Poisson Equations, Uniformization, and Geometrical Optics

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This paper is dedicated to Duong Phong

Abstract. This paper studies the Poisson equation $\Delta_g u = f$ on a variety of noncompact Riemannian manifolds M, with f either compactly supported or possessing a simple asymptotic expansion at infinity. A construction in geometrical optics motivates the study of this equation when M is a compactly perturbed plane and f has compact support. Results in this case in turn motivate a study for higher dimensional M and more general f.

1. Introduction

This paper studies the Poisson equation

$$\Delta_g u = f \tag{1.1}$$

where Δ_g is the Laplace operator on a noncompact Riemannian manifold, in a number of settings.

Our original motivation arose from an issue in geometrical optics, concerning the null bicharacteristics of a variable speed d'Alembertian

$$\partial_t^2 - a(x)^2 \Delta, \tag{1.2}$$

with $t \in \mathbb{R},\ x \in \mathbb{R}^n,\ \Delta = \partial_1^2 + \cdots + \partial_n^2$. We might assume

$$a \in C^\infty(\mathbb{R}^n), \quad a > 0, \quad a(x) = 1 \ \text{ for } \ |x| \geq R, \tag{1.3}$$

for some $R \in (0, \infty)$. To leading order, the operator (1.2) agrees with

$$\partial_t^2 - \Delta_g, \tag{1.4}$$

where Δ_g is the Laplace-Beltrami operator on $M = \mathbb{R}^n$, endowed with the metric tensor

$$g_{jk} = a(x)^{-2}\delta_{jk}, \tag{1.5}$$

and in particular the two operators have the same null bicharacteristics, and hence propagate singularities along the same rays. These rays correspond naturally to orbits of the geodesic flow on S^*M, with metric tensor g_{jk}.

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When it comes to constructing examples that have periodic orbits with prescribed geometric properties, the setting (1.4) is quite convenient, as it allows one's geometrical intuition to take hold. We take g_{jk} to be an arbitrary compactly supported perturbation of the flat metric on $\mathbb{R}^n$:

$$g_{jk} \in C^\infty(\mathbb{R}^n), \quad \text{positive definite}, \quad g_{jk}(x) = \delta_{jk} \ \text{ for } \ |x| \geq R. \tag{1.6}$$

For example, we can take a sphere S^n, cut out a disk about its south pole, cut out a disk about the origin in $\mathbb{R}^n$, and attach these two spaces by a tube, obtaining a Riemannian manifold, diffeomorphic to $\mathbb{R}^n$, with closed geodesics of a certain type. This leads to the question of what such a construction might say about (1.2). That is to say, does there exist a function $a(x)$, satisfying, not quite (1.3), but

$$a \in C^\infty(\mathbb{R}^n), \quad a > 0, \quad a(x) \sim 1 \ \text{ as } \ |x| \to \infty, \tag{1.7}$$

such that $(\mathbb{R}^n, a(x)^{-2}\delta_{jk})$ is isometric to (M, g)? Certainly this will fail in general if $n \geq 3$, since (M, g) will typically not be locally conformally flat. As we will see, it does succeed when $n = 2$.

Here is our first task, in case $n = 2$. Let g_{jk} be a metric tensor on $\mathbb{R}^2$, satisfying (1.6). We desire to find

$$u \in C^\infty(\mathbb{R}^2), \quad u(x) \to 0 \ \text{ as } \ |x| \to \infty, \tag{1.8}$$

such that the new metric tensor

$$\tilde{g}_{jk} = e^{2u} g_{jk} \ \text{ has zero curvature.} \tag{1.9}$$

Then $(\mathbb{R}^2, \tilde{g}_{jk})$ is flat, complete, and simply connected. In such a case, one can choose a base point, and the exponential map yields a global isometry from $(\mathbb{R}^2, \delta_{jk})$ onto $(\mathbb{R}^2, \tilde{g}_{jk})$. Generally, if $k(x)$ denotes the Gauss curvature of $(\mathbb{R}^2, g_{jk})$, then the Gauss curvature $K(x)$ of $(\mathbb{R}^2, e^{2u} g_{jk})$ is given by

$$K(x) = (-\Delta_g u + k(x)) e^{-2u}. \tag{1.10}$$

If we want $K \equiv 0$, we want to solve the linear equation

$$\Delta_g u = k, \tag{1.11}$$

and we want a solution satisfying (1.8). In case $g_{jk} = \delta_{jk}$, we would solve (1.11) for a general function $k \in C_0^\infty(\mathbb{R}^2)$ by convolving k with the fundamental solution

$$E_0(x) = \frac{1}{2\pi} \log |x|. \tag{1.12}$$

Typically, $k * E_0(x)$ has a log blow-up as $|x| \to \infty$, unless k integrates to zero. Fortunately, k in (1.11) has this property. In fact, the Gauss-Bonnet theorem implies

$$\int_M k(x)\, dV(x) = 0, \tag{1.13}$$

where $M = \mathbb{R}^2$ and $dV(x) = \sqrt{g(x)}\, dx$ is the area element associated to the metric tensor g_{jk}, with $g(x) = \det(g_{jk})$.

In §2 we will show that if (M, g_{jk}) is a compactly supported perturbation of $(\mathbb{R}^2, \delta_{jk})$ and $k \in C_0^\infty(M)$ satisfies (1.13), then (1.11) has a solution u satisfying (1.8). In fact, u has a complete asymptotic expansion in negative powers of r. This finer behavior is of potential significance for applications to scattering theory. We find the solution u by solving a certain nonlocal boundary problem on a compact domain in M. We will not require M to be diffeomorphic to $\mathbb{R}^2$; we could add handles to the plane. Of course, in such a case, the Gauss-Bonnet theorem implies that (1.13) fails if $k(x)$ is the Gauss curvature of M, but it is still of intrinsic interest to have this solvability result, under the hypothesis (1.13). In §2 we also show there is a Green function, behaving like $\log|x|$ at infinity.

Results of §2 suggest a number of further problems, some of which are treated in §§3–6. For one, since the solution u to (1.11) obtained in §2 has a simple asymptotic expansion at infinity, it is natural to extend the class of right hand sides of (1.11), and consider k with such an expansion. In §3 we study (1.11) when M is an n-dimensional, asymptotically Euclidean, Riemannian manifold and k has an asymptotic expansion in terms of powers r^{-k-2}, $k \in \mathbb{N}$, and obtain u, with a more complicated asymptotic expansion, involving also powers of $\log r$, such that (1.11) holds asymptotically. This leads to the problem of solving (1.11) when $k \in \mathcal{S}(M)$, i.e., k and all its covariant derivatives vanish rapidly at infinity. We plan to take this problem up elsewhere.

Sections 4–6 tackle (1.11) where M is a general complete, n-dimensional Riemannian manifold (sometimes with nonempty boundary) assuming $k \in C_0^\infty(M)$. Our analysis parallels that of §2, but it is necessarily more elaborate in this case. We take a smoothly bounded, compact $\overline{\Omega} \subset M$, containing the support of k, and construct the solution to (1.11) via a nonlocal boundary problem on Ω. In §4 we construct the Poisson integral on functions in $H^s(\partial\Omega)$, for $s \geq 1/2$, yielding harmonic functions on $X = M \setminus \overline{\Omega}$, and analyze the Dirichlet-to-Neumann map as a pseudodifferential operator on $\partial\Omega$. In §5 we formulate and solve the nonlocal boundary problem mentioned above. We obtain a general criterion on when (1.13) is needed to get $u \in L^\infty(M)$. In §6, we return to the Poisson integral and, using the results of §5, extend it to act on $H^s(\partial\Omega)$ for all $s \in \mathbb{R}$.

In Appendix A, we return to the 2-dimensional setting and give a more general criterion for $(\mathbb{R}^2, g_{jk})$ to be conformally equivalent to $(\mathbb{R}^2, \delta_{jk})$ than done in §2. The proof uses the uniformization theorem and a Liouville theorem. This argument provides less information about the resulting conformal factor $a(x)^{-2}$ than what we get in §2.

