# Concavity of log det on $\mathcal{P}_n$

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In the set $M(n,\mathbb{R})$ of real $n \times n$ matrices, consider

$$\begin{aligned} \operatorname{Sym}(n) &= \{X \in M(n,\mathbb{R}) : X^t = X\}, \\ \mathcal{P}_n &= \{X \in \operatorname{Sym}(n) : X \text{ positive definite}\}. \end{aligned} \tag{1}$$

Note that $\mathcal{P}_n$ is an open, convex cone in $\operatorname{Sym}(n)$. We consider the function

$$F : \mathcal{P}_n \longrightarrow \mathbb{R}, \quad F(X) = \log\det X, \tag{2}$$

and aim to prove the following.

**Proposition.** *The function $F$ is concave.*

To see this, take $X \in \mathcal{P}_n$, $Y \in \operatorname{Sym}(n)$, $Y \neq 0$. It suffices to show that, for arbitrary such $X$ and $Y$, the function

$$\varphi_{X,Y}(t) = \log\det(X + tY) \tag{3}$$

satisfies

$$\varphi''_{X,Y}(0) < 0. \tag{3A}$$

To make this computation, set $X^{-1} = A^2$, $A \in \mathcal{P}_n$. Then

$$\begin{aligned} \det(X + tY) &= \det(X)\det(I + tX^{-1}Y) \\ &= (\det X)\det(I + tAYA), \end{aligned} \tag{4}$$

so

$$\log\det(X + tY) = \log\det X + \log\det(I + tAYA). \tag{5}$$

We set $L = AYA \in \operatorname{Sym}(n)$. Note that if

$$\operatorname{Spec} L = \{\lambda_1, \dots, \lambda_n\}, \tag{6}$$

then

$$\det(I + tL) = \prod_{j=1}^{n} (1 + t\lambda_j), \tag{7}$$

so

$$\log\det(I+tL)=\sum_{j=1}^{n}\log(1+t\lambda_j). \tag{8}$$

Hence

$$\frac{d^2}{dt^2}\log\det(I+tL)\Big|_{t=0}=-\sum_{j=1}^{n}\lambda_j^2=-\operatorname{Tr}L^2<0. \tag{9}$$

This gives the asserted concavity.

Remark. We also have

$$\frac{d}{dt}\log\det(I+tL)\Big|_{t=0}=\sum\lambda_j=\operatorname{Tr}L, \tag{10}$$

so

$$\begin{aligned}\frac{d}{dt}\log\det(X+tY)\Big|_{t=0}&=\operatorname{Tr}AYA\\&=\operatorname{Tr}X^{-1}Y.\end{aligned} \tag{11}$$

This identity readily extends to $X\in G\ell_+(n,\mathbb{R})$, $Y\in M(n,\mathbb{R})$. Furthermore, writing

$$\log(1+z)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}z^k,\quad |z|<1, \tag{12}$$

we have from (8) that

$$\log\det(I+tL)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}(\operatorname{Tr}L^k)t^k,\quad |t|\cdot\|L\|<1, \tag{13}$$

and hence

$$\begin{gathered}\log\det(X+tY)=\log\det X+\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\operatorname{Tr}(X^{-1}Y)^k t^k,\\ |t|\cdot\|X^{-1}Y\|<1,\end{gathered} \tag{14}$$

for $X\in G\ell_+(n,\mathbb{R})$, $Y\in M(n,\mathbb{R})$.