.6}$$

for the solution to (3.1)–(3.2), when $n = 2$ and $B = D$.

Moving from dimension 2 to dimension $n \geq 3$, we are motivated to try a formula for the solution to (3.1) of the form

$$u(x) = C_n(1 - |x|^2) \int\limits_{S^{n-1}} \frac{f(y)}{|x - y|^n}\, dS(y). \tag{3.7}$$

We will show that this works, and along the way calculate the constant C_n. First we will show that, for each $f \in C(S^{n-1})$, the function u is harmonic on B. This is a consequence of the following.

Lemma 3.1. *For a given $y \in S^{n-1}$ (i.e., $|y| = 1$), set*

$$v(x) = (1 - |x|^2)|x - y|^{-n}. \tag{3.8}$$

Then v is harmonic on $\mathbb{R}^n \setminus \{y\}$.

Proof. It suffices to show that $w(x) = v(x + y)$ is harmonic on $\mathbb{R}^n \setminus \{0\}$. Since $1 - |x + y|^2 = -(2x \cdot y + |x|^2)$ provided $|y| = 1$, we have

$$-w(x) = 2(y \cdot x)|x|^{-n} + |x|^{2-n}. \tag{3.9}$$

We have already seen that $|x|^{2-n}$ is harmonic on $\mathbb{R}^n \setminus 0$, as a consequence of the formula for Δ acting on radial functions. Now applying $\partial/\partial x_j$ to a smooth harmonic function on an open set in $\mathbb{R}^n$ gives another, so the following are harmonic on $\mathbb{R}^n \setminus 0$:

$$w_j(x) = \frac{\partial}{\partial x_j}|x|^{2-n} = (2 - n)x_n|x|^{-n}. \tag{3.10}$$

For $n = 2$, we take instead

$$\frac{\partial}{\partial x_j} \log |x| = x_j|x|^{-2}. \tag{3.11}$$

Thus the first term on the right side of (3.9) is a linear combination of these functions, so the lemma is proved.

To justify (3.7), it remains to show that if u is given by this formula, and C_n is chosen correctly, then $u = f$ on S^{n-1}. Note that if we write $x = r\omega$, $\omega \in S^{n-1}$, then (3.7) yields

$$u(r\omega) = \int_{S^{n-1}} p(r, \omega, y)f(y)\, dS(y), \tag{3.12}$$

where

$$p(r, \omega, y) = C_n(1 - r^2)|r\omega - y|^{-n}. \tag{3.13}$$

It is clear that

$$p(r, \omega, y) \longrightarrow 0, \text{ as } r \nearrow 1, \text{ if } \omega \neq y, \tag{3.14}$$

with uniform convergence on each compact subset of $\{(\omega, y) \in S^{n-1} \times S^{n-1} : \omega \neq y\}$. We claim that

$$\int_{S^{n-1}} p(r, \omega, y)\, dS(y) = C'_n, \tag{3.15}$$

a constant independent of r and ω. The independence of ω follows by rotational symmetry. Thus we can integrate with respect to ω. But Lemma 3.1 implies that

$$p(r,x,y) = C_n(1-|rx|^2)|rx-y|^{-n} \tag{3.16}$$

is harmonic in x, for $|x| < 1/r$, so the mean value property for harmonic functions gives

$$\frac{1}{A_{n-1}} \int_{S^{n-1}} p(r,\omega,y)\, dS(\omega) = C_n, \tag{3.17}$$

for all $r < 1,\ y \in S^{n-1}$. This implies (3.15), with $C_n' = C_n A_{n-1}$.

By (3.15) and the fact that $p(r,\omega,y)$ is highly peaked near $\omega = y \in S^{n-1}$ as $r \nearrow 1$, the standard approximate identity argument yields that the limit of (3.12) as $r \nearrow 1$ is equal to $C_n A_{n-1} f(\omega)$, for each $f \in C(S^{n-1})$. This justifies the formula (3.7) and fixes the constant: $C_n = 1/A_{n-1}$. We record the conclusion.

Proposition 3.2. *Given $f \in C(S^{n-1})$, the solution in $C(\overline{B}) \cap C^2(B)$ to (3.1) is given by the Poisson integral formula*

$$u(x) = \frac{1-|x|^2}{A_{n-1}} \int_{S^{n-1}} \frac{f(y)}{|x-y|^n}\, dS(y). \tag{3.18}$$

Furthermore, this solution is unique.