Our analysis makes some use of pseudodifferential operator calculus. To fix notation, if Y is a compact manifold, we denote by $OPS^m(Y)$ the set of "classical" pseudodifferential operators of order m on Y.

We end this introduction with a brief discussion of previous work done on the Poisson equation (1.11), on a complete Riemannian manifold M. Work of [**6**] yielded a solution $u \in C^\infty(M)$, given $k \in C_0^\infty(M)$, for any such M. This paper showed the existence of a "Green function" G on $M \times M$. The approach was non-constructive. A constructive approach was given in [**4**] (building on some unpublished work of Schoen and Yau). This paper also studied conditions guaranteeing that there is a positive Green function. In such a case, estimates of [**4**] imply the solution u is bounded. Various geometric conditions on M are given that yield the existence of a positive Green function. Other important papers on this topic include [**5**], [**8**], [**9**], and [**13**]. There are results on conditions on the Ricci tensor of M that guarantee the existence of a positive Green function. Also, [**8**], [**9**], and [**13**] explore when (1.11), for a class of Kähler manifolds, with k equal to the scalar curvature, help to understand the Poincaré-Lelong equation, yielding important insights into natural classes of complete, noncompact, Kähler manifolds.

2. Solving $\Delta_g u = f$ on compactly perturbed planes

Let (M, g) be a two-dimensional Riemannian manifold. We assume M is connected and that there exist a compact $K \subset M$ and $R \in (0, \infty)$ such that $M \setminus K$ is isometric with $\mathbb{R}^2 \setminus \overline{B_R(0)}$. We denote the Laplace-Beltrami operator of (M, g) by Δ_g. We aim to prove the following.

PROPOSITION 2.1. *Given $f \in C_0^\infty(M)$ such that*

$$\int_M f(x)\, dV(x) = 0, \tag{2.1}$$

there exists a unique solution u to

$$\Delta_g u = f, \tag{2.2}$$

satisfying

$$u \in C^\infty(M), \quad u(x) \to 0 \quad \text{as} \quad x \to \infty. \tag{2.3}$$

To start, we can take a compact, smoothly bounded $\overline{\Omega} \subset M$ such that

$$K \subset \Omega, \quad \operatorname{supp} f \subset \Omega, \quad \text{and } M \setminus \Omega \text{ isometric to } \mathbb{R}^2 \setminus B_S(0), \tag{2.4}$$

for some $S \in (R, \infty)$. Rescaling, we can assume $S = 1$. We will simply identify $M \setminus \Omega$ with $\mathbb{R}^2 \setminus B_1(0)$. We will construct u on Ω to solve a certain nonlocal boundary problem (see (2.8) below). With $v = u|_{\partial\Omega}$ (and $\partial\Omega$ identified with $\partial B_1(0) = S^1$) we define u on $\mathbb{R}^2 \setminus B_1(0)$ to be

$$u(x) = \sum_{k=-\infty}^{\infty} \hat{v}(k) r^{-|k|} e^{ik\theta}, \quad x = re^{i\theta}, \quad r > 1, \tag{2.5}$$

where

$$\hat{v}(k) = \frac{1}{2\pi} \int_{S^1} v(\theta) e^{-ik\theta}\, d\theta. \tag{2.6}$$

Note that, for $|x| > 1$, and $x = (x_1, x_2)$ identified with $z = x_1 + ix_2$,

$$u(x) = \sum_{k=0}^{\infty} \hat{v}(k) \overline{z}^{-k} + \sum_{k=1}^{\infty} \hat{v}(-k) z^{-k} \tag{2.7}$$

is harmonic. To fit this function together with a function on Ω and solve (2.2), we want u on Ω to solve

$$\Delta_g u = f \ \text{ on } \ \Omega, \quad \partial_\nu u = -\Lambda u \ \text{ on } \ \partial\Omega, \tag{2.8}$$

where ν is the outward-pointing unit normal to $\partial\Omega$, and Λ is the operator defined on functions on $\partial\Omega = S^1$ by

$$\Lambda v(\theta) = \sum_{k=-\infty}^{\infty} |k| \hat{v}(k) e^{ik\theta}. \tag{2.9}$$

Note that if u is given on $\mathbb{R}^2 \setminus B_1(0)$, then $\partial_r u = -\Lambda v$ on S^1. If we can solve (2.8), then using (2.5) with $v = u|_{\partial\Omega}$ produces a function that solves (2.6) on $M \setminus \partial\Omega$ and has the property that neither u nor ∇u have a jump across $\partial\Omega$, so in fact u solves (2.6) on all of M.

To proceed, take $k \in \mathbb{N}$ and define a family of operators

$$L_\tau : H^{k+2}(\Omega) \longrightarrow H^k(\Omega) \oplus H^{k+1/2}(\partial\Omega), \tag{2.10}$$

for $\tau \in \mathbb{C}$, by

$$L_\tau u = (\Delta_g u, \partial_\nu u + \tau \Lambda u). \tag{2.11}$$

LEMMA 2.2. *When $\tau \neq -1$, L_τ in* (2.10) *is Fredholm, of index zero.*

Proof. We show that L_τ defines a regular, elliptic boundary problem when $\tau \neq -1$. One method (cf. [**10**], Chapter 5, §5) reduces this to studying solutions to

$$\Delta_g u = 0 \;\text{ on }\; \Omega, \quad \partial_\nu + \tau\Lambda u = h \;\text{ on }\; \partial\Omega, \tag{2.12}$$

and looking for w on $\partial\Omega$ such that (2.12) is solved (mod C^∞) by

$$u = \mathrm{PI}_0\, w, \tag{2.13}$$

where $\mathrm{PI}_0\, w$ solves the Dirichlet problem for Δ_g on $\overline{\Omega}$, with boundary data w. If $M = \mathbb{R}^2$ with its flat metric, then $\partial_\nu \mathrm{PI}_0\, w = \Lambda w$. In the current setting, local regularity results for the Dirichlet problem imply that if (2.13) holds, then near $\partial\Omega$ on $\overline{\Omega}$, u differs from its counterpart with Ω replaced by $B_S(0)$ by a function that is C^∞ near $\partial\Omega = \partial B_S(0)$. Hence

$$\partial_\nu u = \Lambda_0 w, \quad \Lambda_0 - \Lambda \in OPS^0(\partial\Omega), \tag{2.14}$$

so

$$\partial_\nu u + \tau\Lambda u = (\Lambda_0 + \tau\Lambda)u, \quad \Lambda_0 + \tau\Lambda = (1+\tau)\Lambda \bmod OPS^0(\partial\Omega), \tag{2.15}$$

so $\Lambda_0 + \tau\Lambda$ is elliptic in $OPS^1(\partial\Omega)$ whenever $\tau \neq -1$. Such ellipticity implies L_τ in (2.10) is Fredholm whenever $\tau \neq -1$. Since $\mathbb{C} \setminus \{-1\}$ is connected, the index is constant on this set. When $\tau = 0$ we have the Neumann boundary problem, which is regular and self adjoint, hence Fredholm of index 0. $\square$

Of course the case of direct interest in (2.8) is $\tau = +1$. We now examine the null space $\mathcal{N}(L_1)$.

