Spectral Asymptotics of $S^2 \times S^2$

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Abstract. We show that the spectral counting function of $S^2 \times S^2$ satisfies

$$\mathcal{N}(S^2 \times S^2, R) = \frac{1}{2} R^4 + O(R^{3-1/3}).$$

1. Introduction

If M is a smooth, compact Riemannian manifold of dimension n, equipped with Laplace-Beltrami operator Δ, the Weyl formula specifies the asymptotic behavior as $R \to +\infty$ of

$$\mathcal{N}(M, R) = \sum_{\lambda_k \le R} \dim V_k, \tag{1.1}$$

where

$$V_k = \{u \in C^\infty(M) : \Delta u = -\lambda_k^2 u\}, \tag{1.2}$$

and

$$0 = \lambda_0^2 < \lambda_1^2 < \lambda_2^2 < \cdots < \lambda_j^2 < \cdots \nearrow +\infty \tag{1.3}$$

are the eigenvalues of $-\Delta$. The general formula is

$$\mathcal{N}(M, R) = C_n \operatorname{Vol}(M) R^n + O(R^{n-1}), \quad C_n = \frac{(4\pi)^{-n/2}}{\Gamma(\frac{n}{2} + 1)}. \tag{1.4}$$

See [H].

As is well known, the remainder estimate in (1.4) is sharp for spheres. For example, when $M = S^2$, the unit sphere $S^2 \subset \mathbb{R}^3$, and we take

$$\Lambda = \Big(-\Delta + \frac{1}{4} I\Big)^{1/2}, \tag{1.5}$$

we have

$$\operatorname{Spec} \Lambda = \Big\{k + \frac{1}{2} : k \in \mathbb{Z}^+\Big\}, \tag{1.6}$$

or equivalently, with λ_k as in (1.2),

$$\lambda_k^2 = k(k+1). \tag{1.7}$$

Furthermore,

$$\dim V_k = 2k+1, \tag{1.8}$$

so the remainder estimate $O(R)$ cannot be improved in such a case.

For some time, people have looked for conditions on M that allow one to improve the remainder estimate $O(R^{n-1})$. In [DG], it is shown that one can take $o(R^{n-1})$ in case the geodesic flow on M has "not too many" periodic orbits. In [B] it is shown that if either $\dim M = 2$ and there are no conjugate points, or $\dim M \geq 3$ and the sectional curvatures are all ≤ 0, then one can take

$$O(R^{n-1}/\log R) \tag{1.9}$$

as a remainder estimate in (1.4). Recently [CG] has obtained such an improved remainder estimate for a much broader class of Riemannian manifolds. Among the results obtained there is the result that the remainder estimate (1.9) holds when M is a Cartesian product

$$M = X \times Y \tag{1.10}$$

of two Riemannian manifolds (each of positive dimension), with the product metric.

In another vein, it has long been known that one can get a much better remainder estimate when $M = \mathbb{T}^n$ is the flat torus,

$$\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n. \tag{1.11}$$

In such a case, the eigenvalues of $-\Delta$ are the square-lengths of the points of $\mathbb{Z}^n$, counted with multiplicity. One has remainder estimates of the form

$$O(R^{n-1-\alpha}), \tag{1.12}$$

with $\alpha > 0$. Just how big one can take α is an open problem. In case $n = 2$, it is known that one can take $\alpha > 1/3$ but one cannot take $\alpha = 1/2$; see [He], p. 450.

In this note, we examine the Cartesian product

$$M = S^2 \times S^2, \tag{1.13}$$

and observe the following.

Proposition 1.1. *We have the Weyl formula*

$$\mathcal{N}(S^2 \times S^2, R) = C_4 \operatorname{Vol}(S^2 \times S^2) R^4 + O(R^{3-1/3}). \tag{1.14}$$

The analysis proceeds as follows. In §2 we use (1.5)–(1.8) to express the left side of (1.14) (or rather a closely related quantity) as a "weighted lattice point count." In §3 we show how results of [RT] lead to an approximation to this weighted count, yielding the right side of (1.14).

Note added. I have learned that the paper [IW] has the result (1.14), in the more general setting of products of spheres, $S^{d_1} \times \cdots \times S^{d_n}$. The proof given there also involves a weighted lattice point count.

2. Analysis of $\mathcal{N}(S^2 \times S^2)$ as a weighted lattice point count

Let us write the Laplace operator Δ on $S^2 \times S^2$ as $\Delta = \Delta_1 + \Delta_2$, where Δ_j are the Laplace operators on the two factors. Define $\Lambda_j = (-\Delta_j + 1/4)^{1/2}$, parallel to (1.5), so

$$-\Delta = \Lambda_1^2 + \Lambda_2^2 - \frac{1}{2} I. \tag{2.1}$$

We see that the spaces

$$V_{k\ell} = \Big\{ u(x)v(y) : \Lambda_1 u = \Big(k + \frac{1}{2}\Big) u,\ \Lambda_2 v = \Big(\ell + \frac{1}{2}\Big) v \Big\}, \tag{2.2}$$

defined for $k, \ell \in \mathbb{Z}^+$, yield an orthogonal decomposition

$$L^2(S^2 \times S^2) = \bigoplus_{k,\ell \in \mathbb{Z}^+} V_{k\ell}, \tag{2.3}$$

and

$$\begin{aligned} w \in V_{k\ell} \Longrightarrow -\Delta w = \lambda_{k\ell}^2 w,& \\ \lambda_{k\ell}^2 = \Big(k + \frac{1}{2}\Big)^2 + \Big(\ell + \frac{1}{2}\Big)^2 - \frac{1}{2}.& \end{aligned} \tag{2.4}$$

