Multidimensional Toeplitz Operators with Locally Sectorial Symbols

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Abstract

We study a class of Toeplitz operators with discontinuous symbols, stimulated by classical work of R. Douglas and H. Widom. We extend the notion of a locally sectorial symbol from the setting of scalar Toeplitz operators on the circle to systems, acting on sections of vector bundles over a class of multidimensional domains with minimal smoothness, known as uniformly rectifiable domains, and establish Fredholm properties in this expanded setting.

1 Introduction

Let P be the orthogonal projection on $L^2(\mathbb{T})$ given by

$$P\Big(\sum_{k=-\infty}^{\infty} a_k e^{ik\theta}\Big) = \sum_{k=0}^{\infty} a_k e^{ik\theta}, \tag{1.1}$$

where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. If we have a complex-valued $f \in L^\infty(\mathbb{T})$, then the Toeplitz operator T_f can be defined on $L^2(\mathbb{T})$ by

$$T_f u = PfPu + (I-P)u. \tag{1.2}$$

Let us assume f is bounded away from 0, so

$$f, f^{-1} \in L^\infty(\mathbb{T}). \tag{1.3}$$

The most basic result in the theory is that if in addition f is continuous, then T_f is Fredholm on $L^2(\mathbb{T}^1)$, and its index is the negative of the winding

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number of $f : \mathbb{T} \to \mathbb{C} \setminus 0$. In [7] it was shown that T_f is Fredholm whenever f satisfies (1.3) and is *locally sectorial*, i.e., has the property that for each $\theta_0 \in \mathbb{T}$, there is a neighborhood of θ_0 on which $\arg f$ varies by less than π. Furthermore, the notion of winding number extends to this setting, to yield $\iota(f) \in \mathbb{Z}$, and

$$\operatorname{Index} T_f = -\iota(f). \tag{1.4}$$

Here we will discuss some generalizations of this result. These include both Toeplitz operators acting on functions on $\mathbb{T}$, for broader classes of symbols f, discussed in §3, and classes of Toeplitz operators acting on functions on $\partial\Omega$, when Ω is a bounded, uniformly rectifiable domain in $\mathbb{R}^n$ (or in an n-dimensional manifold), discussed in §4.

Before tackling these extensions, we devote §2 to a discussion of the proof of the Douglas-Widom result. The approach taken here is adapted from [7], but there are some differences, which will suggest ways to extend the result. As in [7], we start with a factorization

$$f(\theta) = F(\theta)G(\theta), \quad \theta \in \mathbb{T}. \tag{1.5}$$

Here, the factors satisfy

$$F \in C(\mathbb{T}),\ F(\theta) \neq 0, \quad \operatorname{Re} G(\theta) \geq b > 0, \quad \forall\, \theta \in \mathbb{T}. \tag{1.6}$$

This is a slightly different specification from that of [7], and it allows us to use an endgame that brings in the Lax-Milgram theorem, in place of results of [3].

As indicated above, in §3 we treat further classes of Toeplitz operators on $\mathbb{T}$. For one, we take $f \in L^\infty(\mathbb{T})$, having a factorization (1.5), with G as in (1.6) but the hypothesis on F generalized to

$$F \in L^\infty \cap \operatorname{vmo}(\mathbb{T}), \quad F^{-1} \in L^\infty(\mathbb{T}). \tag{1.7}$$

This hypothesis actually implies $F^{-1} \in L^\infty \cap \operatorname{vmo}(\mathbb{T})$. Here $\operatorname{vmo}(\mathbb{T})$ denotes the space of functions with vanishing mean oscillation, a class introduced by [13] and widely studied since. Applicability of the more general hypothesis (1.7) to Fredholm properties of T_f makes use of commutator estimates of [4], whose ramifications have also had wide use in analysis.

Another extension has $f(\theta)$ taking values in the space $M(\ell, \mathbb{C})$ of complex $\ell \times \ell$ matrices, with u in (1.2) taking values in $\mathbb{C}^\ell$. In this situation, our condition on f is that it has a factorization (1.5) with

$$F \in C(\mathbb{T}, G\ell(\ell, \mathbb{C})), \quad G(\theta) + G(\theta)^* \geq bI > 0, \quad \forall\, \theta \in \mathbb{T}. \tag{1.8}$$

We can further generalize the hypothesis on F, as in (1.7), i.e., $F, F^{-1} \in L^\infty \cap \mathrm{vmo}(\mathbb{T})$. In such a case, index computations are aided by results of [2].

A third extension discussed in §3 is that, given such f, there exist $p_0 \in (1,2)$ and $p_1 \in (2,\infty)$ such that T_f is Fredholm on $L^p(\mathbb{T})$ for $p \in (p_0, p_1)$.

The space $\mathbb{T}$ is naturally identified with the boundary $\partial\mathcal{D}$ of the unit disk $\mathcal{D}$ in $\mathbb{C}$, with P representing the orthogonal projection of $L^2(\partial\mathcal{D})$ onto the subspace of $L^2(\partial\mathcal{D})$ consisting of boundary values of functions holomorphic on $\mathcal{D}$. Natural extensions have been pursued, with $\mathcal{D}$ replaced by more general bounded open sets Ω in $\mathbb{C}$. These extensions become quite nontrivial when $\partial\Omega$ is not smooth. Of particular note is the work of [11], extending the analysis of P to Szegö projectors on functions on $\partial\Omega$, for various classes of bounded domains $\Omega \subset \mathbb{C}$ with finite perimeter, such as Lipschitz domains. Our treatment here of Toeplitz operators associated to such domains will be folded into §4.