LEMMA 2.3. *Given $u \in H^2(\Omega)$,*

$$u \in \mathcal{N}(L_1) \Longrightarrow u \;\; \textit{is constant.} \tag{2.16}$$

Proof. Without loss of generality, we can assume u is real valued. Green's formula gives, for $u \in H^2(\Omega)$,

$$\int_\Omega |\nabla_g u|^2 \, dV = -\int_\Omega u\, \Delta_g u \, dV + \int_{\partial\Omega} u \frac{\partial u}{\partial \nu} \, dS. \tag{2.17}$$

If $u \in \mathcal{N}(L_1)$, then

$$\int_\Omega |\nabla_g u|^2 \, dV = -(u, \Lambda u)_{L^2(\partial\Omega)}. \tag{2.18}$$

The left side of (2.18) is ≥ 0 and the right side is ≤ 0, so both sides must vanish, implying u is constant. $\square$

From Lemmas 2.2–2.3 we have

$$\mathcal{R}(L_1) \text{ has codimension 1 in } \; H^k(\Omega) \oplus H^{k+1/2}(\partial\Omega). \tag{2.19}$$

Taking $k=0$, we want to identify the annihilator of $\mathcal{R}(L_1)$ in $L^2(\Omega)\oplus H^{-1/2}(\partial\Omega)$, a space we know has dimension 1. To say (w,h) belongs to the annihilator of $\mathcal{R}(L_1)$ is to say that

$$(\Delta_g u, w) + (\partial_\nu u + \Lambda u, h) = 0, \quad \forall\, u \in H^2(\Omega). \tag{2.20}$$

We note that $(w,h) = (1,-1)$ satisfies this condition. In fact, Green's theorem implies

$$(\Delta_g u, 1) = \int\limits_{\partial\Omega} (\partial_\nu u)\, dS, \tag{2.21}$$

and

$$(\Lambda u, 1) = (u, \Lambda 1) = 0. \tag{2.22}$$

The dimension count implies

$$(w,h) = (1,-1) \ \text{ spans the annihilator of } \ \mathcal{R}(L_1). \tag{2.23}$$

Corollary 2.4. *If $f \in L^2(\Omega)$ satisfies* (2.1)*, then $(f,0) \in \mathcal{R}(L_1)$, hence there exists $u \in H^2(\Omega)$ satisfying* (2.8).

If $f \in C_0^\infty(\Omega)$ satisfies (2.1), elliptic regularity yields $u \in C^\infty(\overline{\Omega})$. Fitting in the construction (2.5)–(2.7), we have a smooth solution to (2.2), which tends to a constant limit at infinity. Subtracting this constant gives a solution satisfying (2.3). Uniqueness follows from the maximum principle.

Strengthening the uniqueness result, we have the following Liouville theorem.

Proposition 2.5. *In the setting of Proposition 2.1, if $u \in C^\infty(M)$ is bounded and solves*

$$\Delta_g u = 0 \quad \text{on} \quad M, \tag{2.24}$$

then u is constant.

Proof. On $\mathbb{R}^2 \setminus B_1(0)$, u must have the form (2.5), with $v = u|_{S^1}$, and on Ω, u must solve (2.8), with $f=0$, so $u \in \mathcal{N}(L_1)$. Hence, by Lemma 2.3, u is constant on Ω, hence on $\partial\Omega = S^1$, and the representation (2.5) implies u is equal to the same constant on $\mathbb{R}^2 \setminus B_p(0)$. ☐

See Appendix A for a much more general Liouville theorem.

We now extend the scope of Proposition 2.1.

Proposition 2.6. *In the setting of Proposition 2.1, replace* (2.1) *by*

$$\int\limits_M f(x)\, dV(x) = a. \tag{2.25}$$

Then there exists a unique solution u to $\Delta_g u = f$, satisfying

$$u(x) - \frac{a}{2\pi} \log|x| \to 0 \quad \text{as} \quad x \to \infty. \tag{2.26}$$

Proof. Pick $\varphi \in C^\infty(\mathbb{R}^2)$ such that $\varphi(x) = 0$ for $|x| \le 2$, 1 for $|x| \ge 3$. Define $G \in C^\infty(M)$ by

$$\begin{aligned} G(x) = \frac{\varphi(x)}{2\pi} &\log|x|, \quad && x \in \mathbb{R}^2 \setminus B_1(0), \\ &0, && x \in \Omega. \end{aligned} \tag{2.27}$$

Then (with E_0 as in (1.12))

$$\begin{aligned}\int_M \Delta_g G(x)\, dV(x) &= \int_{\mathbb{R}^2\setminus B_1(0)} \Delta G(x)\, dx \\ &= \int_{\mathbb{R}^2} \Delta E_0(x)\, dx - \int_{\mathbb{R}^2} \Delta\big((1-\varphi)E_0\big)\, dx \\ &= 1.\end{aligned} \tag{2.28}$$

Thus, if we set

$$F(x) = \Delta_g G(x), \tag{2.29}$$

then $F \in C_0^\infty(M)$ and $\int_M (f - aF)\, dV = 0$, so Proposition 2.1 applies, to give $w \in C^\infty(M)$ satisfying

$$\Delta_g w = f - aF \quad \text{on} \quad M, \quad w(x) \to 0 \quad \text{as} \quad x \to \infty. \tag{2.30}$$

Hence

$$\Delta_g(w + aG(x)) = f, \tag{2.31}$$

and $u = w + aG$ is the desired solution. □

3. Asymptotic solutions to $\Delta_g u = f$

Here we look at

$$\Delta_g u = f, \tag{3.1}$$

when the n-dimensional Riemannian manifold M is asymptotically flat, so that, for some compact $K \subset M$, and S diffeomorphic to S^{n-1},

$$M \setminus K \sim (1, \infty) \times S, \tag{3.2}$$

and, on $M \setminus K$,

$$\Delta_g u = \partial_r^2 u + M(r)\partial_r u + r^{-2}\Delta_{S(r)} u, \tag{3.3}$$

where, as $r \to \infty$,

$$M(r) \sim \frac{n-1}{r} + \sum_{\ell \geq 1} a_\ell(\omega) r^{-1-\ell}, \tag{3.4}$$

and

$$\Delta_{S(r)} \sim \Delta_S + \sum_{\ell \geq 1} r^{-\ell} L_\ell. \tag{3.5}$$

Here $\omega \in S$, $a_\ell \in C^\infty(S)$, Δ_S is the Laplace-Beltrami operator on S, and L_ℓ are second-order differential operators on S. Cf. [**1**], p. 18. We take $S = S^{n-1}$, so

$$\operatorname{Spec}(-\Delta_S) = \{\ell^2 + (n-2)\ell : \ell = 0, 1, 2, \dots\}, \tag{3.6}$$

though extensions to other compact, $(n-1)$-dimensional Riemannian manifolds S are possible.

We assume f has the form

$$f \sim \sum_{k \geq 1} r^{-k-2} f_k(\omega), \tag{3.7}$$

as $r \to \infty$, with $f_k \in C^\infty(S)$, and look for

$$u \sim \sum_{k\ge 1} u_k(r,\omega) \tag{3.8}$$

such that

$$\Delta_g u \sim f, \tag{3.9}$$

in the sense that $\Delta_g u - f$ vanishes rapidly, with all derivatives, as $r \to \infty$. In (3.8), we want $u_k(r,\omega)$ to decay roughly like r^{-k} as $r \to \infty$, though as we will see, formulas for $u_k(r,\omega)$ can have a more complicated form than $r^{-k}u_k(\omega)$.

Plugging (3.7)–(3.9) into (3.3)–(3.5) gives

$$\begin{aligned}&\sum_{k\ge 1}\Bigl(\partial_r^2 + \frac{n-1}{r}\partial_r + \frac{1}{r^2}\Delta_S\Bigr)u_k(r,\omega)\\ &\sim \sum_{k\ge 1} r^{-k-2} f_k(\omega) - \sum_{k,\ell\ge 1}\Bigl(a_\ell(\omega) r^{-\ell-1}\partial_r + r^{-\ell-2}L_\ell\Bigr)u_k(r,\omega).\end{aligned} \tag{3.10}$$

We find it convenient to make a change of variable,

$$v_k(s,\omega) = u_k(r,\omega), \quad r = e^s, \tag{3.11}$$

so

$$\begin{aligned} u_k(r,\omega) &= v_k(\log r,\omega),\\ \partial_r u_k(r,\omega) &= \frac{1}{r}\partial_s v_k(\log r,\omega),\\ \partial_r^2 u_k(r,\omega) &= \frac{1}{r^2}\partial_s^2 v_k(\log r,\omega) - \frac{1}{r^2}\partial_s v_k(\log r,\omega),\end{aligned} \tag{3.12}$$

and (3.10) becomes

$$\begin{aligned}&\sum_{k\ge 1}(\partial_s^2 + (n-2)\partial_s + \Delta_S)v_k(s,\omega)\\ &\sim \sum_{k\ge 1} e^{-ks} f_k(\omega) - \sum_{k,j\ge 1} e^{-js}(a_j(\omega)\partial_s + L_j)v_k(s,\omega).\end{aligned} \tag{3.13}$$

We seek $v_k(s,\omega)$ in the form

$$v_k(s,\omega) = p_k(s,\omega)e^{-ks}, \tag{3.14}$$

where $p_k(s,\omega)$ is a polynomial in s, with coefficients in $C^\infty(S)$ (functions of ω).