Recall that uniqueness follows from Proposition 1.1.

Another way to write the conclusion (3.18) of Proposition 3.2 is

$$u(r\omega) = \frac{1-r^2}{A_{n-1}} \int_{S^{n-1}} \frac{f(y)}{(1-2r\omega\cdot y + r^2)^{n/2}}\, dS(y). \tag{3.19}$$

We define the Poisson integral operator

$$\mathrm{PI} : C(\partial B) \longrightarrow C(\overline{B}) \tag{3.20}$$

as $\mathrm{PI}\, f(x) = u(x)$, given by (3.18), for $x \in B$, and $\mathrm{PI}\, f(x) = f(x)$ for $x \in \partial B$. Proposition 3.2 asserts that PI has this mapping property, and in addition

$$\mathrm{PI} : C(\partial B) \longrightarrow \{u \in C^2(B) : \Delta u = 0\}. \tag{3.21}$$

Let us also record the fact that, for $f \in C(\partial B)$,

$$\sup_{\overline{B}} |\,\mathrm{PI}\, f| = \sup_{\partial B} |f|. \tag{3.22}$$

4. Regularity theorems for harmonic functions

We have defined the Poisson integral operator PI on $C(\partial B)$ in (3.18)–(3.21). It is useful to note that one can apply an arbitrary x derivative ∂_x^α to the right side of (3.18), and supplement (3.21) with the result

$$\mathrm{PI} : C(\partial\Omega) \longrightarrow C^\infty(B). \tag{4.1}$$

We also have, for each multiindex α and $r < 1$,

$$\sup_{|x|\le r} |\partial_x^\alpha \, \mathrm{PI}\, f(x)| \le C_{\alpha,r} \sup_{\partial B} |f|. \tag{4.1A}$$

Translation and dilation of variables allow us to work on a general ball $B_R(p)$, of radius R, centered at $p \in \mathbb{R}^n$, obtaining solution operators

$$\begin{gathered} \mathrm{PI}_{p,R} : C(\partial B_R(p)) \longrightarrow C(\overline{B_R(p)}) \cap C^\infty(B_R(p)), \\ u = \mathrm{PI}_{p,R}\, f \Longrightarrow \Delta u = 0 \text{ on } B_R(p),\ u\big|_{\partial B_R(p)} = f, \end{gathered} \tag{4.2}$$

and, by Proposition 1.1, $\mathrm{PI}_{p,R}\, f$ is the unique solution to such a Dirichlet problem. These observations lead to the following regularity theorem.

Proposition 4.1. *If $\Omega \subset \mathbb{R}^n$ is open and*

$$u \in C^2(\Omega), \quad \Delta u = 0 \text{ on } \Omega, \tag{4.3}$$

then

$$u \in C^\infty(\Omega). \tag{4.4}$$

Proof. Take $p \in \Omega$ and pick $R > 0$ such that $\overline{B_R(p)} \subset \Omega$. It suffices to show that, for each such p, R,

$$u \in C^\infty(B_R(p)). \tag{4.5}$$

Indeed, the observations made above imply that, on $B_R(p)$,

$$u = \mathrm{PI}_{p,R}\, f, \quad f = u\big|_{\partial B_R(p)}, \tag{4.6}$$

and the conclusion (4.5) follows from (4.2).

Our next result establishes regularity for a locally uniform limit of a sequence of harmonic functions.

Proposition 4.2. *Let $\Omega \subset \mathbb{R}^n$ be open. Assume*

$$u_k \in C^\infty(\Omega), \quad \Delta u_k = 0, \quad u_k \to u, \ \text{locally uniformly on } \Omega. \tag{4.7}$$

Then

$$u \in C^\infty(\Omega), \quad \Delta u = 0, \tag{4.8}$$

and, for all α,

$$\partial_x^\alpha u_k \longrightarrow \partial_x^\alpha u, \ \text{ locally uniformly on } \ \Omega. \tag{4.9}$$