In §4 we deal with a class of Toeplitz operators introduced in [12], and extend the Fredholm theory of such operators in a fashion parallel to that indicated above. In this setting, we let Ω be a bounded, uniformly rectifiable domain (UR domain) in $\mathbb{R}^n$, or more generally a UR domain in a compact, n-dimensional Riemannian manifold M. (See §4 for a definition of the class of UR domains.) We assume we have a first-order elliptic differential operator $D : C^\infty(M, \mathcal{E}_0) \to C^\infty(M, \mathcal{E}_1)$, acting on sections of a vector bundle $\mathcal{E}_0 \to M$. We consider the space

$$\mathcal{H}^p(\Omega, D) = \{u \in C^1(\Omega, \mathcal{E}_0) : Du = 0,\ \mathcal{N}u \in L^p(\partial\Omega),\ u\big|_{\partial\Omega} \text{ exists a.e.}\}, \tag{1.9}$$

for $p \in (1,\infty)$. Here $\mathcal{N}u$ denotes the non-tangential maximal function. Under a mild unique continuation hypothesis on D (satisfied, for example, when D is of Dirac type, or when D has analytic coefficients), the trace map $\tau(u) = u|_{\partial\Omega}$ is injective and yields an isomorphism:

$$\tau : \mathcal{H}^p(\Omega, D) \xrightarrow{\approx} \mathcal{H}^p(\partial\Omega, D), \tag{1.10}$$

onto a closed linear subspace $\mathcal{H}^p(\partial\Omega, D)$ of $L^p(\partial\Omega, \mathcal{E}_0)$. As described in more detail in §4, the work [12] produced a projection

$$\mathcal{P} : L^p(\partial\Omega, \mathcal{E}_0) \longrightarrow \mathcal{H}^p(\partial\Omega, D), \tag{1.11}$$

and studied Toeplitz operators T_f, of the form (1.2), with P replaced by $\mathcal{P}$, given $u \in L^p(\partial\Omega, \mathcal{E}_0 \otimes \mathbb{C}^\ell)$, with $f \in L^\infty(\partial\Omega, M(\ell, \mathbb{C}))$, obtaining Fredholm properties for

$$f \in C(\partial\Omega, G\ell(\ell, \mathbb{C})), \tag{1.12}$$

and more generally for

$$f, f^{-1} \in L^\infty \cap \mathrm{vmo}(\partial\Omega, M(\ell, \mathbb{C})). \tag{1.13}$$

The paper [12] also looked at the orthogonal projection

$$S : L^2(\partial\Omega, \mathcal{E}_0) \longrightarrow \mathcal{H}^2(\partial\Omega, D), \tag{1.14}$$

and associated Toeplitz operators, which we denote $\mathcal{T}_f$. In case $\Omega = \mathcal{D} \subset \mathbb{C}$ and $D = \partial/\partial\bar{z}$, the operators $\mathcal{P}$ and S coincide (with P in (1.1)), but in general they differ. More on their relation is discussed in §4. As shown in §4, Fredholm results on such Toeplitz operators extend to symbols f, having a factorization (1.5) with

$$F \in C(\partial\Omega, G\ell(\ell, \mathbb{C})), \quad G(x) + G(x)^* \geq bI > 0, \quad \forall\, x \in \partial\Omega, \tag{1.15}$$

and, more generally,

$$F,\ F^{-1} \in L^\infty \cap \mathrm{vmo}(\partial\Omega, M(\ell, \mathbb{C})). \tag{1.16}$$

In Theorem 4.1 we obtain such Fredholm results for T_f under the additional hypothesis that

$$\mathcal{P} - \mathcal{P}^* \ \text{ is compact on } \ L^2(\partial\Omega). \tag{1.17}$$

In Theorem 4.4 we obtain such Fredholm results for $\mathcal{T}_f$ (with no need for (1.17)). The hypothesis (1.17) holds, for example, when D is of Dirac type and Ω is a regular SKT domain. (See §4 for a definition.) Theorem 4.4 holds for general UR domains. We end §4 with the observation that $f \in L^\infty(\partial\Omega, M(\ell, \mathbb{C}))$ has a factorization $f = FG$ satisfying (1.15) provided it satisfies the following version of a local sectorial condition: namely, for each $y \in \partial\Omega$ there exist a neighborhood $\mathcal{O}_y$ of y in $\partial\Omega$ and a matrix $C_y \in M(\ell, \mathbb{C})$ such that

$$\mathrm{Re}\, C_y f(x) \geq bI > 0, \quad \forall\, x \in \mathcal{O}_y, \tag{1.18}$$

where $\mathrm{Re}\, T = (1/2)(T + T^*)$.

In §5 we briefly discuss how to extend Theorems 4.1 and 4.4, to treat $f \in L^\infty(\partial\Omega, M(\ell, \mathbb{C}))$ having a factorization

$$f(x) = F_1(x)G(x)F_2(x), \tag{1.19}$$

with $F_j, F_j^{-1} \in L^\infty \cap \mathrm{vmo}(\partial\Omega, M(\ell, \mathbb{C}))$ and $G(x)$ as in (1.15). We also discuss a variant of (1.18) as a sufficient condition for (1.19).

2 The Douglas-Widom theorem

Let $f \in L^\infty(\mathbb{T})$ be a complex-valued function satisfying (1.3), having the local sectorial property, defined in §1. In preparation for deriving the Fredholm and index properties of T_f established in [7], we begin by discussing further the structure of such a locally sectorial symbol.