The case $k=1$ of (3.13) is

$$(\partial_s^2 + (n-2)\partial_s + \Delta_S)v_1(s,\omega) = e^{-s} f_1(\omega). \tag{3.15}$$

We expand both sides in terms of eigenfunctions of Δ_S In case $S = S^{n-1}$ and (3.6) holds, let

$$V_\ell = \bigl\{h \in C^\infty(S) : -\Delta_S h = [\ell^2 + (n-2)\ell]h\bigr\}. \tag{3.16}$$

If $f_{1\ell}$ is the component of f_1 in V_ℓ, we want to solve

$$(\partial_s^2 + (n-2)\partial_s + \nu_\ell^2)v_{1\ell}(s) = e^{-s}, \quad \nu_\ell^2 = \ell^2 + (n-2)\ell. \tag{3.17}$$

Then

$$v_1(s,\omega) = \sum_\ell v_{1\ell}(s) f_{1\ell}(\omega). \tag{3.18}$$

We rewrite (3.17) as

$$(\partial_s - \ell)(\partial_s + \ell + n - 2)v_{1\ell}(s) = e^{-s}. \tag{3.19}$$

At this point, let us pause and consider solving

$$(\partial_s - \ell)v = p(s)e^{-ks}, \tag{3.20}$$

when $p(s)$ is a polynomial in s and $k \in \mathbb{Z}^+$. In (3.6), $\ell \in \mathbb{Z}^+ \cup \{0\}$, but let us more generally take $\ell \in \mathbb{R}$. We write $v(s) = q(s)e^{-ks}$, so (3.20) becomes

$$(\partial_s - \ell - k)q(s) = p(s), \tag{3.21}$$

with solution

$$q(s) = J_{k+\ell}p(s), \tag{3.22}$$

where the operators J_m, acting on polynomials in s, are given as follows, for $m \in \mathbb{R}$. First,

$$J_0 p(s) = \int_0^s p(\sigma)\, d\sigma. \tag{3.23}$$

If $m \neq 0$, we take

$$\begin{aligned} J_m p(s) &= (\partial_s - m)^{-1} p(s) \\ &= -\frac{1}{m}\Bigl(1 - \frac{1}{m}\partial_s\Bigr)^{-1} p(s) \\ &= -\frac{1}{m}\sum_{j\ge 0}\Bigl(\frac{1}{m}\partial_s\Bigr)^{j} p(s), \end{aligned} \tag{3.24}$$

the last sum being over $j \le K$ if $p(s)$ is a polynomial of degree K. Then (3.20) is solved by

$$v(s) = J_{k+\ell}p(s) \cdot e^{-ks}. \tag{3.25}$$

Returning to (3.18), we have the solution

$$\begin{aligned} v_{1\ell}(s) &= J_{1+2-\ell-n}J_{1+\ell}(1) \cdot e^{-s} \\ &= q_{\ell n}(s)e^{-s}, \end{aligned} \tag{3.26}$$

where $q_{\ell n}(s)$ is a polynomial in s. Note that

$$\ell \ge 0 \Longrightarrow J_{1+\ell}(1) = -\frac{1}{\ell + 1}, \tag{3.27}$$

and

$$J_{3-\ell-n}(1) \tag{3.28}$$

is constant if $\ell + n \neq 3$, and a constant multiple of s if $\ell + n = 3$. In this way, we have a solution $v_1(s,\omega)$ to (3.15).

From here, we find $v_k(s,\omega)$ in (3.17) by induction, for $k \ge 2$. It solves

$$(\partial_s^2 + (n-2)\partial_s + \Delta_S)v_k(s,\omega) = e^{-ks}\varphi_k(s,\omega), \tag{3.29}$$

where $\varphi_k(s,\omega)$ is a polynomial in s, with coefficients in $C^\infty(S)$. Let

$$\{f_\ell^\mu : 1 \le \mu \le \dim V_\ell\} \ \text{ be an orthonormal basis of } \ V_\ell.$$

Write

$$\varphi_k(s,\omega) = \sum_{\ell,\mu} \varphi_{k\ell}^\mu(s) f_\ell^\mu(\omega). \tag{3.30}$$

Then we want to solve

$$(\partial_s^2 + (n-2)\partial_s + \nu_\ell^2) v_{k\ell}^\mu(s) = \varphi_{k\ell}^\mu(s) e^{-ks}, \tag{3.31}$$

to obtain

$$v_k(s,\omega) = \sum_{\ell,\mu} v_{k\ell}^\mu(s) f_\ell^\mu(\omega). \tag{3.32}$$

Equivalently, we solve

$$(\partial_s - \ell)(\partial_s + \ell + n - 2) v_{k\ell}^\mu(s) = \varphi_{k\ell}^\mu(s) e^{-ks}, \tag{3.33}$$

so we take

$$v_{k\ell}^\mu(s) = q_{k\ell}^\mu(s) e^{-ks}, \quad q_{k\ell}^\mu(s) = J_{k+2-\ell-n} J_{k+\ell} \varphi_{k\ell}^\mu(s). \tag{3.34}$$

Thus $q_{k\ell}^\mu(s)$ is a polynomial in s of degree at most 1 more than that of $\varphi_{k\ell}^\mu(s)$. That $v_j(s,\omega)$ in (3.32) is e^{-ks} times a polynomial in s with coefficients in $C^\infty(S)$ is a straightforward consequence of the formulas (3.23)–(3.24). Let us formalize this:

$$v_k(s,\omega) = q_k(s,\omega) e^{-ks}, \tag{3.35}$$

where $q_k(s,\omega)$ is a polynomial in s with coefficients in $C^\infty(S)$. Rewinding (3.8)–(3.11), we have an asymptotic solution to (3.9) of the form

$$u(r,\omega) \sim \sum_{k\ge 1} q_k(\log r, \omega) r^{-k}. \tag{3.36}$$

Borel's theorem on summing asymptotic series yields the following.

Proposition 3.1. *Let M be an asymptotically Euclidean, Riemannian manifold, of dimension n. Take $f \in C^\infty(M)$ having the asymptotic expansion* (3.7)*, with $f_k \in C^\infty(S)$. Then there exists $u \in C^\infty(M)$, having an asymptotic expansion of the form* (3.36)*, where each q_k is a polynomial in $\log r$ with coefficients in $C^\infty(S)$, such that*

$$\Delta_g u - f = h \in \mathcal{S}(M), \tag{3.37}$$

i.e., h and all its covariant derivatives vanish at infinity.

Given this, we are highly motivated to establish solvability of

$$\Delta_g u = f \tag{3.38}$$

given $f \in \mathcal{S}(M)$, perhaps integrating to 0, and investigate asymptotic properties of the solution. Sections 4–5 have results on this for quite general M, but they require $f \in C_0^\infty(M)$. They obtain $u \in C^\infty(M) \cap L^\infty(M)$, but they do not get finer asymptotic results.