Proof. The assumption $u_k \to u$ locally uniformly on Ω implies $u \in C(\Omega)$. To proceed, pick $p \in \Omega$, and $R > 0$ such that $\overline{B_R(p)} \subset \Omega$; hence $u_k \to u$ uniformly on $\overline{B_R(p)}$. Set

$$f_k = u_k\big|_{\partial B_R(p)}, \quad f = u\big|_{\partial B_R(p)}. \tag{4.10}$$

Hence $f_k \to f$ uniformly on $\partial B_R(p)$. Parallel to (3.22), we have

$$\sup_{\overline{B_R(p)}} |\operatorname{PI}_{p,R}(f_k - f)| = \sup_{\partial B_R(p)} |f_k - f|, \tag{4.11}$$

i.e.,

$$\sup_{\overline{B_R(p)}} |u_k - \operatorname{PI}_{p,R} f| = \sup_{\partial B_R(p)} |f_k - f|, \tag{4.12}$$

and taking $k \to \infty$ yields

$$u = \operatorname{PI}_{p,R} f, \quad \text{on } \ B_R(p). \tag{4.13}$$

Since this holds for arbitrary $p \in \Omega$, this gives (4.8). The result (4.9) follows from the fact that, parallel to (4.1A), for $r < R$,

$$\sup_{|x-p|\le r} |\partial_x^\alpha \operatorname{PI}_{p,R}(f_k - f)| \le C_{\alpha,r} \sup_{\partial B_R(p)} |f_k - f|. \tag{4.14}$$

5. Converse of the mean value property

Previous sections have made use of the fact that harmonic functions have the following mean value property.

Proposition 5.1. *Let $\Omega \subset \mathbb{R}^n$ be open, $u \in C^2(\Omega)$ harmonic. Assume $\overline{B_R(x_0)} \subset \Omega$. Then*

$$\operatorname{Avg}_{\partial B_R(x_0)} u = u(x_0), \tag{5.1}$$

and

$$\operatorname{Avg}_{B_R(x_0)} u = u(x_0). \tag{5.2}$$

Just for grins, we recall the proof given in class. Several other proofs are given in [T] and [T2]. To begin, we define ψ on $[0, R]$ by

$$\psi(r) = \int\limits_{S^{n-1}} u(x_0 + r\omega)\, dS(\omega). \tag{5.3}$$

Then

$$\begin{aligned} \psi'(r) &= \int\limits_{S^{n-1}} \omega \cdot \nabla u(x_0 + r\omega)\, dS(\omega) \\ &= r^{n-1} \int\limits_{\partial B_r(x_0)} \partial_\nu u(x)\, dS(x). \end{aligned} \tag{5.4}$$

We use the Green theorem, which implies that, for any smoothly bounded $\overline{\mathcal{O}} \subset \Omega$,

$$\int\limits_{\partial \mathcal{O}} \partial_\nu u\, dS = 0, \quad \text{if } \ \Delta u = 0 \text{ on } \Omega. \tag{5.5}$$

Taking $\mathcal{O} = B_r(x_0)$, we deduce that

$$\psi'(r) = 0, \quad \forall\, r \in (0, R], \tag{5.6}$$

hence $\psi(0) = \psi(r)$, for all $r \in (0, R]$. This gives (5.1). Furthermore,

$$\begin{aligned} \int\limits_{B_R(x_0)} u(x)\, dx &= \int_0^R \int\limits_{S^{n-1}} u(x_0 + r\omega)\, dS(\omega)\, r^{n-1}\, dr \\ &= \int_0^R \psi(r) r^{n-1}\, dr \\ &= u(x_0) A_{n-1} \int_0^R r^{n-1}\, dr \\ &= V(B_R) u(x_0), \end{aligned} \tag{5.7}$$

and we have (5.2).

This result motivates the following concept.

Definition. A function $u \in C(\Omega)$ is said to satisfy the mean value property (MVP) provided (5.2) holds whenever $\overline{B_R(x_0)} \subset \Omega$.

The proof of the strong maximum principle for harmonic functions works without change in the setting of continuous functions satisfying the MVP:

Proposition 5.2. *Let $\Omega \subset \mathbb{R}^n$ be open and connected. Assume $u \in C(\Omega)$ satisfies the MVP, and is real valued. If $x_0 \in \Omega$ and u has a maximum at x_0, the u is constant.*

Proof. Let $X = \{x \in \Omega : u(x) = u_{max}\}$. By hypothesis, $x_0 \in X$. Clearly X is a closed subset of Ω. Now, if $x_1 \in X$ and $\overline{B_r(x_1)} \subset \Omega$, we have

$$\operatorname{Avg}_{B_r(x_1)} u = u(x_1) = u_{max}, \tag{5.5}$$

hence $B_r(x_1) \subset X$, so X is open. If Ω is connected, this forces $X = \Omega$.