Since $\mathbb{T}$ is compact, we can cover it with a finite number of open subsets I_j, $1 \le j \le k$, and pick $\delta > 0$ such that, on each I_j, $\arg f$ varies by an amount that is $\le \pi - \delta$. It then follows that there exist $b > 0$ and $c_j \in \mathbb{C}$ such that

$$\operatorname{Re} c_j f(\theta) \ge b > 0, \quad \text{for} \quad \theta \in I_j. \tag{2.1}$$

Now let $\{\psi_j : 1 \le j \le k\}$ be a continuous partition of unity on $\mathbb{T}$ subordinate to the cover $\{I_j : 1 \le j \le k\}$, so $\operatorname{supp} \psi_j \subset I_j$ and $0 \le \psi_j \le 1$. Then set

$$\Phi(\theta) = \sum_{j=1}^{k} c_j \psi_j(\theta). \tag{2.2}$$

We have $\Phi \in C(\mathbb{T})$ and

$$\operatorname{Re} \Phi(\theta) f(\theta) \ge b > 0, \quad \forall\, \theta \in \mathbb{T}. \tag{2.3}$$

It follows that $\Phi(\theta)$ is bounded away from 0 so, with

$$F = \Phi^{-1} \in C(\mathbb{T}), \quad G = \Phi f, \tag{2.4}$$

we have the following.

Lemma 2.1 *If $f \in L^\infty(\mathbb{T})$ is a complex-valued function satisfying (1.3) and f is locally sectorial, then f has a factorization*

$$f(\theta) = F(\theta) G(\theta), \quad \theta \in \mathbb{T}, \tag{2.5}$$

with

$$F, F^{-1} \in C(\mathbb{T}), \quad \operatorname{Re} G(\theta) \ge b > 0, \quad \forall\, \theta \in \mathbb{T}. \tag{2.6}$$

Fredholm properties of T_{FG} follow from the results given in the next two lemmas.

Lemma 2.2 *Given $F \in C(\mathbb{T}), G \in L^\infty(\mathbb{T})$,*

$$T_{FG} - T_F T_G \quad \text{is compact on } L^2(\mathbb{T}^1). \tag{2.7}$$

Proof. We have

$$T_{FG} - T_F T_G = PFGP - PFPGP \\ = P[P,F]GP, \tag{2.8}$$

and, as is well known, $[P, F]$ is compact on $L^2(\mathbb{T})$ whenever F is continuous. Of course, the other factors on the last line of (2.8) are bounded on $L^2(\mathbb{T})$. □

Lemma 2.3 *Given $G \in L^\infty(\mathbb{T})$, if for some $b > 0$,*

$$\operatorname{Re} G(\theta) \geq b, \quad \forall\, \theta \in \mathbb{T}, \tag{2.9}$$

then T_G is invertible on $L^2(\mathbb{T})$.

Proof. For $u \in L^2(\mathbb{T})$,

$$\operatorname{Re}(T_G u, u)_{L^2} = \operatorname{Re}(GPu, Pu)_{L^2} + \|(I-P)u\|_{L^2}^2 \\ \geq b\|Pu\|_{L^2}^2 + \|(I-P)u\|_{L^2}^2, \tag{2.10}$$

so the invertibility of T_G on $L^2(\mathbb{T})$ follows from the Lax-Milgram theorem. □

In the situation arising from (2.5), we have from the standard theory of Toeplitz operators with continuous symbol that T_F is Fredholm, of index $-\nu_F$, where

$$\nu_F = \text{ winding number of } F : \mathbb{T} \to \mathbb{C} \setminus 0. \tag{2.11}$$

and T_G is invertible on $L^2(\mathbb{T})$, so $T_F T_G$ is Fredholm on $L^2(\mathbb{T}^1)$, of index $-\nu_F$. Given (2.7), we have the following conclusion, due to [7].

Theorem 2.4 *If $f \in L^\infty(\mathbb{T}^1)$ is a complex-valued function satisfying (1.3) and if f is locally sectorial, then*

$$T_f \text{ is Fredholm, of index } -\nu_F, \text{ on } L^2(\mathbb{T}), \tag{2.12}$$

with ν_F as in (2.11).

3 One-dimensional extensions

In the introduction we mentioned three extensions of the results on Toeplitz operators discussed in §2. Here we treat all three simultaneously. We take P as in (1.1), acting on functions with values in $\mathbb{C}^\ell$, and define T_f as in (1.2), for $f \in L^\infty(\mathbb{T}, M(\ell, \mathbb{C}))$.

Theorem 3.1 *Assume f has the factorization $f = FG$, with*

$$
\begin{aligned}
&F,\ F^{-1} \in L^\infty \cap \mathrm{vmo}(\mathbb{T}, M(\ell,\mathbb{C})), \\
&G(\theta) + G(\theta)^* \ge bI > 0, \quad \forall\, \theta \in \mathbb{T}.
\end{aligned} \tag{3.1}
$$

Then there exist $p_0 \in (1,2)$ and $p_1 \in (2,\infty)$ such that

$$
T_f : L^p(\mathbb{T},\mathbb{C}^\ell) \longrightarrow L^p(\mathbb{T},\mathbb{C}^\ell) \ \ \textit{is Fredholm, for} \ \ p \in (p_0,p_1), \tag{3.2}
$$

and

$$
\mathrm{Index}\, T_f = \mathrm{Index}\, T_F. \tag{3.3}
$$

The proof starts with the following extension of Lemma 2.2.