4. Poisson integral on a complete manifold with compact boundary

Let $\overline{X}$ be a complete, n-dimensional Riemannian manifold with compact boundary ∂X, and interior X. We assume X is connected. We want to establish the existence of a map

$$\mathrm{PI} : C^\infty(\partial X) \longrightarrow C^\infty(\overline{X}) \cap L^\infty(X) \tag{4.1}$$

and record properties of the Dirichlet-to-Neumann map Λ, given by

$$\Lambda f = -\partial_\nu \, \mathrm{PI}\, f\big|_{\partial X}, \tag{4.2}$$

where ν is the unit normal to ∂X, pointing inside X. We also define PI on other function spaces on ∂X. We may as well assume $\overline{X}$ is not compact. Let X_k be an increasing sequence of bounded open subsets of X, such that

$$X_k \supset \{x \in X : \operatorname{dist}(x, \partial X) \le k\}. \tag{4.3}$$

Write $\partial X_k = \partial X \cup S_k$. We can assume S_k is smooth. We define

$$P_k : C^\infty(\partial X) \longrightarrow C^\infty(\overline{X}_k) \tag{4.4}$$

by

$$\Delta_g P_k f = 0 \ \text{ on } \ X_k, \quad P_k f = f \text{ on } \partial X, \ P_k f = 0 \text{ on } S_k. \tag{4.5}$$

We then extend $P_k f$ by 0 on $\overline{X} \setminus \overline{X}_k$, defining $P_k : C^\infty(\partial X) \to C(\overline{X})$. If $C_+^\infty(\partial X)$ denotes the class of $f \ge 0$ in $C^\infty(\partial X)$, we have

$$f \in C_+^\infty(\partial X),\ u_k = P_k f \Longrightarrow 0 \le u_k \le u_{k+1} \le \sup f, \tag{4.6}$$

by the maximum principle, and from here and local elliptic regularity results, we have

$$u_k \longrightarrow u \in C^\infty(\overline{X}) \cap L^\infty(X), \tag{4.7}$$

solving

$$\Delta_g u = 0 \ \text{ on } \ X, \quad u\big|_{\partial X} = f. \tag{4.8}$$

We denote the limit by $\mathrm{PI}\, f$. The construction (4.4)–(4.5) gives

$$f, g \in C_+^\infty(\partial X) \Longrightarrow \mathrm{PI}(f+g) = \mathrm{PI}(f) + \mathrm{PI}(g). \tag{4.9}$$

Given a general (real valued) $f \in C^\infty(\partial X)$, set

$$f = f_1 - f_2,\ f_j \in C_+^\infty(\partial X), \tag{4.10}$$

and

$$\mathrm{PI}\, f = \mathrm{PI}\, f_1 - \mathrm{PI}\, f_2. \tag{4.11}$$

It follows from (4.9) that this is independent of the choice of f_j such that (4.10) holds. Note that if $\widetilde{X}_k \nearrow \overline{X}$ also satisfies (4.3) and $\widetilde{P}_k$ is defined analogously to (4.5), then $f \in C_+^\infty(\partial X),\ X_j \subset \widetilde{X}_k \subset X_\ell \Rightarrow P_j f \le \widetilde{P}_k f \le P_\ell f$, so PI is well defined, independently of the choice of $\{X_k\}$.

The convergence (4.7) holds in $C^\infty(\overline{\Omega})$ for each compact $\overline{\Omega} \subset \overline{X}$, given $f \in C^\infty(\partial X)$. The maximum principle yields an extension

$$\mathrm{PI} : C(\partial X) \longrightarrow C(\overline{X}) \cap L^\infty(X). \tag{4.12}$$

Also, standard elliptic regularity results yield

$$\mathrm{PI} : H^s(\partial X) \longrightarrow H^{s+1/2}(\overline{X}_1) \cap C^\infty(X) \cap L^\infty(X), \tag{4.13}$$

for $s > (n-1)/2$. Shortly, we will extend (4.13) to a larger range of s.

Note that, for each $f \in C(\partial X)$, elliptic regularity implies

$$\mathrm{PI}\, f - P_2 f \in C^\infty(\overline{X}_1). \tag{4.14}$$

Also, a parametrix construction yields

$$\Lambda \in OPS^1(\partial X), \tag{4.15}$$

elliptic, with

$$\Lambda - \sqrt{-\Delta_S} \in OPS^0(\partial X), \tag{4.16}$$

where Δ_S denotes the Laplace-Beltrami operator on $S = \partial X$.

We pause to consider the family of special cases

$$\overline{X} = \mathbb{R}^n \setminus B_1, \quad B_1 = \{x \in \mathbb{R}^n : |x| < 1\}. \tag{4.17}$$

Take $n \geq 2$. In spherical polar coordinates $x = r\omega,\ r \in [1, \infty),\ \omega \in S^{n-1}$, we have

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{n-1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\Delta_S u, \tag{4.18}$$

where Δ_S is the Laplace-Beltrami operator on S^{n-1}. If we set

$$A = \Big(-\Delta_S + \frac{(n-2)^2}{4}\Big)^{1/2}, \tag{4.19}$$

we have (cf. [**11**], Chapter 8, §4)

$$\operatorname{Spec} A = \Big\{\frac{n-2}{2} + k : k = 0, 1, 2 \dots\Big\}. \tag{4.20}$$

Separation of variables applied to (4.8) yields

$$\begin{aligned} \mathrm{PI}\, f(r\omega) &= r^{-A-(n-2)/2} f(\omega) \\ &= r^{-B} f(\omega), \end{aligned} \tag{4.21}$$

with

$$\operatorname{Spec} B = \{n - 2 + k : k = 0, 1, 2, \dots\}. \tag{4.22}$$

The definition (4.2) gives

$$\Lambda = B = \Big(-\Delta_s + \frac{(n-2)^2}{4}\Big)^{1/2} + \frac{n-2}{2}, \tag{4.23}$$

a result consistent with (4.16). Note that this is a self-adjoint, positive semi-definite operator, with discrete spectrum, whose smallest eigenvalue is

$$\lambda_0 = n - 2, \tag{4.24}$$

which vanishes if $n = 2$ but is strictly positive if $n \geq 3$. It follows that

$$\mathrm{PI} : C(\partial X) \longrightarrow C_*(\overline{X}) \tag{4.25}$$

if $X = \mathbb{R}^n \setminus B_1$ with $n \geq 3$, where

$$C_*(\overline{X}) = \{u \in C(\overline{X}) : \lim_{x \to \infty} u(x) = 0\}. \tag{4.26}$$

However,

$$\overline{X} = \mathbb{R}^2 \setminus B_1 \Longrightarrow \mathrm{PI}(1) \equiv 1. \tag{4.27}$$

In [**12**] it is shown that (4.25) holds whenever $\overline{X}$ is asymptotically Euclidean and has dimension $n \geq 3$.

Back to generalities, take $f, g \in C^\infty(\partial X)$, set $u_k = P_k f$ as in (4.4)–(4.5), and set $v_k = P_k g$. Green's formula gives

$$\int_{X_k} \nabla u_k \cdot \nabla v_k \, dV = -\int_{\partial X_k} u_k(\partial_\nu v_k)\, dS = -\int_{\partial X} u_k(\partial_\nu v_k)\, dS, \tag{4.28}$$

the negative sign because ν points into X. The smooth convergence of u_k to $u = \mathrm{PI}\, f$ and of v_k to $v = \mathrm{PI}\, g$ implies that the right side of (4.28) converges to

$$-\int_{\partial X} u(\partial_\nu v)\, dS = \int_{\partial X} f(\Lambda g)\, dS. \tag{4.29}$$

Since the left side of (4.28) is symmetric in u_k and v_k, we have

$$\int_{\partial X} f(\Lambda v)\,dS = \int_{\partial X} (\Lambda f) g\,dS, \tag{4.30}$$

for $f, g \in C^\infty(\partial X)$. In concert with (4.15)–(4.16), we deduce that Λ is self-adjoint, with domain $H^1(\partial X)$. Taking $g = \overline{f}$ gives $v_k = \overline{u}_k$, and hence

$$\int_{X_k} |\nabla u_k|^2\,dV = -\int_{\partial X} u_k(\partial_\nu \overline{u}_k)\,dS. \tag{4.31}$$

Taking $k \to \infty$ and applying Fatou's lemma to the left side of (4.31) gives

$$\int_X |\nabla u|^2\,dV \le (f, \Lambda f), \tag{4.32}$$

for $u = \mathrm{PI}\, f$. This implies Λ is positive semidefinite. Also, by (4.15), $(f, \Lambda f) \le C\|f\|^2_{H^{1/2}(\partial X)}$. This leads to the following result.