It follows that the spectral counting function

$$\mathcal{N}(S^2 \times S^2, R) = \sum_{\lambda_{k\ell} \le R} \dim V_{k\ell} \tag{2.5}$$

is related to the weighted lattice point count

$$\widetilde{\mathcal{N}}(R) = \sum_{(k+1/2)^2 + (\ell+1/2)^2 \le R^2} \dim V_{k\ell} \tag{2.6}$$

by

$$\mathcal{N}(S^2 \times S^2, R) = \widetilde{\mathcal{N}}\Big(\sqrt{R^2 + \frac{1}{2}}\Big). \tag{2.7}$$

Since

$$\Big(\sqrt{R^2 + \frac{1}{2}}\Big)^4 = R^4 + R^2 + \frac{1}{4}, \tag{2.8}$$

the asserted result (1.14) is equivalent to the statement that

$$\widetilde{\mathcal{N}}(R) = C_4 \operatorname{Vol}(S^2 \times S^2) R^4 + O(R^{3-1/3}). \tag{2.9}$$

We also note that, by (1.8),

$$\dim V_{k\ell} = (2k+1)(2\ell+1). \tag{2.10}$$

Here is another presentation of the function $\widetilde{\mathcal{N}}(R)$. Define the locally finite measure μ on $\mathbb{R}^2$ by

$$\mu = \sum_{k,\ell \in \mathbb{Z}} \delta_{p_{k\ell}}, \quad p_{k\ell} = \Big(k + \frac{1}{2}, \ell + \frac{1}{2}\Big), \tag{2.11}$$

define the function χ_R on $\mathbb{R}^2$ by

$$\begin{aligned} \chi_R(x) = 1, &\quad \text{if } \ |x| \le R, \\ 0, &\quad \text{otherwise.} \end{aligned} \tag{2.12}$$

and define the functions x_j^+ on $\mathbb{R}^2$ (with $x = (x_1, x_2)$) by

$$\begin{aligned} x_j^+ = x_j, &\quad \text{if } \ x_j \ge 0, \\ 0, &\quad \text{otherwise.} \end{aligned} \tag{2.13}$$

Then

$$\widetilde{\mathcal{N}}(R) = 4 \int\limits_{\mathbb{R}^2} x_1^+ x_2^+ \chi_R(x)\, d\mu(x). \tag{2.14}$$

A crude analysis of (2.14) readily yields

$$\widetilde{\mathcal{N}}(R) = K R^4 + o(R^4), \tag{2.15}$$

with

$$\begin{aligned} K &= 4 \int\limits_{|x| \le 1} x_1^+ x_2^+ \, dx \\ &= 4 \int_0^1 \int_0^{\pi/2} r^2 \sin\theta \cos\theta \, r \, d\theta \, dr \\ &= \int_0^{\pi/2} \sin\theta\cos\theta \, d\theta \\ &= \frac{1}{2}, \end{aligned} \tag{2.16}$$

i.e.,

$$\widetilde{\mathcal{N}}(R) = \frac{1}{2} R^4 + o(R^4), \tag{2.17}$$

consistent with the leading term in (2.9), since

$$C_4 \operatorname{Vol}(S^2 \times S^2) = \frac{(4\pi)^{-2}}{\Gamma(3)} (4\pi)^2 = \frac{1}{2}. \tag{2.18}$$

In §3 we make a finer analysis of (2.14), using material from [RT], and verify the remainder estimate in (2.9).

3. Approximation of $\widetilde{\mathcal{N}}(R)$ as a quadrature problem

Here we describe a result, Proposition 3.2 of [RT], and show that it applies to the formula (2.14) for $\widetilde{\mathcal{N}}(R)$ and yields (2.9) (which in turn implies Proposition 1.1). The general setting of [RT] takes $u \in L^\infty(\mathbb{T}^n)$, $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$, and examines the approximation of the mean value

$$Mu = (2\pi)^{-n} \int_{\mathbb{T}^n} u(x)\, dx \tag{3.1}$$

by

$$\sigma_h u(x) = \nu^{-n} \sum_{\ell \in (\mathbb{Z}/\nu)^n} u(x + h\ell), \quad h = \frac{2\pi}{\nu}, \tag{3.2}$$

where $\nu \in \mathbb{N}$ and $\mathbb{Z}/\nu$ denotes the group of residue classes mod ν. The goal is to examine the difference

$$\rho_h u(x) = \sigma_h u(x) - Mu. \tag{3.3}$$

An entry to this analysis is the basic identity

$$\rho_h u(x) = \sum_{j \in \mathbb{Z}^n \setminus 0} \hat{u}(\nu j) e^{i\nu j \cdot x}, \quad \nu = \frac{2\pi}{h}. \tag{3.4}$$