Lemma 3.2 *Given $F \in L^\infty \cap \mathrm{vmo}(\mathbb{T}, M(\ell,\mathbb{C}))$ and $G \in L^\infty(\mathbb{T}, M(\ell,\mathbb{C}))$,*

$$
T_{FG} - T_F T_G \ \ \textit{is compact on} \ \ L^p(\mathbb{T},\mathbb{C}^\ell), \quad \forall\, p \in (1,\infty). \tag{3.4}
$$

Proof. As in (2.8), we have

$$
T_{FG} - T_F T_G = P[P,F]GP. \tag{3.5}
$$

All the factors on the right side are bounded on $L^p(\mathbb{T},\mathbb{C}^\ell)$ for each $p \in (1,\infty)$. It is a consequence of commutator estimates of [4] that, if $F \in L^\infty \cap \mathrm{vmo}(\mathbb{T}, M(\ell,\mathbb{C}))$, then

$$
[P,F] \ \ \text{is compact on} \ \ L^p(\mathbb{T},\mathbb{C}^\ell), \quad \forall\, p \in (1,\infty). \tag{3.6}
$$

Such commutator estimates in a more general setting, which will be directly applicable in §4, are given in §2.4 of [8]. ☐

The next lemma helps to finish the proof of Theorem 3.1.

Lemma 3.3 *In the setting of Theorem 3.1, there exist $p_0 \in (1,2)$ and $p_1 \in (2,\infty)$ such that*

$$
T_G \ \ \textit{is invertible on} \ \ L^p(\mathbb{T},\mathbb{C}^\ell), \ \ \textit{for} \ \ p \in (p_0,p_1). \tag{3.7}
$$

Proof. We start with $p = 2$. Parallel to (2.10), for $u \in L^2(\mathbb{T},\mathbb{C}^\ell)$,

$$
\begin{aligned}
\mathrm{Re}(T_G u, u)_{L^2} &= \mathrm{Re}(GPu, Pu)_{L^2} + \|(I-P)u\|_{L^2}^2 \\
&\ge \frac{b}{2}\|Pu\|_{L^2}^2 + \|(I-P)u\|_{L^2}^2,
\end{aligned} \tag{3.8}
$$

so the invertibility of T_G on $L^2(\mathbb{T}, \mathbb{C}^\ell)$ follows from the Lax-Milgram theorem. The invertibility on $L^p(\mathbb{T}, \mathbb{C}^\ell)$ then follows from an extrapolation result of [15]. □

The last ingredient in the proof of Theorem 3.1 is the fact that, when F satisfies (3.1), then

$$T_F \text{ is Fredholm on } L^p(\mathbb{T}, \mathbb{C}^\ell), \quad \forall\, p \in (1, \infty). \tag{3.9}$$

In fact, Lemma 3.2 implies that, for all $p \in (1, \infty)$,

$$T_F T_{F^{-1}} - I \text{ and } T_{F^{-1}} T_F - I \text{ are compact on } L^p(\mathbb{T}, \mathbb{C}^\ell). \tag{3.10}$$

REMARK. When F satisfies (3.1), it is fairly easy to show that the index of T_F on $L^p(\mathbb{T}, \mathbb{C}^\ell)$ is independent of $p \in (1, \infty)$; cf. (4.2.16)–(4.2.18) of [12]. In case $F \in C(\mathbb{T}, G\ell(\ell, \mathbb{C}))$, it is well known that $\operatorname{Index} T_F$ is equal to the negative of the winding number of $\det F : \mathbb{T} \to \mathbb{C} \setminus 0$. Extensions to more general F satisfying (3.1) are given in [2] and, in a more general setting, applicable to material in §4, in §4.2 of [12].

We end this section with the observation that the partition of unity argument used to prove Lemma 2.1 also establishes the following.

Proposition 3.4 *Take $f \in L^\infty(\mathbb{T}, M(\ell, \mathbb{C}))$ and assume that for each $\xi \in \mathbb{T}$ there is a neighborhood $\mathcal{O}_\xi$ of ξ and $C_\xi \in M(\ell, \mathbb{C})$ such that*

$$\operatorname{Re} C_\xi f(\theta) \ge bI > 0, \quad \forall\, \theta \in \mathcal{O}_\xi, \tag{3.11}$$

where $\operatorname{Re} T = (1/2)(T + T^*)$. Then f has a factorization $f = FG$ with

$$F \in C(\mathbb{T}, G\ell(\ell, \mathbb{C})) \tag{3.12}$$

and G as in (3.1).

4 Higher-dimensional variants

As advertised in the introduction, here we discuss a class of Toeplitz operators that act on functions on $\partial\Omega$, when Ω is a UR domain in a compact, n-dimensional Riemannian manifold M. This is to say, $\nabla \chi_\Omega$ is a finite vector measure on M, supported on $\partial\Omega$, whose total variation σ coincides with

$(n-1)$-dimensional Hausdorff measure on $\partial\Omega$, and satisfies the following additional properties. First is Ahlfors regularity:

$$c_0 r^{n-1} \le \sigma(B_r(x_0) \cap \partial\Omega) \le c_1 r^{n-1}, \quad \forall\, x_0 \in \partial\Omega, \tag{4.1}$$

for $r \in (0,1]$. We call such Ω an Ahlfors regular domain. Such Ω is said to be a UR domain if, in addition, $\partial\Omega$ contains "big pieces of Lipschitz surfaces," in the sense that there exist $\varepsilon, L \in (0,\infty)$ such that, for each $x \in \partial\Omega$, $R \in (0,1]$, there is a Lipschitz map $\varphi : B_R^{n-1} \to M$, with Lipschitz constant $\le L$, such that

$$\mathcal{H}^{n-1}(\partial\Omega \cap B_R(x) \cap \varphi(B_R^{n-1})) \ge \varepsilon R^{n-1}. \tag{4.2}$$

Here, B_R^{n-1} is a ball of radius R in $\mathbb{R}^{n-1}$.