Proposition 4.1. *Assume $\overline{X}$ is a complete Riemannian manifold with compact boundary ∂X. The map* PI *extends uniquely from $C^\infty(\partial X)$ to*

$$\mathrm{PI} : H^{1/2}(\partial X) \longrightarrow \Big\{ u \in C^\infty(X) : \int_X |\nabla u|^2\,dV < \infty \Big\}. \tag{4.33}$$

Proof. Given $f \in H^{1/2}(\partial X)$, we take $f_j \in C^\infty(\partial X)$ such that $f_j \to f$ in $H^{1/2}$ norm, and set $u_j = \mathrm{PI}\, f_j$. Also set $u_{jk} = P_k f_j$. We have

$$\|\nabla u_j\|^2_{L^2(X)} \le (f_j, \Lambda f_j) \le C_0 \|f\|^2_{H^{1/2}(\partial X)}. \tag{4.34}$$

Also

$$\|\nabla(u_j - u_{jk})\|_{L^2(X_k)} \le C_k \|f\|_{H^{1/2}(\partial X)}, \tag{4.35}$$

and

$$u_j - u_{jk}\big|_{\partial X} = 0, \tag{4.36}$$

so, by Poincaré's inequality,

$$\|u_j - u_{jk}\|_{L^2(X_k)} \le \widetilde{C}_k \|f\|_{H^{1/2}(\partial X)}. \tag{4.37}$$

These uniform estimates readily yield the extension (4.33). □

An interpolation argument then extends (4.13) from $s > (n-1)/2$ to $s \ge 1/2$, with $L^\infty(X)$ replaced by $L^\infty(X^\#)$, where $X^\# = \{x \in X : \mathrm{dist}(x, \partial X) \ge 1\}$. Further extensions are possible, as we will see in §6.

Remark 1. The result (4.32) suggests the following problem.

Determine when one has equality in (4.32).

Remark 2. As we have seen in (4.17)–(4.27), when $\overline{X} = \mathbb{R}^n \setminus B_1$,

$$\mathrm{PI} : C(\partial X) \longrightarrow C_*(\overline{X}), \tag{4.38}$$

when $n \ge 3$, but not when $n = 2$. Also, $\mathcal{N}(\Lambda) = 0$ when $n \ge 3$, but $\mathcal{N}(\Lambda) = \mathrm{Span}(1)$ when $n = 2$. In general, we can deduce the following, from (4.32).

Proposition 4.2. *If $f \in \mathcal{N}(\Lambda)$, then* $\mathrm{PI}\, f$ *is constant.*

The conclusion implies f is constant. The converse need not hold, i.e., $\mathrm{PI}\,1$ might not be constant. It is constant if $\overline{X} = \mathbb{R}^2 \setminus B_1$; cf. (4.27). Perhaps $\mathrm{PI}\,1 = 1$ whenever $\overline{X}$ is asymptotically Euclidean, of dimension $n = 2$. By Proposition 4.2,

$$\mathrm{PI}(1) \neq 1 \Longrightarrow \mathcal{N}(\Lambda) = 0. \tag{4.39}$$

The implication

$$\mathrm{PI}(1) = 1 \Longrightarrow \mathcal{N}(\Lambda) \supset \mathrm{Span}(1) \tag{4.40}$$

follows directly from the definition (4.2). This together with Proposition 4.2 gives

$$\mathrm{PI}(1) = 1 \Longrightarrow \mathcal{N}(\Lambda) = \mathrm{Span}(1). \tag{4.41}$$

5. Solving $\Delta_g u = f$ on a complete Riemannian manifold

Let M be a complete Riemannian manifold, of dimension n. Assume M is connected. Given $f \in C_0^\infty(M)$, we desire to find u such that

$$\Delta_g u = f, \quad u \in C^\infty(M) \cap L^\infty(M). \tag{5.1}$$

This is easily done if $M = \mathbb{R}^n$, for all such f, if $n \geq 3$; for $n = 2$ one can find such u provided

$$\int\limits_M f\,dV = 0. \tag{5.2}$$

We will study solvability of (5.1) under the general hypothesis stated above, and look into when (5.2) is required.

To start, given $f \in C_0^\infty(M)$, pick a smoothly bounded, connected, open set Ω such that

$$\mathrm{supp}\, f \subset \Omega \tag{5.3}$$

and $\overline{\Omega}$ is compact. Set

$$\overline{X} = M \setminus \Omega. \tag{5.4}$$

We want to find $v \in C^\infty(\overline{\Omega})$ such that

$$\Delta_g v = f \ \text{ on } \ \Omega, \quad \partial_\nu v = -\Lambda v \ \text{ on } \ \partial\Omega, \tag{5.5}$$

where ν is the unit normal to $\partial\Omega = \partial X$ pointing out of Ω (and into X), and Λ is the Dirichlet-to-Neumann map associated to X, discussed in §4 (cf. (4.2)). If we have such a solution to (5.5), a solution to (5.1) is given by

$$\begin{aligned} u(x) = v(x), &\qquad x \in \overline{\Omega}, \\ \mathrm{PI}\, v\big|_{\partial X}, &\quad x \in X, \end{aligned} \tag{5.6}$$

with PI as in (4.1). This clearly solves $\Delta_g u = f$ on $M \setminus \partial\Omega$, and has the property that neither u nor ∇u have a jump across $\partial\Omega$, so in fact (5.1) holds.

There is one minor point to address. In §4, we assumed X was connected. Here, we do not want to impose this restriction. We allow X to have connected components X_j, $1 \leq j \leq K$. Then we have

$$\begin{aligned} \mathrm{PI}_j : C^\infty(\partial X_j) &\longrightarrow C^\infty(\overline{X}_j) \cap L^\infty(X_j), \\ \Lambda_j f = -\partial_\nu \,\mathrm{PI}_j\, f, &\quad \Lambda_j \in OPS^1(\partial X_j), \end{aligned} \tag{5.7}$$

and, in the obvious sense,

$$\mathrm{PI} = \mathrm{PI}_1 \oplus \cdots \oplus \mathrm{PI}_K, \quad \Lambda = \Lambda_1 \oplus \cdots \oplus \Lambda_K. \tag{5.8}$$

We also bring in

$$\begin{aligned} &\mathrm{PI}_0 : C^\infty(\partial\Omega) \longrightarrow C^\infty(\overline{\Omega}), \\ &\Lambda_0 f = \partial_\nu \, \mathrm{PI}_0 \, f, \quad \Lambda_0 \in OPS^1(\partial\Omega). \end{aligned} \tag{5.9}$$

Note the absence of a minus sign, since ν points out of Ω. As in (4.16), we have

$$\Lambda_0 - \sqrt{-\Delta_S} \in OPS^0(\partial\Omega), \tag{5.10}$$

where Δ_S is the Laplace-Beltrami operator on $\partial\Omega = \partial X$. Hence

$$\Lambda_0 - \Lambda \in OPS^0(\partial\Omega). \tag{5.11}$$

To proceed, take $k \in \mathbb{Z}^+$ and define a family of operators

$$L_\tau : H^{k+2}(\Omega) \longrightarrow H^k(\Omega) \oplus H^{k+1/2}(\partial\Omega) \tag{5.12}$$

for $\tau \in \mathbb{C}$, by

$$L_\tau v = (\Delta_g v, \partial_\nu v + \tau \Lambda v). \tag{5.13}$$

Lemma 5.1. *When $\tau \neq -1$, L_τ in* (5.12) *is Fredholm, of index zero.*

Proof. The proof is the same as that of Lemma 2.2, except that (5.9)–(5.11) replaces (2.14). □

Of course, the case of direct interest in (5.13) is $\tau = +1$.