To state the result, we bring in the space F^s, defined for $s > 0$ by

$$\begin{aligned} u \in F^s &\Longleftrightarrow |\hat{u}(j)| \le C(1 + |j|)^{-s}, \\ \|u\|_{F^s} &= \sup_{j \in \mathbb{Z}^n} (1 + |j|)^s |\hat{u}(j)|. \end{aligned} \tag{3.5}$$

A key example is the following. Suppose

(3.6) $\Omega \subset \mathbb{T}^n$ is open, and $\partial\Omega = \Sigma$ is smooth and has positive Gauss curvature.

Then

$$\chi_\Omega \in F^{(n+1)/2}, \tag{3.7}$$

where $\chi_\Omega = 1$ on $\overline{\Omega}$, 0 on $\mathbb{T}^n \setminus \overline{\Omega}$. We denote by $\mathrm{PLip}(\Omega)$ the set of functions on $\mathbb{T}^n$ that are Lipschitz on each component of $\mathbb{T}^n \setminus \Sigma$, with a jump across Σ. Here is Proposition 3.2 of [RT].

Proposition 3.1. *Assume Ω satisfies (3.6), and assume $u \in L^\infty(\mathbb{T}^n)$ satisfies*

$$u \in F^s \cap \mathrm{PLip}(\Omega), \quad s = \frac{n+1}{2}. \tag{3.8}$$

Then

$$\|\rho_h u\|_{L^\infty} \le C h^{2n/(n+1)} \Big(\|u\|_{F^s} + \|u\|_{L^\infty} + \|u\|_{\mathrm{PLip}} \Big). \tag{3.9}$$

In case $n = 2$, (3.9) becomes

$$\|\rho_h u\|_{L^\infty} \le C h^{4/3} \Big(\|u\|_{F^s} + \|u\|_{L^\infty} + \|u\|_{\mathrm{PLip}} \Big). \tag{3.10}$$

To apply this result to $\widetilde{\mathcal{N}}(R)$, let us modify (2.11) to

$$\mu_h = \Big(\frac{h}{2\pi}\Big)^2 \sum_{k,\ell \in \mathbb{Z}} \delta_{hp_{k\ell}}, \quad h = \frac{2\pi}{\nu}, \tag{3.11}$$

with $\nu \in \mathbb{N}$. Then

$$\widetilde{\mathcal{N}}(R) = (2\pi)^2 h^{-4} \int_{\mathbb{R}^2} 4x_1^+ x_2^+ \chi_r(x)\, d\mu_h(x), \quad r = hR. \tag{3.12}$$

As $R \to \infty$, pick $\nu = \nu(R) \in \mathbb{N}$ such that $2\pi R/\nu(R) \in [1/2, 3/2]$. Then the disk $D_r = \{x \in \mathbb{R}^2 : |x| \le r\}$ can be regarded as a subset of $\mathbb{T}^2$, and we can write

$$\widetilde{\mathcal{N}}(R) = (2\pi)^{-2} h^{-4} \int_{\mathbb{T}^2} u_r(x)\, d\mu_h(x), \tag{3.13}$$

with

$$u_r(x) = 4x_1^+ x_2^+ \chi_r(x). \tag{3.14}$$

Comparison with (3.2) gives

$$\widetilde{\mathcal{N}}(R) = (2\pi)^2 h^{-4} \sigma_h u_r(h/2, h/2). \tag{3.15}$$

Now, parallel to (2.16), we have

$$M u_r = (2\pi)^{-2} \frac{r^4}{2} = (2\pi)^{-2} \cdot \frac{1}{2} \cdot h^4 R^4, \tag{3.16}$$

so (compare (2.17))

$$(2\pi)^2 h^{-4} M u_r = \frac{1}{2} R^4. \tag{3.17}$$

Consequently, our desired result (2.9) is equivalent to the statment that

$$|\rho_h u_r(h/2, h/2)| \leq C h^{4/3}. \tag{3.18}$$

We claim that actually

$$\|\rho_h u_r\|_{L^\infty} \leq C h^{4/3}. \tag{3.19}$$

To verify (3.19), it suffices to check that u_r satisfies the hypotheses of Proposition 3.1, uniformly in $r \in [1/2, 3/2]$. Clearly $\Omega = D_r$ satisfies (3.6), and $u_r \in \mathrm{PLip}(D_r)$. It remains to show that

$$u_r \in F^{3/2}. \tag{3.20}$$

Since we know that $\chi_r \in F^{3/2}$, this is a special case of the following.

Lemma 3.2. *Assume $s \in (0, 2]$ and $v \in F^s$ has support in a disk D_r with $r < \pi$. Then*

$$x_j^+ v \in F^s. \tag{3.21}$$

This follows from a convolution estimate that we leave to the reader.

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