Let $D : C^\infty(M,\mathcal{E}_0) \to C^\infty(M,\mathcal{E}_1)$ be a first order elliptic differential operator, and, for $1 < p < \infty$, define $\mathcal{H}^p(\Omega, D)$ as in (1.9), yielding a closed linear subspace $\mathcal{H}^p(\partial\Omega, D) \subset L^p(\partial\Omega, \mathcal{E}_0)$ under the trace isomorphism (1.10). A construction of the projection $\mathcal{P}$ in (1.11) arises via the following Cauchy transform:

$$\mathcal{C}g(x) = i \int\limits_{\partial\Omega} E(x,y)\sigma_D(y,\nu(y))g(y)\, d\sigma(y), \quad x \in \Omega, \tag{4.3}$$

and associated principal value singular integral operator

$$Cg(x) = i\, \mathrm{PV} \int\limits_{\partial\Omega} E(x,y)\sigma_D(y,\nu(y))g(y)\, d\sigma(y), \quad x \in \partial\Omega. \tag{4.4}$$

Here σ_D is the principal symbol of D, $\nu(y)$ the outward unit normal to $\partial\Omega$ at y (well defined for σ-a.e. y), and $E(x,y)$ a fundamental solution to D, defined on a neighborhood of $\overline{\Omega}$. Results initiated in the seminal work [5] and pursued further in [8] imply that C exists as a bounded operator on $L^p(\partial\Omega)$ for $1 < p < \infty$, and further results of [8] yield nontangential maximal function estimates on $\mathcal{C}g$, for $g \in L^p(\partial\Omega)$, and also pointwise nontangential limits

$$\mathcal{C}g\big|_{\partial\Omega}(x) = \Bigl(\frac{1}{2}I + C\Bigr)g(x), \quad \text{for } \sigma\text{-a.e. } x \in \partial\Omega. \tag{4.5}$$

In (4.3)–(4.5), g belongs to $L^p(\partial\Omega,\mathcal{E}_0)$, or more generally to $L^p(\partial\Omega,\mathcal{E}_0 \otimes \mathbb{C}^\ell)$, but to lighten the notation we say $g \in L^p(\partial\Omega)$. As shown in §2.3 of [12], we have

$$\mathcal{C} : L^p(\partial\Omega) \longrightarrow \mathcal{H}^p(\Omega, D), \tag{4.6}$$

and

$$\mathcal{C}g\big|_{\partial\Omega} = \Big(\frac{1}{2}I + C\Big) = \mathcal{P}g \Longrightarrow \mathcal{P}^2 = \mathcal{P}, \tag{4.7}$$

and $\mathcal{P}$ is a projection of $L^p(\partial\Omega)$ onto $\mathcal{H}^p(\partial\Omega, D)$.

We are now in a position to define Toeplitz operators. Given $f \in L^\infty(\partial\Omega, M(\ell, \mathbb{C}))$, T_f acts on sections of $\mathcal{E}_0 \otimes \mathbb{C}^\ell$ by

$$T_f u = \mathcal{P} f \mathcal{P} u + (I - \mathcal{P})u. \tag{4.8}$$

We have $T_f : L^p(\partial\Omega) \to L^p(\partial\Omega)$ for $p \in (1, \infty)$. The following result extends Theorem 3.1.

Theorem 4.1 *Assume $f \in L^\infty(\partial\Omega, M(\ell, \mathbb{C}))$ has the factorization $f = FG$ with*

$$\begin{aligned} &F,\ F^{-1} \in L^\infty \cap \mathrm{vmo}(\partial\Omega, M(\ell, \mathbb{C})), \\ &G(x) + G(x)^* \geq bI > 0, \quad \forall\, x \in \partial\Omega. \end{aligned} \tag{4.9}$$

Assume in addition that

$$\mathcal{P} - \mathcal{P}^* \quad \textit{is compact on} \quad L^2(\partial\Omega). \tag{4.10}$$

Then there exist $p_0 \in (1, 2)$ and $p_1 \in (2, \infty)$ such that

$$T_f \quad \textit{is Fredholm on } L^p(\partial\Omega), \textit{ for } \ p \in (p_0, p_1), \tag{4.11}$$

and

$$\mathrm{Index}\, T_f = \mathrm{Index}\, T_F. \tag{4.12}$$

The proof of Theorem 4.1 follows along the lines of that of Theorem 3.1, except that it depends on results from [8] and [12], rather than on more classical results. It starts with the following lemma.

Lemma 4.2 *Given $F \in L^\infty \cap \mathrm{vmo}(\partial\Omega, M(\ell, \mathbb{C}))$ and $G \in L^\infty(\partial\Omega, M(\ell, \mathbb{C}))$,*

$$T_{FG} - T_F T_G \quad \textit{is compact on} \quad L^p(\partial\Omega), \quad \forall\, p \in (1, \infty). \tag{4.13}$$

Proof. An identity parallel to (3.5) reduces the proof to establishing compactness of

$$[\mathcal{P}, F] = [C, F] \tag{4.14}$$

on $L^p(\partial\Omega)$, for $p \in (1, \infty)$. In this case, such compactness follows from commutator estimates in §2.4 of [8]. □

We proceed with the following extension of Lemma 3.3.