Lemma 5.2. *Given $v \in H^2(\Omega)$,*

$$v \in \mathcal{N}(L_1) \Longrightarrow v \quad \textit{is constant.} \tag{5.14}$$

Proof. The proof is the same as that of Lemma 2.3. □

For the constant function 1 to belong to $\mathcal{N}(L_1)$, it is necessary and sufficient that $\Lambda 1 = 0$, i.e.,

$$\Lambda_j 1 = 0, \quad \forall\, j \in \{1, \dots, K\}, \tag{5.15}$$

with Λ_j as in (5.7). This leads to the following.

Lemma 5.3. *If* (5.15) *holds, then* $\mathcal{N}(L_1) = \mathrm{Span}(1)$. *If* (5.15) *fails, then* $\mathcal{N}(L_1) = 0$.

Remark. In light of (4.39)–(4.41), we see that (5.15) is equivalent to

$$\mathrm{PI}_j(1) = 1, \quad \forall\, j \in \{1, \dots, K\}. \tag{5.16}$$

We are ready for our first existence result.

Proposition 5.4. *If* (5.15) *fails, then* (5.1) *has a solution for all $f \in C_0^\infty(\Omega)$.*

Proof. By Lemmas 5.1–5.3, L_1 is an isomorphism in (5.12). Hence, for each $f \in C_0^\infty(\Omega)$, there is a unique $v \in H^{k+1}(\Omega)$ such that $L_1 v = (f, 0)$. Elliptic regularity implies $v \in C^\infty(\overline{\Omega})$. Then the construction (5.6) produces the desired solution u. □

The following result complements Proposition 5.4.

Proposition 5.5. *If* (5.15) *holds, then* (5.1) *has a solution for all $f \in C_0^\infty(\Omega)$ satisfying* (5.2).

Proof. By Lemmas 5.1–5.3, L_1 is Fredholm of index 0 in (5.12), and

$$\mathcal{N}(L_1) = \mathrm{Span}(1). \tag{5.17}$$

Hence

$$\mathcal{R}(L_1) \text{ has codimension one in } H^k(\Omega) \oplus H^{k+1/2}(\partial\Omega). \tag{5.18}$$

Taking $k = 0$, we want to identify the annihilator of $\mathcal{R}(L_1)$ in $L^2(\Omega) \oplus H^{-1/2}(\partial\Omega)$, a space we know has dimension 1. To say (w, h) belongs to the annihilator of $\mathcal{R}(L_1)$ is to say that

$$(\Delta_g v, w) + (\partial_\nu v + \Lambda v, h) = 0, \quad \forall\, v \in H^2(\Omega). \tag{5.19}$$

We note that $(w, h) = (1, -1)$ satisfies this condition. In fact, Green's theorem implies

$$(\Delta_g v, 1) = \int\limits_{\partial\Omega} \partial_\nu v \, dS, \tag{5.20}$$

and

$$(\Lambda v, 1) = (v, \Lambda 1) = 0, \tag{5.21}$$

the latter identity by (5.15). The dimension count implies

$$(w, h) = (1, -1) \text{ spans the annihilator of } \mathcal{R}(L_1). \tag{5.22}$$

Hence, given $f \in C_0^\infty(\Omega)$,

$$\int\limits_{\Omega} f \, dV = 0 \Longrightarrow (f, 0) \in \mathcal{R}(L_1), \tag{5.23}$$

so there exists $v \in H^2(\Omega)$ such that $L_1 v = (f, 0)$. The end of the proof follows as in Proposition 5.4. □

6. Rougher boundary values

Let $\overline{M}$ be a complete, n-dimensional Riemannian manifold with compact boundary ∂M, and interior M. We assume M is connected. As shown in §4, we have

$$\mathrm{PI} : H^s(\partial M) \longrightarrow H^{s+1/2}(M^b) \cap C^\infty(M) \cap L^\infty(M^\#), \tag{6.1}$$

provided $s \geq 1/2$. Here,

$$M^b = \{x \in M : \mathrm{dist}(x, \partial M) < 1\}, \quad M^\# = M \setminus M^b. \tag{6.2}$$

Our goal is to extend (6.1) to all $s \in \mathbb{R}$.

To begin, a standard parametrix construction (cf. [**10**], Chapter 9, §2) yields

$$\widetilde{P} : H^s(\partial M) \longrightarrow H^{s+1/2}(M) \cap C^\infty(M), \tag{6.3}$$

defined simultaneously for all $s \in \mathbb{R}$, such that

$$\begin{aligned} h \in H^s(\partial M) \Longrightarrow \ &\mathrm{supp}\, \widetilde{P}h \subset \overline{M}^b, \ \text{ and} \\ &f = \Delta_g \widetilde{P}h \in C^\infty(\overline{M}). \end{aligned} \tag{6.4}$$

We construct PI in the form

$$\mathrm{PI}\, h = \widetilde{P}h - Qh, \tag{6.5}$$

where $u = Qh$ satisfies

$$\Delta_g u = f, \quad u \in C^\infty(\overline{M}) \cap L^\infty(M), \quad u = 0 \ \text{ on } \ \partial M. \tag{6.6}$$

This is like (5.1) except that now M has a boundary and we impose a Dirichlet boundary condition. We parallel the construction of §5. In (6.6), we can take arbitrary $f \in C_0^\infty(\overline{M})$ (enlarging M^b).

Let $\Omega \subset M$ be a smoothly bounded, connected open set that contains M^b, with compact closure $\overline{\Omega}$. Set $X = \overline{M} \setminus \overline{\Omega}$. We have $\partial\Omega = \partial M \cup \partial X$, and the construction of §4 gives

$$\mathrm{PI}_1 : C^\infty(\partial X) \longrightarrow C^\infty(\overline{X}) \cap L^\infty(X), \tag{6.7}$$

extending to $H^s(\partial X)$ for $s \geq 1/2$. We also have

$$\mathrm{PI}_0 : C^\infty(\partial X) \longrightarrow C^\infty(\overline{\Omega}), \tag{6.8}$$

given by

$$u = \mathrm{PI}_0\, h \;\text{ solves }\; \Delta_g u = 0 \;\text{ on }\; \Omega, \quad u\big|_{\partial M} = 0, \quad u\big|_{\partial X} = h. \tag{6.9}$$

We define Λ_0 and Λ_1 by

$$\Lambda_1 h = -\partial_\nu PI_1 h, \quad \Lambda_0 h = \partial_\nu \,\mathrm{PI}_0\, h, \tag{6.10}$$

where ν is the unit normal to ∂X pointing into X (out of Ω). Then $\Lambda_0,\ \Lambda_1 \in OPS^1(\partial X)$ are elliptic, and

$$\Lambda_0 - \Lambda_1 \in OPS^0(\partial X). \tag{6.11}$$

We want to find $v \in C^\infty(\overline{\Omega})$ such that

$$\Delta_g v = f, \quad v\big|_{\partial M} = 0, \quad \partial_\nu v = -\Lambda_1 v \;\text{ on }\; \partial X, \tag{6.12}$$

given $f \in C^\infty(\overline{M})$, supported on $\overline{M}^b$. If we have such a solution to (6.12), a solution to (6.6) is given by

$$\begin{aligned} u(x) = v(x), &\qquad x \in \overline{\Omega}, \\ \mathrm{PI}_1\, v\big|_{\partial X}, &\quad x \in X. \end{aligned} \tag{6.13}$$

To proceed, take $k \in \mathbb{Z}^+$ and define a family of maps

$$L_\tau : H_b^{k+1}(\Omega) \longrightarrow H^k(\Omega) \oplus H^{k+1/2}(\partial X) \tag{6.14}$$

for $\tau \in \mathbb{C}$, by

$$L_\tau v = (\Delta_g v, \partial_\nu v + \tau\Lambda_1 v). \tag{6.15}$$

Here,

$$H_b^{k+2}(\Omega) = \{v \in H^{k+2}(\Omega) : v = 0 \text{ on } \partial M\}. \tag{6.16}$$

The argument used in Lemmas 2.2 and 5.1 gives the following.