We have the following immediate consequence.

Corollary 5.3. *Let $\Omega \subset \mathbb{R}^n$ be bounded and open, $u \in C(\overline{\Omega})$. If u satisfies the MVP on Ω, then*

$$\sup_{\Omega} u = \sup_{\partial\Omega} u, \tag{5.11}$$

and

$$\sup_{\Omega} |u| = \sup_{\partial\Omega} |u|. \tag{5.12}$$

Here is our advertised converse to the mean value property.

Proposition 5.4. *Let $\Omega \subset \mathbb{R}^n$ be open, $u \in C(\Omega)$. Assume u satisfies the MVP. Then $u \in C^\infty(\Omega)$ and $\Delta u = 0$.*

Proof. It suffices to show that if the ball $\overline{B} \subset \Omega$, then u is harmonic on B. To this end, set

$$v = \mathrm{PI}\, f, \quad f = u\big|_{\partial B}. \tag{5.13}$$

Then $v \in C^\infty(B)$ and $\Delta v = 0$ on B. Since both u and v satisfy the MVP on B, so does $w = u - v$. Also $w|_{\partial B} = 0$, so Corollary 5.3 implies $w = 0$ on B, i.e., $u = v$ on B, so u is harmonic on B.

6. Schwarz reflection principle

Let $\Omega \subset \mathbb{R}^n$ be a connected open set with the property that

$$x \in \Omega \Longrightarrow \rho(x) \in \Omega, \tag{6.1}$$

where

$$\rho(x_1, \dots, x_{n-1}, x_n) = (x_1, \dots, x_{n-1}, -x_n). \tag{6.2}$$

Set

$$\Omega^+ = \{x \in \Omega : x_n > 0\}, \quad \Omega^- = \{x \in \Omega : x_n < 0\}, \quad \Sigma = \{x \in \Omega : x_n = 0\}. \tag{6.3}$$

The Schwarz reflection principle states the following.

Proposition 6.1. *Assume $u \in C(\Omega^+ \cup \Sigma)$,*

$$u \text{ is harmonic on } \Omega^+, \quad \text{and} \quad u = 0 \text{ on } \Sigma. \tag{6.4}$$

Define v on Ω by

$$\begin{aligned} v(x) = u(x), &\quad \text{for } x \in \Omega^+ \cup \Sigma, \\ -u(\rho(x)), &\quad \text{for } x \in \Omega^-. \end{aligned} \tag{6.5}$$

Then v is harmonic on Ω.

Proof. Clearly $v \in C(\Omega)$, and v is harmonic on Ω^+ and on Ω^-. It suffices to show that, if

$$p \in \Sigma, \quad \overline{B_r(p)} \subset \Omega, \tag{6.6}$$

then v is harmonic on $B_r(p)$. To this end, set

$$f = v\big|_{\partial B_r(p)}, \quad w = \mathrm{PI}_{p,r}\, f, \tag{6.7}$$

so

$$w \in C(\overline{B_r(p)}), \quad w\big|_{\partial B_r(p)} = f, \quad w \text{ is harmonic on } B_r(p). \tag{6.8}$$

Now $\rho : B_r(p) \to B_r(p)$, and $\rho : \partial B_r(p) \to \partial B_r(p)$, and we have

$$f \circ \rho = -f. \tag{6.9}$$

It follows by symmetry that

$$w \circ \rho = -w. \tag{6.10}$$

Also, of course, $v \circ \rho = -v$. Therefore, if we set

$$\mathcal{O}_\pm = B_r(p) \cap \Omega^\pm, \tag{6.11}$$

we have

$$v - w \in C(\overline{\mathcal{O}}_+), \quad v - w = 0 \text{ on } \partial\mathcal{O}_+, \tag{6.12}$$

and $v - w$ is harmonic on $\mathcal{O}_+$. It follows from Proposition 1.1 that

$$v = w \text{ on } \mathcal{O}_+, \tag{6.13}$$

and then, by (6.10) and its analogue for v,

$$v = w \text{ on } \mathcal{O}_-, \quad \text{hence on } \ B_r(p). \tag{6.14}$$

Thus v is harmonic on $B_r(p)$, and Proposition 6.1 is proved.