Lemma 4.3 *In the setting of Theorem 4.1 (particularly assuming* (4.10)*), there exist* $p_0 \in (1,2)$ *and* $p_1 \in (2,\infty)$ *such that*

$$T_G \ \text{ is Fredholm of index } 0 \text{ on } \ L^p(\partial\Omega), \quad \text{for } \ p \in (p_0, p_1). \tag{4.15}$$

Proof. It is convenient to bring in S, the orthogonal projection of $L^2(\partial\Omega, \mathcal{E}_0 \otimes \mathbb{C}^\ell)$ onto $\mathcal{H}^2(\partial\Omega, D)$, which will be discussed at further length below. The main fact that we need to know is that (4.10) implies

$$\mathcal{P} - S \ \text{ is compact on } L^p(\partial\Omega), \text{ for } \ p \in (q_0, q_1), \tag{4.16}$$

for some $q_0 \in (1,2)$, $q_1 \in (2,\infty)$. This is a consequence of (4.23)–(4.24) below. Consequently, for such p, T_G differs from

$$\mathcal{T}_G = SGS + (I - S) \tag{4.17}$$

by an operator that is compact on $L^p(\partial\Omega)$ for $p \in (q_0, q_1)$. Now, a straightforward variant of (3.11) yields the invertibility of $\mathcal{T}_G$ on $L^2(\partial\Omega)$. In fact, we have

$$\begin{aligned} \operatorname{Re}(\mathcal{T}_G u, u)_{L^2} &= \operatorname{Re}(GSu, Su)_{L^2} + \|(I-S)u\|_{L^2}^2 \\ &\geq \frac{1}{2} b \|Su\|_{L^2}^2 + \|(I-S)u\|_{L^2}^2, \end{aligned} \tag{4.18}$$

so the invertibility of $\mathcal{T}_G$ on $L^2(\partial\Omega)$ follows from the Lax-Milgram theorem. As seen below, $\mathcal{T}_G$ is bounded on $L^p(\partial\Omega)$ for $p \in (q_0, q_1)$. Hence it follows from [Sn] that there exist $p_0 \in (q_0, 2)$ and $p_1 \in (2, p_1)$ such that

$$\mathcal{T}_G \ \text{ is invertible on } L^p(\partial\Omega), \text{ for } \ p \in (p_0, p_1). \tag{4.19}$$

Also, (4.16) implies $T_G - \mathcal{T}_G$ is compact on $L^p(\partial\Omega)$ for $p \in (p_0, p_1)$, so we have (4.15). □

As in §3, the proof of Theorem 4.1 is finished by the fact that, when F satisfies (4.9), then

$$T_F \ \text{ is Fredholm on } \ L^p(\partial\Omega), \quad \forall\, p \in (1,\infty). \tag{4.20}$$

Indeed, one has compactness results parallel to (3.10), this time by Lemma 4.2.

Again it is fairly easy to show that the index of T_F on $L^p(\partial\Omega)$ is independent of $p \in (1,\infty)$, via (4.2.16)–(4.2.18) of [12]. On the other hand, it is worth noting that the evaluation of $\operatorname{Index} T_F$ in this setting is very much more subtle than that of §3. We will briefly indicate results of [12] that have been brought to bear on this problem.

For one, given F as in (4.9), a construction given in §4.2 of [12] produces

$$\Phi \in C(\partial\Omega, G\ell(\ell,\mathbb{C})) \ \text{ such that } \ \operatorname{Index} T_F = \operatorname{Index} T_\Phi. \tag{4.21}$$

This construction takes off from material in [2], on the degree of BMO maps from one smooth compact manifold X to another, Y, but here we need to replace X by the rough surface $\partial\Omega$, which involves substantial technical complications.

As for the problem of computing $\operatorname{Index} T_F$ when $F \in C(\partial\Omega, G\ell(\ell,\mathbb{C}))$, results on localization and cobordism invariance in §§4.6–4.8 of [12] allow one to identify $\operatorname{Index} T_F$ with the index of such a Toeplitz operator when Ω is replaced by a domain $\widetilde{\Omega} \subset M$ with smooth boundary. In the smooth case, a homotopy argument allows us to replace $F \in C(\partial\widetilde{\Omega}, G\ell(\ell,\mathbb{C}))$ by a smooth $\widetilde{F} \in C^\infty(\partial\widetilde{\Omega}, G\ell(\ell,\mathbb{C}))$. Then $T_{\widetilde{F}}$ is a pseudodifferential operator on $\partial\widetilde{\Omega}$, and the Atiyah-Singer formula can be applied to its index. In fact, the index problem for general such Toeplitz operators in the smooth case can be seen to be essentially equivalent to the general Atiyah-Singer problem. Results of this nature are given in §§4–6 of [1].