Lemma 6.1. *When $\tau \neq -1$, L_τ in* (6.14) *is Fredholm, of index zero.*

Then the argument used in Lemmas 2.3 and 5.2 gives the following.

Lemma 6.2. *For $k \geq 0$,*

$$\mathcal{N}(L_1) = 0. \tag{6.17}$$

Hence L_1 is an isomorphism in (6.14).

Proof. The argument in Lemma 2.3 works here to show that any $v \in \mathcal{N}(L_1)$ must be constant. Then the constraint $v|_{\partial M} = 0$ yields (6.17). The isomorphism property follows from the index 0 property and (6.17). $\square$

Solvability of (6.12) follows directly from Lemma 6.2 and elliptic regularity. Then (6.13) produces the solution to (6.6), and the extension of (6.1) to all $s \in \mathbb{R}$ is complete.

Appendix A. Harnack estimates, Liouville theorems, and uniformization

The first Liouville theorem we establish is the following.

PROPOSITION A.1. *Let $G(x) = (g_{ij}(x))$ be a continuous symmetric $n \times n$ matrix function, defining a metric tensor on $\mathbb{R}^n$. Assume there exist $B_0, B_1 \in (0, \infty)$ such that*

$$B_0 I \le G(x) \le B_1 I, \quad \forall\, x \in \mathbb{R}^n. \tag{A.1}$$

If u is a bounded solution to

$$\Delta_g u = 0 \quad \text{on} \quad \mathbb{R}^n, \tag{A.2}$$

then u is constant.

Before proving this, we deduce the following result.

PROPOSITION A.2. *In the setting of Proposition A.1, if $n = 2$ and g_{jk} is Hölder continuous, then $(\mathbb{R}^2, g_{jk})$ is conformally equivalent to the flat plane $(\mathbb{R}^2, \delta_{jk})$.*

Proof. Under these hypotheses, there are local isothermal coordinates, so $(\mathbb{R}^2, g_{jk})$ has the structure of a Riemann surface. By the uniformization theorem, it is conformally equivalent to

$$\text{the flat plane} \tag{A.3}$$

or

$$\text{the Poincaré disk.} \tag{A.4}$$

(See [**2**] for a careful treatment of the uniformization theorem. For a PDE proof, see [**7**].) The case (A.4) holds if and only if there is a nonconstant bounded harmonic function on $(\mathbb{R}^2, g_{jk})$; otherwise the case (A.3) holds. By Proposition A.1, we know case (A.4) cannot hold. $\square$

REMARK. While the setting of Proposition A.2 is much more general than that of (1.8)–(1.11), as carried out in §2, Proposition A.2 does not imply the results given there, since we have no large x asymptotics on the conformal diffeomorphism of $(\mathbb{R}^2, g_{jk})$ with $(\mathbb{R}^2, \delta_{jk})$ given by Proposition A.2.

The proof of Proposition A.1 (which is perhaps well known) makes use of Harnack's inequality. See [**3**], pp. 44–45, for a related argument. We use the following form of Harnack's inequality, which follows from Corollary 8.21 of [**3**].

PROPOSITION A.3. *Let $A(x) = (a^{jk}(x))$ be a continuous, symmetric, $n \times n$ matrix function on $B_2(0) \subset \mathbb{R}^n$. Assume there exist $A_0, A_1 \in (0, \infty)$ such that*

$$A_0 I \le A(x) \le A_1 I, \quad \forall\, x \in B_2(0). \tag{A.5}$$

There exists $C = C(A_0, A_1, n)$ with the property that, if u is a solution to

$$\partial_j a^{jk}(x)\partial_k u = 0 \quad on \quad B_2(0), \quad u \geq 0 \quad on \quad B_2(0), \tag{A.6}$$

then

$$\sup_{B_1(0)} u(x) \leq C \inf_{B_1(0)} u(x). \tag{A.7}$$

Proof of Proposition A.1. To begin, adding a constant to u, we can arrange

$$u \geq 0 \quad \text{on} \quad \mathbb{R}^n, \quad \inf_{\mathbb{R}^n} u = 0. \tag{A.8}$$

Then the goal is to show that $u \equiv 0$. Note that (A.2) is equivalent to

$$\partial_j a^{jk}(x)\partial_k u = 0, \quad a^{jk}(x) = g(x)^{1/2} g^{jk}(x), \tag{A.9}$$

where $(g^{jk}(x)) = (g_{jk}(x))^{-1}$, $g = \det G$. The hypothesis (A.1) implies (A.5), for all $x \in \mathbb{R}^n$. Now, for $R > 0$, define v_R in $B_2(0)$ by

$$v_R(x) = u(Rx). \tag{A.10}$$

We have

$$\partial_j a^{jk}(Rx)\partial_k v_R(x) = 0 \quad \text{on} \quad B_2(0). \tag{A.11}$$

Now this replacement of (A.6) has the same ellipticity constants as in (A.5), so Proposition A.3 implies that there exists $C = C(A_0, A_1, n)$ (independent of R) such that

$$\sup_{B_1(0)} v_R \leq C \inf_{B_1(0)} v_R, \tag{A.12}$$

hence

$$\sup_{B_R(0)} u \leq C \inf_{B_R(0)} u. \tag{A.13}$$

Taking $R \to \infty$ yields $\sup_{\mathbb{R}^n} u = 0$, hence $u \equiv 0$, as desired. □

Note that Proposition A.1 does not imply Proposition 2.5, since the latter allows for nontrivial topology. The following extension of Proposition A.1 is strictly stronger than Proposition 2.5.

Proposition A.4. *In the setting of Proposition A.1, cut out $B_1(0)$ from $\mathbb{R}^n$ and glue in Ω, a compact Riemannian manifold with boundary $\partial\Omega \approx S^{n-1}$, to form a Riemannian manifold with continuous metric tensor (M, g_{jk}), agreeing with $(\mathbb{R}^n, g_{jk})$ on $|x| \geq 1$. If u is a bounded solution of*

$$\Delta_g u = 0 \quad on \quad M, \tag{A.14}$$

then u is constant.

Proof. As in the proof of Proposition A.1, we can add a constant to u and arrange

$$u \geq 0 \quad \text{on} \quad M, \quad \inf_M u = 0. \tag{A.15}$$

Then the goal is to show $u \equiv 0$. Note that there must exist $x_\nu \in M \setminus \Omega \approx \mathbb{R}^n \setminus B_1(0)$ such that

$$|x_\nu| = R_\nu + 1 \to \infty, \quad u(x_\nu) = \varepsilon_\nu \to 0 \tag{A.16}$$

($|x_\nu|$ denotes the Euclidean norm on $\mathbb{R}^n$), since otherwise u would have to assume its minimum at a point of M (hence $u \equiv 0$). Now a Harnack inequality argument like that used in the proof of Proposition A.1 gives

$$\sup_{B_{R_\nu/2}(x_\nu)} u \leq C\varepsilon_\nu. \tag{A.17}$$

Then (assuming $R_\nu > 2$) we can cover

$$\mathcal{A}_\nu = \{x \in \mathbb{R}^n : R_\nu \leq |x| \leq R_\nu + 1\} \tag{A.18}$$

by M_n balls of radius $R_\nu/2$, and invoke the Harnack estimate repeatedly to get

$$\sup_{\mathcal{A}_\nu} u \leq \widetilde{C}\varepsilon_\nu. \tag{A.19}$$

That $u \equiv 0$ then follows by the maximum principle. □

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