7. Liouville theorems

Here we establish the following form of Liouville's theorem.

Proposition 7.1. *If $u \in C^2(\mathbb{R}^n)$ is harmonic on all of $\mathbb{R}^n$ and bounded, then u is constant.*

Proof. Pick any two points $p, q \in \mathbb{R}^n$. We have, for each $r > 0$,

$$u(p) - u(q) = \frac{1}{V(B_r(0))} \Bigl(\int\limits_{B_r(p)} u(x)\, dx - \int\limits_{B_r(q)} u(x)\, dx \Bigr). \tag{7.1}$$

Note that $V(B_r(0)) = C_n r^n$. Thus

$$|u(p) - u(q)| \le \frac{C_n}{r^n} \int\limits_{\Delta(p,q,r)} |u(x)|\, dx, \tag{7.2}$$

where

$$\Delta(p,q,r) = B_r(p) \triangle B_r(q) = \Bigl(B_r(p) \setminus B_r(q) \Bigr) \cup \Bigl(B_r(q) \setminus B_r(p) \Bigr). \tag{7.3}$$

Now, if $a = |p - q|$, then

$$\Delta(p,q,r) \subset B_{r+a}(p) \setminus B_{r-a}(p),$$

hence

$$V(\Delta(p,q,r)) \le C(p,q) r^{n-1}, \quad \text{for} \;\; r \ge a. \tag{7.4}$$

It follows that, if $|u(x)| \le M$ for all $x \in \mathbb{R}^n$, then

$$|u(p) - u(q)| \le M C_n C(p,q) r^{-1}, \quad \forall\, r \ge a. \tag{7.5}$$

Taking $r \to \infty$, we obtain $u(p) - u(q) = 0$, so u is constant.

Alternative approach

Another approach to Liouville's theorem involves estimating ∇u when u is harmonic. To start, suppose

$$u \in C(\overline{B}), \quad \text{harmonic on the interior,} \tag{7.6}$$

with $B = B_1(0)$. We take the Poisson integral formula (3.18) and differentiate. Using

$$\nabla_x |x-y|^{-n} = -n \frac{x-y}{|x-y|^{n+2}}, \tag{7.7}$$

and evaluating the resulting integral at $x = 0$, yields

$$\nabla u(0) = \frac{n}{A_{n-1}} \int\limits_{S^{n-1}} u(y) y \, dS(y). \tag{7.8}$$

Hence

$$\begin{aligned} |\nabla u(0)| &\le \frac{n}{A_{n-1}} \int\limits_{S^{n-1}} |u(y)| \, dS(y) \\ &= n \operatorname{Avg}_{\partial B} |u|. \end{aligned} \tag{7.9}$$

Translating and dilating coordinates, we have the following.

Proposition 7.2. *Let $u \in C(\overline{B_R(p)})$ be harmonic on the interior. Then*

$$|\nabla u(p)| \le \frac{n}{R} \operatorname{Avg}_{\partial B_R(p)} |u|. \tag{7.10}$$

We now give the

Second proof of Proposition 7.1. If u is harmonic on $\mathbb{R}^n$ and $|u(x)| \le M$ for all x, then (7.10) gives, for each $p \in \mathbb{R}^n$, $R > 0$,

$$|\nabla u(p)| \le \frac{n}{R} \sup_{|x-p|=R} |u| \le \frac{nM}{R}. \tag{7.11}$$

Taking $R \to \infty$ gives

$$\nabla u(p) = 0, \quad \forall \, p, \tag{7.12}$$

which implies u is constant.

Going further, suppose that, in Proposition 7.2, u is also ≥ 0 on $B_R(p)$. Then

$$\operatorname{Avg}_{\partial B_R(p)} |u| = \operatorname{Avg}_{\partial B_R(p)} u = u(p),$$

so we have the following.

Proposition 7.3. *Let $u \in C(\overline{B_R(p)})$ be harmonic on the interior and ≥ 0. Then*

$$|\nabla u(p)| \leq \frac{n}{R} u(p). \tag{7.13}$$

With this in hand, we have the following sharper Liouville theorem.

