Dirichlet Problem for Wave Equations

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Let M be a compact, n-dimensional Riemannian manifold, with Laplace operator Δ. We consider the following Dirichlet problem for the wave equation,

$$\begin{aligned} (\partial_t^2 - \Delta)u &= 0, \quad 0 \le t \le T, \\ u(0,x) = f(x), &\quad u(T,x) = g(x), \end{aligned} \tag{1}$$

and aim to establish the following.

Proposition 1. *Take $s > n$. Then, for a.e. $T \in (0,\infty)$, (1) has a unique solution for each $f, g \in H^s(M)$, satisfying*

$$\|u(t,\cdot)\|_{L^2(M)} \le C\Big(\|f\|_{H^s(M)} + \|g\|_{H^s(M)}\Big). \tag{2}$$

To get this, we let $\{u_k : k \ge 0\}$ be an orthonormal basis of $L^2(M)$, consisting of eigenfunctions of Δ,

$$\Delta u_k = -\lambda_k^2 u_k, \quad 0 \le \lambda_0 < \lambda_1 < \cdots \nearrow \infty. \tag{3}$$

(Assume M is connected.) We seek a solution

$$u(t,x) = (a_0 + b_0 t)u_0 + \sum_{k\ge1} (a_k e^{it\lambda_k} + b_k e^{-it\lambda_k})u_k. \tag{4}$$

We write

$$f = \sum \hat{f}(k)u_k, \quad g = \sum \hat{g}(k)u_k, \tag{5}$$

with $\hat{f}(k) = (f, u_k)$. The boundary condition in (1) requires

$$a_0 = \hat{f}(0), \quad a_0 + b_0 T = \hat{g}(0), \tag{6}$$

and, for $k \ge 1$,

$$\begin{aligned} a_k + b_k &= \hat{f}(k), \\ e^{iT\lambda_k} a_k + e^{-iT\lambda_k} b_k &= \hat{g}(k). \end{aligned} \tag{7}$$

From (7) we get

$$b_k = \frac{e^{iT\lambda_k}\hat{f}(k) - \hat{g}(k)}{2i \sin T\lambda_k}. \tag{8}$$

Thus our task is to show that, for a.e. $T > 0$, $|\sin T\lambda_k|$ stays away from 0, in a quantifiable way, for $k \geq 1$ (hence for $\lambda_k \geq \lambda_1 = b > 0$).

To this end, let us take $A \geq 1$ and $\gamma \in (0,1)$, and estimate

$$\int_0^A |\sin t\lambda|^{-\gamma}\, dt, \quad \text{for} \;\; \lambda \geq b. \tag{9}$$

A change of variable shows that (9) is equal to

$$\frac{1}{\lambda}\int_0^{A\lambda} |\sin s|^{-\gamma}\, ds \leq C, \tag{10}$$

where $C = C_{\gamma,b}A$. Hence, if we fix $\sigma > 0$ and set

$$X_{k,N} = \Big\{ t \in (0,A] : |\sin t\lambda_k|^{-\gamma} \geq N\lambda_k^{\sigma} \Big\}, \tag{11}$$

we have, by Chebecheff's inequality,

$$m(X_{k,N}) \leq \frac{C}{N}\lambda_k^{-\sigma}. \tag{12}$$

It follows that, for each N,

$$m\Big(\bigcup_{k\geq 1} X_{k,N}\Big) \leq \frac{C}{N}\sum_{k\geq 1}\lambda_k^{-\sigma} = \frac{C'}{N}, \tag{13}$$

provided

$$C\sum_{k\geq 1}\lambda_k^{-\sigma} = C' < \infty. \tag{14}$$

We then get

$$m\Big([0,A] \setminus \bigcup_{k\geq 1} X_{k,N}\Big) \geq A - \frac{C'}{N}. \tag{15}$$

Now, for T in the set measured in (15),

$$|\sin T\lambda_k|^{-1} \leq N^{1/\gamma}\lambda_k^{\sigma/\gamma}, \quad \forall\, k \geq 1. \tag{16}$$

It follows that, for such T,

$$|b_k| \leq N^{1/\gamma}\big(|\hat{f}(k)| + |\hat{g}(k)|\big)\lambda_k^{\sigma/\gamma}, \tag{17}$$

with a similar estimate on $|a_k|$, so (2) holds as long as

$$\sum_{k\geq 1} \lambda_k^{-s} < \infty. \tag{19}$$

In view of the Weyl asymptotic formula

$$\lambda_k \sim C(M)k^{1/n}, \quad k \nearrow \infty, \tag{19}$$

this holds as long as $s > n$.

REMARK 1. Once one has (17), we also have, for each $r \in \mathbb{R}$,

$$\|u(t,\cdot)\|_{H^r(M)} \leq C\Big(\|f\|_{H^{r+s}(M)} + \|g\|_{H^{r+s}(M)}\Big). \tag{20}$$

In addition,

$$\|\partial_t u(t,\cdot)\|_{H^{r-1}(M)} \leq \text{ RHS (20)}. \tag{21}$$

Furthermore, the solution to (1) holds for all $t \in \mathbb{R}$, not just for $t \in [0,T]$.

REMARK 2. Note that the assertion that (18) holds for all $s > n$ is equivalent to the assertion that

$$(-\Delta + 1)^{-s/2} \text{ is Hilbert-Schmidt on } L^2(M), \text{ for } s > n/2, \tag{22}$$

hence a consequence of the fact that any continuous linear operator

$$T : L^2(M) \longrightarrow H^s(M), \quad s > \frac{n}{2}, \tag{23}$$

is Hilbert-Schmidt on $L^2(M)$. This is more elementary than a derivation of eigenvalue asymptotics (19).