Local Regularity of Solutions to $\Delta u = f$

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Let M be a compact Riemannian manifold, with Laplace operator Δ. As seen in §5.1 of [T], we have an isomorphism

$$L : H^1(M) \xrightarrow{\approx} H^{-1}(M), \quad L = -\Delta + 1. \tag{1}$$

More generally,

$$L : H^{k+1}(M) \xrightarrow{\approx} H^{k-1}(M), \tag{2}$$

for $k \in \mathbb{Z}^+$, and then, by duality, for all $k \in \mathbb{Z}$. Furthermore, by interpolation (cf. [T], Chapter 5, (1.53)–(1.54)),

$$L : H^{s+1}(M) \xrightarrow{\approx} H^{s-1}(M), \quad \forall\, s \in \mathbb{R}. \tag{3}$$

Here, we establish a local regularity result. Here is our first version.

Proposition 1. *If $\Omega \subset M$ is open, then, for each $s \in \mathbb{R}$,*

$$u \in H^s_{\text{loc}}(\Omega),\ \Delta u = f \in H^{s-1}_{\text{loc}}(\Omega) \Longrightarrow u \in H^{s+1}_{\text{loc}}(\Omega). \tag{4}$$

Here, by definition,

$$u \in H^s_{\text{loc}}(\Omega) \Longleftrightarrow \varphi u \in H^s(M),\ \forall\, \varphi \in C_0^\infty(\Omega). \tag{5}$$

Proof. Assume the hypothesis in (4) holds. Take $\varphi \in C_0^\infty(\Omega)$, so $\varphi u \in H^s(M)$. Then

$$\Delta(\varphi u) = \varphi \Delta u + [\Delta, \varphi] u, \tag{6}$$

where $[\Delta, \varphi] = X_\varphi$ is a first-order differential operator, whose coefficients are supported in $\operatorname{supp} \varphi \subset\subset \Omega$. Hence

$$u \in H^s_{\text{loc}}(\Omega) \Longrightarrow X_\varphi u \in H^{s-1}(M). \tag{7}$$

We have

$$(\Delta - 1)(\varphi u) = \varphi \Delta u + X_\varphi u - \varphi u \in H^{s-1}(M), \tag{8}$$

so the isomorphism (3) implies $\varphi u \in H^{s+1}(M)$. This proves Proposition 1.

Note that applying L^{-1} to (8) yields the estimate

$$\begin{aligned} \|\varphi u\|_{H^{s+1}(M)} &\le C\|\varphi \Delta u\|_{H^{s-1}(M)} + C\|X_\varphi u - \varphi u\|_{H^{s-1}(M)} \\ &\le C\|\varphi \Delta u\|_{H^{s-1}(M)} + C\|\psi u\|_{H^s(M)}, \end{aligned} \tag{9}$$

provided $\psi \in C_0^\infty$ satisfies $\psi > 0$ on $\operatorname{supp} \varphi$.

Iterating this argument yields the following.

Proposition 2. *In the setting of Proposition 1, if* $s < r \in \mathbb{R}$*, then*

$$u \in H^s_{\mathrm{loc}}(\Omega),\ \Delta u = f \in H^{r-1}_{\mathrm{loc}}(\Omega) \Longrightarrow u \in H^{r+1}_{\mathrm{loc}}(\Omega), \tag{10}$$

and, for $\varphi, \psi \in C_0^\infty(\Omega),\ \psi > 0$ *on* $\operatorname{supp}\varphi$,

$$\|\varphi u\|_{H^{r+1}(M)} \le C\|\varphi\Delta u\|_{H^{r-1}(M)} + C\|\psi u\|_{H^s(M)}. \tag{11}$$

Proof. Given that $s < r$, the hypotheses of (10) yield

$$u \in H^{s+1}_{\mathrm{loc}}(\Omega), \tag{12}$$

as an application of (4). Now if $s + 1 \le r$, one can again apply (4), with s replaced by $s + 1$, to get $u \in H^{s+2}_{\mathrm{loc}}(\Omega)$. Continue until you get $u \in H^{s+\ell}_{\mathrm{loc}}(\Omega)$ and $s + \ell \ge r$. Then you can apply (4) one last time, with s replaced by r, to get $u \in H^{r+1}_{\mathrm{loc}}(\Omega)$. We leave it to the reader to carry out the estimate (11).

Reference

[T] M. Taylor, Partial Differential Equations, Vol. 1, Springer, NY, 1996 (2nd ed. 2011).