Proposition 7.4. *If $u \in C^2(\mathbb{R}^n)$ is harmonic and ≥ 0 on all of $\mathbb{R}^n$, then u is constant.*

Proof. We see that (7.13) holds for each $p \in \mathbb{R}^n$ and each $R > 0$. Taking $R \to \infty$ again gives (7.12), so u is constant.

Corollary 7.5. *Each function that is harmonic on all $\mathbb{R}^n$ and bounded from below by some $a \in \mathbb{R}$ is constant.*

To obtain other extensions of Liouville's theorem, we return to the setting of (7.6) and derive estimates on higher derivatives of u. Recalling the Poison integral formula,

$$u(x) = \frac{1}{A_{n-1}} \int_{S^{n-1}} P_n(x,y) u(y)\, dS(y), \quad P_n(x,y) = \frac{1 - |x|^2}{|x-y|^n}, \tag{7.14}$$

we have, for $y \in S^{n-1}$,

$$\partial_x^\alpha P_n(0,y) = p_{n\alpha}(y), \quad p_{n\alpha} \in C^\infty(S^{n-1}), \tag{7.15}$$

and

$$\partial^\alpha u(0) = \frac{1}{A_{n-1}} \int_{S^{n-1}} p_{n\alpha}(y) u(y)\, dS(y), \tag{7.16}$$

so

$$|\partial^\alpha u(0)| \leq K_\alpha \operatorname{Avg}_{\partial B} |u|, \quad K_\alpha = \sup_{|y|=1} |p_{n\alpha}(y)|. \tag{7.17}$$

Translating and dilating coordinates, we have the following.

Proposition 7.6. *Let $u \in C(\overline{B_R(p)})$ be harmonic on the interior. Then*

$$|\partial^\alpha u(p)| \leq \frac{K_\alpha}{R^{|\alpha|}} \operatorname{Avg}_{\partial B_R(p)} |u|. \tag{7.18}$$

This estimate enables us to establish the following extension of Liouville's theorem.

Proposition 7.7. *Let u be harmonic on $\mathbb{R}^n$, and assume there exist C_0, C_1, and k such that*

$$|u(x)| \le C_0 + C_1|x|^k, \quad \forall\, x \in \mathbb{R}^n. \tag{7.19}$$

Then $u(x)$ is a polynomial in x, of degree $\le k$.

Proof. For each $x \in \mathbb{R}^n,\ R > 0$, (7.18) applies. The estimate (7.19) implies

$$\sup_{|x-p|\le R} |u(x)| \le C_0 + C_1(R + |p|)^k, \tag{7.20}$$

and then (7.18) implies

$$|\partial^\alpha u(p)| \le \frac{K_\alpha}{R^{k+1}}\big[C_0 + C_1(R + |p|)^k\big], \quad \text{if } \ |\alpha| = k+1. \tag{7.21}$$

Taking $R \to \infty$ yields

$$\partial^\alpha u(p) = 0, \quad \forall\, p \in \mathbb{R}^n,\ |\alpha| = k+1, \tag{7.22}$$

which in turn implies $u(x)$ is a polynomial of degree $\le k$.

8. Removable singularity theorem

The following is a removable singularity theorem for harmonic functions. It extends in several ways the familiar removable singularity theorem for holomorphic functions on planar domains. Take $B = B_1(0) \subset \mathbb{R}^n$.

Proposition 8.1. *Assume* $u \in C^2(B \setminus 0) \cap C(\overline{B} \setminus 0)$ *is harmonic on* $B \setminus 0$ *and bounded, i.e., there exists* $M < \infty$ *such that*

$$|u(x)| \le M, \quad \forall\, x \in \overline{B} \setminus 0. \tag{8.1}$$

Then u *can be extended (in a unique fashion) to be harmonic on all of* B.

Proof. Let $f = u|_{\partial B}$ and set

$$v = \mathrm{PI}\, f, \quad v \in C(\overline{B}) \cap C^2(B). \tag{8.2}$$

We claim that $v = u$ on $B \setminus 0$. To see this, consider $w = u - v$ on $B \setminus 0$. we have

$$w \in C(\overline{B} \setminus 0) \cap C^2(B \setminus 0), \quad \Delta w = 0 \text{ on } B \setminus 0, \quad w\big|_{\partial B} = 0.$$

Also $|w| \le 2M$ on $\overline{B} \setminus 0$. We claim $w \equiv 0$.

