## Concavity of log det on $\mathcal{P}_{n}$

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In the set $M(n, \mathbb{R})$ of real $n \times n$ matrices, consider

$$
\begin{align*}
\operatorname{Sym}(n) & =\left\{X \in M(n, \mathbb{R}): X^{t}=X\right\} \\
\mathcal{P}_{n} & =\{X \in \operatorname{Sym}(n): X \text { positive definite }\} \tag{1}
\end{align*}
$$

Note that $\mathcal{P}_{n}$ is an open, convex cone in $\operatorname{Sym}(n)$. We consider the function

$$
\begin{equation*}
F: \mathcal{P}_{n} \longrightarrow \mathbb{R}, \quad F(X)=\log \operatorname{det} X \tag{2}
\end{equation*}
$$

and aim to prove the following.
Proposition. The function $F$ is concave.
To see this, take $X \in \mathcal{P}_{n}, Y \in \operatorname{Sym}(n), Y \neq 0$. It suffices to show that, for arbitrary such $X$ and $Y$, the function

$$
\begin{equation*}
\varphi_{X, Y}(t)=\log \operatorname{det}(X+t Y) \tag{3}
\end{equation*}
$$

satisfies

$$
\begin{equation*}
\varphi_{X, Y}^{\prime \prime}(0)<0 . \tag{3A}
\end{equation*}
$$

To make this computation, set $X^{-1}=A^{2}, A \in \mathcal{P}_{n}$. Then

$$
\begin{align*}
\operatorname{det}(X+t Y) & =\operatorname{det}(X) \operatorname{det}\left(I+t X^{-1} Y\right)  \tag{4}\\
& =(\operatorname{det} X) \operatorname{det}(I+t A Y A),
\end{align*}
$$

so

$$
\begin{equation*}
\log \operatorname{det}(X+t Y)=\log \operatorname{det} X+\log \operatorname{det}(I+t A Y A) \tag{5}
\end{equation*}
$$

We set $L=A Y A \in \operatorname{Sym}(n)$. Note that if

$$
\begin{equation*}
\operatorname{Spec} L=\left\{\lambda_{1}, \ldots, \lambda_{n}\right\}, \tag{6}
\end{equation*}
$$

then

$$
\begin{equation*}
\operatorname{det}(I+t L)=\prod_{j=1}^{n}\left(1+t \lambda_{j}\right) \tag{7}
\end{equation*}
$$

so

$$
\begin{equation*}
\log \operatorname{det}(I+t L)=\sum_{j=1}^{n} \log \left(1+t \lambda_{j}\right) \tag{8}
\end{equation*}
$$

Hence

$$
\begin{equation*}
\left.\frac{d^{2}}{d t^{2}} \log \operatorname{det}(I+t L)\right|_{t=0}=-\sum_{j=1}^{n} \lambda_{j}^{2}=-\operatorname{Tr} L^{2}<0 \tag{9}
\end{equation*}
$$

This gives the asserted concavity.
Remark. We also have

$$
\begin{equation*}
\left.\frac{d}{d t} \log \operatorname{det}(I+t L)\right|_{t=0}=\sum \lambda_{j}=\operatorname{Tr} L \tag{10}
\end{equation*}
$$

so

$$
\begin{align*}
\left.\frac{d}{d t} \log \operatorname{det}(X+t Y)\right|_{t=0} & =\operatorname{Tr} A Y A  \tag{11}\\
& =\operatorname{Tr} X^{-1} Y
\end{align*}
$$

This identity readily extends to $X \in G \ell_{+}(n, \mathbb{R}), Y \in M(n, \mathbb{R})$. Furthermore, writing

$$
\begin{equation*}
\log (1+z)=\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} z^{k}, \quad|z|<1 \tag{12}
\end{equation*}
$$

we have from (8) that

$$
\begin{equation*}
\log \operatorname{det}(I+t L)=\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\left(\operatorname{Tr} L^{k}\right) t^{k}, \quad|t| \cdot\|L\|<1 \tag{13}
\end{equation*}
$$

and hence

$$
\begin{gather*}
\log \operatorname{det}(X+t Y)=\log \operatorname{det} X+\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \operatorname{Tr}\left(X^{-1} Y\right)^{k} t^{k},  \tag{14}\\
|t| \cdot\left\|X^{-1} Y\right\|<1,
\end{gather*}
$$

for $X \in G \ell_{+}(n, \mathbb{R}), Y \in M(n, \mathbb{R})$.