We next consider the orthogonal projection S of $L^2(\partial\Omega, \mathcal{E}_0 \otimes \mathbb{C}^\ell)$ onto $\mathcal{H}^2(\partial\Omega, D)$, and associated Toeplitz operators

$$\mathcal{T}_f u = SfSu + (I-S)u, \quad f \in L^\infty(\partial\Omega, M(\ell,\mathbb{C})), \tag{4.22}$$

which is bounded on $L^2(\partial\Omega)$. A key ingredient in the analysis of these operators is the following identity relating S and $\mathcal{P}$ (cf. [12], (3.2.5), taking a cue from [10]):

$$\mathcal{P} = S(I+A), \quad A = \mathcal{P} - \mathcal{P}^* = C - C^*. \tag{4.23}$$

Note that A is skew-adjoint on $L^2(\partial\Omega)$, so $I+A$ is invertible on this space. Also, A is bounded on $L^p(\partial\Omega)$ for all $p \in (1,\infty)$, so again Sneiberg's theorem implies that there exist $q_0 \in (1,2)$ and $q_1 \in (2,\infty)$ such that

$$I + A \ \text{ is invertible on } L^p(\partial\Omega), \text{ for } \ p \in (q_0, q_1). \tag{4.24}$$

Thus $S = \mathcal{P}(I+A)^{-1}$ is bounded on $L^p(\partial\Omega)$ for such p, and hence so is $\mathcal{T}_f$. We will establish the following variant of Theorem 4.1.

Theorem 4.4 *Assume $f \in L^\infty(\partial\Omega, M(\ell,\mathbb{C}))$ has the factorization $f = FG$, with F and G as in* (4.9)*. Then there exist $p_0 \in (1,2)$ and $p_1 \in (2,\infty)$ such that*

$$\mathcal{T}_f \quad \text{is Fredholm on } L^p(\partial\Omega), \text{ for } \ p \in (p_0,p_1), \tag{4.25}$$

and

$$\operatorname{Index} \mathcal{T}_f = \operatorname{Index} \mathcal{T}_F. \tag{4.26}$$

Before starting the proof, we note that actually

$$\operatorname{Index} \mathcal{T}_F = \operatorname{Index} T_F \tag{4.27}$$

on $L^p(\partial\Omega)$ for $p \in (q_0,q_1)$, for F as in (4.9), as seen in Proposition 4.4.3 of [12].

The proof of Theorem 4.4 starts with the following variant of Lemma 4.2.

Lemma 4.5 *Given $F \in L^\infty \cap \mathrm{vmo}(\partial\Omega, M(\ell,\mathbb{C}))$ and $G \in L^\infty(\partial\Omega, M(\ell,\mathbb{C}))$, with q_j as in* (4.24)*,*

$$\mathcal{T}_{FG} - \mathcal{T}_F\mathcal{T}_G \quad \text{is compact on} \quad L^p(\partial\Omega), \quad \forall\, p \in (q_0,q_1). \tag{4.28}$$

Proof. As in previous variants, we start with the identity

$$\mathcal{T}_{FG} - \mathcal{T}_F\mathcal{T}_G = S[S,F]GS. \tag{4.29}$$

Then it suffices to establish compactness of $[S,F]$ on $L^p(\partial\Omega)$. To get this, we take the commutator of multiplication by F with (4.23) to get

$$[S,F](I+A) = [\mathcal{P},F] - S[A,F]. \tag{4.30}$$

As before, $[\mathcal{P},F]$ is compact on $L^p(\partial\Omega)$ for all $p \in (1,\infty)$, and, by the same reasoning so is $[C,F]$. Taking adjoints yields compactness of $[C^*,F]$ on $L^p(\partial\Omega)$ for all $p \in (1,\infty)$, so (4.30) is compact on $L^p(\partial\Omega)$ for all $p \in (1,\infty)$. The conclusion (4.28) then follows from (4.24). □

Next, the following result was already established in the proof of Lemma 4.3.

Lemma 4.6 *In the setting of Theorem 4.4, there exist $p_0 \in (1,2)$ and $p_1 \in (2,\infty)$ such that*

$$\mathcal{T}_G \quad \text{is invertible on } L^p(\partial\Omega), \text{ for } \ p \in (p_0,p_1). \tag{4.31}$$

As in (4.20), the proof of Theorem 4.4 is finished by the fact that, when F satisfies (4.9), then

$$\mathcal{T}_F \text{ is Fredholm on } L^p(\partial\Omega), \quad \forall\, p \in (q_0, q_1), \tag{4.32}$$

with q_j as in (4.28). The result (4.32) follows from Lemma 4.5, which implies that $\mathcal{T}_{F^{-1}}$ is a two-sided Fredholm inverse of $\mathcal{T}_F$ for such p. We hence have Theorem 4.4, after perhaps adjusting p_j to ensure that $p_0 \geq q_0$ and $p_1 \leq q_1$.

We close with some remarks on the hypothesis (4.10) in Theorem 4.1 and comparison with Theorem 4.4. As shown in §3.2 of [12], (4.10) holds provided D is of Dirac type and Ω is a regular SKT domain. This class of domains was introduced and studied in [14] and [9], where they were called chord-arc domains with vanishing constant. The label "regular SKT domains" was proposed in [8], where it was shown that this class of domains can be characterized as follows:

$$\begin{aligned} &\Omega \text{ is an Ahlfors regular domain,} \\ &\Omega \text{ satisfies a two-sided local John condition, and} \\ &\text{the unit normal } \nu \text{ belongs to } \mathrm{vmo}(\partial\Omega). \end{aligned} \tag{4.33}$$

In such a case, Ω is a UR domain and also an NTA domain. Details on this can be found in §§4.1–4.2 of [8]. As shown in [8], a domain whose boundary is locally the graph of a function with gradient in vmo is a regular SKT domain.

On the other hand, Theorem 4.4 applies to arbitrary relatively compact UR domains. In this sense, Theorem 4.4 is a more satisfactory extension of Theorem 3.1 than Theorem 4.1 is.

Finally, we note the following extension of Proposition 3.4, whose proof again uses the same partition of unity argument.