To show this, we can assume w is real valued. Now bring in the function

$$H \in C(\overline{B} \setminus 0) \cap C^2(B \setminus 0),$$

given by

$$\begin{aligned} H(x) &= |x|^{2-n} - 1, \quad \text{if } n \ge 3, \\ &\log \frac{1}{|x|}, \qquad\quad \text{if } n = 2. \end{aligned} \tag{8.3}$$

We see that H is harmonic on $B \setminus 0$, $H \ge 0$ on $B \setminus 0$, $H = 0$ on ∂B, and $H(x) \to +\infty$ as $x \to 0$. Hence, for each $\varepsilon > 0$, there exists $\delta_0 > 0$ such that

$$\varepsilon H - w \ge 0 \ \text{ on } \ \partial B_\delta(0), \quad \forall\, \delta \in (0, \delta_0]. \tag{8.4}$$

The maximum principle implies

$$\varepsilon H - w \ge 0, \tag{8.5}$$

on $B \setminus B_\delta(0)$. Taking $\delta \searrow 0$ yields (8.5) on $B \setminus 0$. Then taking $\varepsilon \searrow 0$ yields

$$w \le 0 \ \text{ on } \ B \setminus 0. \tag{8.6}$$

A similar argument gives $w \ge 0$ on $B \setminus 0$, hence $w \equiv 0$, and the proof is complete.

9. Harnack estimates

Harnack inequalities deal with harmonic functions that satisfy one-sided bounds. To start, assume $u \in C(\overline{B})$ is harmonic on $B = B_1(0)$ and ≥ 0. Thus $u|_{\partial B} = f \geq 0$ and u is given by (3.18). Hence, for $x \in B$,

$$\begin{aligned} u(x) &\geq (1-|x|^2) \cdot \min_{|y|=1} |x-y|^{-n} \operatorname{Avg}_{\partial B} f \\ &= \frac{1-|x|^2}{(1+|x|)^n} u(0), \end{aligned} \tag{9.1}$$

so

$$u(x) \geq \frac{1-|x|}{(1+|x|)^{n-1}} u(0), \quad \forall\, x \in B. \tag{9.2}$$

If we omit the hypothesis that u is continuous on $\overline{B}$, and apply this reasoning to $u_b(x) = u(bx)$ and let $b \nearrow 1$, we obtain (9.2) for this more general class. Going further, we can apply translations and dilations, and obtain the following result, known as Harnack's inequality.

Proposition 9.1. *Assume u is harmonic on $B_R(x_0)$ and ≥ 0 there. Then, for all $x_1 \in B_R(x_0)$,*

$$u(x_0) \geq \frac{1 - R^{-1}|x_1 - x_0|}{(1 + R^{-1}|x_1 - x_0|)^{n-1}}\, u(x_0). \tag{9.3}$$

It is useful to complement Harnack's lower bound with an upper bound. Again, we assume $u \in C(\overline{B})$ is harmonic on B and ≥ 0, and complement (9.1) with

$$\begin{aligned} u(x) &\leq (1-|x|^2) \cdot \max_{|y|=1} |x-y|^{-n} \operatorname{Avg}_{\partial B} f \\ &= \frac{1-|x|^2}{(1-|x|)^n}\, u(0), \end{aligned} \tag{9.4}$$

so

$$u(x) \leq \frac{1+|x|}{(1-|x|)^{n-1}}\, u(0), \quad \forall\, x \in B. \tag{9.5}$$

We can remove the hypothesis of continuity on $\overline{B}$ by the dilation argument used above. Further translation and dilation gives the following complement to Proposition 9.1.

Proposition 9.2. *Assume u is harmonic on $B_R(x_0)$ and ≥ 0 there. Then, for all $x_1 \in B_R(x_0)$,*

$$u(x_1) \leq \frac{1 + R^{-1}|x_1 - x_0|}{(1 - R^{-1}|x_1 - x_0|)^{n-1}}\, u(x_0). \tag{9.6}$$

As a first illustration of the use of Harnack's estimate, we give a second proof of the Liouville theorem, Proposition 7.4, whose statement we recall.

Proposition 9.3. *Assume $u \in C^2(\mathbb{R}^n)$ is harmonic and ≥ 0 on all of $\mathbb{R}^n$. Then u is constant.*

Proof. Given $x_0, x_1 \in \mathbb{R}^n$, we can take $R > |x_1 - x_0|$ and apply (9.3). Taking $R \to \infty$ then gives

$$u(x_1) \geq u(x_0), \quad \forall\, x_0, x_1 \in \mathbb{R}^n.$$

Reversing roles gives $u(x_0) \geq u(x_1)$, so u is constant.

We will proceed in §10 with a further extension of Liouville's theorem.

10. Further Liouville theorems

Here we show how the Harnack estimates lead to further versions of Liouville's theorem. The following will provide a key step.

Proposition 10.1. *For each $n \geq 2$, there exist constants $K_n \in (0, \infty)$ with the following property. Let u be harmonic on $B_R(0) \subset \mathbb{R}^n$. Assume*

$$u(0) = 0, \quad u(x) \leq M \quad \text{on} \quad B_R(0). \tag{10.1}$$

Then

$$u(x) \geq -K_n M \quad \text{on} \quad B_{R/2}(0). \tag{10.2}$$

Proof. Apply Proposition 9.3, with M replaced by $M - u$, which is ≥ 0 on $B_R(0)$, and equal to M at $x_0 = 0$. We see that

$$|x_1| = \frac{R}{2} \Longrightarrow M - u(x_1) \leq \frac{3}{2} 2^{n-1} M, \tag{10.3}$$

so (10.2) holds with $K_n = 3 \cdot 2^{n-2} - 1$.

As a first application of Proposition 10.1, we present a

Third proof of Proposition 7.4. If $v \geq 0$ is harmonic on all of $\mathbb{R}^n$, then $u(x) = v(0) - v(x)$ satisfies (10.1), with $M = v(0)$, for all R, so (10.2) implies $u(x) \geq -K_n v(0)$ for all $x \in \mathbb{R}^n$. Hence u is harmonic and bounded on $\mathbb{R}^n$, so the fact that u is constant follows from the first Liouville theorem, Proposition 7.1.

The following result leads to an important sharpening of Proposition 7.7

Proposition 10.2. *Assume that u is harmonic on $\mathbb{R}^n$ and that there exist $C_0, C_1 \in (0, \infty)$ and $k \in \mathbb{N}$ such that*

$$u(x) \leq C_0 + C_1 |x|^k, \quad \forall\, x \in \mathbb{R}^n. \tag{10.4}$$

Then there exist $C_2, C_3 \in (0, \infty)$ such that

$$u(x) \geq -C_2 - C_3 |x|^k, \quad \forall\, x \in \mathbb{R}^n. \tag{10.5}$$

Proof. Apply Proposition 10.1 to

$$u(x) - u(0), \quad M = C_0 + |u(0)| + C_1 R^k.$$

Putting together Propositions 10.2 and 7.7, we have the following.

Corollary 10.3. *If u is harmonic on $\mathbb{R}^n$ and satisfies (10.4), then $u(x)$ is a polynomial in x, of degree $\le k$.*

Here is a basic application of Corollary 10.3 to complex function theory.

Proposition 10.4. *Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic. Assume there is a bound*

$$|e^{f(z)}| \le C e^{A|z|}, \quad \forall\, z \in \mathbb{C}. \tag{10.6}$$

Then $f(z)$ has the form

$$f(z) = az + b. \tag{10.7}$$

Proof. The bound (10.6) implies

$$\operatorname{Re} f(z) \le A|z| + A'. \tag{10.8}$$

Hence, by Corollary 10.3,

$$\operatorname{Re} f(z) = \alpha x + \beta y + \gamma, \tag{10.9}$$

with $z = x + iy$. Consequently the harmonic conjugate $v(x, y)$ of $u(x, y) = \operatorname{Re} f(z)$ is also a first-order polynomial in x, y. This yields (10.7).

References

[T] M. Taylor, Partial Differential Equations, Vols. 1–3, Springer, New York 1996 (2nd ed., 2011).

[T2] M. Taylor, Introduction to Analysis in Several Variables – Advanced Calculus, AMS 2020.