Proposition 4.7 *Take $f \in L^\infty(\partial\Omega, M(\ell,\mathbb{C}))$ and assume that for each $y \in \partial\Omega$ there are a neighborhood $\mathcal{O}_y$ of y in $\partial\Omega$ and $C_y \in M(\ell,\mathbb{C})$ such that*

$$\operatorname{Re} C_y f(x) \geq bI > 0, \quad \forall\, x \in \mathcal{O}_y. \tag{4.34}$$

Then f has a factorization $f = FG$, with

$$F \in C(\partial\Omega, G\ell(\ell,\mathbb{C})), \tag{4.35}$$

and G as in (4.9).

5 Further extension

We can extend Theorems 4.1 and 4.4 as follows. Assume $f \in L^\infty(\partial\Omega, M(\ell, \mathbb{C}))$ has a factorization

$$f(x) = F_1(x)G(x)F_2(x), \tag{5.1}$$

with

$$F_j, F_j^{-1} \in L^\infty \cap \mathrm{vmo}(\partial\Omega, M(\ell, \mathbb{C})), \tag{5.2}$$

and G as in (4.9). Then the conclusions of Theorems 4.1 and 4.4 continue to hold, with the index identity (4.12) replaced by

$$\operatorname{Index} T_f = \operatorname{Index} T_{F_1} + \operatorname{Index} T_{F_2}, \tag{5.3}$$

and a similar replacement for (4.26). The key to seeing this is simply to extend Lemmas 4.2 and 4.5 as follows.

Lemma 5.1 *Given $F_j \in L^\infty \cap \mathrm{vmo}(\partial\Omega, M(\ell, \mathbb{C}))$ and $G \in L^\infty(\partial\Omega, M(\ell, \mathbb{C}))$, we have*

$$T_{F_1GF_2} - T_{F_1}T_GT_{F_2} \quad \textit{compact on} \quad L^p(\partial\Omega), \quad \forall\, p \in (1, \infty), \tag{5.4}$$

and

$$\mathcal{T}_{F_1GF_2} - \mathcal{T}_{F_1}\mathcal{T}_G\mathcal{T}_{F_2} \quad \textit{compact on} \quad L^p(\partial\Omega), \quad \forall\, p \in (q_0, q_1), \tag{5.5}$$

with q_j as in (4.24).

Proof. Lemmas 4.2 and 4.5, with F_1 in place of F and GF_2 in place of G, imply the desired compactness of $T_{F_1GF_2} - T_{F_1}T_{GF_2}$ and of $\mathcal{T}_{F_1GF_2} - \mathcal{T}_{F_1}\mathcal{T}_{GF_2}$. To finish Lemma 5.1 off, it suffices to get analogues of Leemmas 4.2 and 4.5 for $T_{GF} - T_GT_F$ and $\mathcal{T}_{GF} - \mathcal{T}_G\mathcal{T}_F$. For this, we can just replace the identity (3.5) by

$$T_{GF} - T_GT_F = -\mathcal{P}G[\mathcal{P}, F]\mathcal{P}, \tag{5.6}$$

and (4.24) by

$$\mathcal{T}_{GF} - \mathcal{T}_G\mathcal{T}_F = -SG[S, F]S. \tag{5.7}$$

Then the compactness results on $[\mathcal{P}, F]$ and $[S, F]$ established before apply to finish the proof. $\square$

In connection with the factorization (5.1), we mention the following variant of Proposition 4.7.

Proposition 5.2 *Take* $f \in L^\infty(\partial\Omega, M(\ell,\mathbb{C}))$ *and assume that there exists* $b > 0$ *and an open covering* $\{\mathcal{O}_j : 1 \le j \le N\}$ *of* $\partial\Omega$ *and* $C_j, \widetilde{C}_k \in M(\ell,\mathbb{C})$ *such that*

$$\operatorname{Re} C_j f(x) \widetilde{C}_k \ge bI > 0, \quad \forall\, x \in \mathcal{O}_j \cap \mathcal{O}_k. \tag{5.8}$$

Then f *has a factorization* (5.1) *with*

$$F_j \in C(\partial\Omega, G\ell(\ell,\mathbb{C})) \tag{5.9}$$

and G *as in* (4.9).

Proof. Let $\{\psi_j : 1 \le j \le N\}$ be a continuous partition of unity on $\partial\Omega$ subordinate to the cover $\{\mathcal{O}_j : 1 \le j \le N\}$, so $\operatorname{supp} \psi_j \subset \mathcal{O}_j$ and $0 \le \psi_j \le 1$. Then set

$$\Phi_1(x) = \sum_{j=1}^N C_j \psi_j(x), \quad \Phi_2(x) = \sum_{k=1}^N \widetilde{C}_k \psi_k(x). \tag{5.10}$$

We have $\Phi_j \in C(\partial\Omega, M(\ell,\mathbb{C}))$ and

$$\begin{aligned} \operatorname{Re} \Phi_1(x) f(x) \Phi_2(x) &= \sum_{j,k=1}^N \operatorname{Re} C_j f(x) \widetilde{C}_k \psi_j(x) \psi_k(x) \\ &\ge bI. \end{aligned} \tag{5.11}$$

It follows that both $\Phi_1(x)$ and $\Phi_2(x)$ are invertible for each $x \in \partial\Omega$, and $F_j(x) = \Phi_j(x)^{-1}$ satisfy (5.9), and that (5.1) holds with

$$G(x) = \Phi_1(x) f(x) \Phi_2(x). \tag{5.12}$$